Yoga with Cobordisms

April 13, 2025

Contents

Abstract

These notes were prepared for a talk delivered in Prague on December 16, 2024, focusing on the concepts of *smooth functorial field theory* and the *(geometric) cobordism hypothesis*. To begin, let us clarify what this document does *not* aim to achieve: it does not provide exhaustive motivation or detail for every concept discussed. While some effort has been made to offer context, the exposition remains intentionally limited in its treatment of the underlying physics. This choice reflects both the speaker's background and the assumption that readers may have little familiarity with physics. For a deeper exploration of the physical context, we recommend the overview available at QFTs on the nLab. The content of these notes is as follows: we discuss the notion of *smooth symmetric monoidal dualic*[1] (∞, d)-*categories*, carefully introducing and motivating each term, with brief references to their physical relevance. We then explore the concept of cobordisms, supported by illustrative diagrams, and conclude with a discussion of the *Cobordism Hypothesis*, as rigorously formulated by Lurie in [3]. Time permitting, we will touch on recent developments by Grady and Pavlov concerning the *Geometric Cobordism Hypothesis*, as presented in [1] and [2].

[1]The term *dualic* is non-standard and probably only used by the author. Usually the property of being dualic is referred to has having all (higher) duals.

1 A bedtime Story

The quality of a physical theory depends on how well its predictions align with experimental results. In terms of this measure of 'goodness,' quantum field theory is widely regarded as the most successful theory in physics. However, Quantum Field Theory has been quite elusive when it comes to its underlying mathematical description. The definition for a QFT one usually finds in a textbook on the matter is given in terms of the Feynman path integral. The path integral, in essence, is a framework that allows for integrating over the entire space of possible physical field configurations, such as the different ways an electromagnetic field can vary throughout space and time. It provides a way to calculate quantities in quantum field theory by summing contributions from all possible histories or states of the system, weighted by an exponential factor involving the action. This concept, however, is fundamentally ill-defined, as it can be shown that the path integral, with all the desired properties physicists attribute to it, cannot rigorously exist in general.

Despite this, physicists approach quantum field theory by cleverly approximating this nonexistent path integral. Remarkably, these approximations yield predictions that align with experimental results to an unprecedented degree of accuracy. Despite the tremendous success of QFT, from a mathematical perspective, one might expect a rigorous framework that defines the path integral properly. To this end, two main approaches to axiomatizing QFT have been developed: algebraic quantum field theory and functorial field theory.

In this talk (or in these notes), we will focus exclusively on functorial field theory. There is a funny story[2] with regards to the origin of functorial quantum field theory, which goes as follows. During a lecture on *Conformal Field Theory* (a special case of QFT), Witten presented a set of axioms that the Feynman path integral should satisfy. After Witten's presentation, Segal, who was in the audience, casually remarked:

Is that not just a ***symmetric monoidal functor*** *from some category of cobordisms to the category of vector spaces?*

This remark was reportedly quite amusing to an audience predominantly composed of physicists as Segal appeared to have taken these *straightforward* axioms and reframed them in a manner that was, at the time, perceived as either incomprehensible or intimidating. Conceived from this very idea, *functorial quantum field theory* has become a very active area of research, driving the development of numerous groundbreaking new mathematics (among them e.g. the theory of (∞, n)-categories). Notably, it stands as a cornerstone in the intersection of mathematics and physics, playing a significant role in modern mathematical physics.

[2] This story originates from Dmitri Pavlov, who shared it at the beginning of one of his talks on the Geometric Cobordism Hypothesis. See Lecture series on the Geometric Cobordism Hypothesis Video #1.

2 A Cooking manual on smooth QFT

Let us discuss what a quantum field theory (QFT) should be, framed in the most conceptual terms possible. Physics aims to describe phenomena occurring in spacetime, which constitutes the framework of our observable universe. We will not delve into what this means precisely, but, very crudely speaking, a *Quantum Field Theory* $\mathcal{Q}$ ought to assign values to patches of spacetime. With regards to that, it seems plausible that a QFT should be a morphism of *theories*:

$$\Big\{\text{Geometric Theory of Spacetime}\Big\} \xrightarrow{\mathcal{Q}} \Big\{\text{Algebraic Theory of Values}\Big\}$$

where the left hand domain should be thought of as containing information about our observable universe i.e. spacetime patches (think e.g. of smooth manifolds), while the right hand codomain should be thought of as a theory of algebraic entities (think e.g. of Hilbert spaces or chain complexes). What we really mean by *theory* is a *category*. And in fact, when we say morphism of theories we must therefore mean that a QFT is some sort of functor between categories.

- **Daunting Ingredients, Delightful Outcome**

  1. As sketched above, a QFT should be a functor between some geometric category of spacetime to some algebraic category. A good question would now be: *Why would that be a functor?* Physical theories are thought to satisfy something that might be referred to as being *local*. By that we mean that if we can calculate the values of our QFT locally on some patches of spacetime, then we can glue the values together to obtain the value of the whole spacetime. But this locality condition will precisely correspond to *functoriality* of our assignment! More precisely, given a spacetime of dimension d, we may smoothly cut this up into as many finite smooth sub-pieces and we would expect that if we applied our QFT $\mathcal{Q}$ to the composition of patches of spacetime, then the result would be the same as evaluating our QFT on the whole spacetime. We have the catchphrase: *All global effects arise by integrating up local effects.* This makes it obvious that since we want to cut up our given d-dimensional spacetimes in all possible d-many directions, and since glueing of spacetime patches should be considered as the right composition operation in this geometric category of spacetimes, that we cannot use just ordinary categories to encode something like this. In fact, we shall consider (∞, d)-categories of spacetimes as well as algebraic (∞, d)-categories of values and a QFT should then be a functor $\mathcal{Q}$ between such (∞, d)-categories.
  2. In Quantum Field Theory we have the notion of *Quantum entanglement* and *superposition*. These phenomena fundamentally arise from the *non-cartesianness* of the *tensor product* of e.g. Hilbert spaces (which are used in the formulation of QM). In category theory there is a natural generalization of the *tensor product*. We realize therefore that both the (∞, d)-category of spacetimes and the one of values ought to be *symmetric monoidal* (both

categories have a tensor product functor) and a QFT $\mathscr{Q}$ *must* then be a *symmetric monoidal functor* (a functor compatible with the tensor functor) between these entities.

