RICCI FLOW

CONTENTS

ABSTRACT

Some stuff about Ricci Flow.

1 INFORMAL DISCUSSION ABOUT THE POINCARÉ CONJECTURE - CA. 15 MIN - EDUARD

2 UNIFORMIZATION THEOREM (QUICK EXPLANATION) - CA. 5 MIN - EDUARD

3 CRASHCOURSE IN RIEMANNIAN GEOMETRY - 30 MIN - ALEX

We shall start our journey by first recalling some facts about Riemannian geometry.

3.1 *Bundles and metrics*

Let $\mathscr{M}$ be a smooth (abstract) manifold. The tangent bundle $T\mathscr{M}$ is defined as the disjoint union

$$T\mathscr{M} := \bigsqcup_{m\in\mathscr{M}} T_m\mathscr{M} \tag{1}$$

where $T_m\mathscr{M}$ is the tangent space of M at the point $m \in \mathscr{M}$. The manifold $\mathscr{M}$ canonically induces a smooth manifold structure onto $T\mathscr{M}$. Quite similarly, the cotangent bundle is given by

$$T^*\mathscr{M} := \bigsqcup_{m\in\mathscr{M}} T_m^*\mathscr{M} \tag{2}$$

where $T_m^*\mathscr{M}$ is the dual space of $T_m\mathscr{M}$ and $T\mathscr{M}$ also comes endowed with a smooth manifold structure.

Definition 1. Let $\mathscr{M}$ be an n-dimensional smooth manifold. A Riemmanian metric g on $\mathscr{M}$ is a smooth section of $T^*\mathscr{M} \otimes T^*\mathscr{M}$ defining a positive definite symmetric bilinear form on $T_p\mathscr{M}$ for each $p \in \mathscr{M}$. In other words, g is a smooth map

$$\mathscr{M} \xrightarrow{g} T^*\mathscr{M} \otimes T^*\mathscr{M} \qquad m \mapsto g_m \in T_m^*\mathscr{M} \otimes T_m^*\mathscr{M} \tag{3}$$

where g_m is a positive definite symmetric bilinear form $T_m\mathscr{M} \times T_m\mathscr{M} \to \mathbb{R}$.

If we have local coordinates $\{x^i\}$ in a neighborhood U of some point of $\mathscr{M}$, then the coordinate vector fields $\{\partial_i\}$ form a local basis for $T\mathscr{M}$ and the 1-forms $\{dx^i\}$ form a dual basis for $T^*\mathscr{M}$, that is,

$$dx^i(\partial_j) = \delta^i_j \tag{4}$$

The metric g may then be written in local coordinates as

$$g = \underbrace{g_{ij}}_{g(\partial_i, \partial_j)} dx^i \otimes dx^j \tag{5}$$

The inverse g^{-1} will be locally written as (g^{ij}) which is the inverse of the positive definite matrix (g_{ij}). The pair $(\mathscr{M}, g)$ is called a Riemannian manifold.

Being Riemannian is not some rare property in the world of manifolds. Quite on the contrary actually:

Proposition 1. *Any smooth manifold admits a Riemmanian metric.*

Proving this result is rather easy: Pick any covering of $\mathscr{M}$ by coordinate neighborhoods $\{U_\alpha\}$ and a partition of unity $\{\varphi_\alpha\}$ subordinate to the covering. On each U_α we have a metric

$$g_\alpha = \delta_{ij} dx^i \otimes dx^j \tag{6}$$

in the corresponding local coordinates. Define

$$g = \sum \varphi_\alpha g_\alpha \tag{7}$$

then g is a Riemannian metric on $\mathscr{M}$.

3.2 *Levi Civita Connection*

The next ingredient we will need is the notion of a covariant derivative. Let us denote by $\mathscr{X} := \mathscr{X}(\mathscr{M})$ the space of vector fields for some fixed manifold $\mathscr{M}$. Moreover, let $\mathscr{T} := \mathscr{T}(\mathscr{M})$ denote the space of tensor fields (of any valence) on $\mathscr{M}$.

Definition 2. A connection on $\mathscr{M}$ is an $\mathbb{R}$-bilinear operator

$$\nabla\colon \mathscr{X} \times \mathscr{T} \to \mathscr{T} \qquad (X, T) \mapsto \nabla_X T \tag{8}$$

such that

- valence$(\nabla_X T) =$ valence(T),
- $(X, T) \mapsto \nabla_X T$ is $\mathscr{C}^\infty$-linear in the variable X,
- ∇ satisfies the Leibniz rule: $\forall X \in \mathscr{X} \forall T \in \mathscr{T} \forall f \in \mathscr{C}^\infty$:

$$\nabla_X(fT) = X(f)T + f\nabla_X T \tag{9}$$

The fundamental theorem of Riemannian geometry then reads:

Theorem 1 (Levi Civita Connection)**.** *Let $(\mathscr{M}, g)$ be a Riemannian manifold. Then there exists a unique connection such that*

- ∇ *is torsionless:* $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0$ *for all* $X, Y \in \mathscr{X}$.
- ∇ *preserves the metric:* $\nabla g = 0$.

