PDE'S - THE OMINOUS EXERCISE 43

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1 SOME QUICK WORDS:

This solution has taken quite some time to devise. The crucial hint, given to me by Irfan, to "vectorize" the problem was key to solve the exercise. This is because it yields a much stronger version of Meyers Serrin, and it thus basically minimized the difficulty of the exercise tremendously. Without further ado, here is the solution:

2 EXERCISE 43:

We are given smooth bounded functions $a^{jk} \in \mathscr{C}_b^\infty$ such that the operator $a^{jk}\partial_j\partial_k$ is elliptic. Moreover, we may assume that there exists a distributional solution $F \in L^2$ to the PDE

$$a^{jk}\partial_j\partial_k\phi_F = \phi_G \tag{1}$$

where $G \in H^1$. We now prove that this already implies that $F \in H^2$ (i.e. $F \in L^2$ has a representative $(f_l) \subset \mathscr{S}$ such that (f_l) is also Cauchy in H^2). In order to prove this statement we first need to improve Theorem 1.7 from the lecture notes.

Theorem 1. *Let $a^{jk}, b^j, c \in \mathscr{C}_b^\infty$ and assume that the operator $a^{jk}\partial_j\partial_k + b^j\partial_j + c$ is elliptic and that $F \in L^2$ satisfies*

$$(a^{jk}\partial_j\partial_k + b^j\partial_j + c)\phi_F = \phi_G \tag{2}$$

for some $G \in L^2$. Then $F \in H^1$, i.e. $F \in L^2$ has a representative that is also Cauchy in H^1.

Proof. By Meyers-Serrin (with non-constant coefficients) F has a representative $(f_l) \subset \mathscr{S}$ which satisfies

$$\|(a^{jk}\partial_j\partial_k + b^j\partial_j + c)(f_l - f_k)\|_{L^2} \longrightarrow 0 \tag{3}$$

Now by ellipticity we obtain

$$\gamma\|f\|_{\dot{H}^1}^2 \leq \left|(a^{jk}\partial_j f \mid \partial_k f)_{L^2}\right| \leq \underbrace{\left|(\partial_k a^{jk}\partial_j f \mid f)_{L^2}\right|}_{A} + \underbrace{\left|(a^{jk}\partial_j\partial_k f \mid f)_{L^2}\right|}_{B} \tag{4}$$

We may estimate B by the same procedure as in the proof of Theorem 1.7 with the only difference being that we apply the statement of exercise 41 not for $\varepsilon = \frac{\gamma}{2}$, but for $\varepsilon = \frac{\gamma}{3}$. For A we do the following

$$A \leq \|\partial_k a^{jk}\partial_j f\|_{L^2}\|f\|_{L^2} \underset{\text{exercise 41}}{\leq} \frac{\gamma}{3}\|f\|_{\dot{H}^1}^2 + C_\gamma\|f\|_{L^2}^2 \tag{5}$$

and therefore we obtain

$$\gamma\|f\|_{\dot{H}^1}^2 \leq \frac{2\gamma}{3}\|f\|_{\dot{H}^1}^2 + 2C_\gamma\|f\|_{L^2}^2 + \|(a^{jk}\partial_j\partial_k + b^j\partial_j + c)f\|_{L^2}\|f\|_{L^2} \tag{6}$$

By plugging in $(f_k - f_l)$ for f we see that $(f_l) \subset \mathscr{S}$ is indeed Cauchy in H^1, since $\|.\|_{H^1} \simeq \|.\|_{L^2} + \|.\|_{\dot{H}^1}$. □

We now get back to our case, where G is even in H^1. By Theorem 1, we deduce that $F \in L^2$ must have a representative $(f_l) \subset \mathscr{S}$ such that (f_l) is Cauchy in H^1. For (f_l) to be Cauchy in H^1 means that, besides being Cauchy in L^2, we also have that all the sequences $(\partial_i f_l) \subset \mathscr{S}$ are Cauchy in L^2, thus these sequences define elements $\partial_i F := [(\partial_i f_l)]_{L^2} \in L^2$. Analogously, we obtain $\partial_i G \in L^2$. Now how wonderful would it be if we could simply take the i-th derivative of the equation

$$a^{jk}\partial_j\partial_k\phi_F = \phi_G \tag{7}$$

and end up with yet another elliptic equation so that we could apply yet again Theorem 1? Quite wonderful to be certain. However, it is not that simple as it turns out. We need to be a bit cleverer than that. In order to deduce some kind of ellipticity we need to tread a vectorial path which will give us a lot more freedom in the sense that it will yield a stronger version of Meyers Serrin. To this end, note that we may consider the spaces