3. The notion of *duality* (think of dual vector spaces) is omnipresent in any quantum theory (think e.g. of Dirac's Bra-ket notation). Any ordinary symmetric monoidal category allows for the notion of duals for its objects (if they exist). In fact, if we are given a symmetric monoidal (∞, d)-category then we may also ask for *duals of duals* (higher duals). Asking a symmetric monoidal (∞, d)-category to have all higher duals (meaning every object is *fully dualizable*) leads to the concept of a *symmetric monoidal, dualic* (∞, d)*-category*. It is observed that our given symmetric monoidal (∞, d)-category of spacetimes will always be dualic. Moreover, any symmetric monoidal functor preserves all duality information. Hence a QFT, for the time being, is still just assumed to be a symmetric monoidal functor just like in the previous step. The notion of *fully dualizable objects* is crucial when we arrive at the Cobordism classification theorem.
4. Finally, our field theories ought to be *smooth* (as in practice they are e.g. the assignment is smooth in the time parameter). In order to encode such smoothness for a field theory $\mathscr{Q}$, we would like to have a notion of *smooth symmetric monoidal functor*. But in order for this to make sense, we need the notion of *smooth symmetric monoidal* (∞, d)*-categories* (think of smoothly parametrized families of symmetric monoidal (∞, d)-categories). Thereby a *smooth QFT* may be encoded as a smooth symmetric monoidal functor.

- **Spacetimes and lots of Glueing**

1. We shall sketch the definition of our *smooth symmetric monoidal* (∞, d)*-category of spacetimes*. Usually this is referred to as the symmetric monoidal (∞, d)-category of *cobordisms*, denoted

$$^{\amalg}\mathbf{Cob}_d^{\mathscr{G}}$$

where $\mathscr{G}$ is the given *geometry*[3] that we want to encode on our cobordisms (i.e. spacetimes), while $\amalg$ denotes the tensor product of cobordisms (disjoint union).

2. We shall discuss a certain variant of

$$^{\amalg}\mathbf{Cob}_d^{\mathscr{G}}$$

namely the *framed* cobordism category

$$^{\amalg}\mathbf{Cob}_d^{\mathrm{fr}}$$

In the *Cobordism Theorem* we will restrict to the case where our cobordism category is of the framed type. Also we shall not care for smoothness in our presentation (maybe only in the last section).

[3] This is the most recent definition as provided in [1].

- **The Cobordism Hypothesis**

  1. We shall discuss Lurie's Topological Cobordism Hypothesis and, if time allows, I will try to convince you why the result is true.
  2. If time permits, I will talk on the geometric cobordism hypothesis as stated in [1].

2.1 Higher Categories

As we said in the introduction a QFT should be a functor between some category of spacetimes to some algebraic category of values. We will see later that the category of spacetimes that we want to look at is not an ordinary category. In fact, it will be an (∞, d)-category (where d depends on the dimension of our QFT). The reason for this is that composition in our spacetime category will be something like glueing together patches of d-dimensional spacetimes[4]. But since we are in d-dimensional spacetime there is d different directions in which we can cut up our spacetimes. Moreover, glueing of spacetimes will not result in a uniquely determined spacetime but rather in one that is *unique up to a contractible space of choices*. So we shall start by postulating what an $(\infty, 1)$-category should be. Roughly speaking $(\infty, 1)$-categories are smug little things that look like ordinary categories, but have a fancy *homotopy coherent air* to it. An $(\infty, 1)$-category is an entity that has a collection of objects, a collection of 1-morphisms between these objects, a collection of 2-morphisms between these 1-morphisms, and so on. Furthermore, each morphism layer is assumed to be endowed with a composition operation which is associative and unital up to coherent homotopy. In particular, the 1 in $(\infty, 1)$ means that every k-morphism, for $k \geq 2$, is invertible (up to homotopy). We can make this a little more precise by saying that an $(\infty, 1)$-category is a category weakly enriched in spaces. This means that an $(\infty, 1)$-category $\mathscr{C}$ has a collection of objects $\mathscr{C}_0$ and for each pair of objects X, Y a whole space of morphisms $\mathscr{C}(X, Y)$. The points of the spaces $\mathscr{C}(X, Y)$ are, by definition, the 1-morphisms of $\mathscr{C}$. Paths between points in the mapping spaces $\mathscr{C}(X, Y)$ are the 2-morphisms, homotopies between paths are the 3-morphisms, homotopies between homotopies are the 4-morphisms and so on. Note that paths, homotopies, homotopies of homotopies etc. are automatically invertible (with regards to concatenation of paths, homotopies, and so on) by just reversing the direction of the path, homotopy, etc. The whole of ordinary category theory like e.g. (co)limits, adjunctions, Kan extensions all have analogues in the ∞-world. We can now inductively "define" (∞, d)-categories for $d \geq 2$: An (∞, d)-category is a category weakly enriched in $(\infty, d-1)$-categories. In other words, an (∞, d)-category is something which has a collection of objects along with, for each pair of such objects, an $(\infty, d-1)$-category of morphisms between these two objects.

