PDE'S - EXERCISES 21-27

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In all these exercises we use the abbreviations $H^s := H^s(\mathbb{R}^d)$, $\dot{H}^s := \dot{H}^s(\mathbb{R}^d)$, $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{C}_b := \mathscr{C}_b(\mathbb{R}^d)$, $\mathscr{C}_b^k := \mathscr{C}_b^k(\mathbb{R}^d)$ and $\mathscr{C}_c^\infty := \mathscr{C}_c^\infty(\mathbb{R}^d)$. Moreover, for $r > 0$ we define $B_r := \{x \in \mathbb{R}^d \mid |x| < r\}$ and $S^{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$.

1 EXERCISE 21:

From exercises 17 and 18 we deduce that $F \mapsto \phi_F$ is an antilinear map $H^s \to \mathscr{S}'(\mathbb{R}^d)$. We now need to show that for $s \geq 0$ this map is actually injective. It suffices to show that if $F \in H^s$ satisfies $\phi_F = 0$, then $F = 0$. Let $(f_k) \subset \mathscr{S}$ be a representative for F in H^s. First observe that $g_k := \mathscr{F}^{-1}\big[\langle . \rangle^{2s}\mathscr{F} f_k\big] \in \mathscr{S}$. A rather dull calculation shows $\|f_k - f_l\|_{H^s}^2 \underset{(l\to\infty)}{\longrightarrow} \|f_k\|_{H^s}^2 + \|F\|_{H^s}^2$. Indeed,

$$\|f_k - f_l\|_{H^s}^2 = \big(\langle . \rangle^s \mathscr{F}(f_k - f_l) \mid \langle . \rangle^s \mathscr{F}(f_k - f_l)\big) = \|f_k\|_{H^s}^2 + \|f_l\|_{H^s}^2 - 2\mathfrak{Re}\big(\langle . \rangle^s \mathscr{F} f_k \mid \langle . \rangle^s \mathscr{F} f_l\big) \quad (1)$$

$$= \|f_k\|_{H^s}^2 + \|f_l\|_{H^s}^2 - 2\mathfrak{Re}\big(\langle . \rangle^{2s} \mathscr{F} f_k \mid \mathscr{F} f_l\big) = \|f_k\|_{H^s}^2 + \|f_l\|_{H^s}^2 - 2\mathfrak{Re}\big(\mathscr{F}^{-1}\big[\langle . \rangle^{2s} \mathscr{F} f_k\big] \mid f_l\big) \quad (2)$$

$$= \|f_k\|_{H^s}^2 + \|f_l\|_{H^s}^2 - 2\mathfrak{Re}\big(g_k \mid f_l\big) \underset{(l\to\infty)}{\longrightarrow} \|f_k\|_{H^s}^2 + \|F\|_{H^s}^2 \quad (3)$$

where we have used that $\big(g_k \mid f_l\big) \to 0$ (which holds since $\phi_F = 0$). Using this we obtain

$$0 = \lim_k \lim_l \|f_k - f_l\|_{H^s}^2 = 2\|F\|_{H^s}$$

and therefore $F = 0$, as wanted.

2 EXERCISE 22:

Fix $s, t \in \mathbb{R}$ with $s \leq t$. We have to prove that there exists a linear embedding $H^t \hookrightarrow H^s$. Let $F \in H^t$ and pick some representative $(f_l) \subset \mathscr{S}$ for F. Clearly

$$\|f_l - f_k\|_{H^s} \leq \|f_l - f_k\|_{H^t}$$

and therefore the sequence (f_l) is also Cauchy in H^s and hence defines an element $\big[(f_l)\big]_{H^s} \in H^s$. If $(g_l) \subset \mathscr{S}$ is another representative of F, then of course

$$\|f_l - g_l\|_{H^s} \leq \|f_l - g_l\|_{H^t} \to 0$$

which shows that $\big[(f_l)\big]_{H^s} \in H^s$ is independent of the chosen representative for F. In particular, this yields that the map

$$\iota \colon H^t \to H^s \qquad F = \big[(f_l)\big]_{H^t} \mapsto \big[(f_l)\big]_{H^s}$$

is a well defined map. Of course ι is linear and it is continuous, since if $F = \big[(f_l)\big]_{H^t}$ with $(f_l) \subset \mathscr{S}$ Cauchy then

$$\|\iota(F)\|_{H^s} = \lim_l \|f_l\|_{H^s} \leq \lim_l \|f_l\|_{H^t} = \|F\|_{H^t}$$

