LOCALLY CONVEX SPACES

ALEXANDER ZAHRER

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ABSTRACT

I can illustrate the second approach with the same image of a nut to be opened. The first analogy which came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise you let time pass. The shell becomes more flexible through weeks and months – when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado!

A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration… the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it… yet it finally surrounds the resistant substance.

(Alexander Grothendieck, Récoltes et semailles, 1985–1987, pp. 552-3-1 The Rising Sea)

1 TOPOLOGICAL VECTOR SPACES - BASICS

We recall that if E is a TVS, and $\mathscr{U}$ is a basis of the filter of 0-neighborhoods then for all $z \in E$ and $\lambda \neq 0$ the set $\{z + \lambda U \mid U \in \mathscr{U}\}$ is a basis of the filter of neighborhoods of z. This is indeed pretty obvious as it is quite frankly implied by the linear structure and the topological compatibility given in the definition of a TVS. Now a handy corollary of this result is that if $f\colon E \to F$ is a map between to TVS, then f is continuous if and only if f is continuous at 0. Indeed, if $z \in E$ then let $U \in \mathscr{U}$ be a 0-neighborhood such that $f(U) \subset V$ (V is some 0-neighborhood in F). Of course we then have $f(z+U) \subset f(z) + V$, as wanted. Some more subtle fact is the following:

Corollary 1. *If $\mathscr{U}$ is a 0-neighborhood of a TVS E, then for every nonempty subset $A \subset E$ we have*

$$\overline{A} = \bigcap \{A + U \mid U \in \mathscr{U}\} \tag{1}$$

Proof. We have that $x - \mathscr{U}$ is a basis of the neighborhood filter of $x \in E$ and hence $x \in \overline{A} \iff U \in \mathscr{U}\colon (x-U) \cap A \neq \emptyset \iff \forall U \in \mathscr{U}\colon x \in A + U$. □

Corollary 2. *If $\mathscr{U}$ is a 0-basis of a TVS E, then so is $\{\overline{U} \mid U \in \mathscr{U}\}$.*

Proof. For $U \in \mathscr{U}$, by continuity of addition there is some $V \in \mathscr{U}$ such that $V + V \subset U$. However, by the preceding corollary $\overline{V} \subset V + V$, so $\overline{V} \subset U$. □

So to summarize this, if we are given a 0-filter, then the closures of the sets in the filter also yield a 0-filter, which is quite a nice thing to have. We now get to one of the most important notions of the whole topic, namely balanced and absorbent subsets of a TVS. A subset $A \subset E$ is called

- balanced, if $\mathbb{D}A \subset A$ (this already implies $\mathbb{D}A = A$).
- absorbent, if $\forall x \in E\colon \exists \lambda_0 > 0\colon \forall |\lambda| \geq \lambda_0\colon x \in \lambda A$.

Of course arbitrary intersections of balanced sets are balanced and for every $A \subset E$ there exists a smallest balanced set $\breve{A}$ which contains A. Obviously, $\breve{A} = \mathbb{D}A$. Moreover, we will often consider the closed, balanced hull $\overline{\breve{A}}$ of $\breve{A}$. This is indeed the smallest closed, balanced set containing A, for if $A \subset M$ and M is closed and balanced, then $\breve{A} \subset \breve{M} = M$ and thus $\overline{\breve{A}} \subset \overline{M} = M$ and we certainly have $\overline{\breve{A}}\mathbb{D} = \overline{A\mathbb{D}\mathbb{D}} = \overline{A\mathbb{D}}$, as scalar multiplication is a homeomorphism. We now have the following proposition:

Proposition 1. *If $A \subset E$ is balanced, then $\overline{A}$ is balanced and if $0 \in A$ then also $int(A)$ is balanced.*

Proof. Frankly, $\mathbb{D}A \subset A$ implies $\mathbb{D}\overline{A} \subset \overline{A}$ by continuity of multiplication. Now for $\zeta \in \mathbb{K}\backslash 0$ we have $\zeta \operatorname{int}(A) = \operatorname{int}(\zeta A) \subset A$, which gives the claim. □

So all we need for $\operatorname{int}(A)$ to also be balanced is the condition that $0 \in \operatorname{int}(A)$, as for this particular case we cannot make us of the homeomorphism property. Now let's try to combine these observations and mix in some topological considerations in order to investigate further what really constitutes filters of neighborhoods for a TVS.

