ON SOBOLEV SPACES - FROM EMBEDDINGS TO DUALITY

ALEXANDER ZAHRER

CONTENTS

ABSTRACT

> The unknown thing to be known appeared to me as some stretch of earth or hard marl, resisting penetration... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it... yet it finally surrounds the resistant substance.
>
> (Alexander Grothendieck, Récoltes et semailles, 1985–1987, pp. 552-3-1 The Rising Sea)

In the following we will give a very brief introduction to some of the intricacies of Sobolev spaces. We will start from scratch by first giving some motivation as to how one might have come up with these entities. After having done so, we will prove an embedding theorem and talk a bit about differentiation. Last but not least, we will concern ourselves with the question of duality. Chapter 1-3 are based on the lecture notes [1] of professor Donninger, while chapter 4 is based on the lecture notes [2] of professor Kunzinger.

1 NOTATION AND CONVENTIONS

We will always assume that the underlying vector space for our L^p-spaces, Sobolev spaces, etc. will be $\mathbb{R}^d$, where $d \in \mathbb{N}$ is some fixed dimension. Every integral will be taken over all of $\mathbb{R}^d$ and thus we will simply write $\int$ instead of $\int_{\mathbb{R}^d}$. Moreover, expressions like $L^2(\mathbb{R}^d), \mathscr{C}_b^\infty(\mathbb{R}^d)$ will be shortened to $L^2, \mathscr{C}_b^\infty$ respectively.

2 INTRODUCTION

The concept of differentiation is probably one of the most important ideas in all of mathematics. It is only natural then to search for some kind of generalization of this concept, that is, to define some form of "weaker" differentiability of suitable functions. Of course, accompanied with such considerations of weakly differentiable functions, there is always the question of which spaces might accommodate them. In this section we will both define these spaces, as well as motivate their origin.
Our starting point will be the integral equation

$$\int (D^\alpha \psi) f = (-1)^{|\alpha|} \int \psi (D^\alpha f) \tag{1}$$

for $\psi, f \in \mathscr{S}$ and $\alpha \in \mathbb{N}^d$. Of course, the left hand side of the above equation still makes sense if we drop the differentiability assumption on f, that is, if we solely assume $f \in L^2$. This leads quite naturally to a concept of a weaker derivative in the following way: We say that $f \in L^2$ is weakly differentiable with respect to D^α, if there exists $g \in L^2$ such that

$$\int (D^\alpha \psi) f = (-1)^{|\alpha|} \int \psi g \tag{2}$$

holds for all $\psi \in \mathscr{S}$. It is rather easily shown that, in case f is differentiable, this definition coincides with the usual derivative. Now the question is how we shall define those spaces in which this broader class of differentiable functions live. Of course, with such functional spaces we also need some kind of topology, in the best case we would like to have a Hilbert space structure and we would also like the Schwartz space to be densely embedded in these new spaces. Let $k \in \mathbb{N}$ and denote by H^k the space of L^2-functions which are weakly differentiable with respect to all D^α for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k$. Suppose now that we have already endowed H^k with a norm topology induced by $\|.\|_{H^k}$ and such that $\mathscr{S}$ is dense in H^k. If we then pick some element $\psi \in H^k$ there must be a sequence $\{\psi_k\} \subset \mathscr{S}$ so that $\psi_k \xrightarrow{H^k} \psi$. What we certainly want to expect from such a sequence is then that the derivatives $\{D^\alpha \psi_k\} \subset \mathscr{S}$ will converge to the corresponding weak derivative of ψ in L^2. More precisely, if $D^\alpha \psi \in L^2$ represents the weak derivative of $\psi \in H^k$, then we would like to have that $\|D^\alpha(\psi_k - \psi)\|_{L^2} \xrightarrow{k \to \infty} 0$. Thus we have found the norm which will induce the topology on our space in question, namely,