Notation 2.1. *From now on we will say d-category or category instead of always saying* (∞, d)*-category, or* $(\infty, 1)$*-category (any ordinary category may be viewed as an* $(\infty, 1)$*-category, so this is consistent). Likewise, we will say groupoid or space, but mean* ∞*-groupoid.*

[4] This can be more properly motivated by the concept of *locality* in physics, which may be stated informally by saying "all global effects arise by integrating up local effects".

2.2 Entanglement leads to Monoidality

Certainly the idea that a QFT is *just* a functor (without any extra whistles and bells) is way too naive. For example, given two physical systems we also want to make sense of their *composite* i.e. regard them as a single system, but trivially so, without the two interacting. In classical mechanics this is achieved by taking the *cartesian product* of their *phase spaces* (the space of all those fields on spacetime which solve the equations of motion, or in other words, the *space of trajectories of the system*). In QM however, forming the composite of two physical systems amounts to taking the (non-cartesian) *tensor product* of the spaces of quantum states. The *non-cartesian nature of this tensor product* is the source of the phenomenon of *quantum entanglement*. Encoding this in the language of category theory is achieved by passing to *symmetric monoidal categories*: Recall that a *symmetric monoidal category* $\mathscr{C}$ is a category $\mathscr{C}$ together with a *tensor product functor*

$$\boxtimes\colon \mathscr{C}\times\mathscr{C}\to\mathscr{C}$$

along with a unit object $\mathbb{1}\in\mathscr{C}$ (referred to as the *tensor unit*) such that we have natural isomorphisms (natural in $X, Y, Z\in\mathscr{C}$)

$$(X\boxtimes Y)\boxtimes Z\simeq X\boxtimes(Y\boxtimes Z),\qquad X\boxtimes\mathbb{1}\simeq X\simeq\mathbb{1}\boxtimes X,\qquad X\boxtimes Y\simeq Y\boxtimes X$$

and these isomorphisms are *coherent* (see Symmetric Monoidal Cats NLab). Now, in order to encode entanglement, our assignment from cobordisms to some category of values has to preserve the symmetric monoidal structure: Recall that a *symmetric monoidal functor* between symmetric monoidal categories $\mathscr{Q}\colon\mathscr{C}\to\mathscr{D}$ is a functor from $\mathscr{C}$ to $\mathscr{D}$ that is suitably compatible (see Monoidal Functor Nlab) with the respective tensor functors:

$$\mathscr{Q}(X\boxtimes_{\mathscr{C}} Y)\simeq\mathscr{Q}X\boxtimes_{\mathscr{D}}\mathscr{Q}Y,\qquad\mathscr{Q}(\mathbb{1}_{\mathscr{C}})\simeq\mathbb{1}_{\mathscr{D}}$$

2.3 Duality

In Quantum mechanics the notion of a *dual* is crucial (e.g. *dual vector spaces*). The notion, yet again, has a natural generalization in the world of category theory. In order to speak about this, let us recall the notion of an adjunction internal to any given 2-category:

Definition 2.3.1. Let $\mathscr{A}$ be a 2-category, and suppose we are given objects $X, Y\in\mathscr{A}$ and 1-morphisms $\varphi\colon X\to Y$.

- The morphism φ is said to be *left adjoint* (or to *admit a right adjoint*), if there exists a morphism $\varphi^\vee\colon Y\to X$ and two 2-morphisms referred to as coevaluation and evaluation

$$\mathrm{coev}_\varphi\colon \mathrm{id}_X\to\varphi^\vee\circ\varphi,\qquad \mathrm{ev}_\varphi\colon\varphi\circ\varphi^\vee\to\mathrm{id}_Y$$

so that the triangle identities are satisfied

$$
\begin{tikzcd}
& \varphi \circ \varphi^\vee \circ \varphi \arrow[dr, "\varphi \cdot \mathrm{ev}_\varphi"] \arrow[d, phantom, "\simeq"] & \\
\varphi \arrow[ur, "\mathrm{coev}_\varphi \cdot \varphi"] \arrow[rr, equal] & & \varphi
\end{tikzcd}
\qquad
\begin{tikzcd}
& \varphi^\vee \circ \varphi \circ \varphi^\vee \arrow[dr, "\mathrm{ev}_\varphi \cdot \varphi^\vee"] \arrow[d, phantom, "\simeq"] & \\
\varphi^\vee \arrow[ur, "\varphi^\vee \cdot \mathrm{coev}_\varphi"] \arrow[rr, equal] & & \varphi^\vee
\end{tikzcd}
$$

- Analogously, φ is said to be a *right adjoint* (or to *admit a left adjoint*), if there exists a morphism ${}^\vee\varphi\colon Y \to X$ such that ${}^\vee\varphi$ is left adjoint to φ.

Remark 2.2. Any symmetric monoidal 1-category $\mathscr{C}$ may be interpreted as a 2-category $\mathbf{B}\mathscr{C}$ as follows:

- $\mathbf{B}\mathscr{C}$ has precisely one (dummy) object $*$,
- 1-morphisms in $\mathbf{B}\mathscr{C}$ are the objects in $\mathscr{C}$,
- 2-morphisms in $\mathbf{B}\mathscr{C}$ are given by the 1-morphisms in $\mathscr{C}$, and so on.

With the previous remark, we have the following definition:

Definition 2.3.3. An object $X \in \mathscr{C}$ is called dualizable if X fits into an adjunction when viewed as a 1-morphism in $\mathbf{B}\mathscr{C}$.

