Introduction to mathematical Logic - Task 2

Alexander Zahrer

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"Contrariwise,' continued Tweedledee, 'if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."

(Lewis Carroll, Alice's Adventures in Wonderland / Through the Looking-Glass)

1 Structures and Theories

1.1 Discussion 3.2.7:

We are asked to consider the formal language $\mathcal{L}$ which consists of only the binary relation $<$. Putting $A_n := \{-n, ..., -1\} \cup \mathbb{N}$ (where $A_0 = \mathbb{N}$) we may define structures $\mathfrak{A}_n$ by naturally interpreting $<$. It is then evident that $\{\mathfrak{A}_i\}_{i\in\mathbb{N}}$ is an increasing chain of $\mathcal{L}-$structures. We set $\mathfrak{B} := \bigcup \mathfrak{A}_i$. We first prove:

- $\mathrm{Th}(\mathfrak{A}_0) = \mathrm{Th}(\mathfrak{A}_i)$ for each $i \in \mathbb{N}$

Recall that for some structure $\mathfrak{A}$ we have (by definition) $\mathrm{Th}(\mathfrak{A}) = \{\varphi \mid \mathfrak{A} \models \varphi, \varphi \text{ is an } \mathcal{L} - \text{sentence}\}$. Define the (obviously bijective) maps

$$f_n : A_0 \mapsto A_n \qquad a \mapsto a - n$$

We certainly have that all these maps are isomorphisms, i.e. $f_n : \mathfrak{A}_0 \cong \mathfrak{A}_n$. We now prove a more general result which will eventually imply our statement:

Lemma 1. Suppose we are given two isomorphic $\mathcal{L}$-structures $\mathfrak{M}$ and $\mathfrak{N}$ with isomorphism f. Then for all $\mathcal{L}$-formulas ψ and all assignments $\overline{m}$ we have

$$\mathfrak{M} \models \psi[\overline{m}] \iff \mathfrak{N} \models \psi[f(\overline{m})] \qquad (\star)$$

Proof. We prove this by induction on the complexity of formulas. It is clear that if ψ is basic the statement holds trivially. Now if ψ_1 and ψ_2 are formulas such that $(\star)$ holds, then we certainly also have

$$\mathfrak{M} \models (\psi_1 \wedge \psi_2)[\overline{m}] \iff \mathfrak{N} \models (\psi_1 \wedge \psi_2)[f(\overline{m})]$$

for all $\overline{m}$. Finally, suppose ψ is of the form $\exists y \phi(\overline{x}, y)$ such that ϕ satisfies $(\star)$. By induction hypothesis we have

$$\mathfrak{M} \models \phi[\overline{m}, m] \iff \mathfrak{N} \models \phi[f(\overline{m}), f(m)]$$

However, we also have

$$\mathfrak{M} \models \psi[\overline{m}] \iff \exists m \in M : \mathfrak{M} \models \phi[\overline{m}, m] \iff \exists m \in M : \mathfrak{N} \models \phi[f(\overline{m}), f(m)] \iff \mathfrak{N} \models \psi[f(\overline{m})]$$

which concludes the proof. □

Now if we have an $\mathcal{L}$-sentence ψ, then, since sentences are independent of assignments, the preceding lemma immediately yields

$$\mathfrak{A}_0 \models \psi \iff \mathfrak{A}_n \models \psi$$

for all $n \in \mathbb{N}$ (since $\mathfrak{A}_0 \cong_{f_n} \mathfrak{A}_n$ for all n). However, this is equivalent to $\mathrm{Th}(\mathfrak{A}_0) = \mathrm{Th}(\mathfrak{A}_n)$ for all n, as wanted.

- Find a $\forall\exists-$sentence ψ such that $\mathfrak{B} \models \psi$, but $\mathfrak{A}_0 \not\models \psi$.