$$\|\psi\|_{H^k} = \sum_{|\alpha| \le k} \|D^\alpha \psi\|_{L^2} \tag{3}$$

for $\psi \in \mathscr{S}$. Taking the completion of the normed space $(\mathscr{S}, \|.\|_{H^k})$ yields H^k for $k \in \mathbb{N}$. However, we still cannot be sure if H^k carries a Hilbert space structure. In order to take the next step, we need to remind ourselves of our favorite unitary automorphism on both the Schwartz space $\mathscr{S}$ and on L^2, namely, the Fourier transform $\mathscr{F}$. One of the strengths of the Fourier transform is its interaction with derivatives, that is, $\mathscr{F}(D^\alpha\psi) = (2\pi i.)^\alpha \mathscr{F}\psi$ for $\psi \in \mathscr{S}$. Having this fact ready we can prove that

$$\sum_{|\alpha|\leq k} \|D^\alpha\psi\|_{L^2} \simeq \|\langle.\rangle^k \mathscr{F}\psi\|_{L^2} \tag{4}$$

Indeed, using both Plancherel's Theorem and the Multinomial Theorem we calculate

$$\sum_{|\alpha|\leq k} \|D^\alpha\psi\|_{L^2} = \sum_{|\alpha|\leq k} \|\mathscr{F}(D^\alpha\psi)\|_{L^2} \simeq \sum_{|\alpha|\leq k} \|\xi^\alpha \mathscr{F}\psi\|_{L^2} \lesssim \sum_{|\alpha|\leq k} \|\langle.\rangle^{|\alpha|}\mathscr{F}\psi\|_{L^2} \tag{5}$$

$$\lesssim \|\langle.\rangle^k \mathscr{F}\psi\|_{L^2} \tag{6}$$

Conversely,

$$\|\langle.\rangle^k \mathscr{F}f\|_{L^2} \simeq \|(1+\sum_{j=1}^d |\xi_j|)^k \mathscr{F}\psi\|_{L^2} = \left\| \sum_{|(\beta_0,\ldots,\beta_d)|=k} \binom{k}{\beta} \prod_{j\geq 1} |\xi_j|^{\beta_j} \mathscr{F}\psi \right\| \tag{7}$$

$$\lesssim \sum_{\alpha\in\mathbb{N}^d, |\alpha|\leq k} \|\xi^\alpha \mathscr{F}\psi\|_{L^2} \simeq \sum_{\alpha\in\mathbb{N}^d, |\alpha|\leq k} \|\mathscr{F}(D^\alpha\psi)\|_{L^2} = \sum_{\alpha\in\mathbb{N}^d, |\alpha|\leq k} \|D^\alpha\psi\|_{L^2} \tag{8}$$

as wanted. Thus we can generalize our Sobolev spaces and define H^s even for $s \in \mathbb{R}$ as the completion of the Schwartz space $\mathscr{S}$ with respect to the norm $\|\langle.\rangle^s \mathscr{F}\psi\|_{L^2}$ for $\psi \in \mathscr{S}$. This also shows us that the spaces H^s for $s \in \mathbb{R}$ are Hilbert spaces. The inner product is simply given by

$$(\psi \mid \zeta)_{H^s} = (\langle.\rangle^s \mathscr{F}\psi \mid \langle.\rangle^s \mathscr{F}\zeta)_{L^2} \qquad (\psi, \zeta \in \mathscr{S}) \tag{9}$$

3 SOBOLEV INEQUALITIES AND EMBEDDINGS

From what we have done to define our beloved Sobolev spaces, it is only natural to ask whether or not there is some relationship with the Banach spaces $(\mathscr{C}_b^k, \|.\|_{W^{k,\infty}})$ $(k \in \mathbb{N})$ with $\|f\|_{W^{k,\infty}} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^\infty}$ for $f \in \mathscr{C}_b^k$. To this end, we need to introduce a most important inequality:

Lemma 1. *Let $s > \frac{d}{2}$. Then*

$$\|\psi\|_{L^\infty} \lesssim \|\psi\|_{H^s} \tag{10}$$

for all $\psi \in \mathscr{S}$.