Remark 2.4. Note that a left adjoint of X, as above, is also right adjoint to X. This follows from $\mathscr{C}$ being symmetric monoidal.

Remark 2.5. More concretely, this means that $X \in \mathscr{C}$ is dualizable, if there exists an object $X^\vee \in \mathscr{C}$ along with evaluation and coevaluation maps satsifying the triangle identities:

$$
\begin{tikzcd}
& X \boxtimes X^\vee \boxtimes X \arrow[dr, "\Phi \boxtimes \mathrm{ev}_X"] \arrow[d, phantom, "\simeq"] & \\
X \arrow[ur, "\mathrm{coev}_X \boxtimes X"] \arrow[rr, equal] & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
& X^\vee \boxtimes X \boxtimes X^\vee \arrow[dr, "\mathrm{ev}_X \boxtimes X^\vee"] \arrow[d, phantom, "\simeq"] & \\
X^\vee \arrow[ur, "X^\vee \boxtimes \mathrm{coev}_X"] \arrow[rr, equal] & & X^\vee
\end{tikzcd}
$$

By means of the string diagramatic calculus (a category theoretic version of *Feynman diagrams*), this translates to (which, as we will see, looks a lot like cobordisms):

$$ev_X := \underset{X^\vee \;\; X}{\overset{\mathbb{1}}{\cap}} : X^\vee \boxtimes X \longrightarrow \mathbb{1}$$

$$coev_X := \underset{\mathbb{1}}{\overset{X \;\; X^\vee}{\cup}} : \mathbb{1} \longrightarrow X \boxtimes X^\vee \quad \text{such that}$$

triangle identities hold:

Since we shall be in the setting of an arbitrary d-category, we also have to account for *higher dualities* we want to encode.

Definition 2.3.6. Let $\mathscr{A}$ be any d-category $\mathscr{A}$.

- We say that $\mathscr{A}$ admits *adjoints for* 1*-morphisms*, if every 1-morphism φ in $\mathscr{A}$ has both a left and a right adjoint internally[5] to the underlying[6] 2-category $\mathscr{A}^{\leq 2}$.
- We say that $\mathscr{A}$ *admits adjoints for k-morphisms*, if for every pair of objects $X, Y \in \mathscr{A}$ the $(d-1)$-category $\mathscr{C}(X, Y)$ admits adjoints for $(k-1)$-morphisms.
- We say that $\mathscr{A}$ *has adjoints* if $\mathscr{A}$ admits adjoints for k-morphisms for all $0 < k < d$.

Having this we may define the notion of *symmetric monoidal dualic d-categories*. For this note that any symmetric monoidal d-category $\mathscr{C}$ may always be considered as an $(\infty, d+1)$-category $\mathbf{B}\mathscr{C}$ analogously to the case $d = 1$ in 2.2.

Definition 2.3.7. A symmetric monoidal d-category is called *dualic*[7] if the $(d+1)$-category $\mathbf{B}\mathscr{C}$ admits adjoints in the sense of Definition 2.3.6

Remark 2.8. The 1-category of symmetric monoidal d-categories SymCat_d admits a full subcategory consisting of those symmetric monoidal d-categories which are dualic, denoted by $\mathrm{SymCat}_d^\vee$. The canonical inclusion admits both a left and a right adjoint:

$$\begin{tikzcd} \mathrm{SymCat}_d^\vee \arrow[r, hook] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{SymCat}_d \arrow[l, bend right] \arrow[l, bend left, "\mathrm{Duals}(-)"] \end{tikzcd}$$

[5] In the sense of Definition 2.3.1 from earlier.
[6] $\mathscr{A}^{\leq 2}$ is obtained from $\mathscr{A}$ by forgetting about all non-invertible k-morphisms for all $k \geq 3$.
[7] This is far from standard terminology, and is probably solely used by myself in these talk notes. The standard term is *symmetric monoidal d-category with duals*.

The important functor here is the right adjoint Duals, which comes with a universal comparison natural transformation

$$\mathrm{Duals}(\mathrm{incl}(-)) \xrightarrow{\iota} \mathrm{id}$$

Definition 2.3.9. An object X in a given symmetric monoidal d-category is called fully dualizable, if X is in the essential image of $\iota_{\mathscr{C}} \colon \mathrm{Duals}_{\mathscr{C}} \to \mathscr{C}$.

In more down to earth terms, an object $X \in \mathscr{C}$ is fully dualizable if

- X is dualizable, i.e. X has a dual $X^\vee \in \mathscr{C}$ along with evaluation and coevaluation maps satisfying the triangle identities as described in 2.5,
- The evaluation map $\mathrm{ev}_X \colon X^\vee \boxtimes X \to \mathbb{1}$ viewed as a 1-morphism in $\mathscr{C}$ admits adjoints (both a left and right adjoint) in the sense of 2.3.1:

$$\begin{tikzcd}
X^\vee \boxtimes X \arrow[r, "\mathrm{ev}_X" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathbb{1} \arrow[l, bend right, shift right=2, "{}^\vee\mathrm{ev}_X"'] \arrow[l, bend left, shift left=2, "\mathrm{ev}_X^\vee"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{id}_{X^\vee \boxtimes X} \arrow[d, "\mathrm{coev}_{\mathrm{ev}_X}"] & \mathrm{ev}_X \circ \mathrm{ev}_X^\vee \arrow[d, "\mathrm{ev}_{\mathrm{ev}_X}"] & \mathrm{id}_{X^\vee \boxtimes X} \arrow[d, "\widetilde{\mathrm{coev}}_{\mathrm{ev}_X}"] & \mathrm{ev}_X \circ {}^\vee\mathrm{ev}_X \arrow[d, "\widetilde{\mathrm{ev}}_{\mathrm{ev}_X}"] \\
\mathrm{ev}_X^\vee \circ \mathrm{ev}_X & \mathrm{id}_{\mathbb{1}} & {}^\vee\mathrm{ev}_X \circ \mathrm{ev}_X & \mathrm{id}_{\mathbb{1}}
\end{tikzcd}$$

  Likewise, the coevaluation map, $\mathrm{coev}_X \colon \mathbb{1} \to X \boxtimes X^\vee$ admits adjoints.