3.3 *Musical isomorphism, gradient and the Laplace-Beltrami operator*

3.3.1 *Gradient*

The musical isomorphism in Riemannian geometry states that if we are given a (semi-) Riemannian manifold $(\mathscr{M}, g)$, then there is a $\mathscr{C}^\infty$-linear isomorphism $\mathscr{X} \longrightarrow \Omega^1$ given by

$$X \mapsto g(X, \_) \tag{10}$$

Let us fix a function $\varphi \in \mathscr{C}^\infty$. Then the differential $d\varphi \in \Omega^1$ is a 1-form and we know by the musical isomorphism that there must exist $\nabla\varphi \in \mathscr{X}$ so that

$$g(\nabla_g \varphi, \_) = d\varphi \tag{11}$$

Hence we have found an operator

$$\nabla_g \colon \mathscr{C}^\infty \to \mathscr{X} \tag{12}$$

satisfying

$$g(\nabla_g \varphi, \_) = d\varphi \tag{13}$$

which we will refer to as the gradient of φ. Locally the gradient reads

$$\nabla\varphi = g^{ij} \partial_j \varphi \frac{\partial}{\partial x_i} \tag{14}$$

3.3.2 *Divergence*

Given $\omega \in \Omega^n$ where $n = \dim(\mathscr{M})$, and any $X \in \mathscr{X}$ one can define the $(n-1)$-form $\iota_X \in \Omega^{n-1}$ by

$$\iota_X \omega(X_1, ..., X_{n-1}) := \omega(X, X_1, ..., X_{n-1}) \tag{15}$$

where $X_1, ..., X_{n-1} \in \mathscr{X}$. Since $d(\iota_X) \in \Omega^n$ we know there must exist a scalar $\mathrm{div}_\omega X$ such that

$$d(\iota_X \omega) = \mathrm{div}_\omega X \cdot \omega \tag{16}$$

If $\omega = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$ is the volume form of $(\mathscr{M}, g)$, then the number $\mathrm{div}_g X := \mathrm{div}_{\omega_g} X$ is known as the divergence of X. Thus the divergence is the operator

$$\mathrm{div}_g \colon \mathscr{X} \to \mathscr{C}^\infty \tag{17}$$

satisfying

$$d(\iota_X \omega_g) = \mathrm{div}_g X \cdot \omega \tag{18}$$

In local coordinates the divergence of $X = X^i \partial_i$ reads

$$\mathrm{div}_g X = \frac{1}{\sqrt{|g|}} \partial_i (X^i \sqrt{|g|}) \tag{19}$$

Combining these two definitions yields the Laplace-Beltrami operator Δ_g which is an operator

$$\Delta_g \colon \mathscr{C}^\infty \to \mathscr{C}^\infty \qquad \Delta_g := \mathrm{div}_g \circ \nabla_g \tag{20}$$

Locally the Laplace-Beltrami operator has the form

$$\Delta \varphi = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \varphi \right) \tag{21}$$

4 FROM RIEMANNIAN CURVATURE TO RICCI CURVATURE - 20 MIN - EDUARD

5 DEFINITION OF RICCI FLOW/(EXAMPLES?) - 10 MIN- EDUARD

6 EXISTENCE/UNIQUENESS RICCI - 5-10 MIN - FLOW - ALEX

Recall that for an n-dimensional Riemannian manifold $\mathscr{M}^n$ with metric g_0 the Ricci flow of $(\mathscr{M}^n, g_0)$ is a time dependent family $(g(t))_{t \geq 0}$ of metrics on $\mathscr{M}^n$ satisfying $g(0) = g_0$ and evolving according to the equation

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)} \tag{22}$$

where $\mathrm{Ric}_{g(t)}$ is the Ricci curvature tensor associated with the metric $g(t)$. When written in coordinates, the Ricci flow is a parabolic system of partial differential equations for the components of the metric. Using the special structure of the equations, Hamilton was able to prove short time existence for the Ricci flow:

Theorem 2. *The Ricci flow on a closed manifold has a unique solution in a time interval* $[0,t_0)$ *for some* $t_0 > 0$.

While the flow admits a solution for positive times, it is in general not solvable for negative times (which is a typical feature of parabolic problems).