Proof. For fixed $\xi \in \mathbb{R}^d$ we estimate

$$|\psi(\xi)| = |\mathscr{F}^{-1}\mathscr{F}\psi(\xi)| \leq \int |\mathscr{F}\psi| \leq \underbrace{\|\langle.\rangle^{-s}\|_{L^2}}_{<\infty \text{ for } s>\frac{d}{2}} \|\langle.\rangle^s \mathscr{F}\psi\|_{L^2} \simeq \|\psi\|_{H^s} \tag{11}$$

As this estimate is independent of ξ, we clearly have $\|\psi\|_{L^\infty} \lesssim \|\psi\|_{H^s}$. □

Theorem 1. *Let $s > \frac{d}{2}$ and fix $k \in \mathbb{N}$. Then we have a continuous linear embedding*

$$H^{s+k} \hookrightarrow \mathscr{C}_b^k \tag{12}$$

which is the identity restricted to the Schwartz space $\mathscr{S}$.

Proof. First, let us try to construct an appropriate map $H^{s+k} \to \mathscr{C}_b^k$. So fix some $\psi \in H^{s+k}$. By construction there must exist a sequence of Schwartz functions $\{\psi_j\} \subset \mathscr{S}$ such that $\psi_j \to \psi$ in H^{s+k}. By Lemma 1 we estimate

$$\|D^\alpha \psi_j - D^\alpha \psi_l\|_{L^\infty} \lesssim \|D^\alpha \psi_j - D^\alpha \psi_l\|_{H^s} \lesssim \|\psi_j - \psi_l\|_{H^{s+k}} \tag{13}$$

and hence all $\{D^\alpha \psi_j\} \subset \mathscr{S} \subset \mathscr{C}_b^k$ with $|\alpha| \le k$ are Cauchy with respect to the uniform topology, that is, $\{\psi_j\}$ is Cauchy in $(\mathscr{C}_b^k, \|.\|_{W^{k,\infty}})$ and there must exist some $\widetilde{\psi} \in \mathscr{C}_b^k$ such that $\psi_j \overset{\|\cdot\|_{W^{k,\infty}}}{\longrightarrow} \widetilde{\psi}$. Of course this is independent of the approximating sequence $\{\psi_j\}$, since $\|\varphi_j - \psi_j\|_{L^\infty} \lesssim \|\varphi_j - \psi_j\|_{H^{s+k}}$ for any other approximating sequence $\{\varphi_j\}$. Thus we obtain a well defined (linear) map

$$\zeta : H^{s+k} \to \mathscr{C}_b^k \qquad \psi \mapsto \widetilde{\psi} \tag{14}$$

Continuity of ζ immediately follows by our previous estimate. Suppose now that $\zeta(\psi) = 0$ for some $\psi \in H^{s+k}$. But then of course

$$(\varphi \mid \psi)_{L^2} = \int \varphi \psi^* = \int \varphi \zeta(\psi)^* = 0 \tag{15}$$

for every $\varphi \in \mathscr{S}$. Hence, if $\{\psi_j\} \subset \mathscr{S}$ is again an approximating sequence of ψ in H^{s+k} we estimate

$$\|\psi_j - \psi_l\|_{H^{s+k}}^2 = \|\psi_j\|_{H^{s+k}}^2 + \|\psi_l\|_{H^{s+k}}^2 - 2\mathfrak{Re}\big(\langle . \rangle^{2(s+k)} \mathscr{F}\psi_j \big| \mathscr{F}\psi_l\big)_{L^2} \tag{16}$$

$$= \|\psi_j\|_{H^{s+k}}^2 + \|\psi_l\|_{H^{s+k}}^2 - 2\mathfrak{Re}\,\underbrace{\big(\mathscr{F}^{-1}[\langle . \rangle^{2(s+k)} \mathscr{F}\psi_j] \,\big|\, \psi_l\big)_{L^2}}_{\to 0 \text{ for } l \to \infty} \tag{17}$$

$$\overset{l\to\infty}{\longrightarrow} \|\psi_j\|_{H^{s+k}}^2 + \|\psi\|_{H^{s+k}}^2 \tag{18}$$