- One carries on by demanding that the evaluations $\mathrm{ev}_{\mathrm{ev}_X}, \widetilde{\mathrm{ev}}_{\mathrm{ev}_X}$ all admit adjoints themselves when viewed as 1-morphisms in $\mathscr{C}(\mathbb{1}, \mathbb{1})$ and so on for the other structure maps.
- Continue like this to the $(d-1)$-morphism layer of $\mathscr{C}$.

We shall highlight the necessity for this concept more clearly when we arrive at the *Cobordism hypothesis*, but there is another thing that is very much worth noting. Symmetric monoidal functors $\mathscr{Q} \colon \mathscr{C} \to \mathscr{D}$ preserve fully dualizable objects, that is, $\mathscr{Q}(\mathrm{Duals}_{\mathscr{C}}) \subset \mathrm{Duals}_{\mathscr{D}}$. For example, if X is dualizable, then $\mathscr{Q}X$ is dualizable with evaluation and coevaluation given by:

$$\mathrm{ev}_{\mathscr{Q}X} := \mathscr{Q}\mathrm{ev}_X \colon \mathscr{Q}(X^\vee \boxtimes_{\mathscr{C}} X) \simeq \mathscr{Q}(X^\vee) \boxtimes_{\mathscr{D}} \mathscr{Q}(X) \to \mathbb{1}_{\mathscr{D}} \simeq \mathscr{Q}(\mathbb{1}_{\mathscr{C}})$$

and analogously $\mathrm{coev}_{\mathscr{Q}\mathscr{X}} := \mathscr{Q}\mathrm{coev}_X$. By monoidality this will satisfy the triangle identities. There is one more Proposition we ought to state:

Proposition 2.3.10. *Suppose we are given symmetric monoidal d-categories $\mathscr{C}, \mathscr{D}$ such that $\mathscr{C}$ is dualic. Then the (symmetric monoidal) d-category of symmetric monoidal functors from $\mathscr{C}$ to $\mathscr{D}$*

$$\mathrm{SymCat}(\mathscr{C}, \mathscr{D})$$

is a space (that is, a groupoid).

Idea of the Proof. We have to show that every k-morphisms for $0 < k \leq n$ in the above functor category has an inverse; Let us illustrate the case $k = 1$. To this end, let $\mathscr{Q}, \mathscr{Q}'$ be symmetric monoidal functors from $\mathscr{C}$ to $\mathscr{D}$ and suppose $\zeta \colon \mathscr{Q} \to \mathscr{Q}'$ is a symmetric monoidal natural transformation between the two. It suffices to show that ζ is a componentwise isomorphism. So let $X \in \mathscr{C}$, then by assumption X has a dual $X^\vee$ along with evaluation and coevaluation maps. We note that $\mathscr{Q}X$ and $\mathscr{Q}'X$ are dual to $\mathscr{Q}(X^\vee)$ and $\mathscr{Q}'(X^\vee)$. In particular, we have $\mathscr{Q}(X^\vee)^\vee \simeq \mathscr{Q}X$ and $\mathscr{Q}'(X^\vee)^\vee \simeq \mathscr{Q}'X$. It is then checked that

$$\mathscr{Q}'X \simeq \mathscr{Q}'(X^\vee)^\vee \xrightarrow{\zeta^\vee_{X^\vee}} \mathscr{Q}(X^\vee)^\vee \simeq \mathscr{Q}X$$

is the desired inverse to ζ_X (note here that $\zeta^\vee_{X^\vee}$ denotes the dual morphism of $\zeta_{X^\vee}$). □

Example 2.3.11. Let us list some examples:

- The (ordinary) category of vector spaces over some field $\mathbb{F}$, denoted $\mathrm{Vect}_\mathbb{F}$, is not dualic[8]. An object $X \in \mathrm{Vect}_\mathbb{F}$ has a dual if and only if X is finite dimensional.
- *Fusion Categories* may be identified as the fully dualizable objects in some 3-category of monoidal categories with 1-morphisms bimodule categories. See Nlab for more references.
- Let R be a commutative ring. Consider then the 2-category of R-algebras $\mathbf{Alg}^2_R$, with 1-morphisms given by Bimodules, while 2-morphisms are given by Bimodule maps (intertwiners). Composition of 1-morphisms is given by tensor product of bimodules. One can show that fully dualizable objects in $\mathbf{Alg}^2_R$ are separable algebras which are projective modules over the base ring R.
- In the category (recall our convention: we mean $(\infty, 1)$ now!) of spaces $\mathscr{S}$, the only (fully) dualizable is the point (and everything equivalent to it).
- In the (stable) category of spectra Sp, the full subcategory on finite spectra consists of only dualizable objects. In particular, retracts of finite spectra are dualizable (see here).
- In the 1-category of presentable stable 1-categories $\mathrm{Pr}^L_{\mathrm{st}}$ the dualizable objects are given by retracts of compactly generated, stable categories. I am not sure what the fully dualizable objects are if we view the 2-category of presentable 1-categories - maybe someone knows and tells me?
- The Cobordism categories that we will define later are all dualic.