Corollary 3. *If $\mathscr{U}$ is a 0-basis in E then $\{int(\breve{U}) \mid U \in \mathscr{U}\}, \{\breve{U} \mid U \in \mathscr{U}\}$ and $\{\overline{\breve{U}} \mid U \in \mathscr{U}\}$ are 0-bases in E.*

One could potentially be cheeky here and say that this is what makes the theory of topological vector spaces such a balanced endeavour.

Proof. Let $U \in \mathscr{U}$ and choose some $U_0 \in \mathscr{U}$ such that $\overline{U_0} \subset U$. Now as scalar multiplication is continuous there is some $\lambda_0 > 0$ and $V \in \mathscr{U}$ such that $\mathbb{D}\lambda_0 V \subset U_0$. Now define $V_0 := \lambda_0 V$, then by construction $\breve{V_0} \subset U_0$. Now let $W \in \mathscr{U}$ be such that $W \subset V_0 = \lambda_0 V$, then we have

$$\operatorname{int}(W) \subset \operatorname{int}(\breve{W}) \subset \breve{W} \subset \overline{\breve{W}} \subset \overline{U_0} \tag{2}$$

□

In particular this yields that every 0-neighborhood of a TVS is absorbent. Indeed, if U is a 0-neighborhood in a TVS E, then for fixed $x \in E$ the map $\lambda \mapsto \lambda x$ is continuous at 0, so there is $\lambda_0 > 0$ such that $\lambda x \in U$ for all $|\lambda| \leq \lambda_0$. Hence, by inversion, for all $\lambda \geq \lambda_0^{-1}$ one has $x \in \lambda U$. We now try to give the minimal requirements on a filter basis such that it generates a linear topology.

Theorem 1. *In a TVS E there exists a 0-basis $\mathscr{U}$ such that*

1. *every $U \in \mathscr{U}$ is both balanced and absorbent,*
2. *for every $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $V + V \subset U$.*

Conversely, if $\mathscr{U}$ is a filter basis on a vector space E satisfying these properties, then this gives rise to a unique topology $\mathscr{T}$ on E which has $\mathscr{U}$ as 0-basis.

Clearly we have already established the first part of the theorem. However, the second part is more tricky. Uniqueness of the linear topology follows by uniqueness of its 0-neighborhood filter. Hence we have to show that the family

$$\mathscr{T} = \{A \subset E \mid \forall x \in A \exists U \in \mathscr{U} : x + U \subset A\} \tag{3}$$

indeed gives rise to a linear topology having $\mathscr{U}$ as a 0-basis. It is easily verified that $\mathscr{T}$ is a topology. For any $(x,y) \in E \times E$ and $U \in \mathscr{U}$ we may choose $V \in \mathscr{U}$ such that $V + V \subset U$. But then $(x+V)+(y+V) \subset (x+y)+U$, and hence addition is continuous. The hard part comes now: For $(\zeta, x) \in \mathbb{K} \times E$ and $U \in \mathscr{U}$ we want that for $\lambda - \zeta$ and $y - x$ in suitable neighborhoods of $\mathbb{K}$ and E, respectively, we have

$$\lambda y - \zeta x = (\lambda - \zeta)(y - x) + (\lambda - \zeta)x + \zeta(y - x) \in U \tag{4}$$

