PDE'S - EXERCISES 36-39

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In all these exercises we use the abbreviations $H^s := H^s(\mathbb{R}^d)$, $\dot{H}^s := \dot{H}^s(\mathbb{R}^d)$, $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{C}_b := \mathscr{C}_b(\mathbb{R}^d)$, $\mathscr{C}_b^k := \mathscr{C}_b^k(\mathbb{R}^d)$ and $\mathscr{C}_c^\infty := \mathscr{C}_c^\infty(\mathbb{R}^d)$. Moreover, for $r > 0$ we define $B_r := \{x \in \mathbb{R}^d \mid |x| < r\}$ and $S^{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$.

1 EXERCISE 40:

We shall prove that for $\varepsilon > 0$ we have

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon} \tag{1}$$

for all $a, b \geq 0$. Putting $\tilde{a} := 2\varepsilon a$ and $\tilde{b} := b$ we immediately obtain

$$0 \leq (\tilde{a} - \tilde{b})^2 = 4\varepsilon^2 a^2 - 4\varepsilon ab + b^2 \tag{2}$$

which is exactly what we wanted.

2 EXERCISE 41:

We are given smooth functions $b^j \in \mathscr{C}_b^\infty$ and $\varepsilon > 0$. The task at hand is to prove that

$$\left|(b^j \partial_j f \mid g)_{L^2}\right| \leq \varepsilon \|f\|_{\dot{H}^1}^2 + C_\varepsilon \|g\|_{L^2}^2 \tag{3}$$

for all $f, g \in \mathscr{S}$. Note first that

$$\|f\|_{\dot{H}^1}^2 = \||.|\mathscr{F} f\|_{L_2}^2 = \int |.|^2 |\mathscr{F} f|^2 \simeq \sum_j \int |\mathscr{F}(\partial_j f)|^2 = \sum_j \|\partial_j f\|_{L_2}^2 \tag{4}$$

Thus in particular there is $A > 0$ such that $\sum_j \|\partial_j f\|_{L_2}^2 \leq A \|f\|_{\dot{H}^1}^2$ Using this and exercise 40 we estimate

$$\left|(b^j \partial_j f \mid g)_{L^2}\right| \leq \sum_j \|b^j \partial_j f\|_{L^2} \|g\|_{L^2} \leq \sum_j \|\partial_j f\|_{L^2} (M \|g\|_{L_2}) \tag{5}$$

$$\underset{\text{exercise 40}}{\leq} \sum_j \left(\varepsilon \|\partial_j f\|_{L^2}^2 + \frac{M^2 \|g\|_{L^2}^2}{4\varepsilon}\right) \leq A\varepsilon \|f\|_{\dot{H}^1} + C_\varepsilon \|g\|_{L_2}^2 \tag{6}$$

Rescaling ε appropriately, since A doesn't really bother us in the above estimate, we easily arrive at (3).

3 EXERCISE 42:

We are given functions $a^{jk} \in \mathbb{C}_b^\infty$ such that for every $y \in \mathbb{R}^d$ the matrix $A_y := (a^{jk}(y)) \in M_d(\mathbb{C})$ is self adjoint and satisfies

$$\inf_{y \in \mathbb{R}^d} \min \sigma(A_y) > 0 \tag{7}$$

We shall prove that this implies that the operator $a^{jk}\partial_j\partial_k$ is elliptic. First of all note that if $A \in M_d(\mathbb{C})$ is an $(d \times d)$-matrix, then by the spectral theorem we may write A as $A = O\Lambda O^\dagger$, where $\Lambda \in M_d(\mathbb{C})$ is diagonal (where the elements on the diagonal are the eigenvalues of A) and O is an orthogonal matrix (i.e. $O^\dagger = O^{-1}$). Now assume that A has only real eigenvalues, then we have the estimate

$$\inf_{|x|=1} \langle Ax \mid x\rangle_{\mathbb{C}^d} = \inf_{|x|=1} x^\dagger Ax = \inf_{|x|=1} x^\dagger O\Lambda O^\dagger x \overset{y=O^\dagger x}{=} \inf_{|y|=1} y^\dagger \Lambda y \tag{8}$$