2.4 Smoothness

Time propagation in Quantum mechanics should be smooth in the time parameter. More generally, one would expect that quantum field theories are depending *smoothly*

[8] Being dualic - this is usually called being rigid in the 1-categorical case - for a symmetric monoidal 1-category is the same as saying every object has a dual.

(whatever that might mean we make more precise in a second) on their input. The right notion of smoothness should be along the following lines: Recall that a QFT is expected to be a symmetric monoidal functor between an (∞, d)-category of spacetimes (to be introduced below) and some symmetric monoidal (∞, d)-category of values. We would like to view both the domain and target of our QFT as some sort of *generalized smooth space* in a way that this would allow us to consider *smooth maps* - smooth symmetric monoidal functors - between these. How do we encode such a thing then? *Sheaves of course!* Recall that the (ordinary) category of *cartesian spaces*, denoted by CartSp, has as objects open subsets U of $\mathbb{R}^n$ (for some $n \in \mathbb{N}$) such that U is smoothly diffeomorphic to $\mathbb{R}^n$; morphisms in CartSp are then just smooth maps. Now if any smooth manifold M may be naturally interpreted as the sheaf (of sets) $\mathrm{C}^\infty(-, M)\colon \mathrm{CartSp}^{\mathrm{op}} \to \mathrm{Set}$. Given a second manifold N, the Yoneda Lemma will imply that natural transformations $\mathrm{C}^\infty(-, M) \to \mathrm{C}^\infty(-, N)$ are precisely the same thing as smooth maps $M \to N$.

Since we are secretly in ∞-land, we do want to consider ∞-sheaves instead of just ordinary sheaves. We are now ready to define *smooth categories*:

Definition 2.4.1. A *smooth symmetric monoidal d-category* is a presheaf with values in the 1-category of symmetric monoidal d-categories

$$\begin{aligned} \mathrm{CartSp}^{\mathrm{op}} &\xrightarrow{\quad \mathscr{C} \quad} \mathrm{SymCat}_d \\ U &\longmapsto \mathscr{C}_U \end{aligned}$$

satisfying the *(homotopy) sheaf condition*: For any $U \in \mathrm{CartSp}$ and any *good open cover* (U_i) of U, the canonical map

$$\mathscr{C}_U \xrightarrow{\sim} \operatorname{holim}\left(\prod_i \mathscr{C}_{U_i} \rightrightarrows \prod_{i,j} \mathscr{C}_{U_{ij}} \rightrightarrows\!\!\!\to \prod_{i,j,k} \mathscr{C}_{U_{ijk}} \quad \cdots \right)$$

sending $\varphi \in \mathscr{C}_U$ to $(\varphi|_{U_i})_i$ is an equivalence (of symmetric monoidal d-categories).

The RHS of the above equivalence has as objects the collection of (homotopy coherent) matching families $(\varphi_i)_i \in \prod_i \mathscr{C}_{U_i}$, that is, we have

$$\varphi_i|_{U_{ij}} \simeq \varphi_j|_{U_{ij}}, \qquad \varphi_{ij}|_{U_{ijk}} \simeq \varphi_{ik}|_{U_{ijk}}, \ldots$$

while the LHS is given by the *global elements* $\varphi \in \mathscr{C}_U$. Saying that the canonical map is an equivalence is (morally speaking) the same as saying that given any matching family $(\varphi_i)_i$, there is a *unique* (up to contractible choice) $\varphi \in \mathscr{C}_U$ such that $\varphi|_{U_i} \simeq \varphi_i$ for all i.

3 Fun with Spacetimes: Cobordisms

Theorem 3.0.1 (Lurie)**.** *There is a symmetric monoidal (dualic) d-category of Cobordisms*

$$^{\amalg}\mathbf{Cob}_d$$

which has

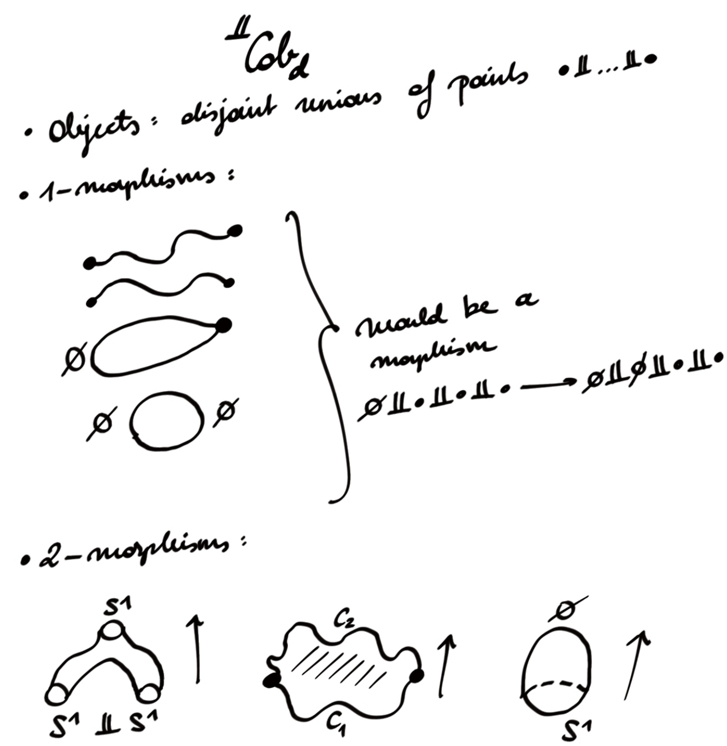
- *objects given by* 0*-manifolds (i.e. disjoint unions of points),*
- 1*-morphisms given by* 1*-manifolds with boundaries (think of disjoint unions of intervals),*
- 2*-morphisms are* 2*-manifolds with corners,*

  ⋮
- *d-morphisms are d-manfiolds with corners,*
- $(d+1)$*-morphisms are smooth diffeomorphisms between d-manifolds with corners (which restrict to the identity on the respective boundaries).*
- $(d+2)$*-morphisms are smooth isotopies between diffeomorphisms between d-manifolds with corners.*
- *and so on.*

The composition operation is given by glueing of manifolds (in any layer) and the tensor product (that then determines the symmetric monoidal structure) is given by taking disjoint unions of manifolds.