$$\mathscr{S}^d := \bigoplus_{t=1}^{d} \mathscr{S} \qquad (L^p)^d := \bigoplus_{t=1}^{d} L^p \qquad (H^s)^d := \bigoplus_{t=1}^{d} H^s \tag{8}$$

where each space is endowed with the respective product topology induced by the norms

$$(L^p)^d \to \mathbb{R} \qquad (F_1,...,F_d) \mapsto \big(\sum_j \|F_j\|_{L^p}^p\big)^{1/p} \tag{9}$$

$$(H^s)^d \to \mathbb{R} \qquad (F_1,...,F_d) \mapsto \big(\sum_j \|F_j\|_{H^s}^2\big)^{1/2} \tag{10}$$

In particular, for $p = 2$ the inner product on $(L^2)^d$ is given by

$$(L^2)^d \times (L^2)^d \ni (F,G) \mapsto \sum_j (F_j \mid G_j)_{L^2} \tag{11}$$

In that setting, for given $F \in (L^p)^d$ or $F \in (H^s)^d$ we may define

$$\phi_F := (\phi_{F_1},...,\phi_{F_d}) \colon \mathscr{S}^d \to \mathbb{C}^d \qquad (f_1,...,f_d) \mapsto (\phi_{F_1}(f_1),...,\phi_{F_d}(f_d)) \tag{12}$$

If we now have a differential operator $\mathscr{D}$ which acts on the cartesian product of d-functions then we may simply write

$$\mathscr{D}\phi_F \tag{13}$$

and we are pretty certain of what that means. To give a concrete example, if the dimension $d = 2$ and we are given the differential operator

$$\mathscr{D}(f_1,f_2) := (\partial_1 f_1, \partial_1 f_1 + \partial_2 f_2) \tag{14}$$

then

$$\mathscr{D}\phi_F = (\partial_1\phi_{F_1}, \partial_1\phi_{F1} + \partial_2\phi_{F_2}) \tag{15}$$

for all $F \in (L^2)^2$. This example should illustrate that differential operators $\mathscr{D}$ in the vectorial setting certainly allow mixed terms of several of the ϕ_{F_i} in each coordinate. Therein lies the strength of the vectorial approach. We can now state a more general Theorem of Meyers Serrin 2.44 as follows:

Theorem 2 (Vectorial Meyers Serrin). *Let $F \in (L^2)^d$ and suppose there exists a $G \in (L^2)^d$ such that*

$$\mathscr{D}\phi_F = \phi_G \tag{16}$$

where $\mathscr{D}$ is some suitable differential operator (linear, elliptic,...), then $F \in (L^2)^d$ has a representative $(f_l) \in \mathscr{S}^d$ so that $(\mathscr{D}f_l) \subset \mathscr{S}^d$ is Cauchy in $(L^2)^d$.

The proof to the above is pretty much the same as in the case proven in the lecture. The sole difference is that one needs to execute the procedure from the notes for every vector entry separately. The benefit of the above version of Meyers Serrin is that prospective differential operators enjoy much more freedom in that formulation. Now what is a vectorial elliptic operator?

Definition 1. Let $a^{jk} \in \mathscr{C}_b^\infty$ and let B^j and C be matrices whose entries are bounded smooth functions. The differential operator $\mathscr{D} = a^{jk}\partial_j\partial_k + B^j\partial_j + C$ which acts on tuples of Schwarz functions $f = (f_1, ..., f_d) \in \mathscr{S}^d$ by

$$a^{jk}\partial_j\partial_k f + B^j\partial_j f + Cf \in \mathscr{S}^d \tag{17}$$

$\big[$note that the term $a^{jk}\partial_j\partial_k f$ is just $(a^{jk}\partial_j\partial_k f_1, ..., a^{jk}\partial_j\partial_k f_d)\big]$ is called elliptic, if there exists $\gamma > 0$ such that for all $f \in \mathscr{S}^d$ we have

$$\sum_i \left| \left(a^{jk}\partial_j f_i \mid \partial_k f_i\right)_{L^2} \right| \geq \gamma \|f\|_{\dot{H}^1}^2 \simeq \gamma \sum_i \|f_i\|_{\dot{H}^1}^2 \tag{18}$$