Consider the sentence ψ given by

$$\forall x \exists y (y < x)$$

Of course ψ holds in the structure $\mathfrak{B}$ with universe $\bigcup A_i = \mathbb{Z}$. However, ψ does not hold in $\mathfrak{A}_0$, since there exists no natural number smaller than 0.

- Is $\mathrm{Th}(\mathfrak{A}_0)$ inductive?

Recall that a theory T is called inductive, if the union of any directed family of models of T is again a model of T. Now consider again the $\forall\exists$-sentence given in the second bullet point of this discussion and call that formula ψ, i.e. consider

$$\psi\colon \forall x_1 \exists x_2 (x_2 < x_1)$$

Since $T := \mathrm{Th}(\mathfrak{A}_0) = \mathrm{Th}(\mathfrak{A}_n)$ is complete for all $n \in \mathbb{N}$ we either have $T \vdash \psi$ or $T \vdash \neg\psi$. However, since $\mathfrak{A}_n$ is by construction a model for T with $\mathfrak{A}_n \not\models \psi$ we must have $T \vdash \neg\psi$. On the other hand we have

$$\bigcup \mathfrak{A}_n = \mathfrak{B} \models \psi$$

and therefore $\mathfrak{B} \not\models T$. Thus $\mathfrak{B}$ is no model of $T = \mathrm{Th}(\mathfrak{A}_0)$ which shows non-inductiveness of $\mathrm{Th}(\mathfrak{A}_0)$.

1.2 Exercise 3.4.5:

- Let T be a model complete theory and let $\mathfrak{M}$ be a model of T which embeds into every model of T. Show that T is complete.

Recall that a theory is said to be model complete if for all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of T with $\mathfrak{M}^1 \subset \mathfrak{M}^2$ we already have that $\mathfrak{M}^1 \prec \mathfrak{M}^2$. We also recall the Robinson test which states that if T is a theory, then T is model complete if and only if whenever $\mathfrak{M}^1 \subset \mathfrak{M}^2$ are models of T and φ is an $\mathcal{L}(M^1)$-existential sentence with $\mathfrak{M}^2 \models \varphi$, then $\mathfrak{M}^1 \models \varphi$. Moreover, T being model complete is also equivalent to the condition that every formula is modulo T equivalent to a universal formula. Having these facts at our disposal makes the exercise relatively simple. Indeed, let T be model complete and let $\mathfrak{M}$ be a model of T which embeds into every model of T. We have to show completeness of T, i.e. for every $\mathcal{L}$-sentence φ we must have

$$(T \vdash \varphi) \vee (T \vdash \neg\varphi)$$

So fix some $\mathcal{L}$-sentence φ. Since T is model complete the sentence φ is equivalent modulo T to a universal sentence. Thus we may assume WLOG that φ is universal. If $T \vdash \varphi$ we are done, so assume $T \nvdash \varphi$. This means that there must exist a model $\mathfrak{N}$ of T such that $\mathfrak{N} \not\models \varphi$. Since φ is a sentence and is thus independent of assignments we must have $\mathfrak{N} \models \neg\varphi$. Now by assumption $\mathfrak{M}$ embeds into $\mathfrak{N}$ and therefore (as $\neg\varphi$ is existential) by the Robinson test $\mathfrak{M} \models \neg\varphi$. Now suppose, by way of contradiction, that $\mathfrak{A} \models T$ is an arbitrary model for T such that $\mathfrak{A} \models \varphi$. Since we can view $\mathfrak{M}$ as a substructure of $\mathfrak{A}$ by assumption, we get by downwards absoluteness of universal formulas that $\mathfrak{M} \models \varphi$. This is a contradiction to $\mathfrak{M} \models \neg\varphi$ and therefore $\mathfrak{A} \models \neg\varphi$. Thus $T \vdash \neg\varphi$, as wanted.

2 Overview on the chapter "Countable models":

We start with a bunch of definitions. Throughout, $\mathcal{L}$ is some formal language.

