PDE'S - EXERCISES 44-47

ALEXANDER ZAHRER

In all these exercises we use the abbreviations $H^s := H^s(\mathbb{R}^d)$, $\dot{H}^s := \dot{H}^s(\mathbb{R}^d)$, $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{C}_b := \mathscr{C}_b(\mathbb{R}^d)$, $\mathscr{C}_b^k := \mathscr{C}_b^k(\mathbb{R}^d)$ and $\mathscr{C}_c^\infty := \mathscr{C}_c^\infty(\mathbb{R}^d)$. Moreover, for $r > 0$ we define $B_r := \{x \in \mathbb{R}^d \mid |x| < r\}$ and $S^{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$.

1 EXERCISE 44:

We are asked to show the uniqueness part in the Lax-Milgram Lemma. By assumption we are given a bilinear map $b\colon H \times H \to \mathbb{R}$, for some real Hilbert space H, which is coercive, i.e. for all $x \in H$ we have $b(x,x) \gtrsim \|x\|^2$. For a given bounded linear functional $\varphi \in H^*$, we have already established the existence of a bounded linear operator $B\colon H \to H$ such that $\varphi = b(A^{-1}w, \_)$ for some $w \in H$. We have to show uniqueness of $A^{-1}w \in H$. So suppose $z \in H$ also satisfies $\varphi = b(z, \_)$, then by coercivity

$$0 = (\varphi - \varphi)(A^{-1}w - z) = b(A^{-1}w - z, A^{-1}w - z) \gtrsim \|A^{-1}w - z\| \tag{1}$$

and therefore $A^{-1}w = z$, as wanted.

2 EXERCISE 45:

Suppose $d \geq 3$ and pick $p \in [2, \frac{2d}{d-2}]$. Suppose that $(f_l) \subset \mathscr{S}$ is Cauchy in H^1. We are to show that the sequence $(f_l|f_l|^{p-1}) \subset \mathscr{C}$ is Cauchy in L^1 and give an explanation as to why this defines an element in L^1. In order to show that this is indeed Cauchy in L^1 we simply estimate

$$\|f_k|f_k|^{p-1} - f_l|f_l|^{p-1}\|_{L^1} \leq \|(f_k - f_l)|f_k|^{p-1}\|_{L^1} + \|f_l(|f_k|^{p-1} - |f_l|^{p-1})\|_{L^1} \tag{2}$$

$$\underset{\text{Hölder}}{\leq} \|f_k - f_l\|_{L^p}\|f_k\|_{L^p}^{p/q} + \|f_l\|_{L^p}\| \underbrace{\left||f_k|^{p-1} - |f_l|^{p-1}\right|}_{\simeq \left||f_k|-|f_l|\right|^{p-1} \leq |f_k - f_l|^{p-1}} \|_{L^q} \tag{3}$$

$$\lesssim \|f_k - f_l\|_{L^p}\|f_k\|_{L^p}^{p/q} + \|f_l\|_{L^p}\|f_k - f_l\|_{L^p}^{p/q} \tag{4}$$

By Corollary 3.2 from the lecture notes we thus obtain

$$\|f_k|f_k|^{p-1} - f_l|f_l|^{p-1}\|_{L^1} \lesssim \|f_k - f_l\|_{H^1}\|f_k\|_{H^1}^{p/q} + \|f_l\|_{H^1}\|f_k - f_l\|_{H^1}^{p/q} \longrightarrow 0 \tag{5}$$