And therefore,

$$0 = \lim_j \|\psi_j - \psi\|_{H^{s+k}} = 2\|\psi\|_{H^{s+k}} \implies \psi = 0 \tag{19}$$

□

Remark 1. *Note that the integral $\int \varphi \psi^*$ in equation (15) makes perfect sense, since this is to be interpreted as the limit*

$$\int \varphi \psi^* = \lim_j \int \varphi \psi_j^* \overset{\text{dominated convergence}}{=} \int \varphi (\lim_j \psi_j^*) = \int \varphi \zeta(\psi)^* \tag{20}$$

for some approximating sequence $\{\psi_j\} \subset \mathscr{S}$ for ψ in H^{s+k}. Moreover, equation (15) in particular tells us that $\psi = 0$ in L^2 and by equation (19) we deduce that $\psi = 0$ in L^2 already implies $\psi = 0$ in H^s. Moreover, this yields a continuous embedding $H^s \hookrightarrow L^2$ for all $s \ge 0$.

The space H^s is called an inhomogenous L^2-based Sobolev space of order s. For $s > -\frac{d}{2}$ we may also define the homogenous Sobolev spaces $\dot{H}^s$ as the completion of the Schwartz space $\mathscr{S}$ with respect to the norm

$$\|\psi\|_{\dot{H}^s} = \||.|^s \mathscr{F}\psi\|_{L^2} \qquad (\psi \in \mathscr{S}) \tag{21}$$

It might be useful to note:

Lemma 2. *We have*

$$\|\psi\|_{H^s} \simeq \|\psi\|_{L^2} + \|\psi\|_{\dot{H}^s} \tag{22}$$

$$\|\psi\|_{H^{s+1}} \simeq \|\psi\|_{L^2} + \sum_i \|\partial_i\psi\|_{\dot{H}^s} \tag{23}$$

for all $\psi \in \mathscr{S}$.

Proof. We have

$$\|\psi\|_{H^s}^2 \simeq \|(1+|.|^s)\mathscr{F}\psi\|_{L^2}^2 = \int |\mathscr{F}\psi|^2 + \int |.|^{2s}|\mathscr{F}\psi|^2 = \|\psi\|_{L^2}^2 + \|\psi\|_{\dot{H}^s}^2 \tag{24}$$

$$\simeq (\|\psi\|_{L^2} + \|\psi\|_{\dot{H}^s})^2 \tag{25}$$

On the other hand

$$\|\psi\|_{H^{s+1}}^2 \simeq \|(1+|.|^{s+1})\mathscr{F}\psi\|_{L^2}^2 \simeq \int \left[(1+\sum_i |\xi_i|^{s+1})|\mathscr{F}\psi|\right]^2 \tag{26}$$

$$\simeq \int \left[|\mathscr{F}\psi| + \sum_i |\xi_i|^s |\mathscr{F}(\partial_i\psi)|\right]^2 \lesssim \int |\mathscr{F}\psi|^2 + \sum_i \int |.|^{2s}|\mathscr{F}(\partial_i\psi)|^2 \tag{27}$$

$$\simeq (\|\psi\|_{L^2} + \sum_i \|\partial_i\psi\|_{\dot{H}^s})^2 \tag{28}$$

and

$$\|\psi\|_{L^2}^2 + \sum_i \|\partial_i\psi\|_{\dot{H}^s}^2 \simeq \int |\mathscr{F}\psi|^2 + \sum_i \int \underbrace{|.|^{2s}|\xi_i|^2}_{\lesssim |.|^{2(s+1)}} |\mathscr{F}\psi|^2 \tag{29}$$

$$\lesssim \int (1+|.|^{s+1})^2 |\mathscr{F}\psi|^2 \simeq \|\psi\|_{H^{s+1}}^2 \tag{30}$$