Definition 1 (Types). *Let Γ be an $\mathcal{L}$-theory.*

1. *An n-type is a maximal set of formulas $p(x_1, ..., x_n)$ which is consistent with Γ, i.e. there exists an $\mathcal{L}$-structure $\mathfrak{M}$ and an assignment b such that $\mathfrak{M} \models \varphi[b]$ for all $\varphi \in \Gamma \cup p(x_1, ..., x_n)$.*
2. *By $S_n(\Gamma)$ we denote the set of all n-types of Γ.*
3. *A partial n-type is a consistent set of formulas in the free variables $x_1, ..., x_n$.*

Now suppose $\mathfrak{M}$ is a fixed $\mathcal{L}$-structure with universe M and suppose $N \subset M$.

1. *A set $s(x_1, ..., x_n)$ of $\mathcal{L}(N)$-formulas is said to be an n-type over N, if $s(\bar{x})$ is maximal finitely satisfiable in $\mathfrak{M}$ (where $\bar{x} = (x_1, ..., x_n)$).*

2. *By $S_n^{\mathfrak{M}}(N)$ we denote the set of n-types over N.*

Let $\mathfrak{M}$ be an $\mathcal{L}$-structure with universe M and take $N \subset M$, $m_1, ..., m_k \in M$ along with a set of $\mathcal{L}(N)$-formulas $\mathfrak{S}(x_1, ..., x_k)$ (dependent only on the variables x_i). Recall that the vector $\overline{m} = (m_1, ..., m_k)$ is said to realize $\mathfrak{S}(\overline{x})$, if $\mathfrak{M}_N \models \sigma(\overline{m})$ for all $\sigma(\overline{x}) \in \mathfrak{S}(\overline{x})$.

Definition 2 (ω-saturation). *A model $\mathfrak{M}$ with universe M is said to be ω-saturated if and only if for every finite subset $N \subset M$ and every $s(x) \in S_1^{\mathfrak{M}}(N)$ the extended structure $\mathfrak{M}_N$ realizes $s(x)$.*

Having defined ω-saturated models one might wonder where its name finds its origin. An answer to this question is the following lemma:

Lemma 2. Let $\mathfrak{M}$ be an ω-saturated model. Suppose $N \underset{\text{finite}}{\subset} M$ and $s(\overline{x}) \in S_n^{\mathfrak{M}}(N)$. Then $\mathfrak{M}_N$ realizes $s(\overline{x})$.

Proof. This is proven by an inductive argument, as one might have suspected. We induce with respect to the order of our types. The case $n = 1$ follows by definition of ω-saturatedness. So suppose the statement holds for $n \geqslant 1$. Fix some arbitrary finite subset $N \subset M$ and pick $s(\overline{x}) \in S_{n+1}^{\mathfrak{M}}(N)$. Note that $\varnothing \vdash (\psi \to \exists x_{n+1}\psi)$ i.e. this formula is valid. Thus for all $\varphi(\overline{x}) \in s(\overline{x})$ we deduce $s(\overline{x}) \vdash \exists x_{n+1}\varphi(\overline{x})$. Now this tells us that the set of formulas $s' := \{\exists x_{n+1}\varphi(\overline{x}) \mid s(\overline{x}) \vdash \varphi(\overline{x})\}$ is contained in the deductive closure of $s(\overline{x})$. We infer that $s' \in S_n^{\mathfrak{M}}(N)$ and therefore by our inductive hypothesis there must exist a vector $\overline{m} = (m_1, ..., m_n) \in M^n$ which realizes s' in the extended structure $\mathfrak{M}_N$. Now consider the set of formulas

$$s'' := \{\varphi(\overline{m}, x_{n+1}) \mid s(\overline{x}) \vdash \varphi(\overline{x})\}$$

which satisfies $s'' \in S_1^{\mathfrak{M}}(N \cup \{m_i\}_{i=1}^n)$. Yet again by our inductive hypothesis we obtain an element $m_{n+1} \in M$ so that $\mathfrak{M}_{N \cup \{m_i\}_{i=1}^n} \models t''(m_{n+1})$. From this we readily obtain that $(\overline{m}, m_{n+1}) \in M^{n+1}$ realizes $s(\overline{x})$ in $\mathfrak{M}_N$. □