$$= \inf_{|y|=1} \sum_j \lambda_j |y_j|^2 \geq \min \sigma(A) \tag{9}$$

Now we get back to our actual exercise. By assumption all A_y are self adjoint (from which it already follows that all eigenvalues are real) and $\min\sigma(A_y) > 0$ for all $y \in \mathbb{R}^d$. In particular $\langle A_y x \mid x\rangle \geq 0$ for all $y \in \mathbb{R}^d$ and for all $x \in \mathbb{C}^d$, by what we have shown. We then estimate

$$\left|(a^{jk}\partial_j f \mid \partial_k f)_{L^2}\right| = \int \langle A_y \nabla f(y) \mid \nabla f(y)\rangle\, dy \geq \int \inf_{|x|=1} \langle A_y x \mid x\rangle |\nabla f(y)|^2\, dy \tag{10}$$

$$\geq \int \min\sigma(A_y)|\nabla f(y)|^2\, dy \geq \inf_{\zeta\in\mathbb{R}^d} \min\sigma(A_\zeta) \sum_j \int |\partial_j f|^2 \simeq \inf_{\zeta\in\mathbb{R}^d} \min\sigma(A_\zeta) \|f\|^2_{\dot{H}^1} \tag{11}$$

4 EXERCISE 43:

We are given smooth bounded functions $a^j \in \mathscr{C}_b^\infty$ such that the operator $a^{jk}\partial_j\partial_k$ is elliptic. Moreover, we may assume that there exists a distributional solution $F \in L^2$ to the PDE

$$a^{jk}\partial_j\partial_k \phi_F = \phi_G \tag{12}$$

where $G \in H^1$. We now prove that this already implies that $F \in H^2$ (i.e. $F \in L^2$ has a representative $(f_l) \subset \mathscr{S}$ such that (f_l) is also Cauchy in H^2). In order to prove this statement we first need to improve Theorem 1.7 from the lecture notes.

Theorem 1. *Let $a^{jk}, b^j, c \in \mathscr{C}_b^\infty$ and assume that the operator $a^{jk}\partial_j\partial_k + b^j\partial_j + c$ is elliptic and that $F \in L^2$ satisfies*

$$(a^{jk}\partial_j\partial_k + b^j\partial_j + c)\phi_F = \phi_G \tag{13}$$

for some $G \in L^2$. Then $F \in H^1$, i.e. $F \in L^2$ has a representative that is also Cauchy in H^1.

Proof. By Meyers-Serrin (with non-constant coefficients) F has a representative $(f_l) \subset \mathscr{S}$ which satisfies

$$\|(a^{jk}\partial_j\partial_k + b^j\partial_j + c)(f_l - f_k)\|_{L^2} \longrightarrow 0 \tag{14}$$

Now by ellipticity we obtain

$$\gamma\|f\|_{\dot{H}^1} \leq \left|(a^{jk}\partial_j f \mid \partial_k f)_{L^2}\right| \leq \underbrace{\left|(\partial_k a^{jk}\partial_j f \mid f)_{L^2}\right|}_{A} + \underbrace{\left|(a^{jk}\partial_j\partial_k f \mid f)_{L^2}\right|}_{B} \tag{15}$$

We may estimate B by the same procedure as in the proof of Theorem 1.7 with the only difference being that we apply the statement of exercise 41 not for $\varepsilon = \frac{\gamma}{2}$, but for $\varepsilon = \frac{\gamma}{3}$. For A we do the following

$$A \leq \|\partial_k a^{jk}\partial_j f\|_{L^2}\|f\|_{L^2} \underset{\text{exercise 41}}{\leq} \frac{\gamma}{3}\|f\|^2_{\dot{H}^1} + C_\gamma\|f\|^2_{L^2} \tag{16}$$

and therefore we obtain

$$\gamma\|f\|_{\dot{H}^1} \leq \frac{2\gamma}{3}\|f\|_{\dot{H}^1} + 2C_\gamma\|f\|^2_{L^2} + \|(a^{jk}\partial_j\partial_k + b^j\partial_j + c)f\|_{L^2}\|f\|_{L^2} \tag{17}$$

