PDE'S - EXERCISES 17-20

ALEXANDER ZAHRER

1 EXERCISE 17:

We have to prove that the map $F \mapsto \phi_F : L^p(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is well defined. Take $f \in \mathscr{S}(\mathbb{R}^d)$ and let $\{f_k\}_k \subset \mathscr{S}(\mathbb{R}^d)$ be a representative for $F \in L^p(\mathbb{R}^d)$. We then observe that

$$|\phi_{f_k}(f) - \phi_{f_l}(f)| = |(f \mid f_k - f_l)| \leq \|f\|_q \|f_k - f_l\|_p \to 0$$

for the appropriate $q \geq 1$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Hence $\{\phi_{f_k}(f)\}_k \subset \mathbb{C}$ is Cauchy and thus convergent for all Schwarz functions $f \in \mathscr{S}(\mathbb{R}^d)$. In particular, if $\{g_k\}$ is another representative for F, then

$$|\phi_{f_k}(f) - \phi_{g_k}(f)| \leq \|f\|_q \|f_k - g_k\|_p \to 0$$

which assures us that ϕ_F is indeed well defined.

2 EXERCISE 18:

We are asked to verify $\phi_{\lambda F+G} = \overline{\lambda}\phi_F + \phi_G$.

$$\lambda\phi_F(f) + \phi_G(f) = \lim\{\lambda\phi_{f_k}(f) + \phi_{g_k}(f)\} = \lim(f \mid \overline{\lambda} f_k + g_k) = \phi_{\overline{\lambda}F+G}$$

3 EXERCISE 19:

We are to show that if $\{F_k\} \subset L^2(\mathbb{R}^d)$ and $F \in L^2(\mathbb{R}^d)$ with $F_k \to F$ in $L^2(\mathbb{R}^d)$, then $\phi_{F_k} \to \phi_F$ in $\mathscr{S}(\mathbb{R}^d)$. So fix some arbitrary Schwarz function $f \in \mathscr{S}(\mathbb{R}^d)$; all we need to show is $\lim_k \phi_{F_k}(f) = \phi_F(f)$. By assumption

$$\|F - F_k\|_2 \to 0$$

so if we take a representative $\{f_k\}_k \subset \mathscr{S}(\mathbb{R}^d)$ of F and representatives $\{f_l^k\}_l$ for each F_k, then

$$0 = \lim_k \|F_k - F\|_2 = \lim_k \lim_l \|f_l^k - f_l\|_2$$

We then deduce

$$|\phi_{F_k}(f) - \phi_F(f)| = \lim_l |(f \mid f_l^k) - (f \mid f_l)| = \lim_l |(f \mid f_l^k - f_l)| \leq \|f\|_2 \|f_l^k - f_l\|_2 = \|f\|_2 \|F_k - F\| \to 0$$

as wanted.

4 EXERCISE 20:

We want to show that for $s \geq 0$ we have

$$(|\nabla|^s f \mid g) = (f \mid |\nabla|^s g)$$

Plugging in definitions and using Plancherel's theorem twice yields

$$(|\nabla|^s f \mid g) = (\mathscr{F}^{-1}(|.|^s \mathscr{F} f) \mid g) = (|.|^s \mathscr{F} f \mid \mathscr{F} g) = (\mathscr{F} f \mid |.|^s \mathscr{F} g) = (f \mid |\nabla|^s g)$$

as wanted.