Recall that an elementary embedding between two structures $\mathfrak{M}$ and $\mathfrak{N}$ is a map $h\colon M \to N$ such that for all formulas $\psi(x_1, ..., x_k)$ and all $m_1, ..., m_k \in M$ we have

$$\mathfrak{M} \models \psi(m_1, ..., m_k) \iff \mathfrak{N} \models \psi(h(m_1), ..., h(m_k))$$

Having that notion in mind one can verify:

Theorem 1. Let $\mathfrak{N}$ be an ω-saturated model and let $\Gamma := \mathrm{Th}(\mathfrak{N})$. If $\mathfrak{M}$ is countable (i.e. its universe M is countable) and $\mathfrak{M} \models \Gamma$, then there is an elementary embedding of $\mathfrak{M}$ into $\mathfrak{N}$.

For brevity we only sketch the idea of the proof. The key idea to proving this theorem is to make use of the completeness of the types

$$\mathrm{tp}^{\mathfrak{M}}(\overline{m}) := \left\{\psi(\overline{x}) \mid \mathfrak{M} \models \psi(\overline{m}), \psi(\overline{x}) \text{ is an } \mathcal{L}\text{-formula}\right\}$$

Indeed, if the universe M of $\mathfrak{M}$ is countable, write $M = \{m_j\}_{j \in \omega}$. Of course $\mathrm{tp}^{\mathfrak{M}}(m_1) \cup \Gamma$ is consistent and by saturatedness of $\mathfrak{N}$ there must exist $n_1 \in N$ such that $\mathfrak{N} \models (\mathrm{tp}^{\mathfrak{M}}(m_1))(n_1)$. This certainly implies $\mathrm{tp}^{\mathfrak{M}}(m_1) \subset \mathrm{tp}^{\mathfrak{N}}(n_1)$. However, these types are complete and therefore the preceding inclusion is actually an equality. One can then continue this game by "finding" $n_j \in N$ inductively such that

$$\mathrm{tp}^{\mathfrak{M}}(\overline{m}) = \mathrm{tp}^{\mathfrak{N}}(\overline{n})$$

for all $k \in \omega$ with $\overline{m} = (m_1, ..., m_k) \in M^k$ and $\overline{n} = (n_1, ..., n_k) \in N^k$. Having all that makes it easy to guess the right candidate for the sought elementary embedding; the map

$$h\colon M \to N \qquad m_j \mapsto n_j$$

will do the job.

Now using the construction of the proof of the preceding theorem in a "Back and Forth" argument, one arrives at the following nice statement:

Corollary 1. Any two countable ω-saturated models of a complete theory Γ are isomorphic.

From that one arrives, though with still a lot of work in between, at the following theorem:

Theorem 2. Let Γ be a countable complete theory. The following are equivalent:

1. Γ has a countable, ω-saturated model.
2. For every natural number n, there are at most countably many n-types $p(\overline{x}) \in S_n(\Gamma)$ extending Γ, i.e. $|S_n(\Gamma)| \leqslant \aleph_0$ for all $n \in \mathbb{N}$.
3. For every model $\mathfrak{M}$ of Γ and every finite subset $N \subset M$, there are at most countably many n-types over N extending $\mathrm{Th}(\mathfrak{M}_N)$, i.e. for all models $\mathfrak{M}$ of Γ and for all finite subsets $N \subset M$ and all natural numbers n we have $|S_n^{\mathfrak{M}}(N)| \leqslant \aleph_0$.