We then set $V_0 := U$ and choose a sequence $(V_k)_{k \in \mathbb{N}}$ in $\mathscr{U}$ such that $V_k + V_k \subset V_{k-1}$ for all k. Now let $n \in \mathbb{N}$ be such that $2^n - 2 \geq |\zeta|$ and $0 < \sigma \leq 1$ such that $[0,\sigma]x \subset V_n$ (which is possible since V_n is absorbent). Then for $\lambda - \zeta \in \sigma\mathbb{D}$ and $y - x \in V_n$ we have

$$(\lambda - \zeta)(y - x) \in V_n \tag{5}$$

$$(\lambda - \zeta)x \in V_n \tag{6}$$

$$\zeta(y - x) \in (2^n - 2)V_n \subset V_n + \ldots + V_n \tag{7}$$

and hence

$$\lambda y - \zeta x \in \underbrace{V_n + \ldots + V_n}_{2^n - 2} + V_n + V_n \subset \ldots \subset V_0 = U \tag{8}$$

So multiplication is continuous. Lastly every 0-neighborhood contains and element of $\mathscr{U}$ by definition. In order to show that each $U \in \mathscr{U}$ is a 0-neighborhood in $\mathscr{T}$, set

$$M := \{x \in U \mid x + V \subset U, V \in \mathscr{U}\} \tag{9}$$

Given $x \in M$ choose $V, W \in \mathscr{U}$ such that $x + V \subset U$ and $W + W \subset V$. Then $x + W + W \subset U$ and thus $x + W \subset M$, hence $M \in \mathscr{T}$. Because $0 \in M \subset U$, U is a 0-neighborhood in $\mathscr{T}$.

Proposition 2. *A TVS E is Hausdorff if and only if for every $x \neq 0$ there exists a neighborhood $U \in \mathscr{U}$ which does not contain x, i.e. if*

$$\{0\} = \bigcap \{U \mid U \in \mathscr{U}\} \tag{10}$$

In particular, by previous observations, a TVS is Hausdorff if and only if $\{0\}$ is closed in E.

Proof. If E is assumed to be Hausdorff and $x \neq 0$, then there are 0-neighborhoods $U, V \in \mathscr{U}$ such that $U \cap (x + V) = \emptyset$. Hence $x \notin U$. Conversely, let $x \neq y$. Then there is a 0-neighborhood U such that $x - y \notin U$. Now choose a 0-neighborhood V such that $V - V \subset U$. Suppose by way of contradiction that $(x+V) \cap (y+V) \neq \emptyset$; then $x + x' = y + y'$ for $x', y' \in V$ and hence $x - y = y' - x' \in V - V \subset U$, a contradiction. □

2 LOCALLY CONVEX SPACES

The concept of convexity, which will also be crucial, is quite a natural one: A subset $A \subset E$ is called convex, if for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have $\alpha A + \beta A \subset A$. For a linear map $f\colon E \to F$ the sets $f(A)$ and $f^{-1}(B)$ are convex, given that $A \subset E$ and $B \subset F$ are assumed to be convex. Of course we have a notion of a convex hull. In particular, if A is a convex subset of a vector space and $x_1, ..., x_n \in A$, $\lambda_1, ..., \lambda_n \geq 0$ such that $\sum \lambda_i = 1$, then $\sum \lambda_i x_i \in A$. The trick to showing this is a straightforward inductive argument. Indeed,

$$\sum_{i=1}^{n} \lambda_i x_i = \alpha\Big(\sum_{i=1}^{n-1} \frac{\lambda_i}{\alpha} x_i\Big) + \lambda_n x_n \tag{11}$$

is a convex combination with $\alpha := \sum_{i=1}^{n-1} \lambda_i$. In particular, it is quite easy to prove that if (A_ι) is a family of convex subsets of a vector space, then the convex hull of $\bigcup A_\iota$ is simply given by

$$\left\{ \sum \lambda_\iota x_\iota \,\middle|\, x_\iota \in A_\iota, \lambda_\iota \geq 0, \sum \lambda_\iota = 1, \lambda_\iota \neq 0 \text{ for only finitely many } \iota \right\} \tag{12}$$

Using this we also easily see that the convex hull of A is the set of all finite linear combinations $\sum \lambda_i x_i$, where $x_i \in A$ and $\lambda_i \geq 0$, $\sum \lambda_i = 1$. Now it is quite easily verified that if A is convex, then its closure $\overline{A}$ is convex as well. The central topic of the subject is the concept of locally convex vector spaces:

Definition 1. A TVS is called LCS if it has a 0-basis consisting of convex sets.