In particular, if we view $\mathscr{S}$ as a subset of both H^s and H^t, then $\iota|_{\mathscr{S}}: \mathscr{S} \subset H^t \to \mathscr{S} \subset H^s$ is just the "identity". It remains to show that $\iota(F) = 0$ implies $F = 0$. But if $F \in H^t$ satisfies $\iota(F) = 0$, then we of course must also have $\phi_{\iota(F)} = 0$ and therefore for all $f \in \mathscr{S}$

$$0 = \phi_{\iota(F)}(f) = \lim_l (f \mid f_l) = \phi_F(f) \qquad (\star)$$

However, in exercise 21 we showed that the antilinear map

$$H^t \mapsto \mathscr{S}' \qquad G \mapsto \phi_G$$

is injective and from $(\star)$ we deduce $\phi_F = 0$. Thus $F = 0$, as wanted.

The second part of exercise 22 asks whether an analogous statement for homogeneous Sobolev spaces is true. The answer to this question is no and we verify this by a counterexample. It suffices to prove that the approximate inequality

$$\int |.|^{2s}|f|^2 \lesssim \int |.|^{2t}|f|^2$$

is not uniformly satisfied for $f \in \mathscr{S}$ (this is sufficient because then $||.||_{\dot{H}^s} \lesssim ||.||_{\dot{H}^t}$ cannot hold and thus the proof in the case of H^s respective H^t from before cannot be executed analogously and in particular we cannot find an embedding $\iota: \dot{H}^t \hookrightarrow \dot{H}^s$ such that $\iota(f) = f \in \dot{H}^s$ for all $f \in \mathscr{S} \subset \dot{H}^t$). To construct our counterexample: Let $(\varphi_l) \subset \mathscr{C}_c^\infty \subset \mathscr{S}$ be a sequence of bump functions such that $\operatorname{supp}(\varphi_l) \subset B_{1/l}$ and $\varphi_l|_{\overline{B_{1/(2l)}}} \equiv 1$ (we know that such a sequence exists from lectures on analysis). We then estimate

$$\int |.|^{2t}|\varphi_l|^2 \leq \int_{B_{1/l}} \frac{1}{l^{2t}} = \frac{\operatorname{Vol}(B_{1/l})}{l^{2t}} = \frac{\operatorname{Vol}(S^{d-1})}{l^{2t+d}d} \tag{4}$$

$$\int |.|^{2s}|\varphi_l|^2 \geq \int_{B_{1/2l}} |.|^{2s} = \operatorname{Vol}(S^{d-1}) \int_0^{1/2l} r^{2s+d-1}\mathrm{dr} = \left(\frac{1}{2l}\right)^{2s+d} \frac{\operatorname{Vol}(S^{d-1})}{2s+d} \tag{5}$$

and therefore we calculate

$$\frac{\int |.|^{2s}|\varphi_l|^2}{\int |.|^{2t}|\varphi_l|^2} \geq \frac{\int_{B_{1/l}} \frac{1}{l^{2t}}}{\int_{B_{1/2l}} |.|^{2s}} \simeq l^{2(t-s)} \underset{(l\to\infty)}{\longrightarrow} \infty \tag{6}$$

3 EXERCISE 23:

It is standard knowledge from functional analysis that the space $(\mathscr{C}_b, ||.||_\infty)$ is a Banach space. Suppose first that the dimension d is equal to 1. Now if we are given a sequence $(f_l) \subset \mathscr{C}_b^1$ satisfying

$$||f_l' - f_k'||_\infty + ||f_l - f_k||_\infty \to 0$$

then since $(f_l), (f_l') \subset \mathscr{C}_b$ we have that there exist unique functions $f, g \in \mathscr{C}_b$ such that $f_l \to f$ and $f_l' \to g$ in the uniform topology. A quick application of the fundamental theorem of calculus along with the fact that uniform convergence allows us to interchange limit and integral shows

$$f(x) \underset{(l\to\infty)}{\longleftarrow} f_l(x) = f_l(0) + \int_0^x f_l'(t)\,dt \underset{(l\to\infty)}{\longrightarrow} f(0) + \int_0^x g(t)\,dt$$

