PDE'S - EXERCISES 36-39

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In all these exercises we use the abbreviations $H^s := H^s(\mathbb{R}^d)$, $\dot{H}^s := \dot{H}^s(\mathbb{R}^d)$, $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{C}_b := \mathscr{C}_b(\mathbb{R}^d)$, $\mathscr{C}_b^k := \mathscr{C}_b^k(\mathbb{R}^d)$ and $\mathscr{C}_c^\infty := \mathscr{C}_c^\infty(\mathbb{R}^d)$. Moreover, for $r > 0$ we define $B_r := \{x \in \mathbb{R}^d \mid |x| < r\}$ and $S^{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$.

1 EXERCISE 36:

We are given a function $\phi \in \mathscr{C}_b^\infty$ and we are asked to "extend" the map $\phi(|\nabla|^2) : \mathscr{S} \to \mathscr{S}$ to an operator on $\mathscr{S}'$. First of all, the map $\phi(|\nabla|^2) : \mathscr{S} \to \mathscr{S}$ is defined by

$$\phi(|\nabla|^2)f := \mathscr{F}^{-1}(\phi(|.|^2)\mathscr{F}f) \tag{1}$$

Note that this is well defined, since we surely have $\phi(|.|^2)\mathscr{F}f \in \mathscr{S}$ for all $f \in \mathscr{S}$ and therefore also $\phi(|\nabla|^2)(f) \in \mathscr{S}$ for all $f \in \mathscr{S}$. Set $\zeta := \overline{\phi(|\nabla|^2)}$, that is, for all $f \in \mathscr{S}$

$$\zeta(f) := \mathscr{F}^{-1}(\overline{\phi(|.|^2)}\mathscr{F}f) \in \mathscr{S} \tag{2}$$

then ζ induces a canonical map

$$\widetilde{\zeta} : \mathscr{S}' \to \{\text{linear maps } \mathscr{S} \to \mathbb{C}\} \qquad \psi \mapsto (\mathscr{S} \to \mathbb{C}, f \mapsto \psi(\zeta(f))) \tag{3}$$

This is our candidate for an extension of $\phi(|\nabla|^2)$ to $\mathscr{S}'$ in the following sense: We are able to embed $\mathscr{S} \hookrightarrow \mathscr{S}'$ by the injection $i : g \mapsto \psi_g = (. \mid g)_{L^2}$. For $f, g \in \mathscr{S}$ we then calculate

$$\begin{aligned}\widetilde{\zeta}(\psi_g)(f) &= \psi_g(\zeta(f)) = (\mathscr{F}^{-1}(\overline{\phi(|.|^2)}\mathscr{F}f) \mid g)_{L^2} = (\overline{\phi(|.|^2)}\mathscr{F}f \mid \mathscr{F}g)_{L^2} &(4)\\ &= (\mathscr{F}f \mid \phi(|.|^2)\mathscr{F}g)_{L^2} = (f \mid \phi(|\nabla|^2)g)_{L^2} = \psi_{\phi(|\nabla|^2)g}(f) &(5)\end{aligned}$$

Thus for all $g \in \mathscr{S}$

$$i \circ \phi(|\nabla|^2)(g) = \psi_{\phi(|\nabla|^2)g} = \widetilde{\zeta}(\psi_g) \tag{6}$$

In that sense $\widetilde{\zeta}$ can truly be thought of as an extension of $\phi(|\nabla|^2)$. All that remains to verify is that $\widetilde{\zeta}$ actually maps into $\mathscr{S}'$, i.e. we have to verify that $\widetilde{\zeta}(\psi) \in \mathscr{S}'$ for all $\psi \in \mathscr{S}'$. Fix $\psi \in \mathscr{S}'$ and $f \in \mathscr{S}$, then since ψ is a distribution there exists $N \in \mathbb{N}$ so that

$$\begin{aligned}|\widetilde{\zeta}(\psi)(f)| = |\psi(\zeta(f))| &\lesssim \sum_{|\alpha| \leq N} \|\langle . \rangle^N \partial^\alpha(\zeta(f))\|_{L^\infty} = \sum_{|\alpha| \leq N} \|\langle . \rangle^N \partial^\alpha(\mathscr{F}^{-1}(\overline{\phi(|.|^2)}\mathscr{F}f))\|_{L^\infty} &(7)\\ = \sum_{|\alpha| \leq N} \|\langle . \rangle^N \mathscr{F}^{-1}\{(2\pi i\xi)^\alpha \overline{\phi(|.|^2)}\mathscr{F}f\}\|_{L^\infty} &= \sum_{|\alpha| \leq N} \|\langle . \rangle^N \mathscr{F}^{-1}\{(\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\}\|_{L^\infty} &(8)\end{aligned}$$