Proof. For brevity we only prove (1) $\implies$ (2). So suppose $\mathfrak{M}$ is a countable, saturated model for Γ. By completeness of Γ we surely must have $\mathrm{Th}(\mathfrak{M}) = \Gamma$ and therefore if $p(\overline{x}) \in S_n(\Gamma)$, then $\mathfrak{M}$ must realize $p(\overline{x})$ by saturatedness. Now distinct $p_1(\overline{x}), p_2(\overline{x}) \in S_n(\Gamma)$ cannot be realized by the same n-tuples of elements of M (this follows by completeness of these types). Thus there is an injection $S_n(\Gamma) \hookrightarrow M^n$. By countability of M^n this yields $|S_n(\Gamma)| \leqslant \aleph_0$. □

Corollary 2. If a countable, complete theory Γ has only countably many countable models, then Γ has a countable, ω-saturated model.

Proof. Each countable model can realize only countably many types and by assumption there are only countably many models for Γ. In particular, every maximal set of n-formulas consistent with Γ must be realized in a countable model. Thus $|S_n(\Gamma)| \leqslant \aleph_0$ for all natural numbers n. The preceding theorem yields the claim. □

We will now point out the necessary notions along with some theorems (without giving proofs or ideas thereof), which are eventually key to proving Vaught's famous "Never Two Theorem".

Definition 3. *An n-formula ψ is said to be n-complete over a theory Γ if and only if $\Gamma \cup \{\psi\}$ is consistent and for all n-formulas φ we either have $\Gamma \cup \{\psi\} \vdash \varphi$ or $\Gamma \cup \{\psi\} \vdash \neg\varphi$. An n-type $p \in S_n(\Gamma)$ is said to be atomic over Γ if it contains an n-complete over Γ formula.*

Definition 4. *Let $\mathfrak{M}$ be a model for a given theory Γ.*

1. *We call the model $\mathfrak{M}$ atomic if for every natural number n and every n-tuple $\overline{m} = (m_1, ..., m_n) \in M^n$ the type $tp^{\mathfrak{M}}(\overline{m})$ is atomic.*
2. *$\mathfrak{M}$ is called prime if for every model $\mathfrak{N} \models \Gamma$ we have an elementary embedding of $\mathfrak{M}$ into $\mathfrak{N}$.*

Theorem 3. Let Γ be a complete theory in the countable language $\mathcal{L}$. If $\mathfrak{M}$ is a countable and atomic model for Γ, then $\mathfrak{M}$ is prime.

A very helpful characterization of atomic models is the following:

Theorem 4. A complete theory Γ has an atomic model if and only if for every natural number n every n-formulas, which is consistent with Γ, is contained in some atomic type.

From the previous theorem one can deduce another nice statement, which yields some first relationship between saturated and atomic models:

Theorem 5. If a theory Γ has a countable, saturated model, then Γ has an atomic model.

Corollary 3. Let Γ be a complete theory and suppose $\mathfrak{M}$ is a countable model for Γ. Then $\mathfrak{M}$ is both ω-saturated and atomic if and only if for every countable model $\mathfrak{N}$ of Γ we have $\mathfrak{M} \cong \mathfrak{N}$.

Definition 5. *A theory Γ is said to be $\aleph_0$-categorical if, up to isomorphism, it has a unique countable model.*

It is now important to note the following:

Lemma 3. A countable, complete theory Γ is $\aleph_0$-categorical if and only if it has a model $\mathfrak{M}$ which is both atomic and ω-saturated.

Theorem 6. Let Γ be a complete theory. Then the following are equivalent:

1. Γ is $\aleph_0$-categorical.
2. All countable models are atomic.

3. All types over Γ are atomic.

4. $\forall n \in \mathbb{N}$: $|S_n(\Gamma)| < \aleph_0$.

5. for each natural number n there is a finite list of n-formulas such that every n-formula is modulo Γ equivalent to a formula from the list.

We conclude this intermezzo of throwing out definitions and statements by proving (at long last):

Theorem 7 (Vaught's Never Two Theorem)**.** A countable, complete theory Γ can not have exactly two non-isomorphic countable models.