Theorem 3. *Each solution of the Ricci flow on a compact manifold can be extended to a maximal time interval* $[0,T)$ *with* $T \leq +\infty$. *If* T *is finite, then we necessarily have*

$$\limsup_{t\to T} M(t) = +\infty \tag{23}$$

where

$$\max_{m\in\mathscr{M}} \sqrt{R_{ijkl}(t)R^{ijkl}(t)} \tag{24}$$

is the maximum of the norm of the Riemannian curvature tensor at time t. *In the world of physics the quantity* $R_{ijkl}R^{ijkl}$ *is usually referred to as the Kretschmann-scalar.*

We describe the above behaviour by saying that the flow becomes singular at time T.

7 EVOLUTION OF CURVATURE, PRESERVATION OF POSITIVITY - ALEX

Theorem 4. *Let* $(g(t))_{t\geq 0}$ *be a solution to the Ricci flow o a closed manifold* $\mathscr{M}^n$.

1. *The minimum of* R *is nondecreasing under the flow. In particular, positive scalar curvature is preserved under the flow (that is, if the initial metric* $g(0)$ *has positive scalar curvature, then so does* $g(t)$ *for all* $t > 0$.*)*

2. *If* $n = 3$ *then positive Ricci curvature and positive sectional curvature are preserved under the flow (this property may fail in dimension greater than 3).*

8 PERELMAN'S FUNCTIONAL - APPLICATIONS TO GR - 10 MIN - ALEX

The Ricci flow has provided many far-reaching insights into long-standing problems in topology and geometry. More surprisingly however is the fact that it also has interesting applications in physics. The first of its occurrences are the low-energy effective action of string theory. There is also some recent applications of the Ricci flow to foundational issues in quantum mechanics (actually Schrödinger quantum mechanics arises from Perelman's functional on a compact Riemann surface).

Let us turn our attention now to the ominous entity that is Perelman's functional: Perelman's functional $\mathfrak{F}[\varphi, g_{ij}]$ on the manifold $\mathscr{M}$ is defined as

$$\mathfrak{F}[\varphi, g] := \int_{\mathscr{M}} e^{-\varphi}\Big[\nabla\varphi\cdot\nabla\varphi + R_g\Big]d\mu_g \tag{25}$$

where g is a metric on $\mathscr{M}$, $\varphi \in \mathscr{C}^\infty$ and $d\mu_g = \sqrt{|g|}d^nx$. The above equation will be our starting point. It may be regarded physically as providing an action functional on the configuration space $\mathscr{M}$ for the two independent fields g_{ij} and φ.

Now some aspects of the functional $\mathscr{F}(\varphi, g)$ are worth mentioning. Setting $\varphi = 0$ identically we have the Einstein–Hilbert functional for gravity on $\mathscr{M}$. This provides a generalisation of gravity provided by the functional $\mathscr{F}(\varphi, g)$ when $\varphi \neq 0$. Indeed, Perelman's functional arises in string theory as the low– energy effective action of the bosonic string.

We first compute the Euler-Lagrange extremals corresponding to the fields g and φ. Next we set the equations of motion so obtained equal to the first order time derivatives of g and φ, respectively. This results in the two evolution equations

$$\frac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i\nabla_j\varphi) \qquad \frac{\partial\varphi}{\partial t} = -\Delta\varphi - R \tag{26}$$

We stress that the RHS of (26), once equated to zero, are the Euler-Lagrange equations of motion corresponding to Perelman's functional, and that the time derivatives on the left-hand sides have been put in by hand. The two equations in (26) are referred to as the gradient flow of $\mathscr{F}$. Perelman's equations are further simplified by an application of what has become known in the mathematical literature as "DeTurck's trick": a specific spatial diffeomorphism is applied to the original equations to obtain the equivalent set of equations:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \qquad \frac{\partial\varphi}{\partial t} = -\Delta\varphi + g(\nabla\varphi, \nabla\varphi) - R \tag{27}$$

In the paper [1] the Ricci flow is used to shed light on the mathematical theory of Hawking's black holes. The hypothesis of the paper is:

The evolution of a black hole follows the dynamical system of Hamilton's Ricci flow.

Moreover, with that assumption one can then define the entropy of black holes as follows:

Definition 3 (The entropy of black holes)**.** The entropy of black holes are defined as Perelman's $\mathscr{F}$-functional and $\mathscr{W}$-functional as follows:

$$\mathscr{F}(g,\varphi) = \int_{\mathscr{M}^3} \Big(R + g(\nabla\varphi,\nabla\varphi)\Big)e^{-f}d\mu_g \tag{28}$$

and

$$\mathscr{W}(g,\varphi,\tau) = \int_{\mathscr{M}^3} \Big(\tau\Big(R + g(\nabla\varphi,\nabla\varphi)\Big) + \varphi - n\Big)(4\pi\tau)^{-\frac{n}{2}}e^{-f}d\mu_g \tag{29}$$

where g is a Riemannian metric, φ is a smooth function on the black hole manifold $\mathscr{M}^3$ and τ is a scale parameter.