Thus any elliptic operator in the usual sense (= definition given in the lecture), is a vectorial elliptic operator. We consider again the equation that is initially given by the exercise

$$a^{jk}\partial_j\partial_k \phi_F = \phi_G \tag{19}$$

Taking the gradient on both sides yields

$$a^{jk}\partial_j\partial_k \nabla\phi_F + \nabla a^{jk}\partial_j\partial_k\phi_F = \nabla\phi_G \tag{20}$$

The crucial step now is to note that equation (20) is elliptic in the gradient $\nabla\phi_F$. Indeed, the matrices B^j are given by $B^j_{ik} = \partial_i a^{jk}$ and $C = 0$, that is, equation (20) boils down to

$$a^{jk}\partial_j\partial_k \nabla\phi_F + B^j\partial_j\nabla\phi_F = \nabla\phi_G \tag{21}$$

In that spirit it is more than time to prove Theorem 1 in the vectorial picture to exploit that fact.

Theorem 3. *Let $\mathscr{D}$ be a vectorial elliptic operator as in the definition and assume that $F \in (L^2)^d$ satisfies*

$$\mathscr{D}\phi_F = \phi_G \tag{22}$$

for some $G \in (L^2)^d$. Then $F \in (H^1)^d$, i.e. $F \in (L^2)^d$ has a representative that is also Cauchy in $(H^1)^d$.

Proof. By the Vectorial Meyers Serrin Theorem 2 $F \in (L^2)^d$ has a representative $(f_l) \in \mathscr{S}^d$ such that $(\mathscr{D}f_l) \subset \mathscr{S}^d$ is Cauchy in $(L^2)^d$. For $h = (h_1, ..., h_d) \in \mathscr{S}^d$ we have by ellipticity that

$$\gamma \|h\|_{\dot{H}^1}^2 \leq \sum_i \left| \left(a^{jk}\partial_j h_i \mid \partial_k h_i\right)_{L^2} \right| \tag{23}$$

Seeing this we conclude that the proof is analogous to that of Theorem 1. □

Now let the magic come forth: We know that equation (20) is elliptic and it is solved by $\nabla F = [(\nabla f_l)]_{(L^2)^d}$. As $\nabla G = [(\nabla g_l)]_{(L^2)^d} \in (L^2)^d$ we may apply Theorem 3 to deduce that ∇F has a representative $f_l^\nabla = (f_l^{\partial_1}, ..., f_l^{\partial_d}) \in \mathscr{S}^d$ which is also Cauchy in $(H^1)^d$. This, however, exactly means that each $(f_l^{\partial_i}) \subset \mathscr{S}$ is Cauchy in H^1 and $\|f_l^{\partial_i} - \partial_i f_l\|_{L^2} \to 0$. So we have established that both F and all $\partial_i F$ have representatives which are Cauchy in H^1. Is this already the statement? Not yet, as we have currently no means to produce, out of thin air, a representative for F which is also Cauchy in H^2. However, we are very close to the truth now and we will continue by constructing the right representative $(\widehat{f}_l) \subset \mathscr{S}$ for F which will be Cauchy in H^2. In order to do that, we quickly mention Proposition 2.42 and Exercise 2.43 stated right under the proposition in the lecture notes.

Proposition 1 (2.42 + Exercise 2.43). Let $\varphi \in \mathscr{S}$ be non-negative such that $\int \varphi = 1$ and set $\varphi_k := k^d \varphi(k.)$. If $(f_l) \subset \mathscr{S}$ is Cauchy in L^p, for $p \in [1,\infty]$, or in any H^s for $s \geq 0$, then the function

$$\widetilde{f}_k := \lim_l (\varphi_k * f_l) \qquad \text{(limit is to be interpreted pointwise)} \tag{24}$$

is in $\mathscr{C}_b^\infty$ and satisfies

$$\|f_l - \widetilde{f}_l\|_\star \to 0 \tag{25}$$

for the respective $\star \in \{H^s, L^p\}$.