We can verify that the sequence $(f_l|f_l|^{p-1}) \subset \mathscr{S}$ defines an element in L^1 by showing that there exists a sequence of Schwarz functions (g_l) which are Cauchy in L^1 and which satisfy $\|f_l|f_l|^{p-1} - g_l\|_{L^1} \longrightarrow 0$. Let $\varphi \in \mathscr{C}_c^\infty$ be such that $\int \varphi = 1$ and set $\varphi_k = k^d\varphi(k.)$. In a quite similar fashion as in Lemma 2.41 and Proposition 2.42 from the lecture notes one can prove that $g_k(x) := \lim_{l \to \infty}(\varphi_k * f_l|f_l|^{p-1})(x)$ exists for all $x \in \mathbb{R}^d$ and that $(g_k) \subset \mathscr{C}_b^\infty$ satisfies $\|f_k|f_k|^{p-1} - g_k\|_{L^1} \longrightarrow 0$ (one might want to use DCT to show that more easily). By means of Exercise 34 we then obtain a sequence $(\hat{g}_k) \subset \mathscr{S}$ such that $\|\hat{g}_k - f_k|f_k|^{p-1}\|_{L^1} \longrightarrow 0$.

3 EXERCISE 46:

We are given $m, \lambda > 0$ and $p > 2$. Our assumption is that $f \in \mathscr{S}$ is a non-trivial solution of the PDE

$$-\Delta\psi + \psi = \psi|\psi|^{p-1} \tag{6}$$

and we are asked to derive that this yields a solution to the equation

$$-\Delta\psi + m\psi = \lambda\psi|\psi|^{p-1} \tag{7}$$

We make the ansatz $\psi := af(.b)$ and calculate

$$-\Delta\psi + m\psi = -a\Delta f(.b)b^2 + maf(.b) = a\big[-\Delta f(.b)b^2 + mf(.b)\big] \tag{8}$$

$$\overset{b=\sqrt{m}}{=} ab\big[-\Delta f(.b) + f(.b)\big] = abf(.b)|f(.b)|^{p-1} = a^{1-p}b\psi|\psi|^{p-1} \tag{9}$$

$$\overset{a=(b/\lambda)^{1/(p-1)}}{=} \lambda\psi|\psi|^{p-1} \tag{10}$$

4 EXERCISE 47:

Let $p \in [1,\infty]$. We shall first prove that $L^p_{\text{rad}} \subset L^p$ is closed. First note that we surely have that the closure of all radial Schwarz functions contains L^p_{rad}, that is, $\overline{\mathscr{S}_{\text{rad}}} \supset L^p_{\text{rad}}$. Suppose, by way of contradiction, that there exists a sequence $(f_l) \subset \mathscr{S}_{\text{rad}}$ which converges to an element $F \in L^p \backslash L^p_{\text{rad}}$. We immediately infer that, if $(g_k) \subset \mathscr{S}$ is a representative for F, then

$$0 = \lim_l \|f_l - F\|_{L^p} = \lim_l \lim_k \|f_l - g_k\|_{L^p} \tag{11}$$

Now let $\varepsilon > 0$, then there exists $L = L(\varepsilon) \in \mathbb{N}$ such that for all $l, n \geq L$ we have

$$\lim_k \|f_l - g_k\|_{L^p} < \varepsilon \qquad \|f_l - f_n\|_{L^p} < \varepsilon \tag{12}$$

Now as $\lim_k \|f_L - g_k\|_{L^p} < \varepsilon$, there exists $\mathbb{N} \ni K = K(\varepsilon) > L(\varepsilon)$ such that for all $k \geq K(\varepsilon)$ we have $\|f_L - g_k\|_{L^p} < \varepsilon$. Thus for all $k, l \geq K(\varepsilon)$ we obtain

$$\|f_l - g_k\|_{L^p} \leq \|f_l - f_L\|_{L^p} + \|f_L - g_k\|_{L^p} < 2\varepsilon \tag{13}$$

In particular, for all $l \geq K(\varepsilon)$ we obtain

$$\|f_l - g_l\|_{L^p} < 2\varepsilon \tag{14}$$

which exactly means $\|f_l - g_l\|_{L^p} \longrightarrow 0$. Therefore, $(f_l) \subset \mathscr{S}_{\text{rad}}$ is a representative of F, which contradicts our assumption.

Upon noting that the above proof works all the same for the norms $\|.\|_{H^s}$ and $\|.\|_{W^{k,p}}$, we conclude that H^s_{rad} resp. $W^{k,p}_{\text{rad}}$ is closed in H^s resp. $W^{k,p}$.