By plugging in $(f_k - f_l)$ for f we see that $(f_l) \subset \mathscr{S}$ is indeed Cauchy in H^1, since $\|.\|_{H^1} \simeq \|.\|_{L^2} + \|.\|_{\dot{H}^1}$. $\square$

We now get back to our case, where G is even in H^1. Combining the more general version of Meyers Serrin (Theorem 2.45 in the lecture notes) with Theorem 1, we deduce that $F \in L^2$ must have a representative $(f_l) \subset \mathscr{S}$ such that both (f_l) and $(a^{jk}\partial_j\partial_k f_l)$ are Cauchy in H^1. Since we have

$$\|f\|_{H^1} \simeq \|f\|_{L^2} + \sum_j \|\partial_j f\|_{L^2} \tag{18}$$

we immediately infer that also all derivatives $(\partial_j f_l) \subset \mathscr{S}$ are Cauchy in L^2. We also deduce from (18) that $(\partial_i[a^{jk}\partial_j\partial_k f_l]) \subset \mathscr{S}$ is Cauchy in L^2 for all i. Now by ellipticity we get

$$\gamma\|f\|_{\dot{H}^2} \simeq \gamma\sum_i \|\partial_i f\|_{\dot{H}^1} \leq \sum_i \left|\left(a^{jk}\partial_j\partial_i f \mid \partial_k\partial_i f\right)_{L^2}\right| \leq \sum_i \left|\left(\partial_k\left[a^{jk}\partial_j\partial_i f\right] \mid \partial_i f\right)_{L^2}\right| \tag{19}$$

$$\leq \underbrace{\sum_i \left|\left(\partial_k a^{jk}\partial_j\partial_i f \mid \partial_i f\right)_{L^2}\right|}_{A} + \underbrace{\sum_i \left|\left(a^{jk}\partial_j\partial_k\partial_i f \mid \partial_i f\right)_{L^2}\right|}_{B} \tag{20}$$

We easily estimate B by a quick application of exercise 41 as follows:

$$B \leq \frac{\gamma}{3}\sum_i \|\partial_i f\|_{\dot{H}^1}^2 + C_\gamma\|a^{jk}\partial_j\partial_k\partial_i f\|_{L^2}^2 \tag{21}$$

Note that if we were able to show that for each i the norm $\|\partial_k a^{jk}\partial_j\partial_i(f_l - f_m)\|_{L^2}$ is bounded, then we would also be able to estimate A and that would conclude the proof. Indeed, by substituting f with $(f_l - f_k)$ in (20) and by using (21) and assuming boundedness of $\|\partial_k a^{jk}\partial_j\partial_i(f_l - f_m)\|_{L^2}$ we obtain

$$\|f_l - f_k\|_{\dot{H}^2} \simeq \frac{2\gamma}{3}\sum_i \|\partial_i(f_l - f_k)\|_{\dot{H}^1} \tag{22}$$

$$\leq C_\gamma\|a^{jk}\partial_j\partial_k\partial_i(f_l - f_k)\|_{L^2} + \sum_i \|\partial_k a^{jk}\partial_j\partial_i(f_l - f_m)\|_{L^2}\|f_l - f_m\|_{L^2} \longrightarrow 0 \tag{23}$$

which proves that the representative $(f_l) \subset \mathscr{S}$ is Cauchy in H^2. The case where all a^{jk} are constants is already handled, since term A vanishes in (20). For the general case where $a^{jk} \in \mathscr{C}_b^\infty$ I have no clue thus far on how to show that

$$\sup_{l,m} \|\partial_k a^{jk}\partial_j\partial_i(f_l - f_m)\|_{L^2} < \infty$$