□

4 ON THE TOPIC OF DIFFERENTIATION

Fix some $\psi \in H^s$ for $s \geq 1$ and suppose that we have weak derivatives $\partial_i\psi$ for $1 \leq i \leq d$. Note that these derivatives exist, since for an approximating sequence $\{\psi_j\}$ of ψ in H^s we have that the sequences $\{\partial_i\psi_j\}_j \subset \mathscr{S}$ are Cauchy in L^2 for all $1 \leq i \leq d$ and thus converge to some $\partial_i\psi \in L^2$. Of course, by construction, all $\partial_i\psi$ are in H^{s-1}, since

$$\|\partial_i\psi_j - \partial_i\psi_l\|_{H^{s-1}} \lesssim \|\psi_j - \psi_l\|_{H^s} \tag{31}$$

So what if we supposed that there were an approximating sequence $\{\psi_j\} \subset \mathscr{S}$ for ψ such that we even had that all $\{\partial_i \psi_j\} \subset \mathscr{S}$ were Cauchy in H^s, that is, if we assumed that all weak derivatives of ψ were also in H^s? Naturally, we would guess (or hope) that, just as in the classical case of the definition of derivatives, this would imply $\psi \in H^{s+1}$. However, in order to show this a bit of machinery is required.

Lemma 3. *Let $\varphi \in \mathscr{S}$ be non-negative such that $\int \varphi = 1$ and set $\varphi_k := k^d \varphi(k.)$. If $(\psi_l) \subset \mathscr{S}$ is Cauchy in H^s for some $s \geq 0$, then the function*

$$\widetilde{\psi}_k := \lim_l (\varphi_k \star \psi_l) \qquad \textit{(limit is to be interpreted pointwise)} \tag{32}$$

is in $\mathscr{C}_b^\infty$ and satisfies

$$\|\psi_l - \widetilde{\psi}_l\|_{H^s} \to 0 \tag{33}$$

We will refrain from proving this in order to save both space and time. However, proving that the functions $\widetilde{\psi}_k$ are well defined and that they are in $\mathscr{C}_b^\infty$ is actually quite easy. Indeed, for all $\alpha \in \mathbb{N}^d$ we may estimate

$$\|D^\alpha(\varphi_k \star \psi_l) - D^\alpha(\varphi_k \star \psi_j)\|_{L^\infty} = \|D^\alpha \varphi_k \star (\psi_l - \psi_j)\|_{L^\infty} \tag{34}$$

$$\leq \|D^\alpha \varphi_k\|_{L^2} \|\psi_l - \psi_j\|_{L^2} \lesssim \|\psi_l - \psi_j\|_{H^s} \tag{35}$$

which shows that $\{\varphi_k \star \psi_l\}_l \subset \mathscr{S}$ is Cauchy in $W^{m,\infty}$ for all $m \in \mathbb{N}$. Thus by completeness $\widetilde{\psi}_k \in \mathscr{C}_b^\infty$.

Remark 2. *If $\{\psi_j\} \subset \mathscr{S}$ is Cauchy in H^s as above, then we note that*

$$\partial_i \widetilde{\psi}_k = \lim_l (\varphi_k \star \partial_i \psi_l) \tag{36}$$

This is a tremendously nice property which we will exploit fully now. First of all observe that if $\{\psi_j\}, \{\zeta_j\} \subset \mathscr{S}$ are two approximating sequences for $\psi \in H^s$, then $\widetilde{\psi}_k = \widetilde{\zeta}_k$ (so we have independence of the chosen approximating sequence). Indeed,

$$\|\varphi_k \star \psi_l - \varphi_k \star \zeta_l\|_{L^2} = \|\varphi_k \star (\psi_l - \zeta_l)\|_{L^2} \leq \|\varphi_k\|_{L^1} \|\psi_l - \zeta_l\|_{L^2} \to 0 \tag{37}$$

and therefore

$$0 = \lim_l \|\varphi_k \star \psi_l - \varphi_k \star \zeta_l\|_{L^2} = \|\widetilde{\psi}_k - \widetilde{\zeta}_k\|_{L^2} \tag{38}$$

which shows $\widetilde{\psi}_k = \widetilde{\zeta}_k$ almost everywhere, and as both functions are continuous equality holds.