and therefore again by the FTC $f' = g$, which verifies completeness of $(\mathscr{C}_b^1, ||.||_{W^{1,\infty}})$ in the case for $d = 1$. Now let's get to the general statement where d and $k \geq 1$ are arbitrary. Let $(f_l) \subset \mathscr{C}_b^k$ be Cauchy with respect to the norm

$$||.||_{W^{k,\infty}} = \sum_{|\alpha|\leq k} ||\partial^\alpha .||_\infty$$

Of course we then must have that for every multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ the sequence $(\partial^\alpha f_l) \subset \mathscr{C}_b$ is Cauchy in $\mathscr{C}_b$ and thus it has a limit $g_\alpha \in \mathscr{C}_b$. Let $x = (x_1, ..., x_d) \in \mathbb{R}^d$. To make notation a tiny bit more convenient we write $(\bar{x},y) = (x_1,...,x_k,y_1,...,y_{d-j})$ for $y \in \mathbb{R}^{d-j}$ and $j \geq 1$. Similarily to what we did before we have by the FTC

$$f_l(x) = f_l(\bar{x},0) + \int_0^{x_d} \partial_d f_l(\bar{x},t_1)\,dt_1 = f_l(\bar{x},0) + x_d \partial_d f_l(\bar{x},0) + \int_0^{x_d}\int_0^{t_1} \partial_d^2 f_l(\bar{x},t_2)\,dt_2 dt_1 = ... \tag{7}$$

$$= \sum_{\nu=0}^{m-1} \frac{x_d^\nu}{\nu!} \partial_d^\nu f_l(\bar{x},0) + \int_0^{x_d}\int_0^{t_1} ... \int_0^{t_{m-1}} \partial_d^m f_l(\bar{x},t_m)\,dt_m...dt_1 \tag{8}$$

It follows immediately by the same arguments as before that sending $l \to \infty$ yields

$$f(x) = \sum_{\nu=0}^{m-1} \frac{x_d^\nu}{\nu!} g_{(0,...,0,\nu)}(\bar{x},0) + \int_0^{x_d}\int_0^{t_1} ... \int_0^{t_{m-1}} g_{(0,...,m)}(\bar{x},t_m)\,dt_m...dt_1$$

Thus $\partial^{(0,...,m)} f = g_{(0,...,m)}$. Taking an arbitrary multi-index $\alpha \in \mathbb{N}^d$ works analogously by an inductive procedure and therefore indeed $\partial^\alpha f = g_\alpha \in \mathscr{C}_b$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$. This proves that $(\mathscr{C}_b^k, ||.||_{W^{k,\infty}})$ is indeed a Banach space for every $k \geq 1$ and for every possible dimension $d \geq 1$.

4 EXERCISE 24:

Let $s > \frac{d}{2}$ and suppose $k \in \mathbb{N}$. We are asked to prove the Sobolev embedding $H^{s+k} \hookrightarrow \mathscr{C}_b^k$. Let $F \in H^{s+k}$ and suppose $(f_l) \subset \mathscr{S}$ is a representative for F in H^{s+k}. Now fix some multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$, then by means of Lemma 2.9 from the lecture we get

$$||\partial^\alpha (f_k - f_l)||_\infty \lesssim ||\partial^\alpha (f_k - f_l)||_{H^s} = ||\langle . \rangle^s \mathscr{F}(\partial^\alpha (f_k - f_l))||_2 \tag{9}$$

$$\simeq ||\langle . \rangle^s \xi^\alpha \mathscr{F}(f_k - f_l)||_2 \lesssim ||\langle . \rangle^{s+k} \mathscr{F}(f_k - f_l)||_2 = ||f_k - f_l||_{H^{s+k}} \tag{10}$$

Hence in total $||\partial^\alpha (f_k - f_l)||_\infty \lesssim ||f_k - f_l||_{H^{s+k}}$ and therefore $(\partial^\alpha f_l) \subset \mathscr{C}_b$ is Cauchy with respect to the uniform topology for all $\alpha \in \mathbb{N}$ with $|\alpha| \leq k$. In particular this means that $(f_l) \subset \mathscr{S} \subset \mathscr{C}_b^k$ is Cauchy with respect to the norm $||.||_{W^{k,\infty}}$ and by Exercise 23 we already know that the space $(\mathscr{C}_b^k, ||.||_{W^{k,\infty}})$ is a Banach space. Thus there exists $f \in \mathscr{C}_b^k$ such that (f_l) converges to f with respect to $||.||_{W^{k,\infty}}$. By means of this procedure define the map

$$\psi \colon H^{s+k} \to \mathscr{C}_b^k \qquad [(f_l)] \mapsto f$$

Of course ψ is well defined, for if $(f_l), (g_l) \subset \mathscr{S}$ are both representatives for $F \in H^{s+k}$, then $||f_l - g_l||_\infty \lesssim ||f_l - g_l||_{H^{s+k}} \to 0$. Moreover, ψ is a linear embedding. Linearity is trivial, so now let $F \in H^{s+k}$ be such that $\psi(F) = 0$. Let $(f_l) \subset \mathscr{S}$ be a representative for F, then $||f_l||_{W^{k,\infty}} \to 0$ (so in particular $||f_l||_\infty \to 0$) and thus for any $f \in \mathscr{S}$ we have

$$|\phi_{f_l}(f)| \leq ||f||_1 ||f_l||_\infty \to 0$$

This yields that $\phi_{f_k} \to 0 \in \mathscr{S}'$. However, $f_l \to F$ in H^{s+k} yields $\phi_{f_l} \to \phi_F$ in $\mathscr{S}'$ and therefore $\phi_F = 0$ (by uniqueness of limits). Making use of Exercise 21 we deduce $F = 0$, which concludes the proof.

5 EXERCISE 25:

Recall that the space of tempered distributions $\mathscr{S}'$ is the space of all those linear functionals $\phi : \mathscr{S} \to \mathbb{C}$ such that there exists $N \in \mathbb{N}$ so that

$$|\phi(f)| \lesssim \sum_{|\alpha| \leq N} ||\langle . \rangle^N \partial^\alpha f||_\infty$$

for all $f \in \mathscr{S}$. For $\alpha \in \mathbb{N}^d$ the distributional derivative $\partial^\alpha \phi$ of some distribution $\phi \in \mathscr{S}'$ evaluated at $f \in \mathscr{S}$ is given by

$$\partial^\alpha \phi(f) = (-1)^{|\alpha|} \phi(\partial^\alpha f)$$

We are asked to verify that $\partial^\alpha \phi \in \mathscr{S}'$. It is clear that $\partial^\alpha \phi$ is again linear. Moreover, for $f \in \mathscr{S}$ we have

$$|\partial^\alpha \phi(f)| = |\phi(\partial^\alpha f)| \lesssim \sum_{|\beta| \leq N} \|\langle . \rangle^N \partial^{\beta+\alpha} f\|_\infty \leq \sum_{|\beta| \leq N+|\alpha|} \|\langle . \rangle^{N+|\alpha|} \partial^\beta f\|_\infty$$

which verifies $\partial^\alpha \phi \in \mathscr{S}'$.

6 EXERCISE 26:

We already made use of the multi-index derivative for tempered distributions in the preceding exercise. The original definition given in the lecture was

$$\partial_j \phi(f) := -\phi(\partial_j f)$$

Now let $\alpha \in \mathbb{N}^d$, then we calculate

$$(\partial^\alpha \phi)(f) = (\partial_1^{\alpha_1} ... \partial_d^{\alpha_d} \phi)(f) = (-1)^{\alpha_1} (\partial_2^{\alpha_2} ... \partial_d^{\alpha_d} \phi)(\partial_1^{\alpha_1} f) = ... = (-1)^{|\alpha|} \phi(\partial^\alpha f)$$

7 EXERCISE 27:

Suppose $s > \frac{d}{2}$ and $g \in H^s \subset \mathscr{C}_b$. We are asked to show that

$$\phi_g(f) = (f \mid g)$$

for all $f \in \mathscr{S}$. We use the notation of exercise 24. By construction of the Sobolev embedding we know that if $F \in H^s$, then $\psi(F) \in \mathscr{C}_b$ is given by $\psi(F) = \lim f_l$ (limit is taken with respect to the uniform topology) for some representative $(f_l) \subset \mathscr{S}$ for F. But from that we easily obtain

$$|\phi_F(f) - (f \mid \psi(F))| = \lim |(f | f_l - f_l)| = 0$$

In other words

$$\phi_F = (. \mid \psi(F))$$

which is exactly what we wanted to prove.