Theorem 5 (Monotonicity formulas of the entropy of black holes)**.** *Under the evolution system of black holes*

$$\begin{cases} \frac{\partial}{\partial t}g_{ij} = -2R_{ij} \\ \frac{\partial}{\partial t}\varphi = -R - \Delta\varphi + g(\nabla\varphi\nabla\varphi) \end{cases}$$

we have that $\mathscr{F}$-entropies of black holes increase monotonically with time t

$$\frac{d}{dt}\mathscr{F}(g(t),\varphi(t)) \geq 0 \tag{30}$$

There is a similar result for $\mathscr{W}$.

Definition 4 (local collapsing Ricci flow)**.** Let $(\mathscr{M},g(t))_{t\in[0,T)}$ be the solution to the Ricci flow for some $T \in (0,\infty]$. The solution $g(t)$ is called locally collapsing at T if there exist sequences $\{x_k\} \subset \mathscr{M}, \{t_k\} \subset [0,T)$ and radii $r_k \in (0,\infty)$ such that

1. $t_k \to T$
2. radius bound: $\frac{r_k^2}{t_k}$ is uniformly bounded for all k.
3. curvature bound: $|\mathrm{Rm}_{g(t_k)}|_{g(t_k)} \leq \frac{1}{r_k^2}$ on $B_{g(t_k)}(x_k,r_k)$ for all k
4. volume collapsing:

$$\lim_{k\to\infty} \frac{Vol_{g(t_k)}B_{g(t_k)}(x_k,r_k)}{r_k^n} = 0 \tag{31}$$

Theorem 6 (Nonexistence of singularity of the black hole)**.** *The theorized singularity of the black hole core with infinitely small size and infinite density cannot exist.*

This idea presents a mathematical interpretation of the following physical analysis of the "singularity" of black holes together with the laws of quantum mechanics. Quantum theory dictates that the event horizon must actually be transformed into a highly energetic region, or " firewall", that would burn the astronaut to a crisp. This was alarming because, although the firewall obeyed quantum rules, it flouted Einstein' s general theory of relativity. According to that theory, someone in free fall should perceive the laws of physics as being identical everywhere in the Universe-whether they are falling into a black hole or floating in empty in- tergalactic space.

In the mentioned paper [1] they also introduced a wonderful mathematical model of the black hole: We consider the manifold

$$\widetilde{\mathscr{M}} := \mathscr{M}^3 \times \mathbb{S}^N \times \mathbb{R}_+ \tag{32}$$

with the following metric as the mathematical model of our black hole (which was first constructed by Perelman):

$$\widetilde{g}_{ij} = g_{ij} \qquad \widetilde{g}_{\alpha\beta} = \tau g_{\alpha\beta} \qquad \widetilde{g}_{00} = \frac{N}{2\tau} + R \qquad \widetilde{g}_{i\alpha} = \widetilde{g}_{i0} = \widetilde{g}_{\alpha 0} = 0 \tag{33}$$

where i, j denote the coordinate indices on the $\mathscr{M}^3$ factor, α, β denote those on the $\mathbb{S}^N$ factor, and the coordinate τ on $\mathbb{R}_+$ has index 0; g_{ij} evolves with τ by the backward Ricci flow

$$\frac{\partial}{\partial \tau} g_{ij} = 2R_{ij} \tag{34}$$

$g_{\alpha\beta}$ is the metric on $\mathbb{S}^N$ of constant curvature $\frac{1}{2N}$. In that manner, a black hole is actually a dynamical system evolving along Ricci flow together with some curled up dimensions.

Theorem 7 (Thurston's Geometrization)**.** *Every closed 3-manifold has a prime decomposition: This means that every closed 3-manifold is the connected sum of prime 3-manifolds (this is essentially unique), which cannot be written as a non-trivial connected sum. There are* 8 *possible geometric structures in 3 dimensions:*

1. *spherical geometry* $\mathbb{S}^3$
2. *Euclidean geometry* $\mathbb{R}^3$
3. *Hyperbolic geometry* $\mathbb{H}^3$
4. *The geometry of* $\mathbb{S}^2 \times \mathbb{S}^1$
5. *The geometry of* $\mathbb{H}^2 \times \mathbb{S}^1$

6. *The geometry of the universal cover of* $SL(2,\mathbb{R})$
7. *Nil geometry*
8. *Sol geometry.*

Based on this we get:

Theorem 8 (Evolution of the Black hole)**.** *Every black hole will evolve with time* t *into a closed 3-manifold with a prime decomposition as above.*

REFERENCES

[1] Qiaofang Xing et al. A mathematical interpretation of hawking's black hole theory by ricci flow. *Journal of Applied Mathematics and Physics*, 5(02):321, 2017.