Introduction to mathematical Logic - Task 1

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Since the dawn of human civilization, our predecessors were concerned with counting items in some sense or another. The concept of a natural number seems, well, pretty natural. By counting up a collection of items we implicity arrange these, that is, we give them an order (i.e. x_n is the n-th element in some sequence). On the other hand, we may also compare the sizes of collections of objects by means of natural numbers, i.e. we may compare collections of objects by their respective cardinalities. As is very well known, mathematicians like to generalize, and thus through the first idea that was mentioned (ordering items) ordinal numbers were born, and through the second idea (comparing cardinalities) cardinal numbers have entered the realm of mathematics.

1 Ordinal numbers

Definition 1. A set ζ is an **ordinal**, if ζ satisfies transitivity, i.e.

$$\forall x(x \in \zeta \rightarrow x \subset \zeta)$$

and the relation $\in$ is a well order on ζ, that is, ζ is both well-founded

$$\forall \xi \subset \zeta(\xi \neq \varnothing \rightarrow (\exists m \in \xi)(\forall x \in \xi)\neg(x \in m)) \tag{1}$$

and it is totally ordered, i.e. it is irreflexive, transitive and satisfies

$$\forall x, y \in \zeta((x \in y) \vee (y \in x) \vee (x = y)) \qquad \textbf{(trichotomy)} \tag{2}$$

We will denote the collection of all ordinals by $\mathbb{ON}$.

Now it is quite clear, that for an ordinal ζ we have $\forall \xi(\xi \in \zeta \rightarrow \xi \in \mathbb{ON})$. Indeed, for $\xi \in \zeta$ we must have by transitivity, that $\xi \subset \zeta$. Now since ζ is well-ordered we must also have, that ξ is well-ordered wrt. $\in$. Now if $x \in \xi \subset \zeta$ and $y \in x$, then of course by transitivity of ζ we get $y \in x \subset \zeta$ and therefore $x, y, \xi \in \zeta$. By transitivity of the relation $\in$ on elements of ζ we get

$$(y \in x \wedge x \in \xi) \rightarrow y \in \xi$$

and therefore $x \subset \xi$, so ξ is transitive as a set and thus an ordinal. It is equally easy to prove, that if ζ and η are ordinals, then $\zeta \cap \eta$ must be an ordinal as well.

Lemma 1. Elements of ordinals are themselves ordinals and the intersection of two ordinals is again an ordinal.

We can utilize this lemma to prove another very helpful characterization of ordinals:

Proposition 1. Let $\xi, \zeta \in \mathbb{ON}$, then

$$\xi \subset \zeta \iff (\xi \in \zeta) \vee (\xi = \zeta)$$

Now it is nice to note, that the class of ordinals $\mathbb{ON}$ actually carries along with it a similar structure as an ordinal itself, i.e.

$$\forall \xi, \zeta, \eta \in \mathbb{ON}((\xi \in \zeta) \wedge (\zeta \in \eta) \rightarrow \xi \in \eta) \qquad \textbf{Transitivity} \tag{3}$$

$$\forall \xi \in \mathbb{ON} \neg(\xi \in \xi) \qquad \textbf{Irreflexivity} \tag{4}$$

$$\forall \zeta, \xi \in \mathbb{ON}((\zeta \in \xi) \vee (\xi \in \zeta) \vee (\xi = \zeta)) \qquad \textbf{Trichotomy} \tag{5}$$

$$\text{If } Z \text{ is a set of ordinals, then } Z \text{ has an } \in\text{-least element} \qquad \textbf{well founded} \tag{6}$$

Considering the above statement, one might be tempted to think, that the collection of all ordinals is an ordinal itself. This, however, turns out to be wrong, since $\mathbb{ON}$ is actually a proper class, i.e. there is no set containing $\mathbb{ON}$ as a subset.

Theorem 1 (Burali-Forty)**.** The class $\mathbb{ON}$ is proper.

Proof. Suppose $\mathbb{ON}$ is no proper class, i.e. there is a set $\mathbf{S}_0$ with $\mathbf{S}_0 \supset \mathbb{ON}$. Now by the axiom of comprehension

$$\mathbf{S} = \{\zeta \in \mathbf{S}_0 \mid \zeta \in \mathbb{ON}\}$$

is a set. Now from what was remarked before, we know that $\mathbf{S}$ is well-ordered by $\in$ and it is transitive as a set, i.e. $\mathbf{S}$ is an ordinal. But then $\mathbf{S} \in \mathbf{S}$ by construction, which contradicts irreflexivity of ordinals (also remarked before). □

We mentioned in the beginning, that the idea to define ordinal numbers originated from the need to generalize the notion of ordering a set. The following lemma is crucial for this generalization procedure:

Lemma 2. If $\zeta, \xi \in \mathbb{ON}$ and if $\psi\colon (\zeta, \in) \to (\xi, \in)$ is an order-isomorphism, then $\zeta = \xi$ and $\psi = \mathbf{id}$.

By means of this observation, one can actually prove a real corner stone concerning the theory of ordinals, which in turn tells us, that ordinals exactly have the desired, generalizing properties:

Theorem 2. Suppose we are given a set $\mathbb{S}$ along with a well-order R on $\mathbb{S}$. Then there is a unique ordinal $\zeta \in \mathbb{ON}$ such that $(\mathbb{S}, R)$ is order-isomorphic to $(\zeta, \in)$.

