Introduction to mathematical Logic - Task 3

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"For last year's words belong to last year's language and next year's words await another voice."

(T.S. Eliot, Four Quartets)

1 Introduction

Let ζ be a cardinal number. A theory is called ζ-categorical, if it has exactly one model up to isomorphism of cardinality ζ. If ζ turns out to be uncountable and if we are given a ζ-categorical theory T, then Morley's Categoricity Theorem states that T is categorical in every uncountable cardinality. Recall that by the Löwenheim-Skolem Theorem we know that if an $\mathcal{L}$-theory T has an infinite model, then T has a model of every cardinality $\zeta \geqslant \max\{|C_{\mathcal{L}}|, \aleph_0\}$. Therefore, Morley's Categoricity Theorem is quite exciting and surprising, as it is rather counterintuitive that a property such as categoricity propagates through all uncountable models, if only one of them is initally assumed to be categorical. The goal of this exposition is to give an outline on how one would be able to derive this very Theorem.

2 Towards Morley's Categoricity Theorem

Morley's Categoricity Theorem is quite an involved Theorem, since one needs a lot of technical Lemmas and Theorems in advance to really be able to devise a proof for it. We will try to give a proof of the Categoricity Theorem by first proving the Baldwein-Lachlan Theorem, which will imply the Categoricity Theorem. Before doing so, we need to ensure some preparation. Throughout, we will assume $\mathcal{L}$ to be some language. Recall that if $\mathfrak{M}$ is an $\mathcal{L}$-structure and $\varphi(\overline{x})$ is an n-formula, then we define

$$\varphi(\mathfrak{M}) := \left\{\overline{m} \in M^n \mid \mathfrak{M} \models \varphi(\overline{m})\right\} \tag{1}$$

Definition 1. *Let $\zeta > \lambda \geqslant \aleph_0$. An $\mathcal{L}$-theory T is said to have a (ζ, λ)-model, if there is a model $\mathfrak{M}$ of T and an $\mathcal{L}$-formula $\varphi(\overline{x})$ such that*

$$|\mathfrak{M}| = \zeta \qquad |\varphi(\mathfrak{M})| = \lambda \tag{2}$$

A pair of models $(\mathfrak{N}, \mathfrak{M})$ for T is said to be a Vaughtian pair of models, if

$$\mathfrak{M} \prec \mathfrak{N} \qquad \mathfrak{M} \neq \mathfrak{N} \tag{3}$$

and if there is an $\mathcal{L}(M)$-formula φ such that $|\varphi(\mathfrak{M})| \geqslant \aleph_0$ and $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$.

It is noteworthy that if an $\mathcal{L}$-theory T has a (ζ, λ)-model, then there is a Vaughtian pair $(\mathfrak{N}, \mathfrak{M})$ of models in T. Indeed, if $\mathfrak{N}$ is a (ζ, λ)-model and $\varphi(\mathfrak{N})$ has cardinality λ, then by the Löwenheim-Skolem Theorem there is a model $\mathfrak{M} \prec \mathfrak{N}$ such that $\varphi(\mathfrak{N}) \subset M$ and $|M| = \lambda$. Since $\varphi(\mathfrak{N}) \subset M$ we must have that $(\mathfrak{N}, \mathfrak{M})$ is Vaughtian.

Definition 2 (ζ-stable)**.** *Let ζ be some cardinal. We say that a theory T is ζ-stable if, given any model $\mathfrak{M}$ of T and any subset $A \subset M$ with $|A| = \zeta$, we have $|S_n^{\mathfrak{M}}(A)| = \zeta$. Recall here that by $S_n^{\mathfrak{M}}(A)$ we mean the set of all sets $s(x_1, ..., x_n)$ of $\mathcal{L}(A)$-formulas which are maximally finitely satisfiable in $\mathfrak{M}$.*

This definition is rather general, however, we will solely use the notion of ω-stability. We now define the notions of algebraic formulas:

Definition 3. *Let $\mathfrak{M}$ be an $\mathcal{L}$-structure and let $A \subset M$.*

1. *An $\mathcal{L}(A)$-formula $\varphi(x)$ is said to be algebraic if $|\varphi(\mathfrak{M})| < \aleph_0$.*