Proposition 3. *A LCS has a 0-basis consisting of balanced convex closed sets.*

Proof. Let W be an arbitrary 0-neighborhood. We have already seen that it contains a closed 0-neighborhood V, which in turn by local convexity, contains a convex 0-neighborhood U. Certainly, $\overline{U} \subset V$ is a closed and convex 0-neighborhood. Now set $B := \bigcap_{|\lambda| \geq 1} \lambda \overline{U}$, which is closed.

- B is a 0-neighborhood: By continuity of multiplication there is $\alpha > 0$ and a 0-neighborhood X such that $\alpha \mathbb{D} X \subset \overline{U}$. Now αX is a 0-neighborhood and $\alpha X \subset B$, since for $|\lambda| \geq 1$ and $x \in X$ we have

$$\underbrace{\lambda^{-1}}_{\in \mathbb{D}} \alpha x \in \alpha \mathbb{D} X \subset \overline{U} \tag{13}$$

- B is balanced: Let $|\mu| \leq 1$, $x \in B$. Then for $|\lambda| \geq 1$ we have $x \in \frac{\lambda}{\mu} \overline{U}$, so $\mu x \in \lambda \overline{U}$ for all $|\lambda| \geq 1$, i.e. $\mu x \in B$.

□

Proposition 4. *Let E be a vector space and $\mathscr{B}$ be a filter basis on E consisting of absorbent, balanced, convex sets. Let $\mathscr{N} := \{\lambda V \mid \lambda > 0, V \in \mathscr{B}\}$. Then there exists a unique topology on E for which E is a locally convex space and which has $\mathscr{N}$ as a* 0*-basis.*

Proof. This is actually quite easy, considering what we have already proven. Indeed, note that for $V \in \mathscr{N}$ we have $\frac{1}{2} V \in \mathscr{N}$ and $\underbrace{\frac{1}{2} V + \frac{1}{2} V}_{=V \text{ by convexity}} \in \mathscr{N}$. Hence by a previous theorem the result follows. □

Corollary 4. *Let E be a vector space and let $\mathfrak{S}$ be a family of absorbent, balanced, convex subsets of E. Let $\mathscr{B}$ be the collection of finite intersections of sets of the form λV, $\lambda > 0$, $V \in \mathfrak{S}$. Then there exists a unique topology on E for which E is a locally convex space and which has $\mathscr{B}$ as a* 0*-basis.*

For a seminorm p we certainly have that the open unit ball and the closed unit ball are balanced, absorbent and convex sets. The most important explicit example of a seminorm we will consider is the Minkowski functional, or the gauge of a set $A \subset E$. For $A \subset E$ define

$$g_A \colon E \to \mathbb{R}_+ \cup \{\infty\} \qquad x \mapsto \inf\{\lambda > 0 \mid x \in \lambda A\} \tag{14}$$

If $0 \in A$, then $g_A(0) = 0$. If A is absorbent, then g_A is finite. If A is balanced, then $g_A(\lambda x) = |\lambda| g_A(x)$ and if A is convex, then g_A even satisfies the triangle inequality. Indeed, the first claim is obvious. If A is balanced, then we have for $\lambda \in \mathbb{D} \backslash 0$ that

$$g_A(\lambda x) = \inf\{\mu > 0 \mid \lambda x \in \mu A\} = \inf\{\mu > 0 \mid |\lambda| x \in \mu A\} = \inf\{\mu > 0 \mid x \in |\lambda|^{-1} \mu A\} = |\lambda| g_A(x) \tag{15}$$