We will now estimate each summand $\|\langle . \rangle^N \mathscr{F}^{-1}\{(\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\}\|_{L^\infty}$ separately for each $|\alpha| \leq N$. We do this as follows:

$$\begin{aligned}\|\langle . \rangle^N \mathscr{F}^{-1}\{\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\}\|_{L^\infty} &\simeq \|(1+|.|)^N \mathscr{F}^{-1}\{\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\}\|_{L^\infty} &(9)\\ &\simeq \|(1+|.|^N)\mathscr{F}^{-1}\{\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\}\|_{L^\infty} &(10)\\ \lesssim \underbrace{\|\mathscr{F}^{-1}\{\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\}\|_{L^\infty}}_{=A} &+ \underbrace{\||.|^{2N}\mathscr{F}^{-1}\{\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\}\|_{L^\infty}}_{=B} &(11)\end{aligned}$$

We then obtain an estimate for A by

$$A \leq \|\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\|_{L^1} \leq \|\langle . \rangle^{-2L}\overline{\phi(|.|^2)}\|_{L^2}\|\langle . \rangle^{2L}\mathscr{F}(\partial^\alpha f)\|_{L^2} \tag{12}$$

$$\lesssim \|\langle . \rangle^{2L}\mathscr{F}(\partial^\alpha f)\|_{L^2} \simeq \|(1+|.|^{2L})\mathscr{F}(\partial^\alpha f)\|_{L^2} \leq \|\mathscr{F}(\partial^\alpha f)\|_{L^2} + \||.|^{2L}\mathscr{F}(\partial^\alpha f)\|_{L^2} \tag{13}$$

$$= \|\partial^\alpha f\|_{L^2} + \||.|^{2L}\mathscr{F}(\partial^\alpha f)\|_{L^2} \tag{14}$$

Moreover, we have

$$\||.|^{2L}\mathscr{F}(\partial^\alpha f)\|_{L^2} \lesssim \sum_j \|\xi_j^{2L}\mathscr{F}(\partial^\alpha f)\|_{L^2} \simeq \sum_j \|\mathscr{F}(\partial_j^{2L}\partial^\alpha f)\|_{L^2} = \sum_j \|\partial_j^{2L}\partial^\alpha f\|_{L^2} \tag{15}$$

$$\lesssim \sum_j \|\langle . \rangle^L \partial_j^{2L}\partial^\alpha f\|_{L^\infty} \tag{16}$$

Now we aim to get an inequality for B:

$$B \lesssim \sum_j \|\xi_j^{2N}\mathscr{F}^{-1}\{\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha)\}\|_{L^\infty} \tag{17}$$

$$\simeq \sum_j \|\mathscr{F}^{-1}\{\partial_j^{2N}[\overline{\varphi(|.|^2)}\mathscr{F}(\partial^\alpha f)]\}\|_{L^\infty} \leq \sum_j \|\partial_j^{2N}[\overline{\varphi(|.|^2)}\mathscr{F}(\partial^\alpha f)]\|_{L^1} \tag{18}$$

Now note that by the Leibniz product rule we may estimate the expression $\partial_j^{2N}[\overline{\varphi(|.|^2)}\mathscr{F}(\partial^\alpha f)]$ to obtain the extremely obvious and famous (not really famous, but this has turned into a joke now) "estimate of the lazy mathematician" which goes as follows:

$$\left|\sum_k \binom{2N}{k} \partial_j^k(\overline{\phi(|.|^2)})\partial_j^{2N-k}\mathscr{F}(\partial^\alpha f)\right| \lesssim \sum_{k,l,n} |\partial_j^n\overline{\phi(|.|^2)}\xi_j^l\partial_j^{2N-k}\mathscr{F}(\partial^\alpha f)| \tag{19}$$

where l,n run over index sets such that all derivatives of $\partial_j^k(\overline{\phi(|.|^2)})$ are obtained (that is l,n are in particular dependent on k). By recalling that $\phi \in \mathscr{C}_b^\infty$ and by using (19) in the inequality (18) we infer that

$$B \lesssim \sum_{j,k,l,n} \|\partial_j^n\overline{\phi(|.|^2)}\xi_j^l\partial_j^{2N-k}\mathscr{F}(\partial^\alpha f)\|_{L^1} \lesssim \sum_{j,k,l,n} \|\langle . \rangle^{2L}\mathscr{F}(\xi_j^{2N-k}\partial^\alpha f)\|_{L^2} \tag{20}$$

And we have seen in the estimate of A that this boils down to an estimate of the form

$$B \lesssim \sum_{j,k,l,n,m} \|\langle . \rangle^L \partial_m^{2L}\partial^\alpha f\|_{L^\infty} \tag{21}$$