2. *An element $m \in M$ is said to be algebraic over A if there is an $\mathcal{L}(A)$-formula $\varphi(x)$ which is algebraic such that $\mathfrak{M} \models \varphi(a)$.*

3. *An element $m \in M$ is said to be algebraic if it is algebraic over the empty set.*

4. *The algebraic closure of A is the set $acl(A) = \{m \in M \mid m \text{ is algebraic over } A\}$.*

Having defined these notions one can prove:

Lemma 1. Let $\varphi(x)$ be an algebraic $\mathcal{L}(A)$-formula and let $\mathfrak{M} \prec \mathfrak{N}$ with $A \subset M$, then

1. $\varphi(\mathfrak{M}) = \varphi(\mathfrak{N})$

2. $\mathfrak{M}$ is algebraically closed, that is, $\operatorname{acl}(M) = M$.

From this we may also easily arrive at $\operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$, which is of course what we would expect from any kind of closure.

Our next ingredient are strongly minimal sets. From now on, we will assume that T is a complete theory with infinite models, unless we explicitly state otherwise.

Definition 4. *Let $\mathfrak{M}$ be a model of T and let $D \subset M^n$ be a definable set, that is, $D = \{\overline{m} \in M^n \mid \varphi(\overline{m})\}$.*

1. *D is said to be minimal (or that $\varphi(\overline{x})$ is minimal) in $\mathfrak{M}$ if for every definable $Y \subset D^n$ either Y or $D\backslash Y$ is finite. Equivalently, one may say that for every $\mathcal{L}(M)$-formula $\psi(\overline{x})$, the set $(\varphi \wedge \psi)(\mathfrak{M})$ is either finite or co-finite.*

2. *The set D and the formula $\varphi(\overline{x})$ are said to be strongly minimal, if for every elementary extension $\mathfrak{N}$ of $\mathfrak{M}$, the formula $\varphi(\overline{x})$ is minimal in $\mathfrak{N}$.*

3. *A theory T is said to be strongly minimal if the universe of every model $\mathfrak{M}$ of T is a strongly minimal set. That is, every definable set in $\mathfrak{M}$ is either finite, or co-finite.*

Definition 5 (Independent Sets)**.** *Let $\mathfrak{M}$ be a model for a theory T and suppose $D \subset M$ is a strongly minimal subset.*

1. *We say that a subset $A \subset D$ is independent if $a \notin acl(A\backslash\{a\})$ for all $a \in A$. If $C \subset D$, we say that A is independent over C if $a \notin acl(C \cup (A\backslash\{a\}))$ for all $a \in A$.*

2. *A subset A of Y, where $Y \subset D$ and D is strongly minimal, is a basis for Y if A is independent and $acl(A) = acl(Y)$.*

One may then prove:

Theorem 1. Let $D \subset \mathfrak{M}$ be strongly minimal and let $A, B \subset D$ be independent with $A \subset \operatorname{acl}(B)$.

1. Let $A_0 \subset A$, $B_0 \subset B$ and let $A_0 \cup B_0$ be a basis for $\operatorname{acl}(B)$, $a \in A\backslash A_0$. Then there is $b \in B_0$ such that $A_0 \cup \{a\} \cup (B\backslash\{b\})$ is a basis for $\operatorname{acl}(B)$.

2. $|A| \leqslant |B|$ and therefore if A and B are bases for some $Y \subset D$, then $|A| = |B|$.

By the preceding theorem it is clear that if $D \subset \mathfrak{M}$ is strongly minimal and $Y \subset D$, then $\dim(Y) := |A|$, for some basis A of Y, is well defined. The idea of the above is that we can actually generalize the concept of dimension of, for example, vector spaces to the much broader setting of first-order formulas. A definition of dimension, however, would only make complete sense if two models sharing the same dimension stood in some kind of relationship and the best such relationship would need them to be isomorphic. This is indeed the case:

Theorem 2. Suppose T is a strongly minimal theory. If $\mathfrak{M}, \mathfrak{N} \models T$, then

$$\mathfrak{M} \cong \mathfrak{N} \iff \dim(\mathfrak{M}) = \dim(\mathfrak{N}) \tag{4}$$

Furthermore, if $\varphi(x)$ is a strongly minimal formula with parameters from A, where $A = \varnothing$ or $A \subset M_0$ for some elementary substructure $\mathfrak{M}_0$ of $\mathfrak{M}$ and $\mathfrak{N}$, then there is a bijective, partial elementary map $f\colon \varphi(\mathfrak{M}) \to \varphi(\mathfrak{N})$.