If A is convex, then for all $\alpha, \beta \geq 0$ we have $(\alpha + \beta)A = \alpha A + \beta A$ and then if $\varepsilon > 0$ there are $\zeta, \sigma > 0$ such that

$$g_A(x) \leq \zeta < g_A(x) + \varepsilon \qquad x \in \zeta A \tag{16}$$
$$g_A(y) \leq \sigma < g_A(y) + \varepsilon \qquad y \in \sigma A \tag{17}$$

hence $g_A(x+y) \leq \zeta + \sigma < g_A(x) + g_A(y) + 2\varepsilon$. Hence the gauge of an absorbent, convex, balanced set is a seminorm with

$$p_{<1} \subset A \subset p_{\leq 1} \tag{18}$$

A seminorm, in general, satisfies

$$\operatorname{int}(p_{\leq 1}) \subset p_{<1} \subset p_{\leq 1} \subset \overline{p_{<1}} \tag{19}$$

which is not hard to show. For a continuous seminorm even more is true; $p_{<1}$ is open, $p_{\leq 1}$ is closed, $\overline{p}_{<1} = p_{\leq 1}$ and $\operatorname{int}(p_{\leq 1}) = p_{<1}$. Moreover, the following equivalences for a seminorm on a TVS are easily established:

1. $p_{<1}$ is open
2. $p_{\leq 1}$ is closed
3. p is continuous at 0
4. p is continuous

We call an absorbent, balanced, closed, convex subset a barrel. Barrels are quite characteristic of LCS as every LCS has a 0-basis consisting of barrels, as we will see soon.

Proposition 5. *Let A be a barrel in a TVS E. Then there is a unique seminorm p on E whose closed unit ball equals A.*

For this proposition the most reasonable guess to the desired seminorm is simply the gauge of the barrel A. Indeed, if p is the gauge of A, then $p_{<1} \subset A \subset p_{\leq 1} \subset \overline{p_{<1}}$. Taking the closure of that chain of inclusions leads to a chain of equalities, thus $A = p_{\leq 1}$. In particular, we observe that the Minkowski functional of a barrel must always be continuous.

The next step is to relate the theory of seminorms with that of locally convex spaces. If $\mathscr{P}$ is a family of seminorms on E, then by some corollary the family of finite intersections of the sets $\varepsilon p_{\leq 1}$, with $\varepsilon > 0$ and $p \in \mathscr{P}$, gives rise to a locally convex topology $\mathscr{T}$ on E, having as 0-basis the sets

$$\{x \mid p_k(x) \leq \varepsilon_k, 1 \leq k \leq n\} \tag{20}$$

A locally convex topology can indeed always be defined in that manner: We already know that if E is a LCS, then it has a 0-basis of barrels V. Each gauge g_V is a continuous seminorm on E, and the locally convex topology defined by $(g_V)_V$ certainly agrees with the original topology. It is also noteworthy that a LCS is Hausdorff if and only if for all $x \neq 0$ there exists $p \in \mathscr{P}$ such that $p(x) \neq 0$. Another nice characterisation is that if $f \colon E \to F$ is a linear map between LCS, then f is continuous if and only if for all continuous seminorms q on F there exists a continuous seminorm p on E such that $q \circ f \leq p$.

Definition 2. Let $\mathscr{P}$ be a family of continuous seminorms on E. We say that $\mathscr{P}$ is a basis of continuous seminorms on E if for every continuous seminorm p on E there exists $p' \in \mathscr{P}$ and $C > 0$ such that $p \leq Cp'$.