Proof. Suppose, by way of contradiction, that Γ is a countable, complete theory with exactly two non-isomorphic models. By corollary 2 Γ must have a countable, ω-saturated model $\mathfrak{N}$. Now by Theorem 5 we infer that there must exist a (countable?) atomic model $\mathfrak{M}$ for Γ as well. By assumption, as Γ has two non-isomorphic models, Γ is not $\aleph_0$ categorical. This however yields by Theorem 6 that the countable model $\mathfrak{N}$ is not atomic and by corollary 3 we must have that $\mathfrak{M}$ cannot be ω-saturated. Hence, since $\mathfrak{N}$ is not atomic, there are $n_1, ..., n_k \in N$ such that $\mathrm{tp}^{\mathfrak{N}}(n_1, ..., n_k)$ is not atomic. We try to use all this accumulated information now: First expand our language to the language $\tilde{\mathcal{L}} = \mathcal{L} \cup \{n_1, ..., n_k\}$. Denote by $\tilde{\mathfrak{N}}$ the associated expansion of $\mathfrak{N}$ to $\tilde{\mathcal{L}}$. We infer that $\tilde{\mathfrak{N}}$ is a countable, ω-saturated model for $\tilde{\Gamma} := \mathrm{Th}(\tilde{\mathfrak{N}})$. Thus, again by Theorem 5, $\tilde{\Gamma}$ has an atomic model $\tilde{\mathfrak{M}}$. We now claim that $\tilde{\Gamma}$ is not $\aleph_0$-categorical. Indeed, since Γ is not $\aleph_0$-categorical we have by Theorem 6 that there must exist an infinite sequence of $\mathcal{L}$-formulas $\{\varphi_i\}_{i\in\omega}$ such that for all distinct $i \neq j$ we have $\Gamma \nvDash (\varphi_i \longleftrightarrow \varphi_j)$. However, this also yields $\tilde{\Gamma} \nvDash (\varphi_i \longleftrightarrow \varphi_j)$. Thus by lemma 3 $\tilde{\mathfrak{M}}$ is not ω-saturated and therefore $\tilde{\mathfrak{M}} \upharpoonright \mathcal{L}$ also has no chance to be ω-saturated. Note that $\tilde{\mathfrak{M}} \upharpoonright \mathcal{L}$ is also a model for Γ and we clearly must have $\tilde{\mathfrak{M}} \upharpoonright \mathcal{L} \ncong \mathfrak{N}$. If we manage to show that $\tilde{\mathfrak{M}} \upharpoonright \mathcal{L}$ is not atomic then this would yield $\tilde{\mathfrak{M}} \upharpoonright \mathcal{L} \ncong \mathfrak{M}$, which is a contradiction, since we thus would have three non-isomorphic countable models for Γ. By Theorem 3 we have that there must exist an elementary embedding $f : \tilde{\mathfrak{M}} \hookrightarrow \tilde{\mathfrak{N}}$ so that $f(n_j^{\tilde{\mathfrak{M}}}) = n_j^{\tilde{\mathfrak{N}}} = n_j$. Thus if φ is a k-formula in $\mathcal{L}$ we obtain

$$\tilde{\mathfrak{M}} \upharpoonright \mathcal{L} \models \varphi(\bar{n}^{\tilde{\mathfrak{M}}}) \iff \mathfrak{N} \models \varphi(\bar{n})$$

But this readily implies

$$\mathrm{tp}^{\tilde{\mathfrak{M}}\upharpoonright\mathcal{L}}(\bar{n}^{\tilde{\mathfrak{M}}}) = \mathrm{tp}^{\mathfrak{N}}(\bar{n})$$

and this is not atomic over Γ. Therefore $\tilde{\mathfrak{M}} \upharpoonright \mathcal{L}$ cannot be atomic. □

Remark 1. I may be well mistaken (please let me know if I am wrong), but I think Theorem 2.2.2 from the lecture notes must be strengthened to yield not only an atomic model, but a countable, atomic model. Otherwise the above is not a proof, since we cannot conclude a contradiction.