All this was achieved by using some sufficiently large natural number L. We see from the above that we were able to estimate both A and B by finite sums of expressions of the form $\|\langle . \rangle \partial^\beta f\|_{L^\infty}$. Thus we may simply estimate

$$\|\langle . \rangle^N\mathscr{F}^{-1}\{(\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\}\|_{L^\infty} \lesssim \sum_{|\beta|\leq K_\alpha} \|\langle . \rangle^{K_\alpha}\partial^\beta f\|_{L^\infty} \tag{22}$$

and therefore we may also estimate all the summands at once by

$$\sum_{|\alpha|\leq N} \|\langle . \rangle^N\mathscr{F}^{-1}\{(\overline{\phi(|.|^2)}\mathscr{F}(\partial^\alpha f)\}\|_{L^\infty} \lesssim \sum_{|\beta|\leq K} \|\langle . \rangle^K\partial^\beta f\|_{L^\infty} \tag{23}$$

for some sufficiently large K. Hence in total we have

$$|\widetilde{\zeta}(\psi)(f)| \lesssim \sum_{|\beta|\leq K} \|\langle . \rangle^K\partial^\beta f\|_{L^\infty} \tag{24}$$

which yields $\widetilde{\zeta}(\psi) \in \mathscr{S}'$, as wanted.

2 EXERCISE 37:

We are given a function $g \in \mathscr{C}_b$ such that $(f \mid g)_{L^2} = 0$ for all $f \in \mathscr{S}$. We shall prove that $g = 0$: For the sake of contradiction, assume $g \neq 0$. Without loss of generality $\mathfrak{Re}(g) \neq 0$, that is, there exists an open set $U \subset \mathbb{R}^d$ such that the restriction of $\mathfrak{Re}(g)$ to U satisfies $|\mathfrak{Re}(g)|_U| > 0$. We know from lectures on mathematical analysis that there exists a smooth bump function $f \in \mathscr{C}_c^\infty \subset \mathscr{S}$ such that $\operatorname{supp}(f) \subset U, 0 \leq f \leq 1$ and $f|_K \equiv 1$ for some compact set $K \subset U$. Therefore,

$$|(f \mid g)_{L^2}| = \left| \int f\overline{g} \right| = \left| \int_U f\overline{g} \right| \geq \left| \int_K \mathfrak{Re}(g) \right| > 0 \tag{25}$$

which contradicts $(f \mid g)_{L^2} = 0$.

3 EXERCISE 38:

We are given a tempered distribution $\phi \in \mathscr{S}'$ and a function $g \in \mathscr{C}_b^\infty$ and we shall prove that the map $g\phi$, defined by

$$g\phi(f) := \phi(\overline{g}f) \tag{26}$$

for all $f \in \mathscr{S}$, is a tempered distribution. Certainly enough, $g\phi$ defines a linear functional on the Schwarz space and by applying Leibniz's product rule in the multivariate setting we arrive at

$$|g\phi(f)| = |\phi(\overline{g}f)| \lesssim \sum_{|\alpha| \leq N} \|\langle\cdot\rangle^N \partial^\alpha(\overline{g}f)\|_{L^\infty} = \sum_{|\alpha| \leq N} \|\langle\cdot\rangle^N \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\overline{g}\partial^\beta f\|_{L^\infty} \tag{27}$$

$$\lesssim \sum_{|\alpha| \leq N} \sum_{\beta \leq \alpha} \|\langle\cdot\rangle^N \partial^\beta f\|_{L^\infty} \simeq \sum_{|\alpha| \leq N} \|\langle\cdot\rangle^N \partial^\alpha f\|_{L^\infty} \tag{28}$$

This exactly means $g\phi \in \mathscr{S}'$.

4 EXERCISE 39:

Let $(f_l) \subset \mathscr{S}$ be Cauchy in H^1 and suppose that $g \in \mathscr{C}_b^\infty$. We are asked to prove that (gf_l) is Cauchy in H^1: First of all, it is clear that $(gf_l) \subset \mathscr{S}$, since $g \in \mathscr{C}_b^\infty$. Furthermore, by exercise 9 we know that

$$\|f_k - f_l\|_{H^1} \simeq \sum_{|\alpha| \leq 1} \|\partial^\alpha(f_k - f_l)\|_{L^2} \tag{29}$$

and therefore both $(f_l) \subset \mathscr{S}$ and all derivatives $(\partial_j f_l) \subset \mathscr{S}$ are Cauchy in L^2. We then define $M := \max_{j=1}^{d}(\|g\|_{L^\infty}, \|\partial_j g\|_{L^\infty})$ and immediately estimate

$$\|g(f_k - f_l)\|_{H^1} \simeq \|g(f_k - f_l)\|_{L^2} + \sum_{j=1}^{d} \|\partial_j g(f_k - f_l) + g\partial_j(f_k - f_l)\|_{L^2} \tag{30}$$

$$\leq M\left(\|f_k - f_l\| + \sum_{j=1}^{d} \left(\|f_k - f_l\|_{L^2} + \|\partial_j(f_k - f_l)\|_{L^2}\right) \right) \longrightarrow 0 \tag{31}$$