Proposition 6. *Let E,F be LCS with bases of continuous seminorms $\mathscr{P}$ and $\mathscr{Q}$, respectively, $f\colon E \to F$ a linear mapping. Then f is continuous if and only if for every $q \in \mathscr{Q}$ there is a seminorm $p \in \mathscr{P}$ and $C > 0$ such that $q \circ f \leq Cp$.*

Balanced convex sets are certainly absolutely convex sets and vice versa. The balanced convex hull of $\bigcup A_i$ (where each A_i is absolutely convex) is

$$\left\{ \sum \lambda_i x_i \middle| \sum |\lambda_i| \leq 1, x_i \in A_i \right\} \tag{21}$$

3 COMPLETENESS

Let $A \subset E$. A filter $\mathscr{F}$ on A is a family of subsets of A such that

- $A \in \mathscr{F}, \emptyset \notin \mathscr{F}$,
- $F, G \in \mathscr{F} \implies F \cap G \in \mathscr{F}$,
- $F \in \mathscr{F}, G \subset F \subset A \implies G \in \mathscr{F}$

A filter (or a filter basis) on A is called a Cauchy filter if for every 0-neighborhood U in E there exists some $F \in \mathscr{F}$ such that $F - F \subset U$. To say that a filter $\mathscr{F}$ converges to some $x \in E$ means that for each neighborhood U of x there is some $F \in \mathscr{F}$ such that $F \subset U$. A sequence in A is called Cauchy if for every 0-neighborhood U there is some natural number N such that $\forall n, m \geq N$ we have $x_n - x_m \in U$. The notion of completeness is now an obvious one: A subset $A \subset E$ is called complete if every Cauchy filter on A has a limit in A and it is called sequentially complete if every Cauchy sequence in A has a limit in A.

Proposition 7. *Let E be a Hausdorff TVS. If $A \subset E$ is complete then it is closed.*

This is rather easily proven. Indeed, for $x \in \overline{A}$ let $\mathscr{F}$ be the neighborhood filter of x. Its trace $\mathscr{F}_A$ on A is a filter on A which is even a Cauchy filter (If U is a 0-neighborhood in E then there exists $F \in \mathscr{F}$ such that $F - F \subset U \implies F \cap A \in \mathscr{F}_A, (F \cap A) - (F \cap A) \subset U$). Hence the filter on E generated by $\mathscr{F}_A$ is a Cauchy filter and by completeness converges to some element of A. Because this filter is finer than $\mathscr{F}$ and $\mathscr{F} \to x$ (we also use the fact that E is Hausdorff), we have $x \in A$.

Another useful fact is that closed subsets $B \subset A$ of a complete set $A \subset E$ are also automatically complete. Indeed, let $B \subset A$ be closed and let $\mathscr{F}$ be a Cauchy filter on B. $\mathscr{F}$ generates a Cauchy filter on A, say $\mathscr{F}'$. $\mathscr{F}' \to x$ for some $x \in A$ by completeness. As x is thus a limit point of $\mathscr{F}$ we must have by closedness that $x \in B$ and $\mathscr{F} \to x$.

Definition 3. Let $A \subset E$. A mapping $f\colon A \to F$ (where E is a top. space and F is a TVS) is called uniformly continuous if for every 0-neighborhood U in F there exists a 0-neighborhood V in E such that whenever $x, y \in A$ with $x - y \in V$ implies $f(x) - f(y) \in U$.

It is clear that if a linear map $f\colon E \to F$ between TVS is continuous at 0, then it is uniformly continuous.

Lemma 1. *Let E,F be TVS, $A \subset E$ and $f\colon A \to F$ uniformly continuous. If $\mathscr{F}$ is a Cauchy filter on A, then $f(\mathscr{F})$ is a Cauchy filter basis in F.*

Proof. If U is a 0-neighborhood in F then there exists a 0-neighborhood V in E such that whenever $x - y \in V$ we have $f(x) - f(y) \in U$. Now there is $G \in \mathscr{F}$ such that $G - G \subset V$ and hence $f(G) - f(G) \subset U, f(G) \in f(\mathscr{F})$. □

It is easily shown that if M is a linear subspace of E then its closure $\overline{M}$ is a linear subspace too. In particular, uniform continuity surely implies continuity. Indeed, if U is a 0-neighborhood in F there is a 0-neighborhood in E such that $x - y \in V$ implies $f(x) - f(y) \in U$. Given $x \in A$, $(x + V) \cap A$ is a neighborhood of x in A and $(x + V) - x \subset V$ implies $f(x + V) - f(x) \subset U$, or equivalently, $f(x + V) \subset U + f(x)$.