For a set $\mathbb{S}$ and a well-order R on $\mathbb{S}$, we write $\mathbf{type}(\mathbb{S}, R)$ for the unique ordinal obtained by the preceding theorem.

2 Cardinal numbers

Cardinals are all about cardinalities. Thus we start by defining a relation that captures the notion of "size". If we have two sets, say X and Y, then we write $X \preceq Y$, if there exists an injection $\psi\colon X \to Y$. Analogously, if ψ is even a bijection between X and Y, then we write $X \approx Y$. If there exists no injection from Y to X and $X \preceq Y$, then we write $X \prec Y$.

Definition 2. An ordinal ζ is called a **cardinal**, if $\forall \xi \in \zeta (\xi \prec \zeta)$. We denote the collection of cardinals by $\mathbb{CD}$. A **limit ordinal** is an ordinal ζ such that $\zeta \neq 0$ and ζ is not a successor ordinal i.e. there is no $\xi \in \mathbb{ON}$ such that $\zeta = S(\xi) := \xi \cup \{\xi\}$. We write ω for the least limit ordinal (=$\mathbb{N}$).

Theorem 3. The following statements are valid:

1. If $\zeta \in \mathbb{CD}$ such that $\zeta \geqslant \omega$, then ζ is a limit ordinal.
2. If we have a set of cardinals $\mathbb{S}$, then $\sup \mathbb{S} := \bigcup \mathbb{S}$ is a cardinal.
3. Every natural number is a cardinal and thus in particular $\omega \in \mathbb{CD}$.

Proof. For brevity we only prove statement (1). Suppose, by way of contradiction, that ζ were a successor ordinal. So $\zeta = \xi \cup \{\xi\}$ for some ordinal ξ. Then consider the function

$$\begin{aligned} &\psi\colon \xi \cup \{\xi\} \to \xi && \psi(\xi) := 0 \\ &\forall n \in \omega : \psi(n) := n + 1 && \forall \eta (\omega \leqslant \eta < \xi) : \psi(\eta) := \eta \end{aligned}$$

By construction ψ is a bijection between ζ and ξ, which is a contradiction to ζ being a cardinal. □

Definition 3. If we have a well-orderable set $\mathbb{S}$, then we define the cardinality of $\mathbb{S}$ as the least ordinal ζ such that $\mathbb{S} \approx \zeta$.

Of course the cardinality of a set is always a cardinal, and by the axiom of choice we also know that every set can be well-ordered, that is, every set's "size" can be associated with a cardinal. We have now accumulated enough information to define the aleph-function.

Definition 4. If $\mathbb{S}$ is a set, then let $\aleph(\mathbb{S})$ denote the least cardinal $\kappa \not\leq \mathbb{S}$.

It can be proven, that such a least cardinal always exists (Hartogs Theorem), so this is well defined. Now by transfinite recursion on $\mathbb{ON}$, define the cardinal numbers $\aleph_\zeta$ by:

1. $\aleph_0 = \omega_0 = \omega$
2. $\aleph_{\zeta+1} = \omega_{\zeta+1} = \aleph(\aleph_\zeta)$
3. $\aleph_\zeta = \omega_\zeta = \sup\{\aleph_\xi \mid \xi < \zeta\}$, if ζ is a limit ordinal.

Definition 5. Suppose we are given a limit ordinal ζ, then the cofinality of ζ is defined by

$$\mathrm{cf}(\zeta) := \min\{\mathbf{type}(X) \mid X \subset \zeta \wedge \sup(X) = \zeta\}$$

We now prove a result of tremendous importance:

Theorem 4 (König)**.** Suppose we are given two families of cardinal numbers, say $\langle \zeta_i \mid i \in I\rangle$ and $\langle \xi_i \mid i \in I\rangle$ along with pairwise disjoint sets X_i, Y_i such that $\zeta_i = |X_i| < |Y_i| = \xi_i$ for all $i \in I$, then

$$\sum_{i\in I} \zeta_i := \left|\bigcup_{i\in I} X_i\right| < \left|\underbrace{\left\{f : I \to \bigcup_{i\in I} Y_i \mid \forall i \in I(f(i) \in Y_i)\right\}}_{\prod_{i\in I} Y_i}\right| =: \prod_{i\in I} \xi_i$$

Proof. WLOG we may assume, that $X_i \subsetneq Y_i$ for all $i \in I$. We now construct an injection

$$\Phi\colon \bigcup X_i \to \prod Y_i$$

For every $i \in I$ choose $y_i \in Y_i \backslash X_i \neq \varnothing$ (axiom of choice). Now if $x \in \bigcup X_i$, then, since all X_i are pairwise disjoint, there exists a unique $j \in I$ such that $x \in X_j$. Now define

$$\Phi(x)(i) = \begin{cases} x & \text{if } i = j \\ y_i & \text{otherwise} \end{cases}$$

By construction Φ is an injection. Now suppose an arbitrary Φ is given as above. For every $i \in I$ let $f(i) \in Y_i \backslash \{\Phi(x)(i) \mid x \in X_i\}$, then $f \in \prod Y_i$, yet by construction $\Phi(x) \neq f$ for all $x \in \bigcup X_i$, which yields the statement. □

Now from König's theorem we immediately obtain, that if $\zeta \geqslant 2$ and λ is infinite, then $\zeta^{\mathrm{cf}(\lambda)} > \lambda$.

References: Lecture notes and the Wikipedia article concerning the Satz von König.