Lemma 4. *Suppose we are given a sequence $(\psi_j) \subset \mathscr{C}^\infty$ which is Cauchy in H^s. Set $\widehat{\psi}_j := \chi(\frac{\cdot}{j})\psi_j$, where $\chi \in \mathscr{C}_c^\infty$ with $\chi = 1$ on $\{|x| \leq 1\}$ and $\chi = 0$ on $\{|x| \geq 2\}$. Then $\{\widehat{\psi}_j\} \subset \mathscr{S}$ is Cauchy in H^s. In particular,*

$$\|\psi_j - \widehat{\psi}_j\|_{H^s} \to 0 \tag{39}$$

Again we will not prove this result, as this is solely technical. We can finally verify what we could have sworn to be true anyways.

Theorem 2. *Let $\psi \in H^s$ with $s \geq 1$ and suppose that all $\partial_i \psi \in H^{s-1}$, $1 \leq i \leq d$, are actually in H^s. Then $\psi \in H^{s+1}$.*

Proof. Let $\{\psi_j\} \subset \mathscr{S}$ be an approximating sequence for $\psi \in H^s$. By assumption we know that the weak derivatives $\partial_i \psi$ admit approximating sequences $\{\psi_j^{\partial_i}\} \subset \mathscr{S}$ with $\|\psi_j^{\partial_i} - \partial_i \psi\|_{H^{s-1}} \to 0$ and $\|\psi_j^{\partial_i} - \partial_i \psi\|_{H^s} \to 0$ for every $1 \leq i \leq d$. In particular,

$$\|\psi_j^{\partial_i} - \partial_i \psi_j\|_{H^{s-1}} \to 0 \tag{40}$$

for every $1 \leq i \leq d$. This means that both $\{\partial_i \psi_j\}$ and $\{\psi_j^{\partial_i}\}$ represent (or converge) to the same element in H^{s-1} and hence by independence of the approximating sequence we infer

$$\partial_i \widetilde{\psi}_k = \lim_j (\varphi_k \star \partial_i \psi_j) = \lim_l (\varphi_k \star \psi_j^{\partial_i}) = \widetilde{\psi^{\partial_i}}_k \tag{41}$$

Hence by lemma 3, $\{\widetilde{\psi^{\partial_i}}_k\} \subset \mathscr{C}_b^\infty$ is Cauchy in H^s and satisfies $\|\psi_l^{\partial_i} - \widetilde{\psi_i^{\partial}}_l\|_{H^s} \to 0$. However, this means

$$\|\widetilde{\psi}_k - \widetilde{\psi}_l\|_{H^{s+1}} \simeq \|\widetilde{\psi}_k - \widetilde{\psi}_l\|_{L^2} + \sum_i \|\partial_i \widetilde{\psi}_k - \partial_i \widetilde{\psi}_j\|_{\dot{H}^s} \tag{42}$$

$$= \|\widetilde{\psi}_k - \widetilde{\psi}_l\|_{L^2} + \sum_i \|\widetilde{\psi^{\partial_i}}_k - \widetilde{\psi^{\partial_i}}_l\|_{\dot{H}^s} \longrightarrow 0 \tag{43}$$

That is, $\{\widetilde{\psi}_j\} \subset \mathscr{C}_b^\infty$ is Cauchy in H^{s+1}. If we now set $\widehat{\psi}_l := \widetilde{\psi}_l \chi_l$ as in lemma 4, then $\{\widehat{\psi}_l\} \subset \mathscr{S}$ is an approximating sequence for ψ in H^s which is also Cauchy in H^{s+1}. Thus $\psi \in H^{s+1}$. □