Proposition 8. *Let E be a TVS, F a complete Hausdorff TVS, and $A \subset E$. If $f: A \to F$ is uniformly continuous then there exists a unique uniformly continuous map $\widehat{f}: \overline{A} \to F$ such that $\widehat{f}|_A = f$. In particular, if A is a linear subspace and f is linear, then $\widehat{f}$ is linear.*

The proof of this proposition is rather tedious, however, the idea is relatively simple: For $x \in \overline{A}$, let $\mathscr{F}$ be its neighborhood filter in E. Then its trace $\mathscr{F}_A$ is a Cauchy filter and thus $f(\mathscr{F}_A)$ is a Cauchy filter basis in F and hence converges to some unique $y \in F$. One then defines $\widehat{f}(x) := y$ and verifies that this indeed agrees with f on A and that it is uniformly continuous.

Definition 4. A Cauchy filter $\mathscr{F}$ on E is called minimal if for any Cauchy filter $\mathscr{G}$ on E, $\mathscr{G} \subset \mathscr{F}$ implies $\mathscr{G} = \mathscr{F}$.

Now one notes that if E is a TVS and $\mathscr{F}$ is a Cauchy filter on E, then there exists a unique minimal Cauchy filter $\mathscr{F}_0$ on E such that $\mathscr{F}_0 \subset \mathscr{F}$. If $\mathscr{B}$ is a basis of $\mathscr{F}$ and $\mathscr{U}$ is a 0-basis in E, $\{F+V \mid F \in \mathscr{B}, V \in \mathscr{U}\}$ is a basis of $\mathscr{F}_0$. Skipping some details here, one arrives at:

Theorem 2. *Let E be a Hausdorff TVS. There exists a complete Hausdorff TVS $\widehat{E}$, called the completion of E, and a mapping $\iota: E \to \widehat{E}$ such that*

1. *ι is a linear homeomorphism onto its image,*
2. *$\iota(E)$ is dense in $\widehat{E}$.*

For any other pair (E_1, ι_1) such that these properties hold there is a linear isomorphism $j: \widehat{E} \to E_1$ such that $\iota_1 = j \circ \iota$. In other words, the completion is unique up to isomorphism.

The gist of the proof is to define $\widehat{E}$ to be the set of all minimal Cauchy filters on E. We can then define a canonical linear structure. From the topological perspective, if U is a 0-neighborhood in E, then we set

$$\widehat{U} := \{\mathscr{F} \in \widehat{E} \mid \exists A \in \mathscr{F} \exists V \in \mathscr{U} : A+V \subset U\} \tag{22}$$

One shows that this gives rise to a linear Hausdorff topology on $\widehat{E}$. Next one defines the embedding $\iota: E \to \widehat{E}$ which sends $x \in E$ to its neighborhood filter. This is injective and linear and one even verifies that it is a homeomorphism if restricted to its image. Finally, one shows that $\widehat{E}$ is complete. Uniqueness up to isomorphism is then clear.

Proposition 9. *Let E be a dense subset of a TVS F. If $\mathscr{U}$ is a 0-basis in E then $\{\overline{U} \mid U \in \mathscr{U}\}$ is a 0-basis in F.*

It is then nice to know that the completion of an LCS is always an LCS too. Moreover, if a family $\mathscr{P}$ of seminorms defines the topology of a LCS E, the family $\{\widehat{p} \mid p \in \mathscr{P}\}$ defines the topology on $\widehat{E}$.

REFERENCES