In order to make use of the concepts of minimal and strongly minimal formulas, we need to establish that there actually always are such objects, or put differently, that our definition isn't one of the empty set.

Lemma 2. Let T be an ω-stable theory and let $\mathfrak{M}$ be a model of T. Then there is a minimal formula in $\mathfrak{M}$.

Having this result along with some more lemmas one arrives at:

Theorem 3. If T is a theory with no Vaughtian pairs, then any minimal formula is strongly minimal.

Combining Lemma 2 and Theorem 3 immediately yields:

Corollary 1. If T is ω-stable and has no Vaughtian pairs, then every model $\mathfrak{M} \models T$ has a strongly minimal formula over $\mathfrak{M}$.

We will also need:

Theorem 4. Let T be a complete theory in a countable language $\mathcal{L}$ with infinite models. If T is ζ-categorical for some $\zeta \geqslant \aleph_1$, then T is ω-stable and has no Vaughtian pairs.

Now recall the following definition:

Definition 6. *Let $\mathfrak{M}$ be a model for T. We say that $\mathfrak{M}$ is a prime model for T if whenever $\mathfrak{N} \models T$ there is an elementary embedding of $\mathfrak{M}$ into $\mathfrak{N}$. Furthermore, given some $A \subset M$, we say that $\mathfrak{M}$ is prime over A if whenever $\mathfrak{N} \models T$ and $f\colon A \to \mathfrak{N}$ is partially elementary there is an elementary embedding $f^*\colon \mathfrak{M} \to \mathfrak{N}$ extending f.*

Our last ingredient is the following:

Lemma 3. Suppose T has no Vaughtian pairs and let $\mathfrak{M}$ be a model for T, and let $X \subset M^n$ be definable. Then no proper elementary submodel of $\mathfrak{M}$ contains X. Moreover, if in addition T is ω-stable, then $\mathfrak{M}$ is prime over X.

Finally we may prove:

Theorem 5 (Baldwein-Lachlan)**.** Let T be a complete theory in a countable language with infinite models and let $\zeta \geqslant \aleph_1$ be a cardinal. Then T is ζ-categorical if and only if T is ω-stable and has no Vaughtian pairs.

Proof. Suppose first that T is ζ-categorical. By Theorem 4 we immediately infer that T is ω-stable and has no Vaughtian pairs. Conversely, suppose T is is ω-stable and has no Vaughtian pairs. Suppose $\mathfrak{M}$ and $\mathfrak{N}$ are two models of T such that $|\mathfrak{M}| = |\mathfrak{N}| = \zeta \geqslant \aleph_1$. Our goal is to show that $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic. By what we have seen in the preceding lemma, T has a prime model $\mathfrak{M}_0$. Moreover, combining yet again Lemma 2 and Theorem 3 we note that there is a strongly minimal formula $\varphi(x)$ with parameters in $\mathfrak{M}_0$. We may now consider $\mathfrak{M}$ and $\mathfrak{N}$ as elementary extensions of $\mathfrak{M}_0$. Thus by Theorem 2 there is a partial elementary bijection $f\colon \varphi(\mathfrak{M}) \longrightarrow \varphi(\mathfrak{N})$. By Lemma 3 $\mathfrak{M}$ is prime over $\varphi(\mathfrak{M})$, so f can be extended to $f^*\colon \mathfrak{M} \to \mathfrak{N}$. However by Lemma 3, we know that $\mathfrak{N}$ has no elementary submodels containing $\varphi(\mathfrak{N})$, so $f^*(\mathfrak{M}) = \mathfrak{N}$. In other words, f^* is a surjective elementary embedding and consequently an isomorphism. This shows that T is ζ-categorical. $\square$

Theorem 6 (Morley's Categoricity Theorem)**.** Let ζ be an uncountable cardinal, then a theory T is $\aleph_1$-categorical if and only if T is ζ-categorical.

Proof. Note that the Baldwein-Lachlan Theorem is independent of the chosen cardinal $\zeta \geqslant \aleph_1$. The result thus immediately follows. $\square$