Let us make more sense of the above by considering an explicit example:

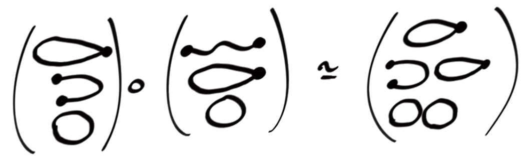
Example 3.0.1. We consider the example of $^{\amalg}\mathbf{Cob}_2$:



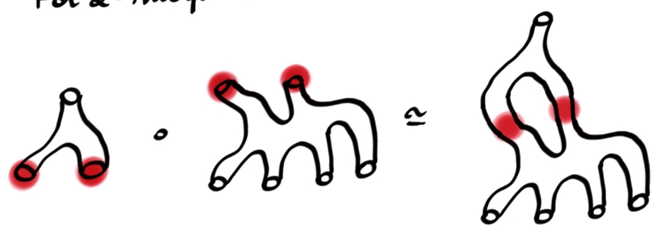
Composition of Cobordisms

For 1-morphisms:

e.g



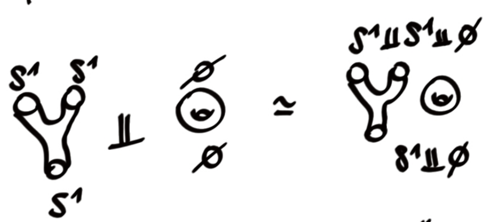
For 2-morphisms:



Tensor Product on Cobordisms

Tensor product on $^{\amalg}\mathrm{Cob}_d$ is taking $\amalg$:

e.g



with monoidal unit $\emptyset \in {}^{\amalg}\mathrm{Cob}_d$.

${}^{\amalg}\mathrm{Cob}_2$ is <u>dualic</u>:

The point $\bullet \in {}^{\amalg}\mathrm{Cob}_2$ is self dual:

$$\bullet^{\vee} \simeq \bullet \text{ with}$$

$$(\mathrm{ev}_\bullet : \bullet \amalg \bullet \longrightarrow \emptyset) := \subset$$

$$(\mathrm{coev}_\bullet : \emptyset \longrightarrow \bullet \amalg \bullet) := \supset$$

The triangle identities trivially hold by construction:

$$S \simeq - \qquad \text{Ƨ} \simeq -$$

Now set ${}^{\vee}\mathrm{ev}_\bullet \simeq \mathrm{ev}_\bullet^{\vee} \simeq \mathrm{coev}_\bullet$ and ${}^{\vee}\mathrm{coev}_\bullet \simeq \mathrm{coev}_\bullet^{\vee} \simeq \mathrm{ev}_\bullet$

$$\mathrm{ev}_{\mathrm{ev}_\bullet} : \underbrace{{}^{\vee}\mathrm{ev}_\bullet \circ \mathrm{ev}_\bullet}_{} \longrightarrow \mathrm{id}_{\bullet\amalg\bullet}$$

$$\supset \circ \subset \; \simeq \; \supset\subset$$

counit of ${}^{\vee}\mathrm{ev}_\bullet \dashv \mathrm{ev}_\bullet$

Hence $\mathrm{ev}_{\mathrm{ev}_\bullet} \simeq$ 

and

$$\mathrm{coev}_{\mathrm{ev}_\bullet} : \mathrm{id}_\emptyset \longrightarrow \underbrace{\mathrm{ev}_\bullet \circ {}^{\vee}\mathrm{ev}_\bullet}_{}$$

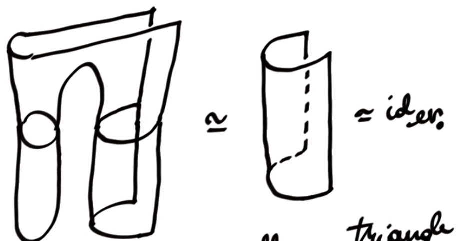
$$\subset \circ \supset \; \simeq \; \bigcirc \; \simeq S^1$$

unit of ${}^{\vee}\mathrm{ev}_\bullet \dashv \mathrm{ev}_\bullet$

Hence $\mathrm{coev}_{\mathrm{ev}_\bullet} \simeq$ 

We must verify

$$\begin{tikzcd}
 & ev_\circ \circ ev^\vee_\circ \circ ev_\circ \arrow[dr, "ev_\circ \cdot ev_{ev_\circ}"] \arrow[d, phantom, "\simeq"] & \\
ev_\circ \simeq id_\emptyset \circ ev_\circ \arrow[ur, "coev_{ev_\circ} \cdot ev_\circ"] \arrow[rr, equal] & & ev_\circ \simeq ev_\circ \circ id_{\mathbb{1}}
\end{tikzcd}$$



Analogously for the other triangle identity.

