PDE'S - EXERCISES 21-27

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In all these exercises we use the abbreviations $H^s := H^s(\mathbb{R}^d)$, $\dot{H}^s := \dot{H}^s(\mathbb{R}^d)$, $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{C}_b := \mathscr{C}_b(\mathbb{R}^d)$, $\mathscr{C}_b^k := \mathscr{C}_b^k(\mathbb{R}^d)$ and $\mathscr{C}_c^\infty := \mathscr{C}_c^\infty(\mathbb{R}^d)$. Moreover, for $r > 0$ we define $B_r := \{x \in \mathbb{R}^d \mid |x| < r\}$ and $S^{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$.

1 EXERCISE 28:

We are asked to verify that the distributional derivative of an H^1 function is an L^2 function. First of all this certainly doesn't refer to H^1 or L^2 elements per se, but rather their embeddings into the space of tempered distributions. Put differently, if $F \in H^1$ then we want to verify that for an arbitrary derivative $\partial_j \phi_F$ we can find $G \in L^2$ such that $\partial_j \phi_F = \phi_G$. So let $F \in H^1$ and pick some representative $(f_l) \subset \mathscr{S}$ for F. Then, by definition, we have for all Schwarz functions $f \in \mathscr{S}$ that

$$\partial_j \phi_F(f) = -\phi_F(\partial_j f) = \lim_l -(\partial_j f \mid f_l) = \lim_l (f \mid \partial_j f_l) \qquad (\Delta)$$

However, we also have

$$\|\partial_j f_l - \partial_j f_k\|_2 \lesssim \|f_l - f_k\|_{H^1} \to 0$$

and therefore $(\partial_j f_l) \subset \mathscr{S}$ defines a unique element $G \in L^2$ (it is clear that G is independent of the chosen representative for F). By construction and by considering (Δ) we immediately get

$$\partial \phi_F = \phi_G$$

as wanted.

2 EXERCISE 29:

We are asked to prove the convolutional inequalities

$$\|f * g\|_\infty \le \|f\|_p \|g\|_q \qquad \|f * g\|_p \le \|f\|_1 \|g\|_q \qquad (\star)$$

for all $f, g \in \mathscr{S}$ and $p, q \ge 1$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Recall that the convolution of $f, g \in \mathscr{S}$ is defined by

$$f * g(x) := \int f(x-y) g(y)\, dy$$

Clearly the convolution operator is symmetric, i.e. $f * g = g * f$ for all $f, g \in \mathscr{S}$. The first of the inequalities in $(\star)$ is pretty straightforward. Indeed,

$$\|f * g\|_\infty \le \sup_x \int |f(x-y)||g(y)|\, dy \le \|f\|_p \|g\|_q$$

For the second inequality we first estimate

$$|f * g(x)| \le \int |f(y)|^{1/q} |f(y)|^{1/p} |g(x-y)|\, dy \underset{\text{Hölder}}{\le} \|f\|_1^{1/q} \left(\int |f(y)||g(x-y)|^p\, dy \right)^{1/p}$$

Using this we obtain

$$\|f * g\|_p \le \|f\|_1^{1/q} \left(\int \int |f(y)||g(x-y)|^p\, dy dx \right)^{1/p} \underset{\text{Fubini}}{=} \|f\|_1^{1/q} \left(\int |f(y)|\, dy \int |g(x)|^p dx \right)^{1/p}$$

Thus $\|f * g\|_p \le \|f\|_1 \|g\|_p$, as wanted.

3 EXERCISE 30:

We have to prove that for every pair $f,g \in \mathscr{S}$ and every multi-index $\alpha \in \mathbb{N}^d$ we have

$$\partial^\alpha(f*g) = (\partial^\alpha f)*g$$

It suffices to prove $\partial_j(f*g) = (\partial_j f)*g$ for all $j = 1,...,d$. In order to show this we simply calculate

$$\frac{(f*g)(x+e_jh)-(f*g)(x)}{h} = \frac{1}{h}\left(\int f(y)g(x+he_j-y)\,dy - \int f(y)g(x-y)\,dy\right) \tag{1}$$

$$\underset{(z=y-he_j)}{=} \frac{1}{h}\left(\int f(z+he_j)g(x-z)\,dz - \int f(y)g(x-y)\,dy\right) \tag{2}$$

$$= \int g(x-z)\left(\frac{f(z+he_j)-f(z)}{h}\right)dz \underset{(h\to 0)}{\longrightarrow} (\partial_j f)*g(x) \tag{3}$$

where the last step is justified by the dominated convergence theorem.

4 EXERCISE 31:

We need to show that for $f,g \in \mathscr{S}$ we have $f*g \in \mathscr{S}$. First note that a function $h \in \mathscr{C}^\infty$ satisfies $h \in \mathscr{S}$ if and only if for all natural numbers $N \in \mathbb{N}$ we have

$$\|h\|_{(N)} := \max_{|\alpha|\le N} \sup_{x\in\mathbb{R}^d} \left|(1+\|x\|)^N \partial^\alpha h(x)\right| < \infty \qquad (\star\star)$$

We have seen in the previous exercise that $f*g \in \mathscr{C}^\infty$ and that $\partial^\alpha(f*g) = (\partial^\alpha f)*g$. We thus infer by $(\star\star)$ that it is enough to prove $\|f*g\|_{(N)} < \infty$ for every natural number $N \in \mathbb{N}$. However, this is not too hard:

$$|(1+\|x\|)^N f*g(x)| \le \int (1+\|x\|)^N |f(x-y)||g(y)|dy \tag{4}$$

$$\underset{1+\|x\|\le(1+\|x-y\|)(1+\|y\|)}{\le} \int (1+\|x-y\|)^N |f(x-y)|(1+\|y\|)^N|g(y)|dy \tag{5}$$

$$\le \|f\|_{(N)} \int (1+\|y\|)^N|g(y)|dy \tag{6}$$

$$\le \|f\|_{(N)}\|g\|_{(N+n)} \int \frac{dy}{(1+\|y\|)^n} < \infty \tag{7}$$

where n is some natural number big enough such that $\int \frac{dy}{(1+\|y\|)^n} < \infty$ (any $n > d$ works). The above estimate is independent of the variable x and thus

$$\sup_{x\in\mathbb{R}^d} |(1+\|x\|)^N(f*g)(x)| < \infty$$

for arbitrary $f,g \in \mathscr{S}$. Thus in particular, since we can just take the maximum over the finitely many finite numbers $\sup_{x\in\mathbb{R}^d} |(1+\|x\|)^N(\partial^\alpha f*g)(x)|$, we obtain

$$\|f*g\|_{(N)} < \infty$$

and therefore by $(\star\star)$ $f*g \in \mathscr{S}$ as wanted.