Now it is noteworthy that if $\widetilde{f}_k$ is as above, then $\partial_i \widetilde{f}_k = \lim_l (\varphi_k * \partial_i f_l)$. This is quite a nice property which we will exploit fully now. First we also note that if $(f_l), (g_l) \subset \mathscr{S}$ represent the same element in L^p, then $\widetilde{f}_k = \widetilde{g}_k$ for all $k \in \mathbb{N}$. Indeed,

$$\|\varphi_k * f_l - \varphi_k * g_l\|_{L^p} = \|\varphi_k * (f_l - g_l)\|_{L^p} \leq \|\varphi_k\|_{L^1} \|f_l - g_l\|_{L^p} \to 0 \tag{26}$$

and therefore

$$0 = \lim_l \|\varphi_k * f_l - \varphi_k * g_l\|_{L^p} = \|\widetilde{f}_k - \widetilde{g}_k\|_{L^p} \tag{27}$$

which shows $\widetilde{f}_k = \widetilde{g}_k$ almost everywhere, however, as both functions are continuous equality holds.

So we get back to our original task at hand. We have found a representative $(f_l) \subset \mathscr{S}$ for F which is also Cauchy in H^1, and we have found a representative $(f_l^{\partial_i}) \subset \mathscr{S}$ for the equivalence class of the derivatives $[(\partial_i f_l)]_{L^2} \in L^2$ which is also Cauchy in H^1. By the above Proposition we know that $\widetilde{f}_k := \lim_l \varphi_k * f_l \in \mathscr{C}_b^\infty$ satisfies $\|f_l - \widetilde{f}_l\|_{H^1} \to 0$. Moreover, since $(\partial_i f_l)$ and $(f_l^{\partial_i})$ represent the same element in L^2 we have (by independence of the representative)

$$\partial_i \widetilde{f}_k = \lim_l (\varphi_k * \partial_i f_l) = \lim_l (\varphi_k * f_l^{\partial_i}) = \widetilde{f^{\partial_i}}_k \tag{28}$$

However, again by the Proposition above, $(\widetilde{f^{\partial_i}}_l)$ is Cauchy in H^1 and satisfies $\|f_l^{\partial_i} - \widetilde{f^{\partial_i}}_l\|_{H^1} \to 0$. But this means that

$$\|\widetilde{f}_l - \widetilde{f}_k\|_{H^2} \simeq \|\widetilde{f}_l - \widetilde{f}_k\|_{L^2} + \sum_i \|\partial_i \widetilde{f}_l - \partial_i \widetilde{f}_k\|_{\dot{H}^1} = \|\widetilde{f}_l - \widetilde{f}_k\|_{L^2} + \sum_i \|\widetilde{f^{\partial_i}}_l - \widetilde{f^{\partial_i}}_k\|_{\dot{H}^1} \to 0 \tag{29}$$

So $(\widetilde{f}_l) \subset \mathscr{C}_b^\infty$ is Cauchy in H^2. Now let $\widehat{f}_l := \widetilde{f}_l \chi_k$ be as in Exercise 35, then $(\widehat{f}_l) \subset \mathscr{S}$ is a representative for F, which satisfies $\|\widehat{f}_l - \widetilde{f}_l\|_{H^2} \to 0$ (as can be shown analogously as for Exercise 35). Thus F has a representative in H^2, which concludes the proof (finally)!!! In particular, we have found

Corollary 1. Let $s \geq 1$. Whenever $F \in H^s$ has a representative $(f_l) \subset \mathscr{S}$ which is Cauchy in H^s such that $[(\partial_i f_l)]_{H^{s-1}} \in H^{s-1}$ has a representative which is Cauchy in H^s, then $F \in H^s$ has a representative which is Cauchy in H^{s+1}.

The above procedure may be applied inductively. Indeed, the following is true:

Corollary 2 (Regularity for vectorial elliptic Operators). Let $\mathscr{D}$ be a vectorial elliptic operator and assume that $F \in (L^2)^d$ satisfies

$$\mathscr{D}\phi_F = \phi_G \tag{30}$$

for some $G \in (H^n)^d$, where $n \in \mathbb{N}$. Then $F \in (H^{n+1})^d$. In particular, if $n > \frac{d}{2} + 1$, then we may consider, by means of Exercise 24, F as an element $F \in (\mathscr{C}_b^2)^d$ and $G \in (H^n)^d$ as an element $G \in (\mathscr{C}_b^1)^d$ so that

$$\mathscr{D}F = G \tag{31}$$

Proof. Applying Theorem 3 yields a representative (f_l) for F which is Cauchy in $(H^1)^d$. Fix $1 \leq i \leq d$ and consider the equation

$$(\mathscr{D}\phi_F)_i = \phi_{G_i} \tag{32}$$

Taking the gradient of the above equation again yields a vectorial elliptic equation in $\nabla\phi_F$. Now repeat the procedure. The remaining statement is simply an application of Exercise 24. $\square$