Remark 3.2. There is an important *framed* variant ${}^{\amalg}\mathbf{Cob}_d^{\mathrm{fr}}$. For this recall that a *framed m-manifold* is an m-manifold M along with a trivialization of its tangent bundle. More concretely, a framed m-manifold M is *framed*, if there exists a list of smooth vector fields

$$(\xi_1, \ldots, \xi_n)$$

such that at each point $x \in M$, the list of vectors $(\xi_i(x))_i$ is a basis of the tangent space $\mathrm{T}_x M$ at x. More generally, for $n \geq m$, an *n-framing* of the m-manifold M is a trivialization of the stablilized tangent bundle $\mathrm{T}M \oplus \underline{\mathbb{R}^{n-m}}$, where $\underline{\mathbb{R}^{n-m}}$ is the trivial bundle with fiber $\mathbb{R}^{n-m}$. With all that established, the *symmetric monoidal d-category of framed Cobordisms*, denoted by ${}^{\amalg}\mathbf{Cob}_d^{\mathrm{fr}}$, is defined in the same way as its variant 3.0.1, except that we require that all manifolds be equipped with a d-framing.

Remark 3.3. More generally even, Grady & Pavlov in [1] define

$${}^{\amalg}\mathbf{Cob}_d^{\mathcal{G}}$$

for any given *geometric structure* $\mathcal{G}$.

4 Where is the Quantum?

4.1 Topological QFTs and the Topological Cobordism Hypothesis

With this we may then define quantum field theories:

Definition 4.1.1. Let $\mathscr{V}$ be some symmetric monoidal d-category. A *d-dimensional framed functorial quantum field theory with values in $\mathscr{V}$* is a symmetric monoidal functor

$$^{\amalg}\mathbf{Cob}_d^{\mathrm{fr}} \xrightarrow{\mathscr{Q}} \mathscr{V}$$

The symmetric monoidal space (i.e. groupoid) (make a footnote here) of such quantum field theories is denoted by

$$\mathscr{Q}\mathrm{FT}_d^{\mathrm{fr}}(\mathscr{V}) := \mathrm{SymFun}(^{\amalg}\mathbf{Cob}_d^{\mathrm{fr}}, \mathscr{V})$$

Lurie's *topological Cobordism Hypothesis*, nowadays probably more aptly referred to as a Theorem, is then as follows:

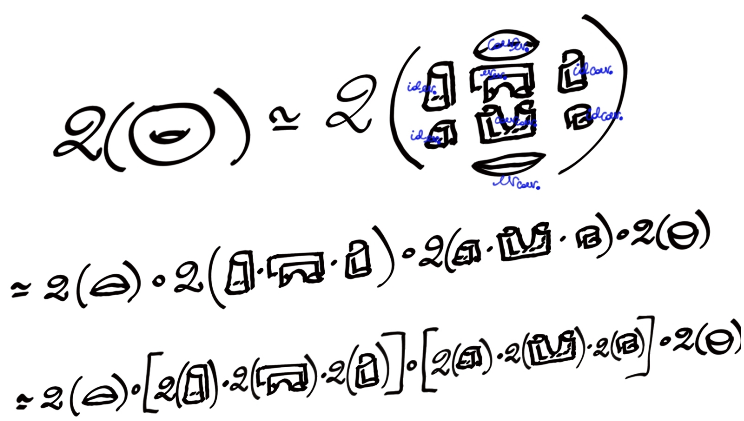
Theorem 4.1.1. *Evaluation at a point* $\bullet \in {}^{\amalg}\mathbf{Cob}_d^{\mathrm{fr}}$ *induces an equivalence*

$$\begin{aligned} \mathscr{Q}\mathrm{FT}_d^{\mathrm{fr}}(\mathscr{V}) &\to \mathrm{Duals}_{\mathscr{V}}^{\simeq} \\ \mathscr{Q} &\mapsto \mathscr{Q}(\bullet) \end{aligned}$$

where $\mathrm{Duals}_{\mathscr{V}}^{\simeq}$ *denotes the maximal subgroupoid of $\mathscr{V}$ whose objects consist of all the fully dualizable objects in $\mathscr{V}$.*

Remark 4.2. Stated differently, the *framed* cobordism Theorem states that ${}^{\amalg}\mathbf{Cob}_d^{\mathrm{fr}}$ is the *free symmetric monoidal, dualic d-category generated from a point*[9].

The idea to the above equivalence is that a *framed field theory* ought to be already completely determined by its value on a point. Indeed, any cobordism may be cut up into into smaller and smaller pieces, until eventually we have a decomposition of our cobordism in terms of (higher) duality information of the point. To illustrate this, consider the following example of the donut (the yummiest of manifolds) and cut it as indicated: Since a field theory (i.e. a symmetric monoidal functor) $\mathscr{Q}$ preserves fully dualizable objects and moreover, $\mathscr{Q}(M \amalg N) \simeq \mathscr{Q}M \boxtimes_{\mathscr{V}} \mathscr{Q}N$, we have (since composition is read from bottom to top):



[9]Note that this statement is really equivalent, even though it looks more like there is some funny truncation business going on here. The reason for the groupoid core popping up on the RHS is that the LHS is always a groupoid by 2.3.10.

4.2 The Geometric Cobordism Hypothesis

To be written, or not; probably don't have time for this in the talk anyways!

References

[1] Daniel Grady and Dmitri Pavlov. The geometric cobordism hypothesis, 2022. URL https://arxiv.org/abs/2111.01095. 1, 4, 5, 15

[2] Daniel Grady and Dmitri Pavlov. Extended field theories are local and have classifying spaces, 2023. URL https://arxiv.org/abs/2011.01208. 1

[3] Jacob Lurie. On the classification of topological field theories, 2009. URL https://arxiv.org/abs/0905.0465. 1

[4] Urs Schreiber. Differential cohomology in a cohesive infinity-topos, 2013. URL https://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos.