BACHELORARBEIT

Aspects of the theory of C*-operator algebras

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Abstract

Mathematical theories often, if not always, are motivated by looking closely at the real world. The first such example of a mathematical theory that should come to mind is, of course, calculus. Just as calculus was derived for the sole purpose of making sense of motion, acceleration, or more generally change, that is, calculus was invented in order to supply classical mechanics with the necessary mathematical tools, functional analysis was born from the depths of hell (or so some people might think) so as to furnish the quantum theorists with the long sought tools and give rigorous descriptions for quantum mechanics in general. One such description is the algebraic formulation of quantum mechanics, that is, the C*-algebraic approach. In this thesis we will derive the basics of the theory of C*-operator algebras. We will, however, not get into the physical theory that is quantum mechanics, as this work is intended to be purely mathematical. If one is interested in applying the theory to describe quantum mechanics see [11].

We will begin our journey by forging a suitable topology, called the weak*-topology, so as to lay the groundwork for Banach algebras. The theory of Banach algebras will be discussed in general at first, but then we will focus on the abelian case and introduce the celebrated Gel'fand transform. After having done so, we will devote ourselves to the study of C*-algebras and we will prove the first Gel'fand Naimark theorem, which roughly states that abelian C*-algebras are nothing more than continuous functions (that is up to *-isomorphism). Our journey then leads us to the derivation of the CFC (=Continuous Functional Calculus) in the setting of C*-algebras, an immensely important tool in functional analysis, especially when applied to spectral theory. By means of the CFC we are then able to give meaning to the notion of positivity of elements in a C*-algebra. After having done so, we will concern ourselves with certain linear functionals called states, so as to be able to derive one of the most iconic statements of the field; the GNS-construction (Gel'fand-Naimark-Segal). The GNS construction will enable us to prove the second Gel'fand-Naimark representation theorem, which roughly states that every C*-algebra is, up to *-isomorphism, just a subset of bounded linear operators on some Hilbert space, which will conclude our journey.

Contents

1 Notation and Preliminaries

1.1 Notation

Throughout the text, if we have a set X and a subset $A \subset X$, we will write A^c to denote the complement of A, that is $A^c := X \backslash A$. Whenever we write $\mathbb{F}$ we mean, that $\mathbb{F}$ can be both the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. If X is a topological space, then $\mathscr{C}(X)$ will always denote the continuous functions mapping from $X \to \mathbb{F}$, $\mathscr{C}_b(X)$ will always denote all bounded continuous functions from $X \to \mathbb{F}$. If we have a normed space $\mathscr{X}$, then its dual space (=all bounded linear functionals mapping $\mathscr{X} \to \mathbb{F}$) will be denoted by $\mathscr{X}^*$ and likewise we will write $\mathscr{X}^{**}$ to mean the bidual space. If we have a topological space X and a subset $A \subset X$, then we will write $cl(A)$ to denote the topological closure of A with respect to the topology on X.

1.2 Functional analysis

The following theorems will be used throughout the whole work. We will state them only for the convenience of the reader and give references on where proofs can be found.

Theorem 1.1 (Hahn-Banach Theorem)**.** *If $\mathscr{X}$ is a normed space, $\mathscr{M}$ is a linear subspace in $\mathscr{X}$, and $m^* : \mathscr{M} \to \mathbb{F}$ is a bounded linear functional, then there is an x^* in $\mathscr{X}^*$ such that $x^*|_{\mathscr{M}} = m^*$ and $\|x^*\| = \|m^*\|$.*

Proof. [4], p.79, Corollary 6.5 ■

Theorem 1.2 (Principle of Uniform Boundedness - PUB)**.** *Let $\mathscr{X}$ be a Banach space and $\mathscr{Y}$ be a normed space. If $\mathscr{A} \subset \mathscr{L}(\mathscr{X}, \mathscr{Y})$ (= bounded linear operators mapping $\mathscr{X} \to \mathscr{Y}$) such that*

$$\forall x \in \mathscr{X} : \ \sup\{\|Ax\| : A \in \mathscr{A}\} < \infty$$

then

$$\sup\{\|A\| : A \in \mathscr{A}\} < \infty$$

Proof. [4], page 95, Theorem 14.1 ■

Theorem 1.3 (The Stone-Weierstrass Theorem)**.** *Let X be a compact space and let $\mathscr{X}$ be a closed subalgebra in $\mathscr{C}(X)$. Suppose the following holds:*

(i). $1 \in \mathscr{X}$

(ii). $\forall x, y \in X$ *such that* $x \neq y$, *we have* $\exists f \in \mathscr{X}$ *with* $f(x) \neq f(y)$, *that is* $\mathscr{X}$ *separates the points of* X

(iii). $f \in \mathscr{X} \implies \overline{f} \in \mathscr{X}$

Then $\mathscr{X} = \mathscr{C}(X)$.

Proof. [4], p. 145, 8.1 ■

1.3 Measure Theory

If (X, Ω, μ) is a measure space, where μ is a complex measure, then we define the variation of μ by

$$|\mu|(E) := \sup\left\{\sum |\mu(E_j)| : \{E_j\} \subset \Omega, \{E_j\} \text{ is a finite partition of } E\right\}$$

for all $E \in \Omega$. It can be shown, that $|\mu|$ defines a finite, positive measure on (X, Ω). Let $B(X)$ be the Borel-sigma algebra of X, then a positive measure μ on $(X, B(X))$ is said to be regular, if

(i). $\mu(K) < \infty$ for all compact subsets $K \subset X$

(ii). for all $A \in B(X)$ we have

$$\mu(A) = \sup\left\{\mu(K)\colon\ K \subset A, K \text{ compact}\right\} = \inf\left\{\mu(O)\colon\ O \supset A, O \text{ open}\right\}$$

A complex-valued Borel-measure is said to be regular, if the variation of μ is regular. If X is a locally compact space, that is for every point $x \in X$ there exists a compact neighbourhood for x, then we will define $M(X)$ to be the space of all complex-valued regular Borel measures on X. If $\mu \in M(X)$ define $\|\mu\| := |\mu|(X)$, then $\|.\|$ is a norm and $(M(X), \|.\|)$ is a Banach space. Another important sort of continuous functions are those which vanish at infinity (we will see later, why those are called that way). These are defined on a locally compact Hausdorff space X by

$$\mathscr{C}_0(X) := \left\{f \in \mathscr{C}(X) \mid \forall \varepsilon > 0\colon\ \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\right\}$$

One can show, that $\mathscr{C}_0(X)$ as defined above yields a Banach space, where the norm is the supremum norm.

Theorem 1.4 (Riesz Representation Theorem)**.** *If X is a locally compact Hausdorff space, then we have the isometric isomorphism $\mathscr{C}_0(X)^* \cong M(X)$. In particular, for every $\Phi \in \mathscr{C}_0(X)^*$, there exists a unique $\mu_\Phi \in M(X)$ such that for all $f \in \mathscr{C}_0(X)$ we have*

$$\Phi(f) = \int_X f d\mu_\Phi$$

and $\|\Phi\| = \|\mu_\Phi\|$.

Proof. [4], p. 383, C.18 ∎

2 Weak Topologies

This chapter is intended to give the necessary background on weak and weak-*-topologies. The material thus presented cannot, by no means, do justice to this subject, as we will only scratch the surface. However, we will still prove the necessary results, so as to be able to reasonably study C*-algebras later. This chapter is based on [3] and [4].

Definition 2.1. *A TVS (Topological vector space) $\mathscr{V}$ is a vector space endowed with a topology with respect to which the maps $\mathbb{F} \times \mathscr{V} \mapsto \mathscr{V}$ given by $(\lambda, x) \mapsto \lambda x$ and $\mathscr{V} \times \mathscr{V} \mapsto \mathscr{V}$ given by $(x, y) \mapsto x + y$, are continuous.*

Now the idea is to topologize a given vector space $\mathscr{V}$ by a family of seminorms $\mathscr{P}$ on $\mathscr{V}$. This can be achieved by defining $\mathscr{T}$ to be the topology that has as a subbase the sets

$$\{x \in \mathscr{V} : \, p(x - x_0) < \varepsilon\}$$

where $x_0 \in \mathscr{V}$, $p \in \mathscr{P}$, and $\varepsilon > 0$. It is then quite clear, that a subset U in $\mathscr{V}$ is open if and only if for all $x_0 \in U$ there are $p_1, ..., p_m \in \mathscr{P}$ and $\varepsilon_1, ..., \varepsilon_m > 0$ such that

$$x_0 \in \bigcap \left\{x \in \mathscr{V} : \, p_j(x - x_0) < \varepsilon_j, 1 \leq j \leq m\right\} \subset U$$

Moreover, note that $\mathscr{V}$ endowed with this topology is a TVS. Indeed, let $f \colon \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ be given by $(x, y) \mapsto x + y$. It suffices to prove, that the f-preimage of any sub-basic open set is open. So let $(x, y) \in \mathscr{V} \times \mathscr{V}$ and let $\varepsilon > 0$. Consider the open sets

$$O_1 = \{w \in \mathscr{V} \mid p(x - w) < \varepsilon/2\} \qquad O_2 = \{z \in \mathscr{V} \mid p(y - z) < \varepsilon/2\}$$

for some fixed $p \in \mathscr{P}$. Then $O_1 \times O_2$ is open with respect to the product topology on $\mathscr{V} \times \mathscr{V}$ and clearly for all $(w, z) \in O_1 \times O_2$ we have

$$p(f(w, z) - f(x, y)) = p((w - x) + (z - y)) \leq p(w - x) + p(z - y) < \varepsilon$$

Therefore f is continuous and similarly we note that $g \colon \mathbb{F} \times \mathscr{V} \to \mathscr{V}$ given by $(\lambda, x) \mapsto \lambda x$ is continuous, so $\mathscr{V}$ is a TVS.

In analysis we always want, that convergent sequences, or more generally, convergent nets have at most one limit. So in the analytic framework, all the spaces that are reasonable to work in should be Hausdorff.

Definition 2.2. *A locally convex space (LCS) is a TVS whose topology is induced by a family of seminorms $\mathscr{P}$, such that*

$$\bigcap_{p \in \mathscr{P}} \{x \in \mathscr{V} \mid p(x) = 0\} = \{0\}$$

Remark 2.1. *As hinted before we want to deal solely with Hausdorff spaces. An LCS is defined precisely in such a way, that will make it Hausdorff, which can be easily seen as follows: Let $x, y \in \mathscr{V}$, $x \neq y$. Then by definition of an LCS we know, that there exists $p \in \mathscr{P}$ such that $p(x - y) > 0$ (since $x - y \neq 0$). Let $\varepsilon > 0$ so that $p(x - y) > \varepsilon > 0$ and set*

$$U := \{z \in \mathscr{V} \mid p(x - z) < \varepsilon/2\} \qquad V := \{z \in \mathscr{V} \mid p(y - z) < \varepsilon/2\}$$

Assume that there is some $z \in U \cap V$, then $\varepsilon < p(x - y) \leq p(x - z) + p(z - y) < \varepsilon$ which yields a contradiction. Thus $U \cap V = \emptyset$ and therefore $\mathscr{V}$ is indeed Hausdorff.

Definition 2.3. *Let $(\mathscr{V}, \|.\|)$ be a normed space and for $x^* \in \mathscr{V}^*$ we define*

$$p_{x^*}(x) = |x^*(x)|$$

for all $x \in \mathscr{V}$. We notice immediately that p_{x^*} is a seminorm on $\mathscr{V}$ for all $x^* \in \mathscr{V}^*$. We set $\mathscr{P} = \{p_{x^*} \mid x^* \in \mathscr{V}^*\}$ and we *will call the topology induced by $\mathscr{P}$ the weak topology on $\mathscr{V}$ and write it as $\sigma(\mathscr{V},\mathscr{V}^*)$. Analogously for $x \in \mathscr{V}$, define*

$$p_x(x^*) = |x^*(x)|$$

for all $x^* \in \mathscr{V}^*$ to get a family $\mathscr{P} = \{p_x \mid x \in \mathscr{V}\}$ of seminorms on $\mathscr{V}^*$ and call the topology thus induced the weak-topology on $\mathscr{V}^*$ and we will write it as $\sigma(\mathscr{V}^*,\mathscr{V})$.*

Recall from functional analysis, that there is a natural inclusion $\mathscr{X} \hookrightarrow \mathscr{X}^{**}$ given by $x \mapsto ev_x$, where $ev_x\colon \mathscr{X}^* \to \mathbb{F}$ is given by $ev_x(x^*) := x^*(x)$. It is not hard to show, that this is an isometric, linear map.

Lemma 2.1. *The weak topology $\sigma(\mathscr{V},\mathscr{V}^*)$ agrees with the initial topology on $\mathscr{V}$ induced by the functionals $\mathscr{V}^*$, that is $\sigma(\mathscr{V},\mathscr{V}^*)$ is the coarsest topology on $\mathscr{V}$, such that all functionals $x^* \in \mathscr{V}^*$ are continuous. Moreover $\sigma(\mathscr{V}^*,\mathscr{V})$ is the inital topology on $\mathscr{V}^*$ with respect to the functionals $\{ev_x \mid x \in \mathscr{V}\} \subset \mathscr{V}^{**}$.*

Proof. For $x^* \in \mathscr{V}^*$, $x \in \mathscr{V}$ and $\varepsilon > 0$, consider the open neighborhood of x given by

$$U := \{z \in \mathscr{V} \mid p_{x^*}(z-x) = |x^*(z) - x^*(x)| < \varepsilon\}$$

This makes it clear, that all linear functionals in $\mathscr{V}^*$ are continuous with respect to $\sigma(\mathscr{V},\mathscr{V}^*)$. On the other hand if $\mathscr{T}$ is a topology on $\mathscr{V}$ such that all linear functionals $\mathscr{V}^*$ are continuous, then clearly $\mathscr{T}$ contains all the sub-basic sets

$$\{x \in \mathscr{V} \mid \underbrace{p_{x^*}(x-x_0)}_{|x^*(x)-x^*(x_0)|} < \varepsilon\}$$

of $\sigma(\mathscr{V},\mathscr{V}^*)$. The second part of the lemma is equally easy to prove, since for one the sets

$$\{x^* \in \mathscr{V}^* \mid p_x(x^* - x_0^*) = |x^*(x) - x_0^*(x)| < \varepsilon\}$$

immediately imply continuity of the family $\{ev_x \mid x \in \mathscr{V}\}$ and on the other hand it is evident, by an analogous argument as before, that this must be the coarsest topology such that the family $\{ev_x \mid x \in \mathscr{V}\}$ is continuous. ■

Remark 2.2. *The weak topology $\sigma(\mathscr{V}^*,\mathscr{V}^{**})$ is stronger than the weak*-topology $\sigma(\mathscr{V}^*,\mathscr{V})$.*

Proposition 2.1. *If $\mathscr{V}$ is a normed vector space, then*

(i). the weak topology on $\mathscr{V}$ is Hausdorff

(ii). the weak-topology on $\mathscr{V}^*$ is Hausdorff*

Proof.

(i). It suffices to prove that

$$S := \bigcap_{x^* \in \mathscr{V}^*} \{x \in \mathscr{V} \mid p_{x^*}(x) = 0\} = \bigcap_{x^* \in \mathscr{V}^*} \{x \in \mathscr{V} \mid x^*(x) = 0\} = \{0\}$$

since in that case $\mathscr{V}$ is an LCS with respect to the weak topology. So assume we have $x^*(x) = 0$ for $x^* \in \mathscr{V}^*$ and $0 \neq x \in \mathscr{V}$. Define $\mathscr{W} = \{\alpha x \mid \alpha \in \mathbb{F}\}$, then $\mathscr{W}$ is a one-dimensional subspace of $\mathscr{V}$ and $f\colon \mathscr{W} \to \mathbb{F}$ given by $\alpha x \mapsto \alpha$ is a linear functional on $\mathscr{W}$. Since all linear maps on finite dimensional spaces are continuous we have $f \in \mathscr{W}^*$. By Theorem 1.1, there exists $F \in \mathscr{V}^*$ with $\|f\| = \|F\|$ and $f = F|_{\mathscr{W}}$. But then $F(x) = 1 \neq 0$, a contradiction. Hence $S = \{0\}$.

(ii). Let $x_1^*, x_2^* \in \mathscr{V}^*$ be two distinct elements. In that case, there exists $x \in \mathscr{V}$ such that we can assume

$$|x_1^*(x)| < \varepsilon < |x_2^*(x)|$$

for some $\varepsilon > 0$. Then, since the weak*-topology coincides with the initial topology on $\mathscr{V}^*$ induced by $\{ev_x \mid x \in \mathscr{V}\}$, we have that

$$x_1^* \in U := \{x^* \in \mathscr{V}^* \mid \underbrace{|x^*(x)|}_{|ev_x(x^*)|} < \varepsilon\} \qquad x_2^* \in V := \{x^* \in \mathscr{V}^* \mid |x^*(x)| > \varepsilon\}$$

are open and $U \cap V = \emptyset$.

■

From now on, if a net $\mathscr{V} \ni x_\lambda \to x \in \mathscr{V}$ weakly, we will write $x_\lambda \rightharpoonup x$ or we will say x_λ converges weakly to x. Similarly for weak*-convergence, we will write $\mathscr{V}^* \ni x_\lambda^* \rightharpoonup^* x^* \in \mathscr{V}^*$. Knowing that the weak and the weak*-topologies both are initial topologies yields some nice attributes straightaway. For one we have, by the universal property of initial topologies, that a function $w\colon \mathscr{W} \to \mathscr{V}$ (where $\mathscr{W}$ is a topological space) is continuous if and only if $x^* \circ w\colon \mathscr{W} \to \mathbb{F}$ is continuous for all $x^* \in \mathscr{V}^*$. Moreover, if $\{x_\lambda\}_{\lambda \in \Lambda} \subset \mathscr{V}$ is a net, then

$$x_\lambda \rightharpoonup x \in \mathscr{V} \iff \forall x^* \in \mathscr{V}^* \colon x^*(x_\lambda) \to x^*(x)$$

Analogously if $\{x_\lambda^*\}_{\lambda \in \Lambda} \subset \mathscr{V}^*$ is a net, then

$$x_\lambda^* \rightharpoonup^* x^* \in \mathscr{V} \iff \forall x \in \mathscr{V} \colon ev_x(x_\lambda^*) = x_\lambda^*(x) \to x^*(x) = ev_x(x^*)$$

This gives a way nicer description of the weak and weak* topology in terms of nets, and since nets characterize a topology completely, this will actually be the description we will work the most with. So now, what about a subset equipped with a weak- or weak*-subspace topology? Let $S \subset \mathscr{V}$. We want to characterize the relative weak topology

$$\sigma(\mathscr{V}, \mathscr{V}^*)|_S = S \cap \sigma(\mathscr{V}, \mathscr{V}^*) := \{S \cap U \mid U \in \sigma(\mathscr{V}, \mathscr{V}^*)\}$$

Recall, that any open set in $\sigma(\mathscr{V}, \mathscr{V}^*)$ is a union of sets of the form

$$\bigcap \{x_j^{*-1}(V_j) \mid 1 \leq j \leq m\}$$

for $x_j^* \in \mathscr{V}^*$ and $V_j \subset \mathbb{F}$ open $(1 \leq j \leq m)$. Obviously

$$S \cap \bigcap \{x_j^{*-1}(V_j) \mid 1 \leq j \leq m\} = \bigcap \{S \cap x_j^{*-1}(V_j) \mid 1 \leq j \leq m\} = \bigcap \{(x_j^*|_S)^{-1}(V_j) \mid 1 \leq j \leq m\}$$

Therefore $\sigma(\mathscr{V}, \mathscr{V}^*)|_S$ is again an initial topology with respect to the family of functionals $\{x^*|_S \colon x^* \in \mathscr{V}^*\}$. Analogously if $S \subset \mathscr{V}^*$, then $S \cap \sigma(\mathscr{V}, \mathscr{V}^*)$ is the initial topology generated by the family $\{ev_x|_S \colon x \in \mathscr{V}\}$.

Corollary 2.1. *Let $\mathscr{X}$ be a Banach space and let $\{x_\lambda\}_\lambda \subset \mathscr{X}$ be a net such that $x_\lambda \rightharpoonup x \in \mathscr{X}$, then $\{x_\lambda\}_\lambda$ is bounded and $\|x\| \leq \liminf \|x_\lambda\|$.*

Proof. Applying the PUB 1.2 to the family of operators $\{ev_{x_\lambda}\}_\lambda$ and using the fact, that $\|ev_{x_\lambda}\| = \|x_\lambda\|$ (since $ev\colon \mathscr{X} \to \mathscr{X}^{**}$ is isometric), we see that $\{x_\lambda\}_\lambda$ is indeed bounded. Now if $x^* \in \mathscr{X}^*$, then

$$|x^*(x_\lambda)| \leq \|x^*\| \|x_\lambda\|$$

and taking the limes inferior on both sides yields

$$|x^*(x)| \leq \|x^*\| \liminf \|x_\lambda\|$$

As a consequence of the Hahn Banach theorem 1.1 we then have

$$\|x\| = \sup_{\|x^*\| \leq 1} |x^*(x)| \leq \liminf \|x_\lambda\|$$

■

2.1 The Banach-Alaoglu Theorem

Up to this point, we didn't even motivate why these topologies are needed in our study of operator algebras. So why all the fuss about weak topologies? From functional analysis, we know that a normed vector space $\mathscr{V}$ is finite dimensional if and only if $B := \{x \in \mathscr{V} \mid \|x\| \leq 1\}$ is compact. Therefore every infinite dimensional normed space will be hard to deal with in terms of compact subsets. Yet, as we know well enough, compactness is essential for analysis. Therefore we need to consider weaker topologies on $\mathscr{V}$, since this means that we will have more compact sets (this is because there are "less" open covers in $\mathscr{V}$). It also turns out that the weak*-topology is exactly that kind of topology we would like to have for a particular object called the maximal ideal space, which we will introduce later. By endowing this space with the weak*-topology, many nice insights can and will be made. We conclude this chapter with a very important result concerning weak*-compactness.

Theorem 2.1. *(Banach Alaoglu) If $\mathscr{V}$ is a normed space, then $B_{\mathscr{V}^*} := \{x^* \in \mathscr{V}^* \mid \|x^*\| \leq 1\} \subset \mathscr{V}^*$ is weak*-compact.*

Proof. Define $P_x := \{\gamma \in \mathbb{F} \mid |\gamma| \leq \|x\|\}$ for all $x \in \mathscr{V}$. Then all P_x are compact in $\mathbb{F}$ and thus by Tychonoff's Theorem

$$P := \prod \{P_x \mid x \in \mathscr{V}\}$$

is compact with respect to the product topology. Now define

$$\Phi\colon B_{\mathscr{V}^*} \to P \qquad x^* \mapsto (\Phi(x^*)(x))_{x\in\mathscr{V}} := (x^*(x))_{x\in\mathscr{V}}$$

Clearly Φ is well defined, since $|\Phi(x^*)(x)| = |x^*(x)| \leq \|x\|$ for all $x \in \mathscr{V}$, so $\Phi(x^*) \in P$. Now the idea here is to show, that $\Phi(B_{\mathscr{V}^*})$ is closed, and therefore compact, in P and that

$$\Phi\colon (B_{\mathscr{V}^*}, \sigma(\mathscr{V}^*, \mathscr{V})|_{B_{\mathscr{V}^*}}) \to \Phi(B_{\mathscr{V}^*})$$

is a homeomorphism (since this implies, that $B_{\mathscr{V}^*}$ is weak*-compact). First we will prove, that Φ defines a continuous map. So take a net $\{x^*_\lambda\}_{\lambda\in\Lambda} \subset B_{\mathscr{V}^*}$ such that $x^*_\lambda \rightharpoonup^* x^* \in B_{\mathscr{V}^*}$. By definition of the relative weak*-convergence and by definition of Φ we have

$$\forall x \in \mathscr{V}: x^*_\lambda(x) \to x^*(x) \iff \forall x \in \mathscr{V}: \Phi(x^*_\lambda)(x) \to \Phi(x^*)(x)$$

But this is equivalent to convergence in the product topology $\Phi(x^*_\lambda) \to \Phi(x^*)$, so Φ is continuous. In addition, if $x^*_1, x^*_2 \in B_{\mathscr{V}^*}$ with $\Phi(x^*_1) = \Phi(x^*_2)$, then for all $x \in \mathscr{V}$ it holds that $x^*_1(x) = x^*_2(x)$, so $x^*_1 = x^*_2$ and therefore Φ is injective. Hence the function

$$\Psi := \Phi^{-1}|_{\Phi(B_{\mathscr{V}^*})}\colon \Phi(B_{\mathscr{V}^*}) \to B_{\mathscr{V}^*}$$

makes sense, and we now set out to prove continuity of Ψ. By the universal property of initial topologies, it suffices to prove, that

$$ev_x|_{B_{\mathscr{V}^*}} \circ \Psi$$

is continuous for all $x \in \mathscr{V}$. Take a net $\{\Phi(y_\lambda)\}_{\lambda\in\Lambda}$ with $y_\lambda, y \in B_{\mathscr{V}^*}$, such that $\Phi(y_\lambda) \to \Phi(y)$. Convergence in the product topology is equivalent to convergence of every coordinate, hence we have

$$\Phi(y_\lambda)(x) = y_\lambda(x) \to y(x) = \Phi(y)(x)$$

for all $x \in \mathscr{V}$. But this exactly means

$$ev_x|_{B_{\mathscr{V}^*}} \circ \Psi(\Phi(y_\lambda)) = y_\lambda(x) \to y(x) = ev_x|_{B_{\mathscr{V}^*}} \circ \Psi(\Phi(y))$$

so $ev_x|_{B_{\mathscr{V}^*}} \circ \Psi$ is continuous for all $x \in \mathscr{V}$ and therefore so is Ψ. We conclude, that Φ is a homeomorphism onto $\Phi(B_{\mathscr{V}^*})$. It remains to prove, that $\Phi(B_{\mathscr{V}^*}) \subset P$ is closed. Let $\{\Phi(x^*_\lambda)\}_{\lambda\in\Lambda} \subset \Phi(B_{\mathscr{V}^*})$ be a net with $\Phi(x^*_\lambda) \to y \in P$. In order to finish the proof we must verify, that $y \in \Phi(B_{\mathscr{V}^*})$, so we need to find $x^* \in B_{\mathscr{V}^*}$ such that $\Phi(x^*) = y$, and such that $x^*_\lambda \rightharpoonup^* x^*$. Yet again by the product topology, we have

$$\Phi(x^*_\lambda) \to y \iff \forall x \in \mathscr{V} : x^*_\lambda(x) \to y(x)$$

So define $x^* : \mathscr{V} \to \mathbb{F}$ by $x^*(x) := y(x)$. Note that $x^* \in B_{\mathscr{V}^*}$, since for all $x, w \in \mathscr{V}, \mu, \gamma \in \mathbb{F}$ we have

$$x^*(\mu x + \gamma w) = y(\mu x + \gamma w) = \lim x^*_\lambda(\mu x + \gamma w) = \lim\{\mu x^*_\lambda(x) + \gamma x^*_\lambda(w)\} = \mu y(x) + \gamma y(w) = \mu x^*(x) + \gamma x^*(w)$$

and

$$|x^*(x)| = |y(x)| = \lim |x^*_\lambda(x)| \leq \|x\|$$

Moreover, since $\forall x \in \mathscr{V} : x^*_\lambda(x) \to y(x) = x^*(x)$, we have $x^*_\lambda \rightharpoonup^* x^*$ and thus $\Phi(x^*_\lambda) \to \Phi(x^*) = y \in \Phi(B_{\mathscr{V}^*})$, so $\Phi(B_{\mathscr{V}^*})$ is closed. ■

3 Banach algebras

In the rest of this work, we will assume that the underlying field of every structure considered is $\mathbb{C}$, unless we explicitly state otherwise. So for example, if we considered a vector space $\mathscr{V}$, then it is always assumed, that we are talking about a $\mathbb{C}$-vector space.

3.1 Definitions and Examples

Before getting to the heart of this exposition, that is the study of C*-algebras, we need to acquaint ourselves with the more general objects called Banach algebras.

Definition 3.1. *An algebra $\mathscr{X}$ is a vector space with an associative, bilinear operation $\mathscr{X} \times \mathscr{X} \to \mathscr{X}$. This operation will be written as $\xi\eta$ for $\xi, \eta \in \mathscr{X}$. A Banach algebra is an algebra $\mathscr{X}$, which is also a Banach space with respect to a norm $\|.\|$, satisfying $\|\xi\eta\| \leq \|\xi\|\|\eta\|$ for all $\xi, \eta \in \mathscr{X}$.*

Example 3.1. *The complex number field $\mathbb{C}$ endowed with the standard norm is obviously a Banach algebra.*

Example 3.2. *Consider $\mathfrak{M}$, the complex vector space of $n \times n$-matrices and equip $\mathfrak{M}$ with some submultiplicative matrix norm $\|.\|$ (take for example the operator norm, or the Frobenius norm). Then, since $(\mathfrak{M}, \|.\|)$ is finite dimensional, it is complete, and thus $\mathfrak{M}$ is a Banach space and if the multiplication is the standard matrix product, $\mathfrak{M}$ is a Banach algebra.*

Example 3.3. *Going a bit bigger than the previous example, if $\mathscr{X}$ is a Banach space (potentially infinite dimensional), then the bounded linear operators $\mathscr{L}(\mathscr{X})$ on $\mathscr{X}$ endowed with the operator norm yield a Banach algebra.*

Example 3.4. *Let $\mathscr{X}$ be a Banach space and let $\mathscr{K}(\mathscr{X})$ denote the space of compact operators on $\mathscr{X}$, that is*

$$\mathscr{K}(\mathscr{X}) := \left\{T \in \mathscr{L}(\mathscr{X}) \mid cl(T(B_{\mathscr{X}})) \subset \mathscr{X} \text{ is compact}\right\}$$

where $B_{\mathscr{X}} := \{x \in \mathscr{X} \mid \|x\| \leq 1\}$. One can verify, that $\mathscr{K}(\mathscr{X})$ is closed in $\mathscr{L}(\mathscr{X})$ and that if $T \in \mathscr{K}(\mathscr{X})$ and $S \in \mathscr{L}(\mathscr{X})$, then $TS, ST \in \mathscr{K}(X)$. This of course tells us, that $\mathscr{K}(\mathscr{X})$ is also a Banach algebra, yet as we know well enough, $\mathscr{K}(\mathscr{X})$ does not contain the identity operator, if $\mathscr{X}$ is infinite dimensional.

Example 3.5. *Let X be a compact space and consider the set $\mathscr{C}(X)$ of continuous functions $X \to \mathbb{C}$. It is well known, that $\mathscr{C}(X)$ is complete with respect to the supremum norm and if we define multiplication in the natural way, we also see*

$$\|fg\| = \sup\{|f(x)g(x)| \colon x \in X\} \leq \sup\{|f(x)| \colon x \in X\} \sup\{|g(x)| \colon x \in X\} = \|f\|\|g\|$$

So $\mathscr{C}(X)$ is a Banach algebra.

One might notice that the definition of an algebra doesn't demand the existence of a unit in an algebra $\mathscr{X}$, in particular after having looked at the example of $\mathscr{K}(\mathscr{X})$. Meaning there does not need to exist $I \in \mathscr{X}$ such that for all $\xi \in \mathscr{X}$, we have $I\xi = \xi I = \xi$. However this property is highly desirable to have in such an object. Moreover if we find ourselves in a Banach algebra, then we would also like to have $\|I\| = 1$, if a unit I exists. This motivates the next definition.

Definition 3.2. *A Banach algebra is called unital, if it has a unit I satisfying $\|I\| = 1$.*

If we have a Banach algebra $\mathscr{X}$ containing a unit, then we will always assume $\mathscr{X}$ to be unital.

So what if a Banach algebra does not contain a unit at all? We will just force one upon it as follows: Consider the vector space

$$\mathscr{X}_I := \mathscr{X} \bigoplus \mathbb{C} = \{(\xi, \lambda) \colon \xi \in \mathscr{X}, \lambda \in \mathbb{C}\}$$

where an element (ξ, λ) will be written as $\xi + \lambda I$. We turn this into an algebra by defining

$$(\xi + \lambda I)(\eta + \mu I) := \xi\eta + \mu\xi + \lambda\eta + \lambda\mu I$$

Moreover we define a norm on $\mathscr{X}_I$ by

$$\|\xi + \lambda I\| := \|\xi\| + |\lambda|$$

Then clearly $\|I\| = 1$ and

$$\|(\xi + \lambda I)(\eta + \mu I)\| \leq \|\xi\|\|\eta\| + \|\mu\xi\| + \|\lambda\eta\| + |\lambda\mu| = \|(\xi + \lambda I)\|\|(\eta + \mu I)\|$$

and it is easily verified that $\mathscr{X}_I$ is a Banach space. Therefore $\mathscr{X}_I$ is a unital Banach algebra. From now on we will abuse notation by writing $\xi + \lambda$ instead of $\xi + \lambda I$, which in particular means, that we will stupidly write 1 instead of I, provided there is no risk of confusion. For the next section, if we have a Banach algebra $\mathscr{X}$, we will always assume it to be unital, unless we explicitly state otherwise.

3.2 Spectrum of a Banach algebra

This subsection is based upon [8] and [4].

Definition 3.3. *Let $\mathscr{X}$ be a unital Banach algebra and take $\xi \in \mathscr{X}$. We define the resolvent of ξ to be the set*

$$\rho_{\mathscr{X}}(\xi) := \{\lambda \in \mathbb{C} \mid \exists (\xi - \lambda)^{-1} \in \mathscr{X}\}$$

Furthermore the function $R_\xi : \rho_{\mathscr{X}}(\xi) \to \mathscr{X}$, given by $\lambda \mapsto (\lambda - \xi)^{-1}$, will be called the resolvent function (we will see that $\rho_{\mathscr{X}}(\xi)$ is never the empty set, so this is well defined). The spectrum of ξ is the set

$$\sigma_{\mathscr{X}}(\xi) := \rho_{\mathscr{X}}(\xi)^c = \{\lambda \in \mathbb{C} \mid \nexists (\xi - \lambda)^{-1} \in \mathscr{X}\}$$

If $\mathscr{X}$ has no unit, we will define $\rho_{\mathscr{X}}(\xi)$ to be $\rho_{\mathscr{X}_I}(\xi)$, and $\sigma_{\mathscr{X}}(\xi)$ will be $\sigma_{\mathscr{X}_I}(\xi)$. If there is no ambiguity, we will just write $\sigma(\xi), \rho(\xi)$ for all the respective terms.

Remark 3.1. *We note that if $\mathscr{X}$ is not unital, we have that no element $\xi \in \mathscr{X}$ is invertible. Hence it follows that $0 \in \sigma_{\mathscr{X}}(\xi)$ for all $\xi \in \mathscr{X}$.*

Example 3.6. *Consider $\mathfrak{M}$, the complex vector space of $n \times n$-matrices endowed with the operator norm. As we have already seen, $\mathfrak{M}$ is a unital Banach algebra. Moreover note that the set of eigenvalues of a matrix $\xi \in \mathfrak{M}$ is exactly the spectrum $\sigma_{\mathfrak{M}}(\xi)$. Indeed, if $\xi - \lambda$ is not invertible for some $\lambda \in \mathbb{C}$, then $ker(\xi - \lambda) \neq 0$ and therefore λ is an eigenvalue of ξ.*

Example 3.7. *Let X be a compact topological space. We know that $\mathscr{C}(X)$ is a unital Banach algebra, and that for $f \in \mathscr{C}(X), \lambda \in \mathbb{C}$ we have $f - \lambda$ is invertible if and only if $\lambda \notin f(X)$. Therefore $\sigma(f) = f(X)$.*

We are now almost ready to prove a Theorem, which will be very helpful as we progress further into the topic, but first some very important lemmas.

Lemma 3.1. *Let $\mathscr{X}$ be a unital Banach algebra and let $\xi \in \mathscr{X}$ with $\|\xi\| < 1$, then $1 - \xi$ is invertible and*

$$(1 - \xi)^{-1} = \sum_{j=0}^{\infty} \xi^j$$

More generally, if $\xi \in \mathscr{X}$ and $\lambda \in \mathbb{C}$ with $|\lambda| > \|\xi\|$, then $(\xi - \lambda)$ is invertible and

$$(\xi - \lambda I)^{-1} = (-1/\lambda)(1 - \xi/\lambda)^{-1}$$

Proof. By completeness of $\mathscr{X}$, since $\sum \xi^j$ converges absolutely for $\|\xi\| < 1$, the series $\sum \xi^j$ converges and its limit is in $\mathscr{X}$. But then

$$\sum_{0 \leq k \leq n} \xi^k (I - \xi) = \sum_{0 \leq k \leq n} \xi^k - \sum_{0 \leq k \leq n} \xi^{k+1} = I - \xi^{n+1}$$

establishes

$$\left\| I - \sum_{0\le k\le n} \xi^k (I-\xi) \right\| = \left\| I - (I-\xi) \sum_{0\le k\le n} \xi^k \right\| \le \|\xi\|^{n+1} \longrightarrow 0 \ (n\to\infty)$$

This implies $(I-\xi)^{-1} = \sum_{k\ge 0} \xi^k$. For the general case we realize that $|\lambda| > \|\xi\|$ yields $\|\xi/\lambda\| < 1$ and therefore $(I-\lambda/\xi)^{-1}$ exists. Thus

$$\left\{(-1/\lambda)(I-\xi/\lambda)^{-1}\right\}(\xi-\lambda I) = I = (\xi-\lambda I)\left\{(-1/\lambda)(I-\xi/\lambda)^{-1}\right\}$$

so we can conclude, that $(\xi-\lambda I)^{-1} = (-1/\lambda)(I-\xi/\lambda)^{-1}$. ■

Lemma 3.2. *If $\mathscr{X}$ is a unital Banach algebra, then the set $\mathscr{G}(\mathscr{X}) := \{\xi \in \mathscr{X} : \exists \xi^{-1} \in \mathscr{X}\}$ is open in $\mathscr{X}$. Moreover the inverse operation $\mathscr{G}(\mathscr{X}) \to \mathscr{G}(\mathscr{X})$ given by $\xi \mapsto \xi^{-1}$ is continuous.*

Proof. Let $\xi \in \mathscr{G}(\mathscr{X})$ be arbitrary and set $\varepsilon := \|\xi^{-1}\|^{-1}$. Then for all $\eta \in B(\xi,\varepsilon) = \{\zeta \in \mathscr{X} : \|\zeta-\xi\| < \varepsilon\}$ we have

$$\|\xi^{-1}(\eta-\xi)\| \le \|\xi^{-1}\|\|\eta-\xi\| < 1$$

so we know by lemma 3.1, that $(\xi^{-1}(\eta-\xi)+I)$ is invertible. Hence

$$\eta = (\eta-\xi)+\xi = \xi(\xi^{-1}(\eta-\xi)+I)$$

has the inverse

$$(\xi^{-1}(\eta-\xi)+I)^{-1}\xi^{-1}$$

implying $B(\xi,\varepsilon) \subset \mathscr{G}(\mathscr{X})$ and proving openness. Next consider the inverse operation $\xi \mapsto \xi^{-1}$. Take $\xi \in \mathscr{G}(\mathscr{X})$ and choose $\eta \in \mathscr{X}$ so that $\|\eta\| < \|\xi^{-1}\|^{-1}$. Then

$$\|\eta\xi^{-1}\| \le \|\eta\|\|\xi^{-1}\| < 1$$

and therefore again by lemma 3.1

$$\xi - \eta = (I-\eta\xi^{-1})\xi \in \mathscr{G}(\mathscr{X})$$

And in fact, since

$$(\xi-\eta)^{-1} = \xi^{-1}(I-\eta\xi^{-1})^{-1} = \xi^{-1}\sum_{j\ge 0}(\eta\xi^{-1})^j$$

we have

$$\|\xi^{-1}-(\xi-\eta)^{-1}\| = \left\|\xi^{-1}(I-\sum_{j\ge 0}(\eta\xi^{-1})^j)\right\|$$
$$\le \|\xi^{-1}\|\sum_{j\ge 1}\|\eta\xi^{-1}\|^j = \|\xi^{-1}\|\frac{\|\eta\xi^{-1}\|}{1-\|\eta\xi^{-1}\|} \to 0$$

as $\|\eta\| \to 0$. Therefrom we see $(\xi-\eta)^{-1} \to \xi^{-1}$ as $\eta \to 0$, so the inverse operation is continuous. ■

Lemma 3.3. *Let $\mathscr{X}$ be a Banach algebra and $x^* \in \mathscr{X}^*$. If $\xi \in \mathscr{X}$, then the function $f\colon \rho(\xi) \to \mathbb{C}$ given by $\lambda \mapsto x^*((\xi-\lambda)^{-1})$ is holomorphic with*

$$f'(\lambda) = x^*((\xi-\lambda)^{-2})$$

Proof. First note, that if $\zeta,\eta \in \mathscr{X}$ are invertible, then

$$\zeta^{-1}(\eta-\zeta)\eta^{-1} = \zeta^{-1}-\eta^{-1} \tag{3.1}$$

So now let $\lambda \in \rho(\xi)$ and $h \in \mathbb{C}$ such that $\lambda+h \in \rho(\xi)$, then by (3.1)

$$\frac{f(\lambda+h)-f(\lambda)}{h} = x^*\left(\frac{(\xi-\lambda-h)^{-1}-(\xi-\lambda)^{-1}}{h}\right) = x^*\left(\frac{(\xi-\lambda-h)^{-1}h(\xi-\lambda)^{-1}}{h}\right)$$

Thus

$$\frac{f(\lambda+h)-f(\lambda)}{h} = x^*((\xi-\lambda-h)^{-1}(\xi-\lambda)^{-1})$$

and by continuity of the inverse operation we obtain $(\xi-\lambda-h)^{-1}(\xi-\lambda)^{-1} \to (\xi-\lambda)^{-2}$. We combine this observation with the fact, that x^* is continuous and obtain

$$\frac{f(\lambda+h)-f(\lambda)}{h} \to x^*((\xi-\lambda)^{-2}) \qquad (h \to 0)$$

■

The following theorem is of immense importance for the theory of Banach algebras and lays the foundation for spectral theory.

Theorem 3.1. *Let $\mathscr{X}$ be a unital Banach algebra. Then for all $\xi \in \mathscr{X}$ we have, that $\sigma_{\mathscr{X}}(\xi)$ is a compact subset of $\mathbb{C}$. Furthermore $\sigma_{\mathscr{X}}(\xi) \neq \emptyset$ for all $\xi \in \mathscr{X}$.*

Remark 3.2. *In particular, since $\sigma_{\mathscr{X}}(\xi)$ is bounded, we have that $\rho_{\mathscr{X}}(\xi) \neq \emptyset$. This in turn implies that the resolvent function is always well defined.*

Proof. By lemma 3.1, we must have $\sigma_{\mathscr{X}}(\xi) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \le \|\xi\|\}$, so $\sigma_{\mathscr{X}}(\xi)$ is bounded. Thus to prove compactness we only need to verify that $\sigma_{\mathscr{X}}(\xi)$ is closed (since the spectrum is a subset of $\mathbb{C}$). Now set $g\colon \mathbb{C} \to \mathscr{X}$, $\lambda \mapsto \lambda - \xi$. Since $\|g(\lambda)-g(\mu)\| = |\lambda-\mu|$, we deduce that g is continuous. By lemma 3.2 we know that $\mathscr{G}(\mathscr{X})$ is open and consequently so is $g^{-1}(\mathscr{G}(\mathscr{X})) = \rho_{\mathscr{X}}(\xi)$ and therefore $\rho_{\mathscr{X}}(\xi)^c = \sigma_{\mathscr{X}}(\xi)$ is closed, as wanted. It remains to prove, that the spectrum is never empty. Observe, that

$$\lim_{|\lambda|\to\infty} 1-\xi/\lambda = 1$$

Now by continuity of the inverse operation we obtain $\lim\limits_{|\lambda|\to\infty}(1-\xi/\lambda)^{-1} = 1$ and therefore

$$\lim_{|\lambda|\to\infty} \|(\xi-\lambda)^{-1}\| = \lim_{|\lambda|\to\infty} |\lambda|^{-1}\|(1-\xi/\lambda)^{-1}\| = 0 \tag{3.2}$$

Fix some $x^* \in \mathscr{X}^*$ and consider the holomorphic function

$$f_{x^*}\colon \rho(\xi) \to \mathbb{C} \qquad \lambda \mapsto x^*((\xi-\lambda)^{-1})$$

as given in lemma 3.3. Suppose by way of contradiction, that $\sigma(\xi) = \emptyset$, then $\rho(\xi) = \mathbb{C}$ and f_{x^*} is an entire function. In particular by (3.2) we obtain, that f_{x^*} is bounded and hence is constant as an entire function by Liouville's theorem. From the limit in (3.2) we conclude, that $f_{x^*} = 0$ and as x^* was arbitrary we have $f_{x^*} = 0$ for all $x^* \in \mathscr{X}^*$. But as a consequence of the Hahn-Banach theorem 1.1 this implies, that $(\xi-\lambda)^{-1} = 0$ for all $\lambda \in \mathbb{C}$, which leads to a contradiction. ■

Remark 3.3. *Note, that if $\mathscr{X} := \mathfrak{M}_2(\mathbb{R})$ = the real Banach algebra of matrices with real valued entries, and if $\xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then $\det(\xi-\lambda) = \lambda^2+1$ which has no roots over $\mathbb{R}$. Hence ξ has no eigenvalues over $\mathbb{R}$ and since the set of eigenvalues and the spectrum of ξ coincide in the case of $\mathfrak{M}_2(\mathbb{R})$ we see, that $\sigma_{\mathfrak{M}_2(\mathbb{R})}(\xi) = \emptyset$. It is therefore essential to point out, that Theorem 3.1 only holds true, if the underlying field is $\mathbb{C}$.*

Theorem 3.2. *(Gel'fand Mazur) Let $\mathscr{X}$ be a Banach algebra such that all elements in $\mathscr{X}$ are invertible except 0, then we have the Banach algebra isomorphism $\mathscr{X} \cong \mathbb{C}$, that is we have an isometric linear, multiplicative isomorphism between $\mathscr{X}$ and $\mathbb{C}$.*

Proof. Let $0 \neq \xi \in \mathscr{X}$. Since $\sigma_{\mathscr{X}}(\xi) \neq \emptyset$, we know that there exists $\lambda_\xi \in \mathbb{C}$, such that $\xi - \lambda_\xi$ is not invertible. By assumption $\xi - \lambda_\xi = 0$. This procedure defines a mapping $\mathscr{X} \to \mathbb{C}$ with $\xi \mapsto \lambda_\xi$, which is surjective (because if $\lambda \in \mathbb{C}$ and $\xi := \lambda I \in \mathscr{X}$, then $\lambda_\xi = \lambda$) and which is an isometry, because

$$\|\xi\| = \|\lambda_\xi\| = |\lambda_\xi|$$

Furthermore note that

$$\xi\eta \mapsto \lambda_{\xi\eta} = \lambda_\xi \lambda_\eta$$
$$\mu\xi + \eta \mapsto \mu\lambda_\xi + \lambda_\eta$$

for all $\mu \in \mathbb{C}$ and for all $\xi, \eta \in \mathscr{X}$. So $\xi \mapsto \lambda_\xi$ is an isometric isomorphism that preserves the multiplication structure of $\mathscr{X}$, thus $\mathscr{X} \cong \mathbb{C}$. ■

3.3 Ideals of Banach algebras

This subsection is based upon [8] and [7], even though we give a different proof for the spectral radius formula, which is taken from [12].

Definition 3.4. *Let $\mathscr{X}$ be a Banach algebra. By a left ideal $\mathscr{I} \subset \mathscr{X}$ of $\mathscr{X}$, we mean a linear subspace of $\mathscr{X}$ with the property*

$$\mathscr{X}\mathscr{I} := \{\xi\zeta \mid \xi \in \mathscr{X}, \zeta \in \mathscr{I}\} \subset \mathscr{I}$$

Analogously a linear subspace $\mathscr{I} \subset \mathscr{X}$ will be called a right ideal, if

$$\mathscr{I}\mathscr{X} := \{\zeta\xi \mid \xi \in \mathscr{X}, \zeta \in \mathscr{I}\} \subset \mathscr{I}$$

Moreover, if $\mathscr{I} \subset \mathscr{X}$ is both a right and a left ideal, $\mathscr{I}$ will be called an ideal of $\mathscr{X}$. An ideal $\mathscr{I} \neq \mathscr{X}$ is called maximal ideal of $\mathscr{X}$, if for any other ideal $\mathscr{J}$ of $\mathscr{X}$ with $\mathscr{J} \supset \mathscr{I}$, it follows that either $\mathscr{I} = \mathscr{J}$ or $\mathscr{J} = \mathscr{X}$.

Remark 3.4. *Note that if an ideal $\mathscr{I} \subset \mathscr{X}$ contains the identity 1 of $\mathscr{X}$, then $\mathscr{I} = \mathscr{X}$. In fact, if $\mathscr{I}$ contains any invertible element $\xi \in \mathscr{X}$, then $\mathscr{I}$ must already be $\mathscr{X}$, since $1 = \xi^{-1}\xi \in \mathscr{X}\mathscr{I} \subset \mathscr{I}$.*

Example 3.8. *If $\mathscr{X}$ is an infinite dimensional Banach space, then $\mathscr{K}(\mathscr{X})$ is a proper, closed ideal of $\mathscr{L}(\mathscr{X})$.*

Ideals always go hand in hand with quotients. Therefore we now give precise meaning to a quotient Banach algebra.

Proposition 3.1. *Let $\mathscr{I}$ be a proper left or right ideal in a unital Banach algebra $\mathscr{X}$, then its norm-closure $cl(\mathscr{I})$ is also a proper left or right ideal in $\mathscr{X}$. In particular, all maximal ideals are closed sets. Moreover, if $\mathscr{I}$ is a closed, proper ideal in $\mathscr{X}$ and*

$$Q\colon \mathscr{X} \to \mathscr{X}/\mathscr{I} \qquad \xi \mapsto \xi + \mathscr{I}$$

is the natural surjection, then the quotient space $\mathscr{X}/\mathscr{I}$, equipped with the norm

$$\|Q(\xi)\| := \inf\{\|\xi + \zeta\| \mid \zeta \in \mathscr{I}\}$$

and endowed with the multiplication structure

$$Q(\xi)Q(\zeta) := Q(\xi\zeta)$$

for all $\xi, \zeta \in \mathscr{X}$, is also a unital Banach algebra.

Proof. For the first part of the proof we restrict ourselves to the case where $\mathscr{I}$ is a left ideal, since the other case is analogous. First of all note that $cl(\mathscr{I})$ cannot be $\mathscr{X}$. Indeed, if $1 \in cl(\mathscr{X})$, then there exists $\xi \in \mathscr{I}$ such that $\|1 - \xi\| < 1$ (by denseness of $\mathscr{I}$ in $cl(\mathscr{I})$) and thus, by lemma 3.1, we know that $1 - (1 - \xi) = \xi \in \mathscr{I}$ is invertible. Hence $\mathscr{I} = \mathscr{X}$, a contradiction. Now if $\xi \in \mathscr{X}$ and $\zeta \in cl(\mathscr{I})$, then yet again by denseness, there exists a net $\{\zeta_i\} \subset \mathscr{I}$ such that $\zeta_i \to \zeta$ and therefore by continuity of the multiplication

$$\mathscr{I} \supset \mathscr{X}\mathscr{I} \ni \xi\zeta_i \to \xi\zeta$$

So by closedness $\xi\zeta \in cl(\mathscr{I})$, and therefore $cl(\mathscr{I}) \supset \mathscr{X}cl(\mathscr{I})$, as wanted. If $\mathscr{I}$ is a maximal ideal, then clearly $\mathscr{I} = cl(\mathscr{I})$. For the second part of the theorem we know from functional analysis, that $\mathscr{X}/\mathscr{I}$ is a Banach space. It is easily seen that if we define multiplication on $\mathscr{X}/\mathscr{I}$ by

$$Q(\xi)Q(\zeta) := Q(\xi\zeta)$$

for all $\xi, \zeta \in \mathscr{Y}$, then this is well defined, since $\mathscr{I}$ is an ideal. Moreover, we have

$$\|Q(\xi\eta)\| \le \|Q(\xi)\|\|Q(\eta)\|$$

Indeed, let $\zeta_1, \zeta_2 \in \mathscr{I}$, then

$$\|Q(\xi\eta)\| \le \|(\xi+\zeta_1)(\eta+\zeta_2)\| \le \|\xi+\zeta_1\|\|\xi+\zeta_2\|$$

so if we take the infimum for $\zeta_1, \zeta_2 \in \mathscr{I}$, we obtain our claim. It remains to prove that $\|Q(1)\| = 1$. By definition $\|Q(1)\| \le \|1\| = 1$ and since $\mathscr{I}$ is proper, there has to be $\xi \in \mathscr{I}^c$, and therefore we have

$$0 < \|Q(\xi)\| \le \|Q(\xi)\|\|Q(1)\|$$

But this implies $1 \le \|Q(1)\|$. ■

Example 3.9. *Let $\mathscr{X}$ be an infinite dimensional Banach space, then $\mathscr{L}(\mathscr{X})/\mathscr{K}(\mathscr{X})$ is a unital Banach algebra.*

Recall from abstract algebra, that if we have a commutative ring with identity $\mathscr{R}$, then a (ring-) ideal $\mathscr{I}$ is maximal if and only if $\mathscr{R}/\mathscr{I}$ is a field. Of course, every ideal in a Banach algebra is also a ring ideal in the usual sense.

Corollary 3.1. *Let $\mathscr{X}$ be an abelian, unital Banach algebra, then the following statements are true:*

(i). $\mathscr{I} \in \{\mathscr{I} \mid \mathscr{I} \text{ is a maximal ideal in } \mathscr{X}\} \implies \mathscr{X}/\mathscr{I} \cong \mathbb{C}$

(ii). If $\varphi\colon \mathscr{X} \to \mathbb{C}$ is a non-zero, multiplicative linear functional $\implies \varphi \in \mathscr{X}^*$ and $\ker(\varphi)$ is a maximal ideal

(iii). The map $\varphi \mapsto \ker(\varphi)$ is a bijection between the set of non-zero, mulplicative linear functionals and $\{\mathscr{I} \mid \mathscr{I} \text{ is a maximal ideal in } \mathscr{X}\}$

Proof. By proposition 3.1 we know that a maximal ideal $\mathscr{I}$ is closed in $\mathscr{X}$. Since $\mathscr{X}$ is a commutative, unital Banach algebra, it is in particular a commutative ring with an identity. So from abstract algebra we know that, since $\mathscr{I}$ is maximal, $\mathscr{X}/\mathscr{I}$ is a field. In particular, by proposition 3.1 $\mathscr{X}/\mathscr{I}$ is a Banach algebra and all elements except $Q(0) = 0 + \mathscr{I}$ are invertible in $\mathscr{X}/\mathscr{I}$ and hence by the Gel'fand Mazur Theorem 3.2 it follows that

$$\mathscr{X}/\mathscr{I} \cong \mathbb{C}$$

which shows (i). Conversely, if

$$\varphi\colon \mathscr{X} \to \mathbb{C}$$

is a non-zero homomorphism, then $\ker(\varphi)$ is a subspace of $\mathscr{X}$ and

$$\mathscr{X}\ker(\varphi) \subset \ker(\varphi) \qquad \ker(\varphi)\mathscr{X} \subset \ker(\varphi)$$

Therefore $\ker(\varphi)$ is an ideal satisfying $\mathscr{X}/\ker(\varphi) \cong \mathbb{C}$, so $\ker(\varphi)$ must be a maximal ideal in $\mathscr{X}$. Yet again by proposition 3.1 $\ker(\varphi)$ is closed. Since φ is a homomorphism, it is in particular a linear functional. From functional analysis we know that a linear functional is continuous if and only if its kernel is closed. Thus φ is continuous, which yields (ii). The inverse of the map given in (iii) is simply $\mathscr{I} \mapsto Q$, where $Q\colon \mathscr{X} \to \mathscr{X}/\mathscr{I} \cong \mathbb{C}$ is the natural surjection. ■

If we remind ourselves of the definition of the spectrum for an element ξ in a unital Banach algebra $\mathscr{X}$, then we might notice that the condition $\xi - \lambda$ is not invertible, for some $\lambda \in \mathbb{C}$, basically means that $\xi - \lambda$ does not have a two-sided inverse in $\mathscr{X}$. First of all, note that if ξ has both a left inverse ζ_1 as well as a right inverse ζ_2, then

$$\zeta_1 = \zeta_1 \xi \zeta_2 = \zeta_2$$

So if $\xi - \lambda$ is not invertible, then it either solely has a right inverse, a left inverse or no kind of inverse at all. Therefore it holds that $(\xi - \lambda)\mathscr{X}$ or $\mathscr{X}(\xi - \lambda)$ is a proper right or left ideal in $\mathscr{X}$. We can now deduce new properties for the spectrum.

Proposition 3.2 (Polynomial mapping property)**.** *Let $\mathscr{X}$ be a Banach algebra and let $\mathbb{C}[Z]$ denote the set of polynomials over $\mathbb{C}$ in the variable Z. If $q \in \mathbb{C}[Z], \xi \in \mathscr{X}$, then $q(\sigma(\xi)) = \sigma(q(\xi))$.*

Proof. Write $q = \sum q_j Z^j$, where the $q_i \in \mathbb{C}$, and take $\lambda \in \sigma(\xi)$. Then $(\xi - \lambda)\mathscr{X}$ or $\mathscr{X}(\xi - \lambda)$ must be a proper right or left ideal $\mathscr{I}$. Now consider

$$q(\xi) - q(\lambda) = \sum_{j=1}^{n} q_j(\xi^j - \lambda^j)$$

and observe that

$$\xi^j - \lambda^j = (\xi - \lambda)\sum_{k=1}^{j} \xi^{j-k}\lambda^{k-1} = \sum_{k=1}^{j} \xi^{j-k}\lambda^{k-1}(\xi - \lambda) \in \mathscr{I}$$

and so $q(\xi) - q(\lambda) \in \mathscr{I}$, since $\mathscr{I}$ is a subspace. We conclude that $q(\xi) - q(\lambda)$ has no two sided inverse (since $\mathscr{I}$ is proper), and so $q(\lambda) \in \sigma(q(\xi))$, yielding $q(\sigma(\xi)) \subset \sigma(q(\xi))$. Conversely take $\lambda \in \sigma(q(\xi))$ and factorize $q - \lambda = \prod(z - \lambda_i)$, where $\lambda_i \in \mathbb{C}$. Then

$$q(\xi) - \lambda = \prod(\xi - \lambda_i)$$

is by definition not invertible, and so there must exist an index j such that $\xi - \lambda_j$ is also not invertible. But this implies $\lambda_j \in \sigma(\xi)$, so we conclude that $\lambda = q(\lambda_j) \in q(\sigma(\xi))$, yielding $\sigma(q(\xi)) \subset q(\sigma(\xi))$. ■

Definition 3.5. *The spectral radius of an element ξ in a unital Banach algebra $\mathscr{X}$ is defined by*

$$r(\xi) := \sup\{|\lambda| \colon \lambda \in \sigma(\xi)\} = \max\{|\lambda| \colon \lambda \in \sigma(\xi)\}$$

Remark 3.5. *It immediately follows from the proof of theorem 3.1, that $r(\xi) \leq \|\xi\| < \infty$.*

The next formula is of immense importance as we will see very often later on. The proof to this theorem is taken from [12], p.163, Theorem 6.6.

Theorem 3.3 (Spectral Radius Formula)**.** *If $\mathscr{X}$ is a Banach algebra, then we have*

$$r(\xi) = \inf_{n\in\mathbb{N}} \|\xi^n\|^{1/n} = \lim_{n\to\infty} \|\xi^n\|^{1/n}$$

for all $\xi \in \mathscr{X}$.

Proof. We will prove

$$\limsup_{n\to\infty} \|\xi^n\|^{1/n} \leq r(\xi) \leq \inf_{n\in\mathbb{N}} \|\xi^n\|^{1/n}$$

which establishes our result. Since $\sigma(\xi)$ is compact and the absolute value function $|.|$ is continuous, there exists $\lambda \in \sigma(\xi)$ such that $|\lambda| = r(\xi)$. By the spectral mapping property 3.2, for the polynomial $p = Z^n$, we have

$$p(\sigma(\xi)) = \sigma(p(\xi)) = \sigma(\xi^n)$$

Hence $r(\xi)^n = |\lambda|^n = r(\xi^n) \leq \|\xi^n\|$ and in particular

$$r(\xi) \leq \inf_{n\in\mathbb{N}} \|\xi^n\|^{1/n} \tag{3.3}$$

Fix $x^* \in \mathscr{X}^*$ and note, that the map $f_{x^*}\colon \{|\lambda| > r(\xi)\} \subset \rho(\xi) \to \mathbb{C}$ given by $\lambda \mapsto x^*((\xi-\lambda)^{-1})$ is holomorphic by lemma 3.3. If $|\lambda| > \|\xi\|$, then we know by lemma 3.1 that the Laurent series

$$f_{x^*}(\lambda) = -\sum_{n\geq 0} \frac{x^*(\xi^n)}{\lambda^{n+1}} \tag{3.4}$$

converges uniformly and absolutely. However by holomorphicity of f_{x^*} on $\{|\lambda| > r(\xi)\}$ and by uniqueness of the Laurent series expansion we also obtain uniform and absolute convergence of (3.4) for $|\lambda| > r(\xi)$. But that means, that if $|\lambda| > r(\xi)$, then the sequences $\left\{x^*\left(\frac{\xi^n}{\lambda^n}\right)\right\}_n$ converge to zero for all $x^* \in \mathscr{X}^*$. In particular $\left\{\frac{\xi^n}{\lambda^n}\right\}_n$ converges weakly to 0 and thus by corollary 2.1 we have, that $\left\{\frac{\xi^n}{\lambda^n}\right\}_n$ is bounded, so there exists some $C(\lambda) > 0$ such that

$$\frac{\|\xi^n\|}{|\lambda|^n} < C(\lambda)$$

for all n. Thus we obtain

$$\limsup_{n\to\infty} \|\xi^n\|^{1/n} \leq \limsup_{n\to\infty} \left\{C(\lambda)^{1/n}|\lambda|\right\} = |\lambda|$$

which yields $\limsup\limits_{n\to\infty} \|\xi^n\|^{1/n} \leq r(\xi)$, since $|\lambda| > r(\xi)$ was arbitrary. ∎

3.4 Abelian Banach algebras - Gel'fand transform

This subsection is based upon [4] and [8].

Corollary 3.1 suggests that there might be a very deep connection between non-zero homomorphisms and maximal ideals. We will investigate this very connection as this will turn out quite fruitful.

Proposition 3.3. *Let $\mathscr{X}$ be an abelian, unital Banach algebra and let $\varphi\colon \mathscr{X} \to \mathbb{C}$ be a non-zero homomorphism. Then $\|\varphi\| = 1$, so in particular $\varphi \in B_{\mathscr{X}^*} = \{x^* \in \mathscr{X}^* \mid \|x^*\| \leq 1\}$.*

Proof. We have already seen in corollary 3.1, that φ is continuous. Since $\varphi \neq 0$, there exists $\xi \in \mathscr{X}$ such that $\varphi(\xi) \neq 0$. But then

$$\varphi(\xi) = \varphi(\xi)\varphi(1) \implies \varphi(1) = 1$$

implying $\|\varphi\| \geq 1$. By way of contradiction, assume $\|\varphi\| > 1$. Then there exists $\xi \in \mathscr{X}$ with $\|\xi\| \leq 1$, such that $|\varphi(\xi)| > 1$. Scale ξ by some $0 < \varepsilon < 1$ so that we still have $|\varphi(\varepsilon\xi)| > 1$. Then $(\varepsilon\xi)^j \to 0$, while

$$|\varphi((\varepsilon\xi)^j)| = |\varphi(\varepsilon\xi)|^j \to \infty$$

which yields a contradiction to φ being continuous. ∎

Definition 3.6. *Let $\mathscr{X}$ be an abelian Banach algebra. We define the maximal ideal space $\Sigma(\mathscr{X})$ of $\mathscr{X}$ to be the set of all non-zero homomorphisms on $\mathscr{X}$ mapping to $\mathbb{C}$, i.e all non-zero multiplicative, linear functionals $\mathscr{X} \to \mathbb{C}$, and we equip $\Sigma(\mathscr{X}) \subset \mathscr{X}^*$ with the relative weak*-topology of $\mathscr{X}^*$. The topology thus defined upon $\Sigma(\mathscr{X})$ is usually referred to as the Gel'fand topology.*

In view of the ideal theory we already went through, an example which gives some first relationship to the preceding and the current section is in order.

Example 3.10. *Suppose we have a non-abelian Banach algebra $\mathscr{X}$. Define the commutator of $\mathscr{X}$ to be the set*

$$[\mathscr{X},\mathscr{X}] := \{\xi\zeta - \zeta\xi \mid \xi,\zeta \in \mathscr{X}\}$$

and let $\mathscr{K}$ be the smallest, closed ideal which contains $[\mathscr{X},\mathscr{X}]$. The ideal $\mathscr{K}$ is called the commutator ideal. By construction, $\mathscr{X}/\mathscr{K}$ is a Banach algebra and we note that all $\xi,\zeta \in \mathscr{X}$ satisfy $(\xi\zeta - \zeta\xi) + \mathscr{K} = 0 + \mathscr{K}$, which is *equivalent to $\xi\zeta + \mathscr{K} = \zeta\xi + \mathscr{K}$. Thus, $\mathscr{X}/\mathscr{K}$ is an abelian Banach algebra. Additionally, it is not hard to see, that $\mathscr{K}$ is the smallest, closed ideal $\mathscr{J}$ such that $\mathscr{X}/\mathscr{J}$ is abelian.*

Corollary 3.2. *Let $\mathscr{X}$ be a non-abelian, unital Banach algebra and let $\varphi\colon \mathscr{X} \to \mathbb{C}$ be a non-zero homomorphism, then it also holds that $\|\varphi\| = 1$.*

Proof. It is easy to see, that $[\mathscr{X}, \mathscr{X}] \subset \ker(\varphi)$ and thus we know, that $\mathscr{X}/\ker(\varphi)$ is an abelian ring with a unit, that is isomorphic to $\mathbb{C}$. Hence $\mathscr{X}/\ker(\varphi)$ is a field and therefore $\ker(\varphi)$ must be a maximal ideal. This implies, that $\ker(\varphi)$ is closed and so φ is continuous. In particular, if $\mathscr{K}$ denotes the commutator ideal, then $\mathscr{K} \subset \ker(\varphi)$. Now let $Q\colon \mathscr{X} \to \mathscr{X}/\mathscr{K}$ denote the quotient map and consider the diagram

$$\begin{tikzcd} \mathscr{X} \arrow[rr, "\varphi"] \arrow[dr, "Q"'] & & \mathbb{C} \\ & \mathscr{X}/\mathscr{K} \arrow[ur, dashed, "\underline{\varphi}"'] & \end{tikzcd}$$

We define $\underline{\varphi}(Q(\xi)) := \varphi(\xi)$ for all $\xi \in \mathscr{X}$. This is well defined, since as we have seen $\ker(\varphi) \supset \mathscr{K}$. Now note, that $\underline{\varphi}$ trivially inherits linearity and multiplicativity from φ and thus is itself a non-zero homomorphism, as $\varphi(\mathscr{X}) = \underline{\varphi}(\mathscr{X}/\mathscr{K})$. But as $\underline{\varphi}$ is a non-zero homomorphism on the unital, abelian Banach algebra $\mathscr{X}/\mathscr{K}$ we have by proposition 3.3, that $\|\underline{\varphi}\| = 1$. Since the natural surjection Q maps the open unit ball $U_{\mathscr{X}} := \{x \in \mathscr{X} \mid \|\xi\| < 1\}$ onto the open unit ball $U_{\mathscr{X}/\mathscr{K}} := \{\xi + \mathscr{K} \in \mathscr{X}/\mathscr{K} \mid \|\xi + \mathscr{K}\| < 1\}$ we know that

$$1 = \|\underline{\varphi}\| = \sup_{\xi + \mathscr{K} \in U_{\mathscr{X}/\mathscr{K}}} \|\underline{\varphi}(\xi + \mathscr{K})\| = \sup_{\xi \in U_{\mathscr{X}}} \|\underline{\varphi}(Q(\xi))\| = \|\varphi\|$$

■

Now we will inspect the maximal ideal space. The next theorem essentially tells us, that there is a very close link between $\Sigma(\mathscr{X})$ and the spectrum.

Theorem 3.4. *Let $\mathscr{X}$ be an abelian, unital Banach algebra. Then $\Sigma(\mathscr{X})$ is a compact Hausdorff space and for all $\xi \in \mathscr{X}$ we have $\sigma(\xi) = \{\varphi(\xi) \mid \varphi \in \Sigma(\mathscr{X})\}$.*

Proof. By proposition 2.1, we know that the weak*-topology is Hausdorff, and since the Hausdorff property is inherited by topological subspaces, we can conclude that $\Sigma(\mathscr{X})$ is Hausdorff. Since $\Sigma(\mathscr{X}) \subset B_{\mathscr{X}^*}$ and $B_{\mathscr{X}^*}$ is weak*-compact by Theorem 2.1, it suffices to show that $\Sigma(\mathscr{X})$ is closed. So let $\{\varphi_i\} \subset \Sigma(\mathscr{X})$ be a net with $\varphi_i \rightharpoonup^* \varphi \in B_{\mathscr{X}^*}$. Observe, that

$$\varphi(\xi\eta) = \lim \varphi_i(\xi\eta) = \lim \varphi_i(\xi)\varphi_i(\eta) = \varphi(\xi)\varphi(\eta)$$
$$\varphi(1) = \lim \varphi_i(1) = 1$$

so $\varphi \in \Sigma(\mathscr{X})$. Now let $\lambda \in \sigma(\xi)$, then $(\xi - \lambda)\mathscr{X}$ is a proper ideal, and by applying Zorn's lemma, we may find a maximal ideal $\mathscr{I}$ in $\mathscr{X}$ satisfying

$$\mathscr{I} \supset (\xi - \lambda)\mathscr{X}$$

By corollary 3.1 we get $\mathscr{X}/\mathscr{I} \cong \mathbb{C}$, so we can interpret the natural surjection as a map $Q\colon \mathscr{X} \to \mathbb{C}$. Since Q is in $\Sigma(\mathscr{X})$ and $Q(\xi - \lambda) = 0$, we have $\lambda \in \{\varphi(\xi) \mid \varphi \in \Sigma(\mathscr{X})\}$. On the other hand, if $\varphi \in \Sigma(\mathscr{X})$, and $\lambda = \varphi(\xi)$, then by construction it follows, that $\xi - \lambda \in \ker(\varphi)$. If $\xi - \lambda$ were invertible, we would have

$$1 = \varphi(\xi - \lambda)\varphi((\xi - \lambda)^{-1}) = 0$$

a contradiction and therefore $\lambda \in \sigma(\xi)$. Thus $\sigma(\xi) = \{\varphi(\xi) \mid \varphi \in \Sigma(\mathscr{X})\}$. ■

Recall that a completely regular space X is a topological Hausdorff space, such that for all non-empty closed sets $Y \subset X$ and for all $x_0 \in X \backslash Y$ there exists a function $f \in \mathscr{C}_b(X)$ such that $f(x_0) = 1$ and $f(Y) = \{0\}$. Moreover note that for a Hausdorff space all singletons are closed, so in particular they are closed for a completely regular space.

We want to characterize the space $\mathscr{C}_b(X)$ in the light of Banach space theory, or more particularly in the light of Gel'fand theory, for which this will be a motivating example. Fix $x \in X$ and consider the function $\gamma_x \colon \mathscr{C}_b(X) \to \mathbb{F}$ defined by $f \mapsto f(x)$. Recall the definition of the Dirac-delta measure and remind yourself of the Riesz-Representation Theorem, since both of these will have a crucial role in the characterization process of continuous functions. Now define the function γ given by $x \mapsto \gamma_x$. Clearly γ_x is a linear functional for all x and we note that

$$\|\gamma_x\| = \sup\{|\gamma_x(f)| \mid f \in \mathscr{C}_b(X), \|f\| = 1\} = \gamma_x(1) = 1$$

so $\gamma_x \in \mathscr{C}_b(X)^*$ for all $x \in X$. Hence $\gamma \colon X \to \mathscr{C}_b(X)^*$ is well defined.

Lemma 3.4. *Consider the map $\gamma \colon X \to \mathscr{C}_b(X)^*$, and let $\gamma(X)$ inherit the relative weak*-topology from $\mathscr{C}_b(X)^*$. Then γ is continuous. Moreover, γ is an embedding, that is $X \cong \gamma(X)$, if and only if X is completely regular.*

Proof. We will start by showing that γ is continuous. So let $\{x_i\} \subset X$ be a net such that $x_i \to x \in X$. Then we have

$$\gamma_{x_i}(f) = f(x_i) \to f(x) = \gamma_x(f)$$

for all $f \in \mathscr{C}_b(X)$, which by definition of the weak*-topology exactly means $\gamma_{x_i} \rightharpoonup^* \gamma_x$, so γ is continuous. Now we deal with the equivalence. If X is completely regular, then for $x, y \in X, x \neq y$, we may find $f \in \mathscr{C}_b(X)$ such that

$$\gamma_x(f) = f(x) = 1 \neq 0 = f(y) = \gamma_y(f)$$

Thus $\gamma_x \neq \gamma_y$ and so γ is injective. Now we show, that γ is an open map, which already suffices to prove that γ is a homeomorphism. So let $U \subset X$ be open and take $\gamma_{x_0} \in \gamma(U) \subset \mathscr{C}_b(X)^*$. By complete regularity, there exists $f \in \mathscr{C}_b(X)$ such that $f(x_0) = 1$ and $f(X \backslash U) = \{0\}$. Now let

$$W := \{x^* \in \mathscr{C}_b(X)^* \mid |x^*(f)| = |ev_f(x^*)| > 0\}$$

then W is weak*-open and therefore

$$W_\gamma := W \cap \gamma(X) = \{\gamma_x \mid |\gamma_x(f)| > 0\}$$

is open with respect to the relative weak*-topology on $\gamma(X)$. Moreover

$$\gamma_{x_0} \in W_\gamma \subset \{\gamma_x \in \gamma(X) \mid \gamma_x(f) = f(x) \neq 0\} \subset \gamma(U)$$

so γ is a homeomorphism onto its image. Conversely, assume that γ induces a homeomorphism onto its image. By Alaoglu's Theorem 2.1 we have that $B_{\mathscr{C}_b(X)^*} = \{x^* \in \mathscr{C}_b(X)^* \mid \|x^*\| \leq 1\}$ is a compact topological Hausdorff space with respect to the relative weak*-topology. From topology we know that a compact Hausdorff space is normal, so in particular by Urysohn's lemma it follows that $B_{\mathscr{C}_b(X)^*}$ must be completely regular. Now complete regularity is inherited by subspaces, since restricting a function to a smaller domain still yields a continuous function in the relative topology with the desired properties. Thus since $\gamma(X) \subset B_{\mathscr{C}_b(X)^*}$, the space $\gamma(X)$ is completely regular and consequently so is X by the homeomorphism assumption. ∎

Now what does the preceding lemma have to do with abelian Banach algebras, or as hinted before, with the Dirac-delta measure? Assume X to be a compact Hausdorff space, then as we have already seen, X is normal and in particular completely regular. Therefore we have an induced homeomorphism $X \cong \gamma(X)$. Can we say more?

Theorem 3.5. *Let X be a compact Hausdorff space and consider $\mathscr{C}(X)$, the space of continuous functions on X. Then X is homeomorphic to the maximal ideal space $\Sigma(\mathscr{C}(X))$. In addition to that, there is a bijective correspondence with all Dirac-measures on B(X) with the maximal ideal space $\Sigma(\mathscr{C}(X))$.*

Proof. By lemma 3.4 it suffices to prove $\gamma(X) = \Sigma(\mathscr{C}(X))$. So let us take $\varphi \in \Sigma(\mathscr{C}(X))$, we will show that $\varphi \in \gamma(X)$, which yields our theorem, since $\gamma(X) \subset \Sigma(\mathscr{C}(X))$ is already established by definition of $\Sigma(\mathscr{C}(X))$. By the Riesz-Representation

Theorem 1.4, there exists $\mu \in M(X)$ such that $\varphi(f) = \int f d\mu$ for all $f \in \mathscr{C}(X)$ and $\|\mu\| = \|\varphi\|$. We claim that μ is a probability measure, that is $\mu(\Delta) \in [0,1]$ for all $\Delta \in B(X)$ and $\mu(X) = 1$: Recall that for $w,z \in \mathbb{C}$ we have $|w+z| = |w|+|z|$ if and only if there is a number $c \geq 0$ such that $w = cz$. Since $\mu(X) = \int d\mu = \varphi(1) = 1$ we have $\mu(X) = \|\mu\|$. So take any set Δ in the Borel sigma algebra $B(X)$ of X, then

$$\begin{aligned}\mu(X) &= \mu(X\backslash\Delta) + \mu(\Delta) = |\mu(X\backslash\Delta) + \mu(\Delta)| \\ &\leq |\mu(X\backslash\Delta)| + |\mu(\Delta)| \leq \|\mu\| = \mu(X)\end{aligned}$$

and therefore

$$|\mu(X\backslash\Delta) + \mu(\Delta)| = |\mu(X\backslash\Delta)| + |\mu(\Delta)|$$

so $\mu(X\backslash\Delta) = c\mu(\Delta)$ for some $c \geq 0$. Thus a set $\Delta \in B(X)$ satisfying $\mu(\Delta) \in \mathbb{C} - [0,\infty)$ would lead to a contradiction. By monotonicity of measures with non-negative values it follows that μ has only values in $[0,1]$. So now it is on us to prove that μ is actually the Dirac-measure for some $x \in X$, since this would mean $\varphi(f) = \int f d\mu = f(x) = \gamma_x(f)$. Observe that the set

$$D := \{w \in X \mid \forall N_w \subset X, \mu(N_w) > 0\}$$

where N_w represents an open neighbourhood of w, cannot be empty. Indeed, if we had $D = \emptyset$, then for all $x \in X$ we could find open neighborhoods N_x satisfying $\mu(N_x) = 0$ for each x, so that by construction we have

$$X = \bigcup \{N_x \mid x \in X\}$$

Since this yields an open covering, by compactness there exist $x_1, ..., x_n \in X$, such that

$$\bigcup \{N_{x_i} \mid 1 \leq i \leq n\} = X$$

and thus

$$1 = \mu(X) = \mu\left(\bigcup \{N_{x_i} \mid 1 \leq i \leq n\}\right) \leq \sum \mu(N_{x_i}) = 0$$

which is an absurdity. So take $x \in D$, then by theorem 3.1 $\ker(\gamma_x)$ is a maximal ideal and likewise is $\ker(\varphi)$. So if we manage to show, that $\ker(\varphi) \subset \ker(\gamma_x)$, then by definition of a maximal ideal, $\ker(\varphi) = \ker(\gamma_x)$ and thus $\varphi = \lambda\gamma_x$ for some $\lambda \in \mathbb{C}$. But we know that $\varphi(1) = \gamma_x(1)$, so $\lambda = 1$ and hence $\varphi = \gamma_x$, which would conclude our proof. So take $g \in \ker(\varphi)$, then since $\ker(\varphi)$ is an ideal, we have $|g|^2 = g\overline{g} \in \ker(\varphi)$. This implies

$$\varphi(|g|^2) = \int |g|^2 d\mu = 0 \tag{3.5}$$

so $g = 0$ μ-a.e. Suppose now, that $g(x) \neq 0$, then if we let $\varepsilon := |g(x)|/2$, we may find an open neighbourhood N_x so that

$$g(N_x) \subset B_\varepsilon(g(x)) \text{ and } \mu(N_x) > 0 \text{ (since } x \in D)$$

But then for all $y \in N_x$

$$|g(y)| \geq |g(x)| - |g(x) - g(y)| \geq |g(x)| - \varepsilon = \frac{|g(x)|}{2} > 0$$

and thus

$$\int |g|^2 d\mu \geq \int_{N_x} |g|^2 d\mu \geq \frac{\mu(N_x)|g(x)|^2}{4} > 0 \tag{3.6}$$

contradicting (3.5). Therefore $g(x) = \gamma_x(g) = 0$, so we conclude that $\ker(\gamma_x) = \ker(\varphi)$. ■

The following corollary is an immediate consequence of the proof from the preceding theorem.

Corollary 3.3. *Let X be a compact Hausdorff space, then all maximal ideals of $\mathscr{C}(X)$ are of the form $\ker(\gamma_x)$ for $x \in X$.*

We want to generalize this idea to arbitrary abelian Banach algebras. The following definition is, from what we have seen, only natural.

Definition 3.7. *Let $\mathscr{X}$ be an abelian Banach algebra. The map $\gamma\colon \mathscr{X} \to \mathscr{C}(\Sigma(\mathscr{X}))$ defined by $\xi \mapsto \gamma_\xi$, where $\gamma_\xi(\varphi) := \varphi(\xi)$, is called the Gel'fand transform on $\mathscr{X}$.*

Remark 3.6. *Note that the Gel'fand transform is well defined. Indeed, if $\{\varphi_i\} \subset \Sigma(\mathscr{X})$ is a net such that $\varphi_i \rightharpoonup^* \varphi \in \Sigma(\mathscr{X})$, then $\gamma_\xi(\varphi_i) = \varphi_i(\xi) \to \varphi(\xi) = \gamma_\xi(\varphi)$, so $\gamma_\xi \in \mathscr{C}(\Sigma(\mathscr{X}))$.*

Theorem 3.6. *For an abelian unital Banach algebra $\mathscr{X}$ the Gel'fand transform $\gamma\colon \mathscr{X} \to \mathscr{C}(\Sigma(\mathscr{X}))$ is a continuous homomorphism with $\|\gamma\| = 1$. Moreover its kernel is the intersection of all maximal ideals of $\mathscr{X}$ and $\|\gamma_\xi\| = r(\xi)$.*

Proof. It is immediate, that γ is a homomorphism and moreover by proposition 3.3 we have $\|\varphi\| = 1$ for all $\varphi \in \Sigma(\mathscr{X})$ and therefore

$$|\gamma_\xi(\varphi)| = |\varphi(\xi)| \le \|\xi\|$$

But this implies $\|\gamma_\xi\| \le \|\xi\|$ and hence

$$\|\gamma\| = \sup\{\|\gamma_\xi\|\colon \xi \in \mathscr{X}, \|\xi\| \le 1\} \le 1$$

Now $\|\gamma_1\| = 1$, so $\|\gamma\| = 1$. Next note that

$$\xi \in \ker(\gamma) \iff \gamma_\xi = 0 \iff \forall \varphi \in \Sigma(\mathscr{X})\colon \varphi(\xi) = 0 \iff \xi \in \bigcap_{\varphi \in \Sigma(\mathscr{X})} \ker(\varphi)$$

Therefore, by corollary 3.1, $\xi \in \ker(\gamma)$ if and only if ξ is contained in every maximal ideal of $\mathscr{X}$. Last but not least, by Theorem 3.4

$$\|\gamma_\xi\| = \sup\{|\varphi(\xi)|\colon \varphi \in \Sigma(\mathscr{X})\} = \sup\{|\lambda|\colon \lambda \in \sigma(\xi)\} = r(\xi)$$

■

4 C*-algebras

4.1 Definitions and Examples

This subsection is based upon [4] and [8].

We have finally accumulated enough information to poke our heads into the world of the ominous creatures that are C*-algebras. Before getting into too much theory, we will give definitions, discuss some examples and deduce some of the easier properties.

Definition 4.1. *An involution on an algebra $\mathscr{X}$ is a map ${}^*: \mathscr{X} \to \mathscr{X}$ satisfying:*

(i). $\xi^{**} = \xi$

(ii). $(\xi + \eta)^* = \xi^* + \eta^*$

(iii). $(\mu\xi)^* = \overline{\mu}\xi^*$

(iv). $(\xi\eta)^* = \eta^*\xi^*$

*for all $\xi, \eta \in \mathscr{X}$ and for all $\mu \in \mathbb{C}$. We will say that an algebra is a *-algebra, if it is equipped with an involution.*

Definition 4.2. *A C*-algebra is a Banach *-algebra $\mathscr{X}$, such that for all $\xi \in \mathscr{X}$ we also have $\|\xi^*\xi\| = \|\xi\|^2$.*

Remark 4.1. *If we have a C*-algebra $\mathscr{X}$, then we deduce that for $\xi \in \mathscr{X}$ we have $\|\xi^*\|^2 = \|\xi^{**}\xi^*\| = \|\xi\xi^*\| \leq \|\xi\|\|\xi^*\|$ implying $\|\xi^*\| \leq \|\xi\|$ and analogously $\|\xi\| \leq \|\xi^*\|$, so $\|\xi\| = \|\xi^*\|$.*

Lemma 4.1. *If $\mathscr{X}$ is a Banach *-algebra satisfying $\|\xi\|^2 \leq \|\xi^*\xi\|$ for all $\xi \in \mathscr{X}$, then $\mathscr{X}$ is a C*-algebra.*

Proof. Let $\xi \in \mathscr{X}$, then we have

$$\|\xi\|^2 \leq \|\xi^*\xi\| \leq \|\xi^*\|\|\xi\| \qquad \|\xi^*\|^2 \leq \|\xi\xi^*\| \leq \|\xi\|\|\xi^*\|$$

This implies $\|\xi\| = \|\xi^*\|$, which, combined with the first inequality, yields that $\|\xi\|^2 = \|\xi^*\xi\|$, as wanted. ∎

Example 4.1. *If $\mathscr{H}$ is a Hilbert space with inner product $\langle | \rangle$ and $\mathscr{X} := \mathscr{L}(\mathscr{H})$ denotes the set of all bounded linear operators on $\mathscr{H}$, then $\mathscr{X}$ is a C*-algebra. We know from elementary functional analysis that the self-adjoint of operators is an involution and that $\mathscr{L}(\mathscr{H})$ is a Banach space. Furthermore we notice that if $\xi \in \mathscr{X}$, $x \in \mathscr{H}$ then*

$$\|\xi x\|^2 = \langle \xi x | \xi x \rangle = \langle x | \xi^*\xi x \rangle \leq \|x\|^2 \|\xi^*\xi\|$$

implying $\|\xi\|^2 \leq \|\xi^*\xi\|$. By the preceding lemma we have $\|\xi\|^2 = \|\xi^*\xi\|$.

Example 4.2. *Let X be a compact topological space and let $\mathscr{C}(X)$ be the set of all continuous functions mapping from $X \to \mathbb{C}$. Then $\mathscr{C}(X)$ is a Banach algebra with respect to the sup norm, and if $f^* := \overline{f}$, then this *-operation defines an involution on $\mathscr{C}(X)$. Moreover*

$$\|f^*f\| = \|f^2\| = \sup\{|f(x)|^2 : x \in X\} = \|f\|^2$$

and thus $\mathscr{C}(X)$ is a C-algebra.*

Example 4.3. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and consider*

$$L^\infty(\mu) = \left\{\phi : X \to \mathbb{C} \mid \phi \text{ measurable and } \|\phi\| < \infty\right\}$$

where

$$\|\phi\| := \inf_{\substack{N \in \mathscr{A} \\ \mu(N)=0}} \sup_{x \in N^c} |\phi(x)|$$

denotes the essential supremum norm. It is easy to see, that if $\phi \in L^\infty(\mu)$ then there exists a null-set $N \in \mathscr{A}$ so that $\|\phi\| = \sup_{x\in N^c} |\phi(x)|$. From courses on functional analysis it is known, that $L^\infty(\mu)$ is a Banach space. So now let $\phi, \psi \in L^\infty(\mu)$, then there exist $N, M \in \mathscr{A}$ with $\mu(N) = \mu(M) = 0$ and

$$\|\phi\| = \sup_{x\in N^c} |\phi(x)| \qquad \|\psi\| = \sup_{x\in M^c} |\psi(x)|$$

Thus

$$\|\varphi\psi\| \leq \sup_{x\in(N\cup M)^c} |\varphi(x)\psi(x)| \leq \left(\sup_{x\in(N\cup M)^c} |\varphi(x)|\right)\left(\sup_{x\in(N\cup M)^c} |\psi(x)|\right) = \|\varphi\|\|\psi\|$$

By a similar procedure we see, that $\|\phi^*\phi\| = \|\phi\|^2$ and thus $L^\infty(\mu)$ is a C-algebra.*

Example 4.4. *Let $\mathscr{H}$ be a Hilbert space and consider the space of compact operators $\mathscr{K}(\mathscr{H})$ on $\mathscr{H}$. Of course $\mathscr{K}(\mathscr{H})$ is a C*-sub-algebra of $\mathscr{L}(\mathscr{H})$ and it is non-unital if and only if $\mathscr{H}$ is infinite dimensional.*

Definition 4.3. *Let $\mathscr{X}$ be a *-Algebra. An element $\xi \in \mathscr{X}$ will be called selfadjoint if $\xi^* = \xi$ and it is called normal if $\xi\xi^* = \xi^*\xi$.*

Remark 4.2. *Let $\mathscr{X}$ be a *-algebra. Each selfadjoint element of $\mathscr{X}$ is normal. Moreover if*

$$\mathscr{X}_{sa} := \{\xi \in \mathscr{X} : \xi = \xi^*\}$$

then $\mathscr{X}_{sa}$ can be realized as a real vector subspace of $\mathscr{X}$. If $\xi, \eta \in \mathscr{X}_{sa}$, we also have

$$(\xi\eta)^* = \eta^*\xi^* = \eta\xi$$

which means $\xi\eta$ is selfadjoint if and only if ξ and η commute.

Some easy observations can be made immediately.

Corollary 4.1. *If $\mathscr{X}$ is a *-algebra, then each element ξ has a unique representation as a sum $\xi = \xi_1 + i\xi_2$, where $\xi_1, \xi_2 \in \mathscr{X}_{sa}$.*

Proof. If $\xi \in \mathscr{X}$ define

$$\xi_1 := \frac{\xi + \xi^*}{2} \qquad \xi_2 := \frac{\xi - \xi^*}{2i}$$

then $\xi = \xi_1 + i\xi_2$ and $\xi_1, \xi_2 \in \mathscr{X}_{sa}$. If on the other hand $\xi = \xi_1 + i\xi_2$ for ξ_1, ξ_2 selfadjoint, then we have

$$\xi^* = \xi_1 - i\xi_2$$

and thus $\frac{\xi+\xi^*}{2} = \xi_1$ and $\frac{\xi-\xi^*}{2i} = \xi_2$. ■

Corollary 4.2. *Let $\mathscr{X}$ be a *-algebra. If e is a left or right unit in $\mathscr{X}$, then e is a unit. The inverse operation and the *-operation commute.*

Proof. If $\mathscr{X}$ has a left identity e (meaning $e\xi = \xi$ for all $\xi \in \mathscr{X}$), then

$$\xi e^* = (e\xi^*)^* = \xi^{**} = \xi$$

so e^* is a right identity, implying $e = e^*$ is an identity for $\mathscr{X}$. The same holds if e is a right identity. Furthermore, if ξ is invertible in $\mathscr{X}$, then

$$(\xi^{-1})^*\xi^* = (\xi\xi^{-1})^* = 1^* = 1$$

and similarly $\xi^*(\xi^{-1})^* = 1$, which means that ξ^* is also invertible and $(\xi^*)^{-1} = (\xi^{-1})^*$. ■

Corollary 4.3. *If $\mathscr{X}$ is a non-trivial C*-algebra with unit, then $\mathscr{X}$ is unital.*

Proof. Apply the involution to $1^* = 1^*1$ to get $1 = 1^*$ and hence

$$\|1\| = \|1^*1\| = \|1\|^2$$

■

4.2 Continuous functions

This subsection is based upon [4].

We will soon classify all abelian C*-algebras up to isomorphism. In order to prepare for this endeavour, we introduce continuous functions into realm of C*-algebras, and we will study these objects quite thoroughly, as they will play a most crucial role in the classification problem, since as it turns out, every abelian C*-algebra will be isometrically *-isomorphic (whatever that means) to such an object.

Definition 4.4. *Let X be a locally compact, Hausdorff topological space and let $\mathscr{X}$ be a Banach space and write the norm on $\mathscr{X}$ as $|.|$ so as to avoid confusion with the following definition. Define the set*

$$\mathscr{C}_b(X,\mathscr{X}) := \{f\colon X \to \mathscr{X} \mid f \text{ is continuous and } \|f\| < \infty\}$$

where $\|f\| := \sup\{|f(x)|\colon x \in X\}$. Moreover, define

$$\mathscr{C}_0(X,\mathscr{X}) := \{f \in \mathscr{C}_b(X,\mathscr{X}) \mid \forall \varepsilon > 0\colon \{|f| \geq \varepsilon\} \text{ is compact}\}$$

where $\{|f| \geq \varepsilon\} := \{x \in X\colon |f(x)| \geq \varepsilon\}$. If $\mathscr{X} = \mathbb{C}$, then we will just write $\mathscr{C}_b(X)$ and $\mathscr{C}_0(X)$.

Remark 4.3. *If X is compact, then clearly $\mathscr{C}(X,\mathscr{X}) = \mathscr{C}_0(X,\mathscr{X}) = \mathscr{C}_b(X,\mathscr{X})$.*

Let X be a locally compact, Hausdorff space and let $\mathscr{T}_X$ denote the topology on X. Recall, that we can define the space $X_\infty = X \cup \{\infty\}$ and equip it with the topology

$$\mathscr{T}_{X_\infty} := \mathscr{T}_X \cup \{(X \backslash C) \cup \{\infty\} \mid C \subset X \text{ compact}\}$$

Note that the space $(X_\infty, \mathscr{T}_{X_\infty})$ is compact, since if we take an open covering $\{U_i\}_{i\in I}$ for X_∞, then there must exist some U_j that is a neighborhood of ∞. Consequently $U_j = (X\backslash C) \cup \{\infty\}$ for some compact set $C \subset X$. Since $\{U_i\}_{i\in I\backslash\{j\}}$ then covers the compact set C, compactness of X_∞ is clear. The topological space $(X_\infty, \mathscr{T}_{X_\infty})$ thus produced, is called the one-point compactification of X.

Proposition 4.1. *Let X be a locally compact Hausdorff space and let X_∞ be the one-point compactification of X. Assume $(\mathscr{X}, |.|)$ to be a Banach space, then $f \in \mathscr{C}_0(X,\mathscr{X})$ if and only if the function $\mathscr{F}(f)\colon X_\infty \to \mathscr{X}$, given by $\mathscr{F}(f)|_X = f$ and $\mathscr{F}(f)(\infty) = 0$, satisfies $\mathscr{F}(f) \in \mathscr{C}(X_\infty, \mathscr{X})$.*

Proof. Suppose first, that $f \in \mathscr{C}_0(X,\mathscr{X})$. We only need to verify, that $\mathscr{F}(f)$ is continuous in ∞, since $\mathscr{T}_X \subset \mathscr{T}_{X_\infty}$. Let $\varepsilon > 0$, then $C := \{|f| \geq \varepsilon\}$ is compact and $|\mathscr{F}(f)(x)| < \varepsilon$ for all $x \in X_\infty \backslash C$. On the other hand, if $\mathscr{F}(f)$ is continuous, then f must also be continuous and there exists a compact set C such that $\|\mathscr{F}(f)(x)\| < \varepsilon$ for all $x \in X_\infty\backslash C$. Therefore $\{|f| \geq \varepsilon\} \subset C$, implying compactness of $\{|f| \geq \varepsilon\}$ (since it is a closed subset of a compact set), hence $f \in \mathscr{C}_0(X,\mathscr{X})$. ■

Example 4.5. *Let X be a locally compact Hausdorff space and let $\mathscr{Y}$ be a C*-algebra, where $|.|$ denotes the norm on $\mathscr{Y}$. It is easy to see, that $\mathscr{C}_b(X,\mathscr{Y})$ is a C*-algebra, if we define the *-operation pointwise, that is $f^*(x) = f(x)^*$ for all $x \in X$. We will show, that $\mathscr{C}_0(X,\mathscr{Y})$ is a C*-subalgebra of $\mathscr{C}_b(X,\mathscr{Y})$. In order to prove this, it is enough to show that $\mathscr{C}_0(X,\mathscr{Y})$ is a closed subalgebra of $\mathscr{C}_b(X,\mathscr{Y})$ and that $\mathscr{C}_0(X,\mathscr{Y})$ is closed under applying the inherited involution. Indeed, let $\{f_n\}_n \subset \mathscr{C}_0(X,\mathscr{Y})$ with $f_n \to f \in \mathscr{C}_b(X,\mathscr{Y})$. Let $\varepsilon > 0$ and choose n big enough, so that $\|f - f_n\| \leq \varepsilon/2$. Now take $x \in X$ such that $|f(x)| \geq \varepsilon$, then*

$$\varepsilon \leq |f(x)| \leq \|f - f_n\| + |f_n(x)| \leq \varepsilon/2 + |f_n(x)|$$

This implies $|f_n(x)| \geq \varepsilon/2$ and thus $\{|f| \geq \varepsilon\} \subset \{|f_n| \geq \varepsilon/2\}$, so $\{|f| \geq \varepsilon\}$ is compact as a closed subset of a compact set. Consequently $\mathscr{C}_0(X,\mathscr{Y})$ is closed. Now if $f,g \in \mathscr{C}_0(X,\mathscr{Y}), \lambda \in \mathbb{C}, \varepsilon > 0$, then by proposition 4.1 it follows, that $\mathscr{F}(f), \mathscr{F}(g)$ are continuous and thus so are the functions

$$\mathscr{F}(f) + \lambda \mathscr{F}(g) \text{ and } \mathscr{F}(f)\mathscr{F}(g) = \mathscr{F}(fg)$$

and again by proposition 4.1

$$f + \lambda g, fg \in \mathscr{C}_0(X,\mathscr{Y})$$

Furthermore since $\{|f^*| \geq \varepsilon\} = \{|f| \geq \varepsilon\}$, we have $f^* \in \mathscr{C}_0(X,\mathscr{Y})$, which proves our claim.

Definition 4.5. *A *-homomorphism ψ between *-algebras $\mathscr{Y},\mathscr{Z}$ is a homomorphism $\mathscr{Y} \to \mathscr{Z}$ satisfying $\psi(\xi^*) = \psi(\xi)^*$. A *-isomorphism between $\mathscr{Y}$ and $\mathscr{Z}$ is a *-homomorphism, which is bijective.*

The next lemma will be very important for the characterization process of abelian C*-algebras and luckily it is very easily proven.

Lemma 4.2. *Let X be a locally compact Hausdorff space, and let $\mathscr{Y}$ be a C*-algebra, then the map $\mathscr{F} : \mathscr{C}_0(X,\mathscr{Y}) \to \mathscr{C}(X_\infty,\mathscr{Y})$ given by $f \mapsto \mathscr{F}(f)$, where $\mathscr{F}(f)$ is given as in proposition 4.1, is an isometric *-homomorphism.*

Proof. Since $\mathscr{F}(f)(\infty) = 0$, it is clear, that $\|f\| = \|\mathscr{F}(f)\|$, so $\mathscr{F}$ is an isometry. We have already seen linearity and multiplicativity of $\mathscr{F}$ in example 4.5. Since we also have $\mathscr{F}(f^*) = \mathscr{F}(f)^*$, our claim follows. ∎

Remark 4.4. *In particular we have the isometric *-isomorphism*

$$\mathscr{C}_0(X,\mathscr{Y}) \cong \mathscr{F}(\mathscr{C}_0(X,\mathscr{Y})) = \{f \in \mathscr{C}(X_\infty,\mathscr{Y}) \mid f(\infty) = 0\}$$

4.3 Unitization process for non-unital C*-algebras

This subsection is based upon [4].

Now we will forget about continuous functions for a bit and turn our attention back to general C*-algebras. If $\mathscr{Y}$ is a C*-algebra, then for $\xi \in \mathscr{Y}$ define $L_\xi : \mathscr{Y} \to \mathscr{Y}$ by $L_\xi(y) := \xi y$. The next lemma essentially tells us, that $L_\xi \in \mathscr{L}(\mathscr{Y}) =$ set of bounded linear operators on $\mathscr{Y}$ with $\|L_\xi\| = \|\xi\|$.

Lemma 4.3. *Let $\mathscr{Y}$ be a C*-algebra, then we have*

$$\|\xi\| = \sup_{y \in S_\mathscr{Y}} \|\xi y\| = \sup_{y \in S_\mathscr{Y}} \|y\xi\|$$

for all $\xi \in \mathscr{Y}$, where $S_\mathscr{Y} := \{y \in \mathscr{Y} \mid \|y\| = 1\}$.

Proof. Note that $\|y\xi\| \leq \|\xi\|$ for all $y \in S_\mathscr{Y}$ and thus $\sup_{y \in S_\mathscr{Y}} \|y\xi\| \leq \|\xi\|$. Now let $y = \frac{\xi^*}{\|\xi\|} \in S_\mathscr{Y}$, then $\|y\xi\| = \frac{\|\xi^*\xi\|}{\|\xi\|} = \|\xi\|$, so $\|\xi\| = \sup_{y \in S_\mathscr{Y}} \|y\xi\|$. The same procedure works for the other equality. ∎

Recall that we can unitize a non-unital Banach algebra. So how about unitizing a C*-algebra? Can we do just as well? Yes we can, but we need a more involved procedure, since the C*-property

$$\|(\xi + \lambda)^*(\xi + \lambda)\| = \|\xi + \lambda\|^2$$

will not be satisfied in general by the norm we defined in the unitization process of Banach algebras. If $\mathscr{Y}$ is a C*-algebra without a unit, then set $\mathscr{Y}_1 := \mathscr{Y} \bigoplus \mathbb{C}$ and for all $\xi + \lambda \in \mathscr{Y}_1$ define

$$\|\xi + \lambda\| := \sup_{y \in S_\mathscr{Y}} \|\xi y + \lambda y\|$$

We will see soon, that this defines a norm, which combined with the *-operation $(\xi+\lambda)^* = \xi^* + \overline{\lambda}$ on $\mathscr{Y}_1$, turns $\mathscr{Y}_1$ into a C*-algebra. For the next proof remind yourself, that if we have a normed space $\mathscr{V}$ and a linear subspace $\mathscr{W} \subset \mathscr{V}$ that is complete, and if moreover $\mathscr{V}/\mathscr{W}$ is complete with respect to the quotient norm, then $\mathscr{V}$ must also be complete. This is usually referred to as "two out of three", since we can assign completeness to any two of the three spaces, and one can then prove, that we will always end up with the third space also being complete.

Lemma 4.4. *Let $\mathscr{Y}$ be a C*-algebra without a unit, then $(\mathscr{Y}_1, \|.\|)$ is a Banach space and a *-algebra.*

Proof. First we will show, that $\|.\|$ is a norm, but the only non-trivial norm-axiom is

$$\|\xi+\lambda\| = 0 \implies \xi+\lambda = 0$$

If ξ or λ is 0, then the statement is clear by lemma 4.3. So assume $\xi, \lambda \neq 0$, then $\|\xi+\lambda\| = 0$ implies $\xi y + \lambda y = 0$ for all $y \in \mathscr{Y}$ (scale the values of y). Therefore $-\lambda^{-1}\xi y = y$ for all $y \in \mathscr{Y}$, so $\mathscr{Y}$ has a left identity and since $\mathscr{Y}$ is a C*-algebra this also yields a right identity, so $-\lambda^{-1}\xi$ is the identity in $\mathscr{Y}$, which is a contradiction to $\mathscr{Y}$ having no unit, so $\|.\|$ is indeed a norm. Let $i\colon \mathscr{Y} \hookrightarrow \mathscr{Y}_1$ given by $\xi \mapsto \xi + 0$ be the natural inclusion. Then by lemma 4.3 we have

$$\|i(\xi)\| = \|L_\xi\| = \|\xi\|$$

so we see that i is isometric. This yields in particular, that $i(\mathscr{Y})$ is complete, since $\mathscr{Y}$ is. Moreover we note, that $\mathscr{Y}_1/i(\mathscr{Y}) \cong \mathbb{C}$ as vector spaces, so $\mathscr{Y}_1/i(\mathscr{Y})$ is of dimension one, and is therefore complete with respect to the natural quotient norm. By "two out of three" we see that $\mathscr{Y}_1$ must be complete. It is obvious, that $\mathscr{Y}_1$ is an algebra and that $(\xi+\lambda)^* = \xi^* + \overline{\lambda}$ defines an involution on $\mathscr{Y}$. ■

Proposition 4.2. *Let $\mathscr{Y}$ be a C*-algebra without unit, then $\mathscr{Y}_1$ is a unital C*-algebra. Moreover, if $\mathscr{Z}$ is a unital C*-algebra and $\psi\colon \mathscr{Y} \to \mathscr{Z}$ is a *-homomorphism, then $\psi_1\colon \mathscr{Y}_1 \to \mathscr{Z}$ defined by $\xi+\lambda \mapsto \psi(\xi)+\lambda$ is the unique extension of ψ to a *-homomorphism $\mathscr{Y}_1 \to \mathscr{Z}$ with $\psi_1(1) = 1$.*

Proof. We have already shown, that $\mathscr{Y}_1$ is a *-algebra and a Banach space, so all that remains to verify for the first part of the proposition is, that $\|.\|$ is submultiplicative and that $\|\xi+\lambda\|^2 = \|(\xi+\lambda)^*(\xi+\lambda)\|$. First of all note, that for $x, y \in S_{\mathscr{Y}}$, we have by lemma 4.3

$$\|x(\xi+\lambda)^* y\| = \|y^*(\xi+\lambda)x^*\| \leq \|(\xi+\lambda)x^*\| \leq \|\xi+\lambda\|$$

and by taking the supremum twice over all $x, y \in S_{\mathscr{Y}}$ we get $\|(\xi+\lambda)^*\| \leq \|\xi+\lambda\|$ and thus $\|\xi+\lambda\| = \|(\xi+\lambda)^*\|$. Now we observe, that

$$\sup_{y \in S_{\mathscr{Y}}} \|(\xi+\lambda)y\| = \sup_{y \in S_{\mathscr{Y}}} \|y(\xi+\lambda)\|$$

Indeed,

$$\begin{aligned} \sup_{y \in S_{\mathscr{Y}}} \|(\xi+\lambda)y\| = \|\xi+\lambda\| = \|(\xi+\lambda)^*\| &= \sup_{y \in S_{\mathscr{Y}}} \|(\xi+\lambda)^* y\| \\ &= \sup_{y \in S_{\mathscr{Y}}} \|y^*(\xi+\lambda)\| = \sup_{y \in S_{\mathscr{Y}}} \|y(\xi+\lambda)\| \end{aligned}$$

And in particular for $\xi, \eta \in \mathscr{Y}, \lambda, \mu \in \mathbb{C}$ it follows

$$\begin{aligned} \|(\xi+\lambda)(\eta+\mu)\| &= \sup_{y \in S_{\mathscr{Y}}} \sup_{x \in S_{\mathscr{Y}}} \|x(\xi+\lambda)(\eta+\mu)y\| \\ &\leq \sup_{y \in S_{\mathscr{Y}}} \sup_{x \in S_{\mathscr{Y}}} \left\{ \|x(\xi+\lambda)\| \|(\eta+\mu)y\| \right\} = \|\xi+\lambda\| \|\eta+\mu\| \end{aligned}$$

showing submultiplicativity of $\|.\|$. But now we see

$$\begin{aligned} \|(\xi+\lambda)^*(\xi+\lambda)\| &= \sup_{y\in S_{\mathscr{Y}}} \|(\xi+\lambda)^*(\xi+\lambda)y\| \\ = \sup_{y\in S_{\mathscr{Y}}}\sup_{x\in S_{\mathscr{Y}}} \|x(\xi+\lambda)^*(\xi+\lambda)y\| &\geq \sup_{y\in S_{\mathscr{Y}}} \|y^*(\xi+\lambda)^*(\xi+\lambda)y\| \\ &= \sup_{y\in S_{\mathscr{Y}}} \|(\xi+\lambda)y\|^2 = \|\xi+\lambda\|^2 \end{aligned}$$

And therefore $\|(\xi+\lambda)\|^2 \leq \|(\xi+\lambda)^*(\xi+\lambda)\| \leq \|\xi+\lambda\|^2$ by submultiplicativity. Thus we have shown, that $\mathscr{Y}_1$ is a C*-algebra. For the second part of the proposition uniqueness is obvious and it is easy to see, that ψ_1 is a *-homomorphism. ∎

Corollary 4.4. *If $\mathscr{Y}$ is a C*-algebra and $\xi \in \mathscr{Y}$ is normal, then $\|\xi\| = r(\xi)$. Moreover, if $\mathscr{Z}$ is another C*-algebra and $\psi\colon \mathscr{Y} \to \mathscr{Z}$ is a homomorphism, then $\sigma_{\mathscr{Z}}(\psi(\xi)) \subset \sigma_{\mathscr{Y}}(\xi)$ and $r(\psi(\xi)) \leq r(\xi)$ for all $\xi \in \mathscr{Y}$.*

Proof. Let $\xi \in \mathscr{Y}$ be normal, then

$$\|\xi^2\|^2 = \|(\xi^2)^*(\xi)^2\| = \|(\xi^*\xi)(\xi^*\xi)\| = \|\xi^*\xi\|^2 = \|\xi\|^4$$

and thus $\|\xi^2\| = \|\xi\|^2$. Carrying on by induction it follows, that $\|\xi^{2^n}\| = \|\xi\|^{2^n}$. But then by the spectral radius formula 3.3

$$r(\xi) = \lim_{n\to\infty} \|\xi^{2^n}\|^{1/2^n} = \|\xi\|$$

For the second claim of the proposition we note, that it suffices to consider the case of $\mathscr{Y}$ and $\mathscr{Z}$ having a unit, since we clearly can extend ψ to a homomorphism $\mathscr{Y}_1 \to \mathscr{Z}_1$ in the same fashion as in proposition 4.2 and this doesn't change the spectrum $\sigma(\psi(\xi))$. Having dealt with that, let $\xi \in \mathscr{Y}$ be arbitrary again, and take $\lambda \in \sigma_{\mathscr{Z}}(\psi(\xi))$. By definition of the spectrum $\psi(\xi)-\lambda = \psi(\xi-\lambda)$ is not invertible and therefore $\xi-\lambda$ cannot be invertible in $\mathscr{Y}$, implying $\sigma_{\mathscr{Z}}(\psi(\xi)) \subset \sigma_{\mathscr{Y}}(\xi)$ and therefore also, by definition of the spectral radius, $r(\psi(\xi)) \leq r(\xi)$. ∎

Proposition 4.3. *If $\mathscr{Y}$ is a C*-algebra, then its norm is uniquely determined by the purely algebraic relation*

$$\|\xi\| = \sqrt{r(\xi^*\xi)}$$

Proof. Let $\xi \in \mathscr{Y}$, then clearly $\xi^*\xi$ is self-adjoint and thus in particular normal. By the preceding corollary it follows, that

$$\|\xi\|^2 = \|\xi^*\xi\| = r(\xi^*\xi)$$

∎

Proposition 4.4. *Suppose we are given a C*-algebra $\mathscr{Y}$ and let ψ be a non-zero homomorphism mapping from $\mathscr{Y} \to \mathbb{C}$. Then ψ is a *-homomorphism, so $\psi(\xi^*) = \overline{\psi(\xi)}$ and $\psi(\xi^*\xi) \geq 0$ for all $\xi \in \mathscr{Y}$.*

Proof. Again by proposition 4.2 we can assume, that $\mathscr{Y}$ has an identity and $\psi(1) = 1$. Let $\xi \in \mathscr{Y}$, then recall that we can decompose ξ into a sum of two unique selfadjoint elements $\xi_1, \xi_2 \in \mathscr{Y}_{sa}$, i.e. $\xi = \xi_1 + i\xi_2$. Now $\xi^* = \xi_1 - i\xi_2$, and if we can show $\psi(\xi_j) \in \mathbb{R}$, then our first claim follows. Therefore we may assume, that ξ itself is selfadjoint and we will show $\psi(\xi) \in \mathbb{R}$. So let $t \in \mathbb{R}$, then by corollary 3.2 we have

$$\begin{aligned} |\psi(\xi+it)|^2 &\leq \|\xi+it\|^2 = \|(\xi+it)^*(\xi+it)\| \\ &= \|(\xi-it)(\xi+it)\| = \|\xi^2+t^2\| \leq \|\xi\|^2+t^2 \end{aligned}$$

But if $\psi(\xi) = a+ib$ for $a,b \in \mathbb{R}$, then

$$\|\xi\|^2+t^2 \geq |\psi(\xi+it)|^2 = |a+i(b+t)|^2 = a^2+b^2+2bt+t^2$$

and therefore $\|\xi\|^2 \geq a^2+b^2+2bt$ for all $t \in \mathbb{R}$ and this can only be, if $b=0$ and thus $\psi(\xi) \in \mathbb{R}$. Now by what we have shown

$$\psi(\xi^*\xi) = \psi(\xi^*)\psi(\xi) = \overline{\psi(\xi)}\psi(\xi) = |\psi(\xi)|^2 \geq 0$$

■

Proposition 4.5. *Suppose $\mathscr{Y}, \mathscr{Z}$ are C*-algebras and $\psi\colon \mathscr{Y} \to \mathscr{Z}$ is a homomorphism, that is a multiplicative linear map, then $\|\psi\| \leq 1$.*

Proof. Utilizing the spectral radius formula 3.3, proposition 4.3 and corollary 4.4 we have

$$\|\psi(\xi)\| = \sqrt{r(\psi(\xi)^*\psi(\xi))} = \lim \sqrt{\|(\psi(\xi)^*\psi(\xi))^n\|^{1/n}} \leq \lim \sqrt{\|(\psi(\xi)^{*n}\|^{1/n}\|\psi(\xi)^n\|^{1/n}}$$
$$= \lim \sqrt{\|\psi(\xi)^n\|^{2/n}} = \sqrt{r(\psi(\xi))^2} \leq \sqrt{r(\xi)^2} = \lim \sqrt{\|\xi^n\|^{2/n}} = \lim \sqrt{\|(\xi^*\xi)^n\|^{1/n}} = \sqrt{r(\xi^*\xi)} = \|\xi\|$$

■

The preceding corollary is quite remarkable, since it tells us that a homomorphism between C*-algebras must always be continuous. We will see later, that the following corollary also holds if $\mathscr{Y}$ is not assumed abelian.

Corollary 4.5. *Suppose $\mathscr{Y}$ is an abelian C*-algebra and $\xi \in \mathscr{Y}$ is selfadjoint, then $\sigma(\xi) \subset \mathbb{R}$.*

Proof. Consider the maximal ideal space $\Sigma(\mathscr{Y})$, which is the set of all non-zero homomorphisms mapping from $\mathscr{Y} \to \mathbb{C}$. By proposition 4.4 we therefore have, that all elements in $\Sigma(\mathscr{Y})$ are *-homomorphisms in particular. Since ξ is selfadjoint we have $\psi(\xi) = \psi(\xi^*) = \overline{\psi(\xi)}$ for all $\psi \in \Sigma(\mathscr{Y})$, so $\psi(\xi) \in \mathbb{R}$ for all $\psi \in \Sigma(\mathscr{Y})$. Thus by Theorem 3.4 we see, that

$$\sigma(\xi) = \{\psi(\xi) \mid \psi \in \Sigma(\mathscr{Y})\} \subset \mathbb{R}$$

■

4.4 Abelian C*-algebras - The first Gel'fand-Naimark Representation Theorem

This subsection is based upon [4].

Recall yet again the definition of the one-point compactification. One can show, that if X is a locally compact Hausdorff space, then the topology $\mathscr{T}_{X_\infty}$ defined upon X_∞ is the unique topology (up to homeomorphism) that turns X_∞ into a compact Hausdorff space and also has X as a topological subspace ([2], p.32, 11.3 Theorem). Now, if $\mathscr{Y}$ is an abelian C*-algebra without unit, then we know, that we can consider the abelian, unital C*-algebra $\mathscr{Y}_1$ as introduced in proposition 4.2. By Theorem 3.4 we then know, that $\Sigma(\mathscr{Y}_1)$ is a compact Hausdorff space with the weak*-subspace topology. Moreover, if $\psi_\infty \in \Sigma(\mathscr{Y}_1)$ is the homomorphism $\xi+\lambda \mapsto \lambda$, then we see that for all $\psi \in \Sigma(\mathscr{Y}_1)\backslash\{\psi_\infty\}$ we have $\psi(\xi+\lambda) = \psi(\xi)+\lambda$ for all $\xi+\lambda \in \mathscr{Y}_1$, so these ψ are fully determined by their values on $\mathscr{Y}$. Conversely, every homomorphism $\Sigma(\mathscr{Y})$ can be uniquely extended to a homomorphism in $\Sigma(\mathscr{Y}_1)$, as we have seen in proposition 4.2. This gives a bijective correspondence between $\Sigma(\mathscr{Y})$ and $\Sigma(\mathscr{Y}_1)\backslash\{\psi_\infty\}$. As the next lemma shows, even more is true.

Lemma 4.5. *Let $\mathscr{Y}$ be C*-algebra without unit, then the topological subspace $\Sigma(\mathscr{Y}_1)\backslash\{\psi_\infty\}$ of $\Sigma(\mathscr{Y}_1)$ is a locally compact Hausdorff space. The map $\Phi\colon \Sigma(\mathscr{Y}_1)\backslash\{\psi_\infty\} \to \Sigma(\mathscr{Y})$ given by $\psi \mapsto \psi|_{\mathscr{Y}}$ is a homeomorphism, so in particular $\Sigma(\mathscr{Y})$ can be seen as a locally compact Hausdorff subspace of $\Sigma(\mathscr{Y}_1)$.*

Proof. Since $\Sigma(\mathscr{Y}_1)$ is compact and Hausdorff, it is in particular locally compact and Hausdorff. These properties are inherited by subspaces, thus $\Sigma(\mathscr{Y}_1)\backslash\{\psi_\infty\}$ is a locally compact Hausdorff space. Next, it is clear, that Φ is a well defined bijection. Let $\{\psi_i\}_i \subset \Sigma(\mathscr{Y}_1)\backslash\{\psi_\infty\}$ be a net with $\psi_i \rightharpoonup^* \psi \in \Sigma(\mathscr{Y}_1)\backslash\{\psi_\infty\}$. By definition

$$\psi_i \rightharpoonup^* \psi \iff \forall \xi+\lambda \in \mathscr{Y}_1\colon \psi_i(\xi)+\lambda \to \psi(\xi)+\lambda$$
$$\iff \forall \xi \in \mathscr{Y}\colon \psi_i(\xi) \to \psi(\xi) \iff \psi|_{\mathscr{Y}} \rightharpoonup^* \psi|_{\mathscr{Y}}$$

From this we see, that Φ is indeed a homeomorphism. ■

Theorem 4.1. *(Gel'fand-Naimark Representation Theorem I) Let $\mathscr{Y}$ be an abelian C*-algebra, then the Gel'fand transform $\gamma\colon \mathscr{Y} \to \mathscr{C}_0(\Sigma(\mathscr{Y}))$ is an isometric *-isomorphism.*

Proof. Suppose first, that we have already shown this Theorem for every abelian, unital C*-algebra $\mathscr{Y}$. If $\mathscr{Y}$ has no unit consider $\mathscr{Y}_1$, then by lemma 4.5 we have that $\Sigma(\mathscr{Y})$ is a subspace of $\Sigma(\mathscr{Y}_1)$ and we moreover know, that $\Sigma(\mathscr{Y}_1)$ is a compact Hausdorff space. By the uniqueness property of the one-point compactification, $\Sigma(\mathscr{Y}_1)$ must be the one-point compactification of $\Sigma(\mathscr{Y}_1)\backslash\{\psi_\infty\} = \Sigma(\mathscr{Y})$. By assumption we have an isometric *-isomorphism $\mathscr{C}(\Sigma(\mathscr{Y}_1)) \cong_{\gamma_1} \mathscr{Y}_1$, where $\gamma_1 : \mathscr{Y}_1 \to \mathscr{C}(\Sigma(\mathscr{Y}_1))$ is the Gel'fand transform. Now by lemma 4.2 we know that the map $\mathscr{F} : \mathscr{C}_0(\Sigma(\mathscr{Y})) \to \mathscr{C}(\Sigma(\mathscr{Y}_1))$ is an isometric *-homomorphism, so in particular we have the isometric *-isomorphism $\mathscr{C}_0(\Sigma(\mathscr{Y})) \cong_{\mathscr{F}} \mathscr{F}(\mathscr{C}_0(\Sigma(\mathscr{Y}))$. Moreover we have, that $\widetilde{\gamma} := \gamma_1|_{\mathscr{Y}} : \mathscr{Y} \to \widetilde{\gamma}(\mathscr{Y})$ is also an isometric *-isomorphism. We then observe that by surjectivity of γ_1 and proposition 4.1 we have

$$\widetilde{\gamma}(\mathscr{Y}) = \{f \mid f \in \mathscr{C}(\Sigma(\mathscr{Y}_1)), f(\psi_\infty) = 0\} = \mathscr{F}(\mathscr{C}_0(\Sigma(\mathscr{Y})))$$

Now the Gel'fand transform $\gamma\colon \mathscr{Y} \to \mathscr{C}_0(\Sigma(\mathscr{Y}))$ given by $\xi \mapsto \gamma_\xi$ is a well-defined isometric *-isomorphism, since for all $\xi \in \mathscr{Y}$ we have

$$\mathscr{F}^{-1} \circ \widetilde{\gamma}(\xi) = \widetilde{\gamma}(\xi)|_{\Sigma(\mathscr{Y})} = \gamma_\xi = \gamma(\xi)$$

for the composition of isometric *-isomorphisms is again an isometric *-isomorphism. Hence our special case is proven and the idea is summarized in the following diagram:

$$\begin{tikzcd}
\mathscr{C}_0(\Sigma(\mathscr{Y})) & \mathscr{F}(\mathscr{C}_0(\Sigma(\mathscr{Y})) \arrow[l, "\mathscr{F}^{-1}"'] \arrow[dr, hook] & \mathscr{Y}_1 \arrow[d, "\gamma_1"] \\
\mathscr{Y} \arrow[u, "\gamma"] \arrow[r, "\widetilde{\gamma}"'] \arrow[urr, hook] \arrow[ur, phantom, "\circlearrowright"] & \widetilde{\gamma}(\mathscr{Y}) \arrow[u, "id"'] \arrow[r, hook] & \mathscr{C}(\Sigma(\mathscr{Y}_1))
\end{tikzcd}$$

To conclude our proof, we now consider the case of $\mathscr{Y}$ being a unital, abelian C*-algebra. First note that for $\xi \in \mathscr{Y}$ we have by proposition 4.4

$$\forall \psi \in \Sigma(\mathscr{Y}) \colon \gamma_{\xi^*}(\psi) = \psi(\xi^*) = \overline{\psi(\xi)} = \gamma_\xi(\psi)^* \tag{4.1}$$

which implies $\gamma(\xi^*) = \gamma(\xi)^*$, so γ is a *-homomorphism. By Theorem 3.6 we have $\|\gamma\| = 1$ and $\|\gamma_\xi\| = r(\xi)$, hence for all selfadjoint $\xi \in \mathscr{Y}_{sa}$ we have $\|\xi\| = r(\xi) = \|\gamma_\xi\|$. Thus

$$\|\xi\|^2 = \|\xi^*\xi\| = \|\gamma_{\xi^*\xi}\| = \|\gamma_\xi^*\gamma_\xi\| = \||\gamma_\xi|^2\| = \|\gamma_\xi\|^2$$

So we see, that γ is an isometric *-homomorphism and therefore $\gamma(\mathscr{Y})$ is a closed subalgebra of $\mathscr{C}(\Sigma(\mathscr{Y}))$. Clearly, $\gamma(1) = 1 \in \gamma(\mathscr{Y})$ and since γ is a *-homomorphism, if $\gamma_\xi \in \gamma(\mathscr{Y})$, then $\gamma(\mathscr{Y}) \ni \gamma_{\xi^*} = \overline{\gamma_\xi}$. Finally if $\psi_1, \psi_2 \in \Sigma(\mathscr{Y})$ are two distinct homomorphisms, then there exists $\xi \in \mathscr{Y}$ so that $\psi_1(\xi) \neq \psi_2(\xi)$. Thus $\gamma_\xi(\psi_1) \neq \gamma_\xi(\psi_2)$. By the Stone Weierstrass Theorem 1.3 we therefore have $\gamma(\mathscr{Y}) = \mathscr{C}(\Sigma(\mathscr{Y}))$, which concludes the proof. ■

4.5 The continuous Functional Calculus

This subsection is based upon [8] and [5].

Corollary 4.3 already hinted at what the spectrum of a selfadjoint element in a general C*-algebra could look like. This idea turns out to be just right, and we will set out to prove this in a bit, but before doing so, we need to introduce some helpful notions. Since we can always turn a C*-algebra, which doesn't have a unit, into a unital C*-algebra, we will assume every C*-algebra to be unital.

Recall that $\mathbb{C}[Z_1,...,Z_n]$ is the algebra of all complex valued polynomials in the variables $Z_1,...,Z_n$.

Definition 4.6. *Let $\mathscr{Y}$ be a C*-algebra, and let $S\subset\mathscr{Y}$. The C*-algebra generated by S and the identity 1 in $\mathscr{Y}$ is the set*

$$\mathscr{C}^*(S) := \bigcap_{\substack{\mathscr{C}\subset\mathscr{Y}\\ \mathscr{C}\text{ is a C*-sub-algebra}\\ S\cup\{1\}\subset\mathscr{C}}} \mathscr{C}$$

Lemma 4.6. *Let $\mathscr{Y}$ be a C*-algebra and let $\emptyset\neq S\subset\mathscr{Y}$. Denote by $\mathfrak{F}$ the set of all finite subsets of S. For $F=\{f_1,...,f_n\}\in\mathfrak{F}$ we define*

$$\mathbb{C}[F] := \{p(f_1,...,f_n,f_1^*,...,f_n^*)\mid p\in\mathbb{C}[Z_1,...,Z_{2n}]\}$$

We then have

$$\mathbb{C}(\mathfrak{F}) := cl\left(\bigcup\{\mathbb{C}[F]\mid F\in\mathfrak{F}\}\right) = \mathscr{C}^*(S)$$

Proof. It is clear, that $\mathbb{C}(\mathfrak{F})\subset\mathscr{C}^*(S)$. On the other hand, we certainly know $\mathbb{C}(\mathfrak{F})$ to be a closed subalgebra, that is closed under applying the inherited involution. Thus $\mathbb{C}(\mathfrak{F})$ is a C*-subalgebra of $\mathscr{Y}$ and by definition of $\mathscr{C}^*(S)$ equality follows. ■

Example 4.6. *If $\xi\in\mathscr{Y}$, then $\mathscr{C}^*(\xi)=cl\big(\{p(\xi,\xi^*)\mid p\in\mathbb{C}[Z_1,Z_2]\}\big)$. Suppose ξ is normal, then $\mathscr{C}^*(\xi)$ is commutative and moreover, if ξ is selfadjoint, then we have $\mathscr{C}^*(\xi)=cl\big(\{p(\xi)\mid p\in\mathbb{C}[Z]\}\big)$.*

Example 4.7. *Of course, the trivial unital C*-subalgebra is $\mathscr{C}^*(1)=\mathbb{C}$.*

So now we have all we need to characterize the spectrum of a selfadjoint element.

Theorem 4.2. *Suppose $\mathscr{Y}$ is a C*-algebra and $\xi\in\mathscr{Y}$ is normal. Then $\sigma_{\mathscr{C}^*(\xi)}(\xi)=\sigma_{\mathscr{Y}}(\xi)$ and in particular, if ξ is selfadjoint, then $\sigma_{\mathscr{Y}}(\xi)\subset\mathbb{R}$.*

Proof. Let $\xi\in\mathscr{Y}$ be normal. Our goal is to show that $\rho_{\mathscr{Y}}(\xi)\subset\rho_{\mathscr{C}^*(\xi)}(\xi)$, since this implies $\rho_{\mathscr{Y}}(\xi)=\rho_{\mathscr{C}^*(\xi)}(\xi)$ since this will also imply that $\sigma_{\mathscr{Y}}(\xi)=\sigma_{\mathscr{C}^*(\xi)}(\xi)$ (as the other set inclusion follows by definition). So let $\lambda\in\rho_{\mathscr{Y}}(\xi)$, then by definition $\zeta:=\xi-\lambda\in\mathscr{G}(\mathscr{Y})=\{\zeta\in\mathscr{Y}\mid\exists\zeta^{-1}\in\mathscr{Y}\}$. It is a straightforward calculation to see, that since ξ is normal, we also have that ζ is normal. Consider the C*-subalgebra $\mathscr{C}^*(\zeta,\zeta^{-1})$, then since $\zeta^*=\zeta\zeta^*\zeta^{-1}$ by normality we have

$$\zeta^{-1}\zeta^* = \zeta^{-1}(\zeta\zeta^*\zeta^{-1}) = \zeta^*\zeta^{-1} \tag{4.2}$$

and taking the adjoint of (4.2) we also get $(\zeta^*)^{-1}\zeta=\zeta(\zeta^*)^{-1}$. That means, that all the elements $\zeta,\zeta^{-1},\zeta^*,(\zeta^{-1})^*$ commute with each other, and hence $\mathscr{C}^*(\zeta,\zeta^{-1})$ is a commutative, unital C*-algebra. By the Gel'fand-Naimark Representation Theorem 4.1 there exists a compact Hausdorff space X, so that we have the isometric *-isomorphism $\mathscr{C}^*(\zeta,\zeta^{-1})\cong_\gamma\mathscr{C}(X)$. Since the element ζ is invertible, the Gel'fand transform of ζ, so the element $f:=\gamma_\zeta$, must also be invertible. As f is invertible, $f(x)\neq 0$ for all $x\in X$ and therefore

$$\Big\|1-\frac{f^*f}{\|f\|^2}\Big\|<1$$

and thus by lemma 3.1 we get

$$\frac{\|f\|^2}{f^*f}=\sum_{k\geq 0}\left(1-\frac{f^*f}{\|f\|^2}\right)^k \iff \frac{1}{f}=\frac{f^*}{\|f\|^2}\sum_{k\geq 0}\left(1-\frac{f^*f}{\|f\|^2}\right)^k$$

$$\iff \gamma^{-1}(f^{-1})=\frac{\gamma^{-1}(f^*)}{\|f\|^2}\sum_{k\geq 0}\left(1-\frac{\gamma^{-1}(f^*f)}{\|f\|^2}\right)^k \iff \zeta^{-1}=\frac{\zeta^*}{\|\zeta\|^2}\sum_{k\geq 0}\left(1-\frac{\zeta^*\zeta}{\|\zeta\|^2}\right)^k \tag{4.3}$$

where we have used, that the inverse of the Gel'fand transform is also an isometric *-isomorphism. From (4.3) we then see by lemma 4.6, that $\zeta^{-1}\in\mathscr{C}^*(\zeta)$, since

$$\left\{\frac{\zeta^*}{\|\zeta\|^2}\sum_{0\leq k\leq j}\big(1-\frac{\zeta^*\zeta}{\|\zeta\|^2}\big)^k\right\}_{j\geq 0}$$

is nothing more than a sequence of polynomials in ζ and ζ^* converging to ζ^{-1}. In particular $\mathscr{C}^*(\zeta) = \mathscr{C}^*(\zeta, \zeta^{-1})$, but that since $\mathscr{C}^*(\zeta) = \mathscr{C}^*(\xi - \lambda) = \mathscr{C}^*(\xi)$, we also have $(\xi - \lambda)^{-1} \in \mathscr{C}^*(\xi)$ and therefore $\lambda \in \rho_{\mathscr{C}^*(\xi)}(\xi)$ which implies $\rho_{\mathscr{Y}}(\xi) \subset \rho_{\mathscr{C}^*(\xi)}(\xi)$ as wanted. In particular if ξ is selfadjoint, we have by corollary 4.3 that $\sigma_{\mathscr{Y}}(\xi) = \sigma_{\mathscr{C}^*(\xi)}(\xi) \subset \mathbb{R}$. ■

Corollary 4.6. *Let $\mathscr{Y}$ be a C*-algebra and let $\xi \in \mathscr{Y}$ be normal. Suppose $\mathscr{Z} \subset \mathscr{Y}$ is a C*-subalgebra containing the identity and ξ, then $\sigma_{\mathscr{Z}}(\xi) = \sigma_{\mathscr{Y}}(\xi)$.*

Proof. By the preceding theorem $\sigma_{\mathscr{C}^*(\xi)}(\xi) = \sigma_{\mathscr{Y}}(\xi)$, so the result follows from

$$\sigma_{\mathscr{C}^*(\xi)}(\xi) \supset \sigma_{\mathscr{Z}}(\xi) \supset \sigma_{\mathscr{Y}}(\xi)$$

■

Therefore we can unambiguously write $\sigma(\xi)$.

Corollary 4.7. *If $\mathscr{Y}$ is a C*-algebra and $\xi \in \mathscr{Y}$ is unitary, that is $\xi^*\xi = \xi\xi^* = 1$, then $\sigma(\xi) \subset \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$.*

Proof. Since $|\gamma_\xi|^2 = \gamma_{\xi^*}\gamma_\xi = \gamma_{\xi^*\xi} = 1$ we have

$$\sigma(\xi) = \gamma_\xi(\Sigma(\mathscr{C}^*(\xi))) \subset \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$$

as wanted. ■

Proposition 4.6. *Let $\mathscr{Y}$ be a C*-algebra and let $\xi \in \mathscr{Y}$ be normal, then the maximal ideal space $\Sigma(\mathscr{C}^*(\xi))$ is homeomorphic to $\sigma(\xi)$.*

Proof. We have already seen, that $\mathscr{C}^*(\xi)$ is a unital, abelian C*-algebra and therefore by the Gel'fand-Naimark Representation Theorem 4.1 we have the isometric *-isomorphism γ: $\mathscr{C}^*(\xi) \to \mathscr{C}(\Sigma(\mathscr{C}^*(\xi)))$. So if ξ is the element, as given in our assumption, then we consider the continuous map $\gamma_\xi : \Sigma(\mathscr{C}^*(\xi)) \to \mathbb{C}$. By corollary 4.3 and by Theorem 3.4 we know that $\gamma_\xi : \Sigma(\mathscr{C}^*(\xi)) \to \sigma(\xi)$ is a well defined continuous surjection. Now assume, that $\psi_1, \psi_2 \in \Sigma(\mathscr{C}^*(\xi))$ satisfy $\psi_1(\xi) = \psi_2(\xi)$. By proposition 4.4 we have $\psi_1(\xi^*) = \psi_2(\xi^*)$, and by the homomorphism structure we obtain

$$\psi_1(\xi^j) = \psi_1(\xi)^j = \psi_2(\xi)^j = \psi_2(\xi^j)$$
$$\psi_1(\xi^{*j}) = \overline{\psi_1(\xi)^j} = \overline{\psi_2(\xi)^j} = \psi_2(\xi^{*j})$$

for all natural numbers j. By linearity of each homomorphism we conclude, that ψ_1 and ψ_2 agree on all polynomials in the variables ξ and ξ^*, i.e.

$$\psi_1|_{\{p(\xi,\xi^*)|p\in\mathbb{C}[Z_1,Z_2]\}} \equiv \psi_2|_{\{p(\xi,\xi^*)|p\in\mathbb{C}[Z_1,Z_2]\}}$$

But by lemma 4.6 the set $\{p(\xi,\xi^*) \mid p \in \mathbb{C}[Z_1,Z_2]\}$ is dense in $\mathscr{C}^*(\xi)$ and by continuity we thus have $\psi_1 = \psi_2$. Hence γ_ξ is injective. It remains to prove continuity of the inverse γ_ξ^{-1}, but this is actually quite easy, since $\Sigma(\mathscr{C}^*(\xi))$ is a compact topological space every closed subset $F \subset \Sigma(\mathscr{C}^*(\xi))$ is compact, and therefore so is $\gamma_\xi(F) \subset \sigma(\xi)$ by continuity. Since $\sigma(\xi)$ is a subspace of $\mathbb{C}$, it is a Hausdorff space, and therefore $\gamma_\xi(F)$ is closed. We conclude, that γ_ξ is a bijective, closed, continuous map and is therefore a homeomorphism. ■

If we have a polynomial $p \in \mathbb{C}[Z]$ and an element $\xi \in \mathscr{Y}$, then it is clear what we mean, when we write $p(\xi)$. However, what about if we want to define what it means for a continuous function f to be evaluated at ξ, i.e. how would we interpret the expression $f(\xi)$? This is called the continuous functional calculus, which works for normal elements, and we have already acquired all tools necessary to actually make sense of the aforementioned.

The set $\mathbb{C}[Z,\overline{Z}]$ will denote the polynomials in the variables Z and $\overline{Z}$, where $\overline{Z}$ is the complex conjugation of Z. Therefore every element in $\mathbb{C}[Z,\overline{Z}]$ is only dependent on one variable and if $\xi \in \mathscr{Y}$, $p = \sum a_{ij} Z^i \overline{Z}^j \in \mathbb{C}[Z,\overline{Z}]$, then write $p(\xi) = \sum a_{ij} \xi^i \xi^{*j}$. Obviously $\mathscr{C}^*(\xi) = cl(\{p(\xi) \mid p \in \mathbb{C}[Z,\overline{Z}]\})$.

Theorem 4.3 (Continuous functional calculus = CFC)**.** *If $\mathscr{Y}$ is a C*-algebra and $\xi \in \mathscr{Y}$ is a normal element, then for each $f \in \mathscr{C}(\sigma(\xi))$ there is an element $f(\xi) \in \mathscr{C}^*(\xi) \subset \mathscr{Y}$ which satisfies the following properties*

$$\sigma(f(\xi)) = f(\sigma(\xi)) \tag{4.4}$$

$$\|f\| = \|f(\xi)\| \tag{4.5}$$

$$\textit{If } f = \sum a_{ij} Z^i \overline{Z}^j \in \mathbb{C}[Z, \overline{Z}]\textit{, then } f(\xi) = \sum a_{ij} \xi^i \xi^{*j} \tag{4.6}$$

Moreover, if $g \in \mathscr{C}(f(\sigma(\xi)))$, then

$$g(f(\xi)) = (g \circ f)(\xi) \tag{4.7}$$

Proof. The idea to the following can be encapsulated in a commutative diagram:

$$\begin{tikzcd}
\mathscr{C}(\sigma(\xi)) \arrow[rr, "f \mapsto f \circ \gamma_\xi"] \arrow[dr, "\Gamma_\xi"'] \arrow[drr, phantom, "\circlearrowleft", pos=0.3] & & \mathscr{C}(\Sigma(\mathscr{C}^*(\xi))) \arrow[dl, "\gamma^{-1}"] \\
& \mathscr{C}^*(\xi) &
\end{tikzcd}$$

By proposition 4.6 we have the homeomorphism $\gamma_\xi : \Sigma(\mathscr{C}^*(\xi)) \to \sigma(\xi)$. Therefore the map

$$\Phi \colon \mathscr{C}(\sigma(\xi)) \to \mathscr{C}(\Sigma(\mathscr{C}^*(\xi))) \qquad f \mapsto f \circ \gamma_\xi$$

is well defined and it is not too hard to verify, that Φ is an isometric *-homomorphism. In particular Φ is bijective, since the inverse is the map $g \mapsto g \circ \gamma_\xi^{-1}$, so Φ is an isometric *-isomorphism. We now use the isometric *-isomorphism

$$\gamma^{-1} \colon \mathscr{C}(\Sigma(\mathscr{C}^*(\xi))) \to \mathscr{C}^*(\xi)$$

to get an isometric *-isomorphism

$$\Gamma_\xi \coloneqq \gamma^{-1} \circ \Phi \colon \mathscr{C}(\sigma(\xi)) \to \mathscr{C}^*(\xi)$$

Now for $f \in \mathscr{C}(\sigma(\xi))$ define the element $f(\xi) \coloneqq \Gamma_\xi(f) \in \mathscr{C}^*(\xi) \subset \mathscr{Y}$. Thus for any normal element ξ we have constructed a mapping $\mathscr{C}(\sigma(\xi)) \to \mathscr{C}^*(\xi)$ with $f \mapsto f(\xi)$ and we will now show, that this satisfies all the properties. So since Γ_ξ is an isometry we have

$$\|f(\xi)\| = \|\Gamma_\xi(f)\| = \|f\|$$

which already yields (4.5). Moreover, if $f = \sum a_{ij} Z^i \overline{Z}^j \in \mathbb{C}[Z, \overline{Z}]$ is a polynomial, then

$$f(\xi) = \Gamma_\xi(f) = \gamma^{-1}(f \circ \gamma_\xi) = \gamma^{-1}\big(\sum a_{ij} \gamma_\xi^i \gamma_\xi^{*j}\big) = \sum a_{ij} \xi^i \xi^{*j}$$

proving (4.6). If $\psi \in \Sigma(\mathscr{C}^*(\xi))$, then

$$\psi(f(\xi)) = \psi(\Gamma_\xi(f)) = \psi\big(\sum a_{ij} \xi^i \xi^{*j}\big) = \sum a_{ij} \psi(\xi)^i \psi(\xi)^{*j} = f(\psi(\xi)) \tag{4.8}$$

Now, $\sigma(\xi)$ is compact and by the Stone Weierstrass Theorem 1.3 $cl(\mathbb{C}[Z, \overline{Z}]) = \mathscr{C}(\sigma(\xi))$ and therefore if $f \in \mathscr{C}(\sigma(\xi))$ is arbitrary, then there exists a sequence $\{f_n\}_n \subset \mathbb{C}[Z, \overline{Z}]$ with $f_n \to f$. Thus by (4.8) and continuity of the elements $\psi \in \Sigma(\mathscr{C}^*(\xi))$

$$\psi(f(\xi)) = \psi(\Gamma_\xi(\lim f_n)) = \lim \psi(\Gamma_\xi(f_n)) = \lim \psi(f_n(\xi)) = \lim f_n(\psi(\xi)) = f(\psi(\xi))$$

From this (4.4) follows immediately, since

$$\sigma(f(\xi)) = \{\psi(f(\xi)) \mid \psi \in \Sigma(\mathscr{C}^*(\xi))\} = \{f(\psi(\xi)) \mid \psi \in \Sigma(\mathscr{C}^*(\xi))\} = f(\sigma(\xi))$$

Lastly, if $g = \sum a_{ij} Z^i \overline{Z}^j \in \mathbb{C}[Z,\overline{Z}] \subset \mathscr{C}(f(\sigma(\xi)))$, then

$$(g \circ f)(\xi) = \Gamma_\xi(g \circ f) = \sum a_{ij} f(\xi)^i f(\xi)^{*j} = g(f(\xi))$$

proving (4.7) by the same approximation argument. ■

We will now deduce some direct consequences of the continuous functional calculus, which shows just how powerful this idea is.

Corollary 4.8. *Suppose $\mathscr{Y}$ and $\mathscr{Z}$ are C*-algebras and $\varphi\colon \mathscr{Y} \to \mathscr{Z}$ is a *-homomorphism. Let $\xi \in \mathscr{Y}$ be normal, then $f(\varphi(\xi)) = \varphi(f(\xi))$ for all $f \in \mathscr{C}(\sigma(\xi))$.*

Proof. As in the proof of the preceding proposition we note, that for all polynomials $f \in \mathbb{C}[Z,\overline{Z}]$ we have

$$f(\varphi(\xi)) = \varphi(f(\xi))$$

By the same approximation argument our statement is shown. ■

4.6 Positive elements

This subsection is based upon [8] and [5].

Let X be a compact Hausdorff space and consider the abelian, unital C*-algebra $\mathscr{C}(X)$. We have already seen, that $\sigma(f) = f(X)$, so it is obvious, that if $f \geq 0$ on X ($f(x) \geq 0$ for all $x \in X$), then $\sigma(f) \subset [0,\infty)$. Conversely, if $\sigma(f) \subset [0,\infty)$, then $f \geq 0$. The partial order, that is induced by $\leq$ is pretty handy and it is desirable to have something similar on an arbitrary C*-algebra that generalizes the natural order of the abelian case. Our observations from before thus lead us to:

Definition 4.7. *If $\mathscr{Y}$ is a C*-algebra, then define the set*

$$\mathscr{Y}_{sa}^+ := \{\xi \in \mathscr{Y}_{sa} \mid \sigma(\xi) \subset [0,\infty)\}$$

The elements $\xi \in \mathscr{Y}_{sa}^+$ are called positive.

Lemma 4.7. *If $\xi \in \mathscr{Y}_{sa}^+$, then ξ has a unique positive n-th root, i.e. there exists a unique $\zeta \in \mathscr{Y}_{sa}^+$ such that $\zeta^n = \xi$.*

Proof. Since $\xi \in \mathscr{Y}_{sa}^+$ we have by definition $\sigma(\xi) \subset [0, r(\xi)]$. Therefore, if $f(x) := \sqrt[n]{x}$, then $f \in \mathscr{C}(\sigma(\xi))$. Now by the CFC 4.3 $\zeta := f(\xi)$ is well defined and moreover ζ is selfadjoint and positive. Indeed, selfadjointness follows from the fact that f is real-valued

$$\zeta^* = \Gamma_\xi(f)^* = \Gamma_\xi(f^*) = \Gamma_\xi(f) = \zeta$$

and positivity follows from $\sigma(\zeta) = f(\sigma(\xi))$. Now let $g = Z^n \in \mathbb{C}[Z]$, then by the commutation relation of the CFC (4.7) we have

$$\xi = (g \circ f)(\xi) = g(f(\xi)) = g(\zeta) = \zeta^n$$

Suppose there were another such n-th root η of ξ, then

$$\eta = (f \circ g)(\eta) = f(\eta^n) = f(\xi) = \zeta$$

so uniqueness is established. ■

Remark 4.5. *For such an n-th root we will usually write $\sqrt[n]{\xi}$. In the case of $n=2$, we will say, that $\sqrt{\xi}$ is the unique positive square root for ξ. Of course this whole procedure also works for $\xi^\alpha := f(\xi)$ with $\alpha > 0$ and $f(x) = x^\alpha$.*

Lemma 4.8. *Let $\xi \in \mathscr{Y}_{sa}$, then we may find $\xi^+, \xi^- \in \mathscr{Y}_{sa}^+$ such that $\xi = \xi^+ - \xi^-$ and $\xi^+\xi^- = 0$.*

Proof. Define the functions

$$f^+ \in \mathscr{C}(\sigma(\xi)) \qquad f^+(x) := \max(0,x)$$
$$f^- \in \mathscr{C}(\sigma(\xi)) \qquad f^-(x) := -\min(0,x)$$

Then $f^+, f^- \geq 0$ and thus if we set $\xi^+ := f^+(\xi)$ and $\xi^- := f^-(\xi)$, then $\xi^+, \xi^- \in \mathscr{Y}_{sa}^+$ and clearly these elements satisfy

$$\xi = (f^+ - f^-)(\xi) = \xi^+ - \xi^-$$
$$\xi^+\xi^- = (f^+f^-)(\xi) = 0$$

■

Here comes our first milestone in terms of positivity.

Proposition 4.7. *If $\xi \in \mathscr{Y}_{sa}$, then we have the following equivalences:*

(i). $\xi \in \mathscr{Y}_{sa}^+$

(ii). $\xi = \zeta^2$ *for some* $\zeta \in \mathscr{Y}_{sa}$

(iii). $\|c - \xi\| \leq c$ *for all* $c \geq \|\xi\|$

(iv). $\|c - \xi\| \leq c$ *for some* $c \geq \|\xi\|$

Proof. By lemma 4.7 we have (i) $\implies$ (ii) and clearly $(iii) \implies (iv)$. Assume now, that (ii) is valid and we want to show, that this implies (iii). By assumption of (ii) we have, that there exists a selfadjoint element $\zeta \in \mathscr{Y}$ so that $\zeta^2 = \xi$. Let $f(x) := x^2$, then $f \in \mathscr{C}(\sigma(\zeta))$ and $f(\zeta) = \xi$. By the CFC (4.5) we have, that $\|f\| = \|\xi\|$ and therefore in particular $0 \leq f \leq \|\xi\|$. Now choose any $c \geq \|\xi\|$, then clearly $0 \leq c - f \leq c$. Observe now that $c - f \in \mathscr{C}(\sigma(\zeta))$, and yet again by the CFC (4.5) we obtain $\|c - f\| = \|(c-f)(\zeta)\| = \|c - \xi\|$. Thus

$$c \geq \|c - f\| = \|c - \xi\|$$

which yields (iii). Now assume, that (iv) holds and let $c \geq \|\xi\|$ be a value for which the inequality holds. Then yet again by the CFC (4.5) we have

$$c \geq \|c - \xi\| = \|(c - id)(\xi)\| = \|c - id\|$$

So $c \geq c - \lambda$ for all $\lambda \in \sigma(\xi) \subset \mathbb{R}$, but this can only be if $\sigma(\xi) \subset [0,\infty)$, so $\xi \in \mathscr{Y}_{sa}^+$, which implies (i). ■

The next step is to show, that we have indeed found a structure, which could provide for a partial order. We will do this by proving the following:

Corollary 4.9. *The set $\mathscr{Y}_{sa}^+$ is a convex cone, i.e.*

(i). If $\xi \in \mathscr{Y}_{sa}^+$ and $c \geq 0$, then $c\xi \in \mathscr{Y}_{sa}^+$

(ii). If $\xi, \zeta \in \mathscr{Y}_{sa}^+$, then $\xi + \zeta \in \mathscr{Y}_{sa}^+$

(iii). If $-\mathscr{Y}_{sa}^+ = \{-\xi \mid \xi \in \mathscr{Y}_{sa}^+\}$, then $\mathscr{Y}_{sa}^+ \cap -\mathscr{Y}_{sa}^+ = \{0\}$

Proof. Since $\sigma(c\xi) = c\sigma(\xi)$, (i) follows immediately. Now take $\xi, \zeta \in \mathscr{Y}_{sa}^{+}$, then by proposition 4.7 there exist elements $c_1 \geq \|\xi\|$ and $c_2 \geq \|\zeta\|$ such that

$$c_1 \geq \|c_1 - \xi\| \qquad c_2 \geq \|c_2 - \zeta\|$$

Now we have that $c_1 + c_2 \geq \|\xi\| + \|\zeta\| \geq \|\xi + \zeta\|$ and in particular

$$\|(c_1 + c_2) - (\xi + \zeta)\| \leq \|c_1 - \xi\| + \|c_2 - \zeta\| \leq c_1 + c_2$$

Hence again by proposition 4.7 we have $\xi + \zeta \in \mathscr{Y}_{sa}^{+}$, which gives (ii). Now assume that $\xi \in \mathscr{Y}_{sa}^{+} \cap -\mathscr{Y}_{sa}^{+}$, then

$$\sigma(\xi) \subset [0, \infty) \qquad -\sigma(\xi) \subset [0, \infty)$$

which is only possible if $\sigma(\xi) = \{0\}$. But by corollary 4.3 we have $\|\xi\| = r(\xi) = 0$ and therefore $\xi = 0$, implying (iii). ■

Now the task of forcing a partial ordering $\preceq$ upon $\mathscr{Y}_{sa}$ is easy and we can do that in a quite natural way. If $\xi, \zeta \in \mathscr{Y}_{sa}^{+}$, then we will write

$$\xi \preceq \zeta \iff \zeta - \xi \in \mathscr{Y}_{sa}^{+}$$

Since we know that $\mathscr{Y}_{sa}^{+}$ is a convex cone, it is easy to verify, that $\preceq$ is a partial ordering (reflexive, antisymmetric and transitive).

So now take an element $\zeta \in \mathscr{Y}$ and consider the element $\xi = \zeta^*\zeta$. Clearly ξ is selfadjoint, but how about positivity? It turns out to be true, that such ξ must always be positive, yet the proof of this is quite involved. What makes this even worse is, that the result seems to have such an obvious proof: Indeed we know, that

$$\sigma(\xi) = \{\psi(\xi) \mid \psi \in \Sigma(\mathscr{C}^*(\xi))\} = \{\psi(\zeta^*\zeta) \mid \psi \in \Sigma(\mathscr{C}^*(\zeta^*\zeta))\}$$

and thus every spectral value of ξ is of the form $\psi(\zeta^*\zeta)$ for some ψ from the maximal ideal space of $\mathscr{C}(\zeta^*\zeta)$. Now one could be misled to think, that $\psi(\zeta^*\zeta) = |\psi(\zeta)|^2$, but $\psi(\zeta)$ does not have to make sense at all, i.e. we don't know if the domain of ψ contains ζ at all. Therefore, very sadly indeed, we need to go about this in a different, slighly annoying manner. We will first prove a lemma, before proving the theorem.

Lemma 4.9. *If $\xi, \eta \in \mathscr{Y}$, then $\sigma(\xi\eta) \cup \{0\} = \sigma(\eta\xi) \cup \{0\}$.*

Proof. Suppose $\lambda \in \rho(\xi\eta) \backslash \{0\}$ and observe that

$$(\eta(\xi\eta - \lambda)^{-1}\xi - 1)(\eta\xi - \lambda) = \eta(\xi\eta - \lambda)^{-1}(\xi\eta - \lambda)\xi - (\eta\xi - \lambda) = \lambda$$

and analogously $(\eta\xi - \lambda)(\eta(\xi\eta - \lambda)^{-1}\xi - 1) = \lambda$. Since $\lambda \neq 0$ we have

$$\lambda^{-1}(\eta(\xi\eta - \lambda)^{-1}\xi - 1) = (\eta\xi - \lambda)^{-1}$$

so $\lambda \in \rho(\eta\xi) \backslash \{0\}$. We conclude that $\sigma(\xi\eta) \cup \{0\} = \sigma(\eta\xi) \cup \{0\}$. ■

Theorem 4.4. *The set of elements $\xi^*\xi$, where $\xi \in \mathscr{Y}$ is equal to the set $\mathscr{Y}_{sa}^{+}$, i.e. $\mathscr{Y}_{sa}^{+} = \{\xi^*\xi \mid \xi \in \mathscr{Y}\}$.*

Proof. From proposition 4.7 (ii) we already know, that $\mathscr{Y}_{sa}^{+} = \{\xi^2 \mid \xi \in \mathscr{Y}_{sa}\}$ and therefore

$$\mathscr{Y}_{sa}^{+} \subset \{\xi^*\xi \mid \xi \in \mathscr{Y}\}$$

is already established. By lemma 4.8 we may write an element $\xi = \zeta^*\zeta \in \mathscr{Y}_{sa}$ as a sum $\xi = \xi^+ - \xi^-$, where $\xi^+, \xi^- \succeq 0$ and $\xi^+\xi^- = 0$. Let $\sqrt{\xi^-}$ be the unique positive square root of ξ^- from lemma 4.7. Observe that, since ξ^- is in particular selfadjoint, we know that the set $\{p(Z) \mid p \in \mathbb{C}[Z]\}$ is dense in $\mathscr{C}(\sigma(\xi^-))$ and therefore there exists a sequence of polynomials $\{p_n\}_n \subset \mathbb{C}[Z]$ with $p_n \to f$, where $f(x) := \sqrt{x}$. Note that we can assume without loss of generality, that all p_n have zero constant coefficient (it is easy to verify, that $\widetilde{p}_n := p_n - p_n(0)$ also satisfies $\widetilde{p}_n \to f$). Now we know, that $\sqrt{\xi^-} = f(\xi^-) = \lim p_n(\xi^-)$. Therefrom we see, that

$$\sqrt{\xi^-}\xi^+ = \lim p_n(\xi^-)\xi^+ = 0$$

since $\xi^-\xi^+ = 0$. We now define $\eta := \zeta\sqrt{\xi^-}$ and note, that

$$-\eta^*\eta = -\sqrt{\xi^-}\zeta^*\zeta\sqrt{\xi^-} = -\sqrt{\xi^-}\xi\sqrt{\xi^-} = -\sqrt{\xi^-}(\xi^+ - \xi^-)\sqrt{\xi^-} = \sqrt{\xi^-}\xi^-\sqrt{\xi^-} = (\xi^-)^2 \tag{4.9}$$

Hence $-\eta^*\eta$ is not only selfadjoint but also positive. Next decompose $\eta = \eta_1 + i\eta_2$, where $\eta_1, \eta_2 \in \mathscr{Y}_{sa}$ and note, that

$$\eta^*\eta + \eta\eta^* = (\eta_1 - i\eta_2)(\eta_1 + i\eta_2) + (\eta_1 + i\eta_2)(\eta_1 - i\eta_2) = 2\eta_1^2 + 2\eta_2^2$$

Since $2\eta_1^2 \succeq 0$ and $2\eta_2^2 \succeq 0$ we have that $\eta^*\eta + \eta\eta^*$ is a sum of positive elements and thus by corollary 4.9 (ii) we have, that $\eta^*\eta + \eta\eta^* \in \mathscr{Y}_{sa}^+$. But this in turn tells us, that

$$\eta\eta^* = (\eta^*\eta + \eta\eta^*) + (-\eta^*\eta)$$

is positive as a sum of positive elements. By lemma 4.9 we therefore know, that $\eta^*\eta \succeq 0$, which tell us that both $\eta^*\eta$ and $-\eta^*\eta$ are positive and that is only possible if $\eta^*\eta = 0$. Now since $\eta^*\eta = (\xi^-)^2$ by (4.9) we have $(\xi^-)^2 = 0$, taking the square root and remembering the uniqueness of this procedure yields, that $\xi^- = 0$. But this means $\xi = \xi^+ \in \mathscr{Y}_{sa}^+$. ■

Remark 4.6. *If $\xi \in \mathscr{Y}$ and $\zeta = \xi^*$, then by the preceding theorem we also have, that $\xi\xi^* = \zeta^*\zeta \succeq 0$, so for positivity it doesn't matter if one writes $\xi^*\xi$ or $\xi\xi^*$.*

4.7 Ideals and Quotients of C*-algebras

This subsection is based upon [6] and [8].

We have already seen, that a non-unital C*-algebra can be unitized. This process is very important as we have often witnessed already. However, sometimes we would like to have some kind of replacement of a unit internally. This will turn out to be an immensely important tool in our new quest to understand quotient C*-algebras.

Definition 4.8. *Let $\mathscr{Y}$ be a non-unital C*-algebra. If $\{e_F\}_{F\in\mathfrak{F}}$ is a net in $\mathscr{Y}$ satisfying*

(i). $\forall F \in \mathfrak{F}$: $e_F^* = e_F$

(ii). $\forall F \in \mathfrak{F}$: $\sigma(e_F) \subset [0,1]$

(iii). $\lim_{F\in\mathfrak{F}} e_F\xi = \lim_{F\in\mathfrak{F}} \xi e_F = \xi$ for all $\xi \in \mathscr{Y}$

then we will call $\{e_F\}_{F\in\mathfrak{F}}$ an approximate identity for $\mathscr{Y}$.

Of course for any object freshly defined the question of existence lurks in the shadows, ready to be unveiled.

Proposition 4.8. *If $\mathscr{Y}$ is a non-unital C*-algebra, then $\mathscr{Y}$ always has an approximate identity.*

Proof. Let

$$\mathfrak{F} := \{F \subset \mathscr{Y} \mid F \text{ finite}\}$$

then $\mathfrak{F}$ can be partially ordered by inclusion. For $F \in \mathfrak{F}$ define the element

$$\xi_F := \sum_{f\in F} ff^*$$

By construction $\xi_F \in \mathscr{Y}_{sa}$ and by theorem 4.4 combined with corollary 4.9 (ii) we even have, that $\xi_F \in \mathscr{Y}_{sa}^+$. Consider the unitization $\mathscr{Y}_1$ of $\mathscr{Y}$ and define $k(x) := ((\#F)^{-1}+x)^{-1}$, where $\#F$ denotes the number of elements in F, then $k \in \mathscr{C}(\sigma(\xi_F))$ and it follows from the CFC, that $k(\xi_F)$ is the inverse of $(\#F)^{-1}+\xi_F$ in $\mathscr{Y}_1$. Now define

$$e_F := \xi_F k(\xi_F)$$

and note that

$$e_F^* = k(\xi_F)^* \xi_F^* = k(\xi_F)\xi_F = (k \cdot id)(\xi_F) = (id \cdot k)(\xi_F) = e_F$$

Now since $k(\xi_F)$ is an element of $\mathscr{Y}_1$, we can write it as $k(\xi_F) = \zeta + \lambda$ for $\zeta \in \mathscr{Y}$ and $\lambda \in \mathbb{C}$. Therefore $e_F = \xi_F k(\xi_F) = \xi_F \zeta + \lambda \xi_F \in \mathscr{Y}$. Now if $g(x) := xk(x)$, then $g \in \mathscr{C}(\sigma(\xi_F))$ and $e_F = g(\xi_F)$ by construction. Moreover, since $\sigma(\xi_F) \subset [0,\infty)$ and $0 \leq g \leq 1$, we have $\sigma(e_F) = g(\sigma(\xi_F)) \subset [0,1]$. Next note that

$$\sum_{f\in F} [(e_F-1)f][(e_F-1)f]^* = \xi_F(e_F-1)^2 = \xi_F(\#F)^{-2}k(\xi_F)^2 = h(\xi_F) \tag{4.10}$$

where

$$h(x) := (\#F)^{-2}\frac{x}{((\#F)^{-1}+x)^2}$$

and observe that $h \in \mathscr{C}(\sigma(e_F))$ attains it maximum at $(\#F)^{-1}$, so

$$0 \leq h \leq h((\#F)^{-1}) = \frac{1}{4(\#F)}$$

Thus $\sigma(h(\xi_F)) \subset \left[0, \frac{1}{4(\#F)}\right]$ and hence by (4.10) we have for all $\zeta \in F$

$$[(e_F-1)\zeta][(e_F-1)\zeta]^* \preceq \sum_{f\in F}[(e_F-1)f][(e_F-1)f]^* \preceq \frac{1}{4(\#F)}$$

and therefrom we see $\|(e_F-1)\zeta\|^2 = r([(e_F-1)\zeta][(e_F-1)\zeta]^*) \leq \frac{1}{4(\#F)}$ and thus

$$\lim_{F\in\mathfrak{F}} \|e_F\zeta - \zeta\| = 0$$

for all $\zeta \in \mathscr{Y}$. Moreover since $\|e_F\zeta - \zeta\| = \|\zeta^* e_F - \zeta^*\|$, we also have

$$\lim_{F\in\mathfrak{F}} \|\zeta e_F - \zeta\| = 0$$

for all $\zeta \in \mathscr{Y}$. Hence $\{e_F\}_{F\in\mathfrak{F}}$ is an approximate identity for $\mathscr{Y}$. ■

Example 4.8. *Consider the non-unital C*-algebra $\mathbb{C}_0(\mathbb{R})$ and let e_n be a non-negative continuous function, that is* 1 *on the interval $[-n,n]$ and vanishes outside $[-n-1,n+1]$. Then $\{e_n\}_{n\in\mathbb{N}}$ is an approximate identity for $\mathscr{C}_0(\mathbb{R})$.*

Example 4.9. *Let $\mathscr{H}$ be a separable Hilbert space and let $\{e_1,e_2,...\}$ be a basis for $\mathscr{H}$. Denote the closed linear span of $\{e_1,...,e_n\}$ by $\bigvee\{e_1,...,e_n\}$ and let P_n be the projection onto $\bigvee\{e_1,...,e_n\}$ for all $n \geq 1$. Of course $P_n \in \mathscr{K}(\mathscr{H})$, since all P_n have finite dimensional range. So now let $\varepsilon > 0$. By compactness of T there exist $x_1,...,x_l \in B_{\mathscr{H}}$ so that*

$$T(B_{\mathscr{H}}) \subset \bigcup_{1\leq j\leq l} B_\varepsilon(Tx_j)$$

Now we may find $N \in \mathbb{N}$ big enough so that for all $n \geq N$ we have

$$\max_{1\leq j\leq l} \|Tx_j - P_nTx_j\| < \varepsilon$$

and thus if $x \in B_{\mathscr{H}}$, then we may find j so that $Tx \in B_\varepsilon(Tx_j)$ and therefore for all $n \geq N$

$$\|Tx - P_nTx\| \leq \|Tx - Tx_j\| + \|Tx_j - P_nTx_j\| + \underbrace{\|P_nTx_j - P_nTx\|}_{\leq \|Tx_j - Tx\|} < 3\varepsilon$$

As this inequality holds for all $x \in B_{\mathscr{H}}$, we have

$$\|T - P_nT\| \leq 3\varepsilon$$

which means $\lim_{n\to\infty} \|T - P_nT\| = 0$. Since T^* is also compact we obtain by the same argument as before, that

$$0 = \lim_{n\to\infty} \|T^* - P_nT^*\| = \lim_{n\to\infty} \|T - TP_n\|$$

Moreover, since all P_n are selfadjoint and $\|1 - P_n\| \leq 1$ for all $n \geq 1$, we have by proposition 4.7, that $P_n \in \mathscr{K}(\mathscr{H})^+_{sa}$ and thus $\{P_n\}_{n\in\mathbb{N}}$ is an approximate identity for $\mathscr{K}(\mathscr{H})$.

We now brew together some helpful lemmas.

Lemma 4.10. *Let $\mathscr{I}$ be a closed ideal in a C*-algebra $\mathscr{Y}$, then $\mathscr{I}$ is selfadjoint, i.e.*

$$\mathscr{I}^* := \{\zeta^* \mid \zeta \in \mathscr{I}\} = \mathscr{I}$$

Proof. It suffices to show, that if $\zeta \in \mathscr{I}$, then we also have $\zeta^* \in \mathscr{I}$. Because $\mathscr{I}$ is assumed to be closed and the *-operation is a homeomorphism, $\mathscr{I}^*$ is also closed in $\mathscr{Y}$. Now if $\zeta \in \mathscr{I}$, then $\zeta\zeta^* \in \mathscr{I}$ and in particular

$$\mathscr{I}^*\mathscr{Y} = (\underbrace{\mathscr{Y}\mathscr{I}}_{\subset \mathscr{I}})^* \subset \mathscr{I}^* \qquad \mathscr{Y}\mathscr{I}^* = (\underbrace{\mathscr{I}\mathscr{Y}}_{\subset \mathscr{I}})^* \subset \mathscr{I}^*$$

implies that $\mathscr{I}^*$ is also a closed ideal, and therefore $\zeta\zeta^* \in \mathscr{I} \cap \mathscr{I}^*$. Thus $\mathscr{J} := \mathscr{I} \cap \mathscr{I}^*$ is a C*-subalgebra of $\mathscr{Y}$. By proposition 4.8 $\mathscr{J}$ has an approximate identity $\{e_F\}_{F\in\mathfrak{F}}$. Note first that $\|e_F\| \leq 1$ for all $F \in \mathfrak{F}$, since $\sigma(e_F) \subset [0,1]$ and $e_F^* = e_F$ for all $F \in \mathfrak{F}$. For $\zeta \in \mathscr{I}$ we know from before, that $\zeta\zeta^* \in \mathscr{J}$. Therefore $\lim_{F\in\mathfrak{F}} \zeta\zeta^* e_F = \zeta\zeta^*$ and thus we calculate

$$\begin{aligned}
\|\zeta^* e_F - \zeta^*\|^2 &= \|(\zeta^* e_F - \zeta^*)^*(\zeta^* e_F - \zeta^*)\| = \|(e_F\zeta - \zeta)(\zeta^* e_F - \zeta^*)\| \\
= \|e_F(\zeta\zeta^* e_F - \zeta\zeta^*) + (\zeta\zeta^* - \zeta\zeta^* e_F)\| &\leq \|e_F\|\|\zeta\zeta^* e_F - \zeta\zeta^*\| + \|\zeta\zeta^* - \zeta\zeta^* e_F\| \\
&\leq 2\|\zeta\zeta^* - \zeta\zeta^* e_F\| \to 0
\end{aligned}$$

But since $e_F \in \mathscr{J}$ for all F, we also have that all $e_F \in \mathscr{I}$, so all $\zeta^* e_F \in \mathscr{I}$. Since $\lim_{F\in\mathfrak{F}} \zeta^* e_F = \zeta^*$ and $\mathscr{I}$ is closed, we have $\zeta^* \in \mathscr{I}$. Hence $\mathscr{I} = \mathscr{I}^*$. ■

Lemma 4.11. *Let $\mathscr{Y}$ be a C*-algebra and $\mathscr{I} \subset \mathscr{Y}$ be a closed ideal. Denote by $Q\colon \mathscr{Y} \to \mathscr{Y}/\mathscr{I}$ the quotient map, and let $\mathscr{Y}/\mathscr{I}$ be equipped with the quotient norm. If $\{e_F\}_{F\in\mathfrak{F}}$ is an approximate identity for $\mathscr{I}$, then*

$$\|Q(\xi)\| = \lim_{F\in\mathscr{F}} \|\xi - \xi e_F\| = \lim_{F\in\mathscr{F}} \|\xi - e_F\xi\|$$

Proof. Embed $\mathscr{Y} \hookrightarrow \mathscr{Y}_1$ isometrically if $\mathscr{Y}$ is non-unital. Since $1 \succeq e_F$ we also have $1 \succeq 1 - e_F \succeq 0$ and in particular, if $f(x) := 1 - x$, then $f \in \mathscr{C}(\sigma(e_F))$ and since $\sigma(e_F) \subset [0,1]$ we have $0 \leq f \leq 1$, so

$$\sigma(1 - e_F) = f(\sigma(e_F)) \subset [0,1]$$

Hence $\|1 - e_F\| \leq 1$ and therefrom for $\xi \in \mathscr{Y}$ fixed and $\zeta \in \mathscr{I}$, we calculate

$$\|\xi - \xi e_F\| = \|(\xi + \zeta)(1 - e_F) - \zeta(1 - e_F)\| \leq \|\xi + \zeta\| + \|\zeta - \zeta e_F\| \to \|\xi + \zeta\|$$

Note that this inequality also holds, if $\mathscr{Y}$ is non-unital, since the embedding $\mathscr{Y} \hookrightarrow \mathscr{Y}_1$ is isometric. Therefore $\lim_{F\in\mathfrak{F}} \|\xi - \xi e_F\| \leq \|Q(\xi)\|$ and on the other hand $\|\xi - \xi e_F\| \geq \|Q(\xi)\|$, since $e_F \in \mathscr{I}$ and $\mathscr{I}$ is an ideal, proving

$$\|Q(\xi)\| = \lim_{F\in\mathscr{F}} \|\xi - \xi e_F\|$$

for all $\xi \in \mathscr{Y}$. Since $\|\xi - \xi e_F\| = \|\xi^* - e_F \xi^*\|$ for all ξ, we also have the second equality. ■

Now we have all that is needed to conquer the realm of quotient C*-algebras.

Theorem 4.5. *If $\mathscr{I}$ is a closed ideal in a C*-algebra $\mathscr{Y}$, then $\mathscr{Y}/\mathscr{I}$ is a C*-algebra, if the involution on $\mathscr{Y}/\mathscr{I}$ is defined by*

$$Q(\xi)^* := Q(\xi^*)$$

for all $\xi \in \mathscr{Y}$.

Proof. First note, that the involution on the quotient is well defined, since if $Q(\xi) = Q(\zeta)$, we have $\xi - \zeta \in \mathscr{I}$ and thus by lemma 4.10 $\xi^* - \zeta^* \in \mathscr{I}$ and consequently $Q(\xi^*) = Q(\zeta^*)$. Since we have already seen that this quotient is a Banach algebra in proposition 3.1, all that remains to prove is that $\|Q(\xi)\|^2 = \|Q(\xi)^* Q(\xi)\|$. But this is not too hard, since by proposition 4.8 there exists an approximate identity $\{e_F\}_{F\in\mathfrak{F}}$ for $\mathscr{I}$ and by lemma 4.11 we then have

$$\begin{aligned}\|Q(\xi)\|^2 = \lim_{F\in\mathfrak{F}} \|\xi - \xi e_F\|^2 = \lim_{F\in\mathfrak{F}} \|(\xi - \xi e_F)^*(\xi - \xi e_F)\| &= \lim_{F\in\mathfrak{F}} \|(1-e_F)(\xi^*\xi - \xi^*\xi e_F)\| \\ &\leq \lim_{F\in\mathfrak{F}} \|\xi^*\xi - \xi^*\xi e_F\| = \|Q(\xi^*\xi)\| = \|Q(\xi)^* Q(\xi)\|\end{aligned} \tag{4.11}$$

Moreover again by lemma 4.11 we have

$$\|Q(\xi)^*\| = \|Q(\xi^*)\| = \lim_{F\in\mathfrak{F}} \|\xi^* - \xi^* e_F\| = \lim_{F\in\mathfrak{F}} \|\xi - e_F\xi\| = \|Q(\xi)\| \tag{4.12}$$

Hence combining (4.11), (4.12) and using submultiplicativity of our quotient norm we have

$$\|Q(\xi)\|^2 \leq \|Q(\xi)^* Q(\xi)\| \leq \|Q(\xi)^*\| \|Q(\xi)\| = \|Q(\xi)\|^2$$

so we are done. ■

Example 4.10. *Let $\mathscr{H}$ be an infinite dimensional Hilbert space. The C*-algebra $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ is called the Calkin algebra.*

All this seems innocent enough, for we have already seen similar statements for Banach algebras. However the next theorem shows yet again the amazing properties C*-algebras possess.

Theorem 4.6. *Let $\mathscr{Y}$ and $\mathscr{Z}$ be C*-algebras and let $\varphi\colon \mathscr{Y} \to \mathscr{Z}$ be an injective *-homomorphism, then φ is isometric.*

Proof. From proposition 4.5 we already know, that $\|\varphi(\xi)\| \leq \|\xi\|$ for all $\xi \in \mathscr{Y}$. Suppose φ were not isometric, so there must exist $0 \neq \xi \in \mathscr{Y}$ so that $\|\varphi(\xi)\| < \|\xi\|$. Let $\zeta := \xi^*\xi \in \mathscr{Y}_{sa}$, then

$$\|\varphi(\zeta)\| = \|\varphi(\xi)^*\varphi(\xi)\| = \|\varphi(\xi)\|^2 < \|\xi\|^2 = \|\zeta\|$$

Now since ζ and $\varphi(\zeta)$ are selfadjoint, we have in particular

$$r(\varphi(\zeta)) = \|\varphi(\zeta)\| < \|\zeta\| = r(\zeta) \tag{4.13}$$

So since $\sigma(\varphi(\zeta)) \subset \sigma(\zeta)$ and by (4.13) we must have the proper inclusion $\sigma(\varphi(\zeta)) \subsetneq \sigma(\zeta)$. Since $\sigma(\zeta)$ is a compact Hausdorff space, Urysohn's Lemma is applicable, so we deduce, that for $\lambda \in \sigma(\zeta)\backslash\sigma(\varphi(\zeta))$ there must exist $f \in \mathscr{C}(\sigma(\zeta))$ such that

$$f(\sigma(\varphi(\zeta))) = \{0\} \qquad f(\lambda) = 1$$

In particular by corollary 4.8 this yields

$$\varphi(f(\zeta)) = f(\varphi(\zeta)) = 0 \tag{4.14}$$

But $\|f(\zeta)\| = \|f\| \geq 1$, so $f(\zeta) \neq 0$ and therefore (4.14) contradicts injectivity of φ, which proves, that φ must be isometric.

∎

Corollary 4.10. *Let $\mathscr{Y}$ and $\mathscr{X}$ be C*-algebras and let φ be a *-homomorphism. Then φ has closed range, so in particular $\varphi(\mathscr{Y})$ is a C*-subalgebra of $\mathscr{X}$.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathscr{Y} \arrow[rr, "\varphi"] \arrow[dr, "Q"'] & \circlearrowleft & \mathscr{X} \\
& \mathscr{Y}/\ker(\varphi) \arrow[ur, dashed, "\underline{\varphi}"'] &
\end{tikzcd}$$

Since φ is continuous as a *-homomorphism, $\ker(\varphi)$ is a closed ideal in $\mathscr{Y}$, so by theorem 4.5 we have that $\mathscr{Y}/\ker(\varphi)$ is a C*-algebra. Let $Q\colon \mathscr{Y} \to \mathscr{Y}/\ker(\varphi)$ be the quotient map and define the function $\underline{\varphi}\colon \mathscr{Y}/\ker(\varphi) \to \mathscr{X}$ given by $\underline{\varphi}(Q(\xi)) := \varphi(\xi)$. By construction $\underline{\varphi}$ is an injective *-homomorphism, so by what we have shown before, $\underline{\varphi}$ is isometric. This yields in particular, that

$$\underline{\varphi}(\mathscr{Y}/\ker(\varphi)) = \varphi(\mathscr{Y})$$

is closed. So since φ is a *-homomorphism $\varphi(\mathscr{Y})$ is a C*-subalgebra of $\mathscr{X}$. ∎

4.8 States

This subsection is based upon [8] and [4].

A state is a linear functional on a C*-algebra, that is characterized, informally said, by being positive on positive elements. States are extremely important in what is to come, and they eventually end up being one of the key ingredients in proving, that the Gel'fand-Naimark-Segal construction, which will be treated in the next chapter, indeed works.

Definition 4.9. *Let Φ be a linear functional on a C*-algebra $\mathscr{Y}$. The functional Φ is said to be positive if and only if $\Phi(\xi) \geq 0$ for all $\xi \in \mathscr{Y}_{sa}^+$, or equivalently put*

$$\xi \succeq 0 \implies \Phi(\xi) \geq 0$$

Proposition 4.9. *(Cauchy-Schwarz inequality for states) Let Φ be a positive linear functional, then*

$$|\Phi(\xi^*\zeta)|^2 \leq \Phi(\xi^*\xi)\Phi(\zeta^*\zeta)$$

Proof. For $\xi, \zeta \in \mathscr{Y}$ define $\langle \xi | \zeta \rangle := \Phi(\zeta^*\xi)$. We claim that this defines a semi-inner product. Indeed, since Φ is positive, the only non-trivial axiom for semi-inner products to verify in this case is $\langle \xi | \zeta \rangle = \overline{\langle \zeta | \xi \rangle}$. In order to prove this, it suffices to prove $\Phi(\xi^*) = \overline{\Phi(\xi)}$, so in particular since every element ξ can be written as $\xi_1 + i\xi_2$ for $\xi_i \in \mathscr{Y}_{sa}$, it is enough to prove $\Phi(\xi) \in \mathbb{R}$ for all $\xi \in \mathscr{Y}_{sa}$. But as every $\xi \in \mathscr{Y}_{sa}$ can be written as $\xi = \xi^+ - \xi^-$ for $\xi^+, \xi^- \in \mathscr{Y}_{sa}^+$ we have

$$\Phi(\xi) = \Phi(\xi^+) - \Phi(\xi^-) \in \mathbb{R}$$

by positivity of Φ, so our claim is proven. By the Cauchy-Schwarz inequality we therefore have

$$|\Phi(\zeta^*\xi)|^2 = |\langle \xi | \zeta \rangle|^2 \leq \langle \xi | \xi \rangle \langle \zeta | \zeta \rangle = \Phi(\xi^*\xi)\Phi(\zeta^*\zeta)$$

∎

Proposition 4.10. *Let $\mathscr{Y}$ be a unital C*-algebra and let Φ be a linear functional on $\mathscr{Y}$. Then Φ is positive if and only if $\|\Phi\| = \Phi(1)$.*

Proof. Suppose first, that Φ is positive and take $\xi \in \mathscr{Y}_{sa}$. Set $g_1(x) := \|\xi\| - x$ and $g_2(x) := \|\xi\| + x$, then $g_1, g_2 \in \mathscr{C}(\sigma(\xi))$ and $g_1, g_2 \geq 0$, therefore $g_i(\xi) \succeq 0$ for $i = 1, 2$, which yields that

$$-\|\xi\| \preceq \xi \preceq \|\xi\|$$

Thus in particular by positivity of Φ we have

$$\Phi(\|\xi\|) = \|\xi\|\Phi(1) \geq |\Phi(\xi)|$$

So now let ξ be arbitrary, then by proposition 4.9 and by what we have shown for selfadjoint elements we have

$$|\Phi(\xi)|^2 \leq \Phi(1)\Phi(\xi^*\xi) \leq \Phi(1)^2\|\xi^*\xi\| = \Phi(1)^2\|\xi\|^2$$

Hence $\|\Phi\| = \Phi(1)$. Conversely, let $\|\Phi\| = \Phi(1)$ and note that $\Phi(1) = 0 \implies \Phi = 0$, for which the statement trivially holds. Hence assume $\Phi(1) \neq 0$, then for $t \in \mathbb{R}$ and $\xi \in \mathscr{Y}_{sa}$ fixed we have (analogously to the proof of proposition 4.4)

$$|\Phi(\xi + it)|^2 \leq \Phi(1)^2(\|\xi\|^2 + t^2)$$

and if $\Phi(\xi) = a + ib$ for $a, b \in \mathbb{R}$, then

$$\Phi(1)^2(\|\xi\|^2 + t^2) \geq |\Phi(\xi + it)|^2 = a^2 + b^2 + 2bt\Phi(1) + t^2\Phi(1)^2$$

and hence $\Phi(1)^2\|\xi\|^2 \geq a^2 + b^2 + 2bt\Phi(1)$ for all $t \in \mathbb{R}$, so $b = 0$ and therefore $\Phi(\xi) \in \mathbb{R}$ for all $\xi \in \mathscr{Y}_{sa}$. Now let $\xi \in \mathscr{Y}_{sa}^+$, then by definition $\sigma(\xi) \subset [0, \infty)$. Choose $0 < \varepsilon < 1$ so small, that $\sigma(\varepsilon\xi) = \varepsilon\sigma(\xi) \subset [0, 1]$ and consider $1 - \varepsilon\xi \in \mathscr{Y}_{sa}$. By selfadjointness we have

$$\|1 - \varepsilon\xi\| = r(1 - \varepsilon\xi) \leq 1$$

and thus

$$\Phi(1) \geq \Phi(1)\|1 - \varepsilon\xi\| \geq |\Phi(1 - \varepsilon\xi)| \tag{4.15}$$

Since $\Phi(\xi) \in \mathbb{R}$ we must have $\Phi(\xi) \geq 0$, since otherwise (4.15) would lead to a contradiction. ■

Definition 4.10. *Let $\mathscr{Y}$ be a unital C*-algebra. A state Φ on $\mathscr{Y}$ is a positive linear functional satisfying $\|\Phi\| = 1$, or equivalently $\Phi(1) = 1$. The set of states on $\mathscr{Y}$ is denoted by $\mathfrak{S}(\mathscr{Y})$.*

Example 4.11. *Let $\mathscr{H}$ be a Hilbert space and consider the set of continuous linear operators on $\mathscr{H}$, namely $\mathscr{Y} = \mathscr{L}(\mathscr{H})$. Let $x \in \mathscr{H}$ be an element with $\|x\| = 1$. Define $\Phi_x(\xi) := \langle \xi x | x \rangle$ for all $\xi \in \mathscr{Y}$. Since every positive element is of the form $\xi^*\xi$ for some $\xi \in \mathscr{Y}$, we have*

$$\Phi_x(\xi^*\xi) = \langle \xi^*\xi x | x \rangle = \langle \xi x | \xi x \rangle = \|\xi x\|^2$$

So Φ_x is positive and $\Phi_x(1) = \langle x | x \rangle = 1$. So each x of norm 1 gives rise to a state on $\mathscr{Y}$.

Example 4.12. *Let X be a compact Hausdorff space and consider the abelian, unital C*-algebra $\mathscr{C}(X)$. Positive elements in $\mathscr{C}(X)$ are all non-negative real-valued functions $f \in \mathscr{C}(X)$. So now if Φ is a state, that is, Φ is a positive linear functional such that $\Phi(1) = 1$, then by the Riesz-Representation Theorem there exists a positive measure $\mu \in M(X)$, so that $\Phi(f) = \int f d\mu$ for all $f \in \mathscr{C}(X)$ and $\|\mu\| = \|\Phi\|$. Since $\Phi(1) = 1 = \mu(X)$, μ is a probability measure. Therefore $\mathfrak{S}(\mathscr{C}(X))$ is in bijective correspondence with the set of probability measures on $B(X)$.*

Proposition 4.11. *Let $\mathscr{Y}$ be a unital C*-algebra and let $\xi \in \mathscr{Y}_{sa}$, then there exists a state $\Phi \in \mathfrak{S}(\xi)$ such that $|\Phi(\xi)| = \|\xi\|$.*

Proof. Consider the abelian, unital C*-subalgebra $\mathscr{C}^*(\xi)$ of $\mathscr{Y}$, then since $\|\xi\| = r(\xi)$ there exists $\lambda \in \sigma(\xi)$ so that $\|\xi\| = |\lambda|$. On the other hand we know by theorem 3.4, that there exists $\psi \in \Sigma(\mathscr{C}^*(\xi))$ such that $\psi(\xi) = \lambda$. By the

Hahn-Banach Theorem 1.1 there exists a bounded linear functional $\Phi \in \mathscr{Y}^*$ so that $\Phi|_{\mathscr{C}^*(\xi)} = \psi$ and $\|\Phi\| = \|\psi\| = 1$. Since $\|\Phi\| = 1 = \psi(1) = \Phi(1)$, we know by proposition 4.10, that $\Phi \in \mathfrak{S}(\mathscr{Y})$. ■

5 Representations and the second Gel'fand Naimark Theorem

This last chapter is based upon [4], [8] and [5].

5.1 Representations

We concern ourselves with representation theory in the study of C*-algebras. This will be quite fruitful as we will see in the end. For simplicity's sake we only deal with unital C*-algebras.

Definition 5.1. *Let $\mathscr{Y}$ be a unital C*-algebra. A representation of $\mathscr{Y}$ is a tuple $(\Pi, \mathscr{H})$ such that $\mathscr{H}$ is a Hilbert space and $\Pi\colon \mathscr{Y} \to \mathscr{L}(\mathscr{H})$ is a *-homomorphism. We will write a triple $(\mathscr{Y}, \Pi, \mathscr{H})$ to mean, that $(\Pi, \mathscr{H})$ is a representation for $\mathscr{Y}$ and we will also refer to $(\mathscr{Y}, \Pi, \mathscr{H})$ as a representation. A representation $(\Pi, \mathscr{H})$ is said to be non-degenerate if a vector $h \in \mathscr{H}$ satisfying $\Pi(\xi)h = 0$ for all $\xi \in \mathscr{Y}$ must already imply, that $h = 0$. In other words, a representation is non-degenerate if the only vector which vanishes on every representative $\Pi(\xi)$ is 0.*

Example 5.1. *Let $\mathscr{H}$ be a Hilbert space and consider the identity map $\mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$. This clearly yields a representation.*

Example 5.2. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space and consider the C*-algebra $L^\infty(\mu)$ and the Hilbert space $L^2(\mu)$. Define $\Pi\colon L^\infty(\mu) \to \mathscr{L}(L^2(\mu))$ given by $\Pi(\phi) := M_\phi$, where $M_\phi f := \phi f$ for all $f \in L^2(\mu)$. It is easily seen, that Π is well defined, linear and multiplicative and moreover*

$$\langle M_\phi f \mid g\rangle = \int \phi f \overline{g} d\mu = \int f \overline{(\overline{\phi} g)} d\mu = \langle f \mid \overline{\phi} g\rangle = \langle f \mid M_{\overline{\phi}} g\rangle$$

for all $f, g \in L^2(\mu)$, so $M_\phi^* = M_{\overline{\phi}}$ and therefore $\Pi(\phi)^* = \Pi(\phi^*)$. Hence $(L^\infty(\mu), \Pi, L^2(\mu))$ is a representation. One can even show, that Π is isometric.

Definition 5.2. *A representation $(\mathscr{Y}, \Pi, \mathscr{H})$ is said to be cyclic, if there exists a vector $\Psi \in \mathscr{H}$ such that*

$$cl(\Pi(\mathscr{Y})\Psi) = \mathscr{H}$$

where $\Pi(\mathscr{Y})\Psi := \{\Pi(\xi)\Psi \mid \xi \in \mathscr{Y}\}$. In that case, Ψ is said to be a cyclic vector for the representation $(\mathscr{Y}, \Pi, \mathscr{H})$.