5 ON THE MATTER OF DUALITY

Let us concern ourselves with what is dual to H^s for $s \in \mathbb{R}$. In the following, $(H^s)'$ will denote the set of continuous linear functionals on H^s, that is,

$$(H^s)' = \{\psi' \mid \psi' : H^s \to \mathbb{C}, \|\psi'\| < \infty\} \tag{44}$$

We define the bilinear form

$$\gamma : \mathscr{S} \times \mathscr{S} \to \mathbb{C} \qquad (\psi, \varphi) \mapsto \int \mathscr{F}\psi(\xi) \mathscr{F}\varphi(-\xi) d\xi \tag{45}$$

and we immediately note that

$$|\gamma(\psi, \varphi)| = \left| \int \langle . \rangle^{-s} \mathscr{F}\psi \langle . \rangle^s \mathscr{F}\varphi \right| \leq \|\psi\|_{H^{-s}} \|\varphi\|_{H^s} \tag{46}$$

for all $\psi, \varphi \in \mathscr{S}$. Thus we may extend γ to a continuous bilinear form $\gamma : H^{-s} \times H^s \to \mathbb{C}$.

Theorem 3. *The continuous bilinear form γ induces an isometric isomorphism*

$$H^{-s} \to (H^s)' \tag{47}$$

Proof. By the previous estimate we know that for $\psi \in H^{-s}$ the linear functional

$$\gamma_\psi : H^s \to \mathbb{C} \qquad \varphi \mapsto \gamma(\psi, \varphi) \tag{48}$$

is continuous with $\|\gamma_\psi\| \le \|\psi\|_{H^{-s}}$. Now let $\{\psi_k\} \subset \mathscr{S}$ be an approximating sequence for $\psi \in H^{-s}$ and define

$$\varphi_j^0 := \mathscr{F}^{-1}\left[\langle . \rangle^{-2s} \mathscr{F}\psi_j(-\xi)\right] \in \mathscr{S} \tag{49}$$

Then the sequence $\{\varphi_j^0\}$ is Cauchy in H^s, since $\langle . \rangle^s \mathscr{F}\varphi_j^0 = \langle . \rangle^{-s}\mathscr{F}\psi_j$ and thus converges to some $\varphi^0 \in H^s$. But then

$$\gamma(\psi, \varphi^0) = \lim_j \int |\mathscr{F}\psi_j|^2 \langle . \rangle^{-2s} = \|\psi\|^2_{H^{-s}} \tag{50}$$

On the other hand, we have

$$\|\varphi^0\|_{H^s} = \left(\int \langle . \rangle^{-2s} |\mathscr{F}\psi|^2 \right)^{1/2} = \|\psi\|_{H^{-s}} \tag{51}$$

In total, $\|\gamma_\psi\| = \|\psi\|_{H^{-s}}$ which shows that $\psi \mapsto \gamma_\psi$ is an isometry. All that is left to show it surjectivitiy. So let $\psi' \in (H^s)'$, then by the Theorem of Fréchet-Riesz there exists some $\varphi \in H^s$ such that $\psi' = (. \mid \varphi)_{H^s}$. If we then pick an approximating sequence $\{\varphi_j\} \subset \mathscr{S}$ for φ in H^s, then define

$$\psi_j := \mathscr{F}^{-1}\left[\{\mathscr{F}(\varphi_j)(-\xi)\}^* \langle . \rangle^{2s}\right] \in \mathscr{S} \tag{52}$$

Just as before, $\{\psi_j\}$ is Cauchy in H^{-s} and thus converges to some $\psi \in H^{-s}$. By construction this ψ satisfies $\psi' = \gamma_\psi$. This completes the proof. □

REFERENCES

[1] Roland Donninger. *Advanced partial differential equations*. Lecture notes, 2021.

[2] Michael Kunzinger. *Theory of distributions*. Lecture notes, 2019.