PDE'S - EXERCISES 48-51

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In all these exercises we use the abbreviations $H^s := H^s(\mathbb{R}^d)$, $\dot{H}^s := \dot{H}^s(\mathbb{R}^d)$, $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{C}_b := \mathscr{C}_b(\mathbb{R}^d)$, $\mathscr{C}_b^k := \mathscr{C}_b^k(\mathbb{R}^d)$ and $\mathscr{C}_c^\infty := \mathscr{C}_c^\infty(\mathbb{R}^d)$. Moreover, for $r > 0$ we define $B_r := \{x \in \mathbb{R}^d \mid |x| < r\}$ and $S^{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$.

1 EXERCISE 48:

The problem with the "proof" given is that even though we know that there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ which converges weakly to some $u \in H$, and we even know that the sequence of norms $(\|u_{k_j}\|)_{j\in\mathbb{N}} = (1)_{j\in\mathbb{N}}$ converges we do not know that $\|u_{k_j}\| \to \|u\|$, which would ultimately be needed to obtain that $(u_{k_j})_{j\in\mathbb{N}}$ converges strongly to u.

2 EXERCISE 49:

We are given a radial $f_0 \in \mathscr{S}_{\mathrm{rad}}$ such that $(f_0 \mid f)_{L^2} = 0$ for all $f \in \mathscr{S}_{\mathrm{rad}}$. We shall deduce that $(f_0 \mid f)_{L^2} = 0$ for all $f \in \mathscr{S}$. However, this exercise seems to be a joke:

$$\|f_0\|_{L^2}^2 = (f_0 \mid f_0)_{L^2} = 0 \implies f_0 = 0 \tag{1}$$

3 EXERCISE 50:

We are given radial $F, G \in L^2_{\mathrm{rad}}$ such that for all $f \in \mathscr{S}_{\mathrm{rad}}$ we have $\phi_F(f) = \phi_G(f)$. We shall now prove that this already implies $F = G$.

Lemma 1. If $H \in L^p$, then $\phi_H : \mathscr{S} \to \mathbb{C}$ extends to a bounded linear functional $\phi_H : L^q \to \mathbb{C}$ (where q is conjugate to p).

Proof. Let $(h_t) \subset \mathscr{S}$ be an arbitrary representative for H. For $K \in L^q$ let $(k_l) \subset \mathscr{S}$ be a representative for K. Now define

$$\phi_H(K) := \lim_{l\to\infty} \phi_H(k_l) \tag{2}$$

This is well defined. Indeed,

$$|\phi_H(k_l) - \phi_H(k_n)| = \lim_{t\to\infty} \left|\left(k_l - k_n \mid h_t\right)_{L^2}\right| \le \|k_l - k_n\|_{L^q}\|H\|_{L^p} \longrightarrow 0 \tag{3}$$

Thus the limit in (2) exists and moreover it is easily seen to be independent of the chosen representative $(k_l) \subset \mathscr{S}$ for K. In particular, we have the estimate

$$|\phi_H(K)| \le \|K\|_{L^q}\|H\|_{L^p} \tag{4}$$

□

By the preceding Lemma we know that $\phi_{F-G} = \phi_F - \phi_G \colon L^2 \to \mathbb{C}$ yields a bounded linear functional and it satisfies $\phi_{F-G}|_{\mathscr{S}_{\text{rad}}} = 0$. In particular, if $H \in L^2_{\text{rad}}$ with a radial representative $(h_t) \subset \mathscr{S}_{\text{rad}}$, then

$$\phi_{F-G}(H) = (H \mid F-G)_{L^2} = \lim_t \underbrace{\phi_{F-G}(h_t)}_{=0 \text{ by assumption}} = 0 \tag{5}$$

Letting $H = F - G \in L^2_{\text{rad}}$ we conclude from (5) that

$$\|F-G\|_{L^2}^2 = 0 \implies F = G \tag{6}$$

4 EXERCISE 51:

Recall that for all $f \in \mathscr{S}$ we define

$$e^{it\Delta} f := \mathscr{F}^{-1}\left(e^{-4\pi^2 it|.|^2} \mathscr{F} f\right) \tag{7}$$

and we call $e^{it\Delta}$ the Schrödinger propagator. Since the Fourier transform maps Schwartz functions to Schwartz functions we certainly have $e^{-4\pi^2 it|.|^2} \mathscr{F} f \in \mathscr{S}$ (as $e^{-4\pi^2 it|.|^2}$ and all its derivatives grow polynomially). Well, taking yet again the inverse Fourier transform of the preceding expression immediately yields $e^{it\Delta} f \in \mathscr{S}$ for all $f \in \mathscr{S}$.