Example 5.3. *The representation $(L^\infty(\mu), \Pi, L^2(\mu))$ from the example before is cyclic. Indeed, since $(X, \mathscr{A}, \mu)$ is σ-finite, we can choose $\Psi \in L^2(\mu)$ such that $\Psi > 0$ μ-a.e. From courses on measure theory it is known, that the linear span of characteristic functions χ_A with $0 < \mu(A) < \infty$ is dense in $L^2(\mu)$. Since $\Pi(\mathscr{Y})$ is a vector space, it suffices to show, that we can approximate any such characteristic function. So let $A \in \mathscr{A}$ with $0 < \mu(A) < \infty$, then $\chi_A \in L^\infty(\mu)$. Define $\varphi_n := \frac{\chi_{\{\Psi \geq 1/n\}}}{\Psi} \nearrow_{n\to\infty} \frac{1}{\Psi}$, then $|\varphi_n(x)| \leq n$ for all $x \in X$ and hence $\varphi_n \in L^\infty(\mu)$. Now define $\phi_n := \chi_A \varphi_n \in L^\infty(\mu)$ and observe, that*

$$\Pi(\phi_n)\Psi = \chi_{A\cap\{\Psi\geq 1/n\}} \frac{\Psi}{\Psi} = \chi_{A\cap\{\Psi\geq 1/n\}} \nearrow \chi_A$$

which shows, that $(L^\infty(\mu), \Pi, L^2(\mu))$ is cyclic.

Let $\{\mathscr{H}_i \mid i \in I\}$ be a family of Hilbert space. Recall, that the direct sum of such a family of Hilbert spaces is defined by

$$\bigoplus_{i\in I} \mathscr{H}_i := \left\{ h\colon I \to \bigcup_{i\in I} \mathscr{H}_i \,\middle|\, h(i) \in \mathscr{H}_i\colon \sum_{i\in I} \|h(i)\|^2 < \infty \right\}$$

where the sum $\sum_{i\in I} \|h(i)\|^2$ is to be interpreted as the limit of the net $\{\sum_{f\in F} \|h(f)\|^2\}_{F\in\mathfrak{F}}$ with $\mathfrak{F}$ denoting the set of finite subsets of I ordered by inclusion. From functional analysis it is known, that

$$\langle h \mid g\rangle := \sum_{i\in I} \langle h(i) \mid g(i)\rangle$$

where the sum is yet again to be interpreted as above, turns $\bigoplus_{i\in I} \mathscr{H}_i$ into a Hilbert space.

Definition 5.3. *Let $\{(\Pi_i, \mathscr{H}_i) \mid i \in I\}$ be a family of representations for a C*-algebra $\mathscr{Y}$. We define the direct sum of these representations to be*

$$(\Pi, \mathscr{H}) := \left(\bigoplus_{i\in I} \Pi_i, \bigoplus_{i\in I} \mathscr{H}_i \right)$$

where $\Pi \colon \mathscr{Y} \to \mathscr{L}(\mathscr{H})$ is given by $\Pi(\xi) = \big(\Pi_i(\xi)\big)_{i\in I}$.

Remark 5.1. *The direct sum of representations as given above is itself a representation. Indeed, Π is well defined, since for $\xi \in \mathscr{Y}$ and $h \in \mathscr{H}$ we have, since all the Π_i are *-homomorphism (and hence $\|\Pi_i(\xi)\| \le \|\xi\|$), that*

$$\|\Pi(\xi)h\|^2 = \sum_{i\in I} \|\Pi_i(\xi)h(i)\|^2 \le \sum_{i\in I} \|\Pi_i(\xi)\|^2 \|(i)\|^2 \le \|\xi\|^2 \|h\|^2$$

so $\|\Pi(\xi)\| \le \|\xi\|$ and thus in particular $\Pi(\xi) \in \mathscr{L}(\mathscr{H})$. Moreover it is clear, that Π is linear and multiplicative, since the Π_i are and in particular

$$\langle \Pi(\xi)h \mid g \rangle = \sum_{i\in I} \langle \Pi_i(\xi)h(i) \mid g(i) \rangle = \sum_{i\in I} \langle h(i) \mid \Pi_i(\xi)^* g(i) \rangle = \langle h \mid \Pi(\xi^*)g \rangle$$

*so $\Pi(\xi^*) = \Pi(\xi)^*$. Thus Π is a well defined *-homomorphism and $(\Pi, \mathscr{H})$ is a representation.*

Recall, that an isomorphism between two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ is an isometric linear bijection $A \colon \mathscr{H}_1 \to \mathscr{H}_2$.

Definition 5.4. *If $(\Pi_1, \mathscr{H}_1)$ and $(\Pi_2, \mathscr{H}_2)$ are representations of $\mathscr{Y}$, then we will say, that $(\Pi_1, \mathscr{H}_1)$ and $(\Pi_2, \mathscr{H}_2)$ are equivalent if there exists an isomorphism $A \colon \mathscr{H}_1 \to \mathscr{H}_2$ such that $A\Pi_1(\xi)A^{-1} = \Pi_2(\xi)$ for all $\xi \in \mathscr{Y}$.*

If $\mathscr{H}$ is a Hilbert space and $M, N \subset \mathscr{H}$ are subsets, then we will write $M \perp N$ to mean $\langle m \mid n \rangle = 0$ for all $m \in M$ and for all $n \in N$.

Proposition 5.1. *Let $(\Pi, \mathscr{H})$ be a non-degenerate representation for $\mathscr{Y}$, then there is a family $\{(\Pi_i, \mathscr{H}_i) \mid i \in I\}$ of cyclic representations for $\mathscr{Y}$ such that $(\Pi, \mathscr{H})$ and $\big(\bigoplus_{i\in I} \Pi_i, \bigoplus_{i\in I} \mathscr{H}_i\big)$ are equivalent.*

Proof. Let

$$\mathscr{E} := \{E \subset \mathscr{H} \mid \forall e, f \in F, e \neq f \colon \Pi(\mathscr{Y})e \perp \Pi(\mathscr{Y})f\}$$

Of course $\mathscr{E}$ is non-empty, since all singletons $\{e\}$ are in $\mathscr{E}$. Now order $\mathscr{E}$ by inclusion and note, that every chain in $\mathscr{E}$ has an upper bound, and thus by Zorn's lemma, there exists a maximal element $E \in \mathscr{E}$. Next define

$$\mathfrak{H} := \bigvee_{e\in E} \Pi(\mathscr{Y})e \qquad \text{(= closed linear span)}$$

and let $h \in \mathfrak{H}^\perp$, then

$$0 = \langle \Pi(\zeta^*\xi)e \mid h \rangle = \langle \Pi(\xi)e \mid \Pi(\zeta)h \rangle$$

for all $e \in E$ and for all $\xi, \zeta \in \mathscr{Y}$. Hence $\Pi(\mathscr{Y})h \perp \Pi(\mathscr{Y})e$ for all $e \in E$. Now if $\Pi(\mathscr{Y})h = \{0\}$, then we have by non-degeneracy of Π, that $0 = h$. If $\Pi(\xi)h \neq \{0\}$, then $h \neq 0$. But then $E \cup \{h\} \in \mathscr{E}$ contradicts the maximality of E. It therefore follows, that

$$\mathfrak{H}^\perp = \{0\} \implies \mathfrak{H} = \mathscr{H}$$

Now for $e \in E$ define $\mathscr{H}_e := cl(\Pi(\mathscr{Y})e)$ and note, that for $\xi \in \mathscr{Y}$ we have $\Pi(\xi)\mathscr{H}_e \subset \mathscr{H}_e$ and so

$$\Pi_e(\xi) := \Pi(\xi)|_{\mathscr{H}_e} \colon \mathscr{H}_e \to \mathscr{H}_e$$

is well defined and $\Pi_e(\xi) \in \mathscr{L}(\mathscr{H}_e)$ for all $\xi \in \mathscr{Y}$. Moreover, $(\mathscr{Y}, \Pi_e, \mathscr{H}_e)$ is a cyclic representation with cyclic vector e. Now consider the linear map

$$A \colon \bigoplus_{e\in E} \mathscr{H}_e \to \mathscr{H} \qquad \big(h_e\big)_{e\in E} \mapsto \sum_{e\in E} h_e$$

From the definition of the norm on $\bigoplus_{e\in E} \mathscr{H}_e$ and from $\mathscr{H}_e \perp \mathscr{H}_f$ for all $e \neq f$, we get for $\big(h_e\big)_{e\in E} \in \bigoplus_{e\in E} \mathscr{H}_e$

$$\|A\big[\big(h_e\big)_{e\in E}\big]\|^2 = \|\sum_{e\in E} h_e\|^2 = \sum_{e\in E} \|h_e\|^2 = \|\big(h_e\big)_{e\in E}\|^2$$

by pythagoras theorem, so A is isometric and thus has closed range. Next note, that

$$\bigcup_{e\in E} \mathscr{H}_e \subset A\bigg(\bigoplus_{e\in E} \mathscr{H}_e\bigg)$$

by construction and hence be closedness of the image of A, we conclude, that $\mathscr{H} = \mathfrak{H} = A\big(\bigoplus_{e\in E} \mathscr{H}_e\big)$, so every element $h \in \mathscr{H}$ can be uniquely written as a sum $\sum_{e\in E} h_e$ for $h_e \in \mathscr{H}_e$. In particular A is an isomorphism. If p_e denotes the projection mapping from $\mathscr{H} \to \mathscr{H}_e$, then

$$A\bigg[\bigoplus_{e\in E} \Pi_e(\xi)\bigg]A^{-1}(h) = A\big[\big(\Pi_e(\xi)p_e(h)\big)_{e\in E}\big] = \sum_{e\in E} \Pi_e(\xi)p_e(h) = \sum_{e\in E} \Pi(\xi)p_e(h) = \Pi(\xi)(h)$$

holds for all $h \in \mathscr{H}$ and thus $A \bigoplus_{e\in E} \Pi_e(\xi)A^{-1} = \Pi(\xi)$ for all $\xi \in \mathscr{Y}$. We conclude therefore, that $(\Pi, \mathscr{H})$ and $\big(\bigoplus_{e\in E} \Pi_e, \bigoplus_{e\in E} \mathscr{H}_e\big)$ are equivalent representations. ■

5.2 GNS-construction

The next theorem is of tremendous impact and it is probably the most iconic theorem in the theory of C*-algebras.

Theorem 5.1. *(Gel'fand-Naimark-Segal Construction) Suppose Φ is a positive linear functional. Then there is a representation $(\mathscr{Y}, \Pi_\Phi, \mathscr{H}_\Phi)$ and a cyclic vector $\Psi_\Phi \in \mathscr{H}_\Phi$ such that*

$$\|\Psi_\Phi\|^2 = \|\Phi\| \qquad \Phi(\xi) = \langle \Pi_\Phi(\xi)\Psi_\Phi \mid \Psi_\Phi \rangle \tag{5.1}$$

Proof. Let

$$\mathscr{N}_\Phi := \big\{\xi \in \mathscr{Y} \mid \Phi(\xi^*\xi) = 0\big\}$$

and note, that if $\xi \in \mathscr{N}_\Phi$ and $\zeta \in \mathscr{Y}$ is arbitrary, then

$$|\Phi(\zeta^*\xi)|^2 \leq \Phi(\zeta^*\zeta)\Phi(\xi^*\xi) = 0$$

so $\mathscr{N}_\Phi = \{\xi \in \mathscr{Y} \mid \forall \zeta \in \mathscr{Y} : \Phi(\zeta^*\xi) = 0\}$. This yields, that $\mathscr{N}_\Phi$ is a linear subspace. Now take $\{\xi_n\}_n \subset \mathscr{N}_\Phi$ with $\xi_n \to \xi \in \mathscr{Y}$, then $\Phi(\xi^*\xi) = \lim \Phi(\xi_n^*\xi_n) = 0$, so $\mathscr{N}_\Phi$ is closed. In particular if $\xi \in \mathscr{N}_\Phi$ and $\zeta \in \mathscr{Y}$, then

$$|\Phi((\zeta\xi)^*\zeta\xi)|^2 = |\Phi(\xi^*\zeta^*\zeta\xi)|^2 = |\Phi((\zeta^*\zeta\xi)^*\xi)|^2 \leq \Phi((\zeta^*\zeta\xi)^*\zeta^*\zeta\xi)\Phi(\xi^*\xi) = 0$$

and therefore $\mathscr{Y}\mathscr{N}_\Phi \subset \mathscr{N}_\Phi$, thus $\mathscr{N}_\Phi$ is a closed left ideal. Let $Q\colon \mathscr{Y} \to \mathscr{Y}/\mathscr{N}_\Phi$ be the canonical surjection and define

$$\langle Q(\xi) \mid Q(\zeta)\rangle := \Phi(\zeta^*\xi)$$

for all $\xi, \zeta \in \mathscr{Y}$. This is well defined, since if $n_1, n_2 \in \mathscr{N}_\Phi$ and $\xi, \zeta \in \mathscr{Y}$ we have

$$\Phi((\zeta + n_2)^*(\xi + n_1)) = \Phi(\zeta^*\xi) + \underbrace{\Phi(\zeta n_1)}_{=0} + \underbrace{\Phi(n_2^*\xi)}_{\overline{\Phi(\xi^* n_2)}=0} + \underbrace{\Phi(n_2^* n_1)}_{=0} = \Phi(\zeta^*\xi)$$

and clearly $\langle | \rangle$ yields an inner product, so $\mathscr{Y}/\mathscr{N}_\Phi$ can be seen as a pre-Hilbert space. Let $\mathscr{H}_\Phi$ denote the Hilbert space, that is generated by completion of $\mathscr{Y}/\mathscr{N}_\Phi$. Now we define

$$\Pi_0\colon \mathscr{Y} \to \mathscr{L}(\mathscr{Y}/\mathscr{N}_\Phi) \qquad \forall \xi, X \in \mathscr{Y} : \Pi_0(\xi)Q(X) := Q(\xi X)$$

The map Π_0 is well defined, since if $\xi, X \in \mathscr{Y}$ and $N \in \mathscr{N}_\Phi$, then

$$Q(\xi(X+N)) = Q(\xi X) + \underbrace{Q(\xi N)}_{=0} = Q(\xi X)$$

The fact, that $\Pi_0(\xi) \in \mathscr{L}(\mathscr{Y}/\mathscr{N}_\Phi)$ follows by first noting, that if $\xi, X \in \mathscr{Y}$ and $\zeta \in \mathscr{Y}_{sa}^+$ is the positive square-root of $\|\xi^*\xi\| - \xi^*\xi$, then

$$\|\xi^*\xi\|X^*X - X^*\xi^*\xi X = X^*(\|\xi^*\xi\| - \xi^*\xi)X = X^*\zeta\zeta X = (\zeta X)^*(\zeta X)$$

So $\|\xi^*\xi\|X^*X - X^*\xi^*\xi X \succeq 0$ and thus

$$\|\xi^*\xi\|\Phi(X^*X) \geq \Phi(X^*\xi^*\xi X) \tag{5.2}$$

by positivity of Φ. But now we utilitze (5.2) to see

$$\begin{aligned}\|\Pi_0(\xi)\|^2 = \sup_{\|Q(X)\|=1} \|\Pi_0(\xi)Q(X)\|^2 = \sup_{\|Q(X)\|=1} \|Q(\xi X)\|^2 = \sup_{\|Q(X)\|=1} \Phi(X^*\xi^*\xi X)\\ \leq \|\xi^*\xi\| \sup_{\|Q(X)\|=1} \Phi(X^*X) = \|\xi\|^2 \sup_{\|Q(X)\|=1} \langle Q(X) \mid Q(X)\rangle = \|\xi\|^2\end{aligned}$$

Hence $\|\Pi_0(\xi)\| \leq \|\xi\|$, so in particular Π_0 is continuous and $\|\Pi_0\| \leq 1$. Furthermore

$$\Pi_0(\xi\zeta) = \Pi_0(\xi)\Pi_0(\zeta) \qquad \Pi_0(\xi + \lambda\zeta) = \Pi_0(\xi) + \lambda\Pi_0(\zeta)$$

and

$$\begin{aligned}\langle \Pi_0(\xi)Q(X) \mid Q(Y)\rangle = \langle Q(\xi X) \mid Q(Y)\rangle = \Phi(Y^*\xi X) = \Phi((\xi^*Y)^*X)\\ = \langle Q(X) \mid Q(\xi^*Y)\rangle = \langle Q(X) \mid \Pi_0(\xi^*)Q(Y)\rangle\end{aligned} \tag{5.3}$$

so $\Pi_0(\xi)^* = \Pi_0(\xi^*)$. Note that we cannot say, that Π_0 is a *-homomorphism, since $\mathscr{Y}/\mathscr{N}_\Phi$ isn't necessarily a Hilbert space, as already mentioned. Yet, since $\Pi_0(\xi)$ is continuous for each ξ, we can extend each $\Pi_0(\xi)$ to a bounded linear operator on the completion $\mathscr{H}_\Phi$. Write this extension as the element $\Pi_\Phi(\xi) \in \mathscr{L}(\mathscr{H}_\Phi)$. It is then obvious, by denseness of $\mathscr{Y}/\mathscr{N}_\Phi$ and continuity of Π_0, that Π_Φ is a *-homomorphism $\mathscr{Y} \to \mathscr{L}(\mathscr{H}_\Phi)$ and thus in particular $\|\Pi_\Phi\| \leq 1$. Now let $\Psi_\Phi := Q(1)$, then

$$\begin{aligned}\langle \Pi_\Phi(\xi)\Psi_\Phi \mid \Psi_\Phi\rangle = \langle Q(\xi) \mid Q(1)\rangle = \Phi(\xi)\\ \|\Psi_\Phi\|^2 = \langle Q(1) \mid Q(1)\rangle = \Phi(1) = \|\Phi\|\end{aligned}$$

and in particular

$$\Pi_\Phi(\mathscr{Y})\Psi_\Phi = \{\Pi_\Phi(\xi)Q(1) \mid \xi \in \mathscr{Y}\} = \{Q(\xi) \mid \xi \in \mathscr{Y}\} = \mathscr{Y}/\mathscr{N}_\Phi \implies cl(\Pi_\Phi(\mathscr{Y})\Psi_\Phi) = \mathscr{H}_\Phi$$

Thus we conclude, that the cyclic representation $(\Pi_\Phi, \mathscr{H}_\Phi)$ with cyclic vector Ψ_Φ satisfies all our conditions. ■

Corollary 5.1. *If $(\Pi, \mathscr{H})$ is a cyclic representation with cyclic vector Ψ, then consider the positive linear functional Φ given by*

$$\Phi(\xi) := \langle \Pi(\xi)\Psi \mid \Psi\rangle$$

If $(\Pi_\Phi, \mathscr{H}_\Phi)$ is the GNS construction with cyclic vector Ψ_Φ as given in theorem 5.1, then $(\Pi_\Phi, \mathscr{H}_\Phi)$ is equivalent to $(\Pi, \mathscr{H})$.

Proof. By the GNS construction 5.1 and by definition of Φ we have

$$\Phi(\xi) = \langle \Pi_\Phi(\xi)\Psi_\Phi \mid \Psi_\Phi \rangle = \langle \Pi(\xi)\Psi \mid \Psi \rangle$$

for all $\xi \in \mathscr{Y}$. And thus we deduce, that

$$\begin{aligned}\|\Pi(\xi)\Psi\|^2 = \langle \Pi(\xi)\Psi \mid \Pi(\xi)\Psi \rangle = \langle \Pi(\xi)^*\Pi(\xi)\Psi \mid \Psi \rangle = \langle \Pi(\xi^*\xi)\Psi \mid \Psi \rangle \\ = \langle \Pi_\Phi(\xi^*\xi)\Psi_\Phi \mid \Psi_\Phi \rangle = \|\Pi_\Phi(\xi)\Psi_\Phi\|^2\end{aligned}$$

So

$$\|\Pi(\xi)\Psi\| = \|\Pi_\Phi(\xi)\Psi_\Phi\| \tag{5.4}$$

Now define

$$A\colon \Pi_\Phi(\mathscr{Y})\Psi_\Phi \to \mathscr{H} \qquad \Pi_\Phi(\xi)\Psi_\Phi \mapsto \Pi(\xi)\Psi$$

Note that A is well defined, since if $\Pi_\Phi(\xi)\Psi_\Phi = \Pi_\Phi(\zeta)\Psi_\Phi$ we have by (5.4), that

$$0 = \|\Pi_\Phi(\xi-\zeta)\Psi_\Phi\| = \|\Pi(\xi-\zeta)\Psi\| = \|\Pi(\xi)\Psi - \Pi(\zeta)\Psi\|$$

so $\Pi(\xi)\Psi = \Pi(\zeta)\Psi$. In particular A is an isometry by (5.4) and it is clearly a linear operator. Moreover, since $\Pi_\Phi(\mathscr{Y})\Psi_\Phi$ is dense in $\mathscr{H}_\Phi$, there is a unique extension of A to an isometric linear isomorphism $\mathscr{H}_\Phi \to \mathscr{H}$. Now let $\xi, \zeta \in \mathscr{Y}$, then

$$A\Pi_\Phi(\xi)\big(\Pi_\Phi(\zeta)\Psi_\Phi\big) = A\Pi_\Phi(\xi\zeta)\Psi_\Phi = \Pi(\xi\zeta)\Psi = \Pi(\xi)\Pi(\zeta)\Psi = \Pi(\xi)A\big(\Pi_\Phi(\zeta)\Psi_\Phi\big)$$

and therefore $A\Pi_\Phi(\xi) = \Pi(\xi)A$ on the dense subset $\Pi_\Phi(\xi)\Psi_\Phi$. We can therefore conclude, that

$$A\Pi_\Phi(\xi)A^{-1} = \Pi(\xi)$$

for all $\xi \in \mathscr{Y}$, so $(\Pi, \mathscr{H})$ and $(\Pi_\Phi, \mathscr{H}_\Phi)$ are indeed equivalent. ■

Corollary 5.2. *Let $(\Pi_1, \mathscr{H}_1), (\Pi_2, \mathscr{H}_2)$ be two cyclic representations with cyclic vectors Ψ_1 and Ψ_2 and define the positive linear functionals*

$$\Phi_i(\xi) := \langle \Pi_i(\xi)\Psi_i \mid \Psi_i \rangle$$

for $i = 1, 2$. If $\Phi_1 = \Phi_2$, then $(\Pi_1, \mathscr{H}_1)$ and $(\Pi_2, \mathscr{H}_2)$ are equivalent.

Proof. By corollary 5.1 we already know, that $(\Pi_i, \mathscr{H}_i)$ is equivalent to $(\Pi_{\Phi_i}, \mathscr{H}_{\Phi_i})$ for $i = 1, 2$. On the other hand, since $\Phi_1 = \Phi_2$ the GNS-constructions of Φ_1, Φ_2 are exactly the same. Hence $(\Pi_1, \mathscr{H}_1)$ and $(\Pi_2, \mathscr{H}_2)$ are equivalent. ■

Corollary 5.3. *Let Φ be a positive linear functional and let $\alpha > 0$, then the GNS-representations $(\Pi_\Phi, \mathscr{H}_\Phi)$ and $(\Pi_{\alpha\Phi}, \mathscr{H}_{\alpha\Phi})$ are equivalent.*

Proof. We use the notation as seen in the proof of the GNS construction 5.1. First note, that

$$\mathscr{N}_\Phi = \{\xi \in \mathscr{Y} \mid \Phi(\xi^*\xi) = 0\} = \{\xi \in \mathscr{Y} \mid \alpha\Phi(\xi^*\xi) = 0\} = \mathscr{N}_{\alpha\Phi}$$

and hence we have the set equality $\mathscr{Y}/\mathscr{N}_\Phi = \mathscr{Y}/\mathscr{N}_{\alpha\Phi}$. In particular, if $\langle | \rangle_\Phi$ and $\langle | \rangle_{\alpha\Phi}$ are the respective scalar products (defined in the proof of the GNS construction), then

$$\langle Q(X) \mid Q(Y) \rangle_\Phi = \Phi(Y^*X) = \alpha^{-1}(\alpha\Phi)(Y^*X) = \alpha^{-1}\langle Q(X) \mid Q(Y) \rangle_{\alpha\Phi}$$

Thus $\|.\|_{\alpha\Phi} = \sqrt{\alpha}\|.\|_\Phi$, so the topologies on $\mathscr{Y}/\mathscr{N}_\Phi$ and $\mathscr{Y}/\mathscr{N}_{\alpha\Phi}$ agree, and thus we have equality of their completions, i.e. $\mathscr{H}_\Phi = \mathscr{H}_{\alpha\Phi}$. Moreover,

$$\Pi_\Phi(\xi)Q(X) = Q(\xi X) = \Pi_{\alpha\Phi}(\xi)Q(X)$$

for all $\xi, X \in \mathscr{Y}$ which yields that $\Pi_\Phi(\xi) = \Pi_{\alpha\Phi}(\xi)$ on the dense subset $\mathscr{Y}/\mathscr{N}_\Phi$ (for both topologies). Therefrom, by continuity we have $\Pi_\Phi(\xi) = \Pi_{\alpha\Phi}(\xi)$ for all $\xi \in \mathscr{Y}$ on all of $\mathscr{H}_\Phi = \mathscr{H}_{\alpha\Phi}$. The equivalence is thus induced by the linear isometric bijection $\mathscr{H}_\Phi \to \mathscr{H}_{\alpha\Phi}$ given by $h \mapsto \sqrt{\alpha}h$. ■

5.3 The second Gel'fand-Naimark Representation Theorem

Think back to the first Gel'fand-Naimark Representation theorem 4.1, which completely classified all abelian C*-algbras. We can now prove something even more powerful which also explains why C*-algebras are spoken of as operator algebras.

Theorem 5.2. *(Gel'fand-Naimark Representation Theorem II) If $\mathscr{Y}$ is a unital C*-algebra, then $\mathscr{Y}$ is isometrically *-isomorphic to a C*-subalgebra of $\mathscr{L}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.*

Proof. We have already seen, that each state $\Phi \in \mathfrak{S}(\mathscr{Y})$ gives rise to a GNS-representation $(\Pi_\Phi, \mathscr{H}_\Phi)$. So define the representation

$$(\Pi, \mathscr{H}) := \left(\bigoplus_{\Phi \in \mathfrak{S}(\mathscr{Y})} \Pi_\Phi, \bigoplus_{\Phi \in \mathfrak{S}(\mathscr{Y})} \mathscr{H}_\Phi \right)$$

If we prove, that the *-homomorphism Π is injective, then by theorem 4.6 Π is an isometric *-homomorphism, and therefore we would be done. So, if $\xi \in \mathscr{Y}$ such that $\Pi(\xi) = 0$, then by definition of the direct sum $\Pi_\Phi(\xi) = 0$ for all $\Phi \in \mathfrak{S}(\mathscr{Y})$. In particular, if Ψ_Φ are the cyclic vectors of the representations of $(\Pi_\Phi, \mathscr{H}_\Phi)$, then

$$\forall \Phi \in \mathfrak{S}(\mathscr{Y}) \colon \Pi_\Phi(\xi)\Psi_\Phi = 0 \implies \forall \Phi \in \mathfrak{S}(\mathscr{Y}) \colon 0 = \|\Pi_\Phi(\xi)\Psi_\Phi\|_\Phi^2 = \Phi(\xi^*\xi)$$

and thus by proposition 4.11 $\xi^*\xi = 0$. Consequently

$$0 = \|\xi^*\xi\| = \|\xi\|^2$$

so $\xi = 0$ and thus Π is injective. ■

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