PDE'S - EXERCISES 32-35

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In all these exercises we use the abbreviations $H^s := H^s(\mathbb{R}^d)$, $\dot{H}^s := \dot{H}^s(\mathbb{R}^d)$, $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$, $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{C}_b := \mathscr{C}_b(\mathbb{R}^d)$, $\mathscr{C}_b^k := \mathscr{C}_b^k(\mathbb{R}^d)$ and $\mathscr{C}_c^\infty := \mathscr{C}_c^\infty(\mathbb{R}^d)$. Moreover, for $r > 0$ we define $B_r := \{x \in \mathbb{R}^d \mid |x| < r\}$ and $S^{d-1} := \{x \in \mathbb{R}^d \mid |x| = 1\}$.

1 EXERCISE 32:

Let $p \in [1,\infty)$ and suppose we are given a sequence $(f_l) \subset \mathscr{S}$ which converges uniformly to $f \in \mathscr{C}$ and which is Cauchy in L^p. We are asked to show that $\|f\|_p \lesssim 1$ and $\|f - f_l\|_p \to 0$. As $(f_l) \subset \mathscr{S}$ is Cauchy in L^p this sequence defines an element $F \in L^p$. Applying Fatou's lemma we obtain

$$\int |f|^p = \int \liminf |f_l|^p \underset{\text{Fatou}}{\leq} \liminf \int |f_l|^p = \|F\|_p < \infty \tag{1}$$

Let $\varepsilon > 0$. As (f_l) is both Cauchy in L^p and converges uniformly to $f \in \mathscr{C}$, we know that there exists a natural number $K \in \mathbb{N}$ such that for all $k, l \geq K$

$$\|f_k - f_l\|_{L^p(\mathbb{R}^d)} < \varepsilon \qquad \|f_k - f\|_\infty < \varepsilon \tag{2}$$

However, as $|f - f_K|^p$ is integrable there also has to exist a positive number $R > 0$ such that

$$\|f_K - f\|_{L^p(\mathbb{R}^d \setminus B_R)} < \varepsilon \tag{3}$$

This yields that for all $k \geq K$ we have

$$\begin{aligned} \|f_k - f\|_{L^p(\mathbb{R}^d)} &\leq \|f_k - f_K\|_{L^p(\mathbb{R}^d)} + \|f_K - f\|_{L^p(\mathbb{R}^d)} < \varepsilon + \|f_K - f\|_{L^p(B_R)} + \|f_K - f\|_{L^p(\mathbb{R}^d \setminus B_R)} \\ &< 2\varepsilon + \|f_K - f_l\|_{L^p(B_R)} + \|f_l - f\|_{L^p(B_R)} < 3\varepsilon + \|f_l - f\|_{L^p(B_R)} \end{aligned} \tag{4}$$

As $\|f_l - f\|_{L^p(B_R)} \to 0$ by uniform convergence (since we only integrate over B_R) we infer that our final inequality (4) turns into (for $l \to \infty$)

$$\forall k \geq K\colon \|f_k - f\|_{L^p(\mathbb{R}^d)} < 3\varepsilon \tag{5}$$

This proves the statement.

2 EXERCISE 33:

We are given the function

$$h_k(x) := \int f(y)\left[g(x - k^{-1}y) - g(x)\right] dy \tag{6}$$

and we are asked to prove that for all natural numbers $k, M \in \mathbb{N}$ (where $k \geq 1$) we have $|h_k| \lesssim k^{-1}\langle . \rangle^{-M}$. Note that we have the equality

$$g(x - k^{-1}y) - g(x) = \int_0^1 \partial_t g(x - tk^{-1}y)\, dt = k^{-1} y^j \int_0^1 \partial_j g(x - tk^{-1}y)\, dt \tag{7}$$

Now we introduce:

Lemma 1 (Stephie's Lemma)**.** *If $M \in \mathbb{N}$ then*

$$\langle x\rangle \leq \langle x-y\rangle\langle y\rangle \tag{8}$$

Proof.

$$\langle x\rangle^2 = 1+|x|^2 \leq 1+(|x-y|+|y|)^2 \simeq 1+|x-y|^2+|y|^2 \tag{9}$$
$$\leq 1+|x-y|^2+|y|^2+|x-y|^2|y|^2 = \langle x-y\rangle^2\langle y\rangle^2 \tag{10}$$

□

Using the Schwarz properties of f,g and both Stephie's 1 lemma and equation (7) we estimate

$$\langle x\rangle^M \left|f(y)\left[g(x-k^{-1}y)-g(x)\right]\right| \underset{\text{Schwarz property of } f}{\lesssim} k^{-1}\langle y\rangle^{-L}\int_0^1 \langle x\rangle^M \left|\partial_j g(x-tk^{-1}y)\right| dt \tag{11}$$
$$\underset{\text{Stephan's Lemma}}{\lesssim} k^{-1}\langle y\rangle^{-L}\int_0^1 \langle x-tk^{-1}y\rangle^M\langle tk^{-1}y\rangle^M \left|\partial_j g(x-tk^{-1}y)\right| dt \tag{12}$$
$$\underset{\text{Schwarz property of } g}{\lesssim} k^{-1}\langle y\rangle^{-L}\int_0^1 \underbrace{\langle tk^{-1}y\rangle^M}_{\leq\langle y\rangle^M} dt \leq k^{-1}\langle y\rangle^{M-L} \tag{13}$$

and therefore, by choosing L large enough, we obtain

$$\left|h_k(x)\langle x\rangle^M\right| \leq k^{-1}\int \langle y\rangle^{M-L}\,dy \lesssim k^{-1} \tag{14}$$

for all $x \in \mathbb{R}^d$.

3 EXERCISE 34:

Fix $p \in [1,\infty)$. We are given a sequence $(f_l) \subset \mathscr{C}^\infty$ which is Cauchy with respect to the L^p-norm. Furthermore, suppose we have a smooth cutoff function $\chi \in \mathscr{C}_c^\infty$ with $0 \leq \chi \leq 1$ and $\chi\restriction_{\overline{B_1}} \equiv 1, \chi\restriction_{B_2^c} \equiv 0$. Set $\widehat{f_k}(x) := \chi(x/k)f(x)$ for all $k \in \mathbb{N}$. We shall now prove that $\widehat{f_k} \in \mathscr{S}$ and $\|\widehat{f_k}-f_k\|_{L^p(\mathbb{R}^d)} \to 0$: For every $k \geq 1$ we certainly have that the functions $\chi_k\colon (x \mapsto \chi(x/k))$ have their support in $\overline{B_{2k}}$. This immediately implies $\widehat{f_k} \in \mathscr{C}_c^\infty \subset \mathscr{S}$. We now introduce another gem of newfound mathematical knowledge:

Lemma 2 (Dave's Lemma)**.** *For a sequence (f_l) that is Cauchy in $L^p(\mathbb{R}^d)$ we have that for all $\varepsilon > 0$ there exists an $R > 0$ and $K \in \mathbb{N}$ such that for all $k \geq K$ we have*

$$\|f_k\|_{L^p(\mathbb{R}^d\setminus B_R)} < \varepsilon \tag{15}$$

Proof. For given $\varepsilon > 0$ we have that by the Cauchy property there has to exist a natural number $K \in \mathbb{N}$ such that $\|f_k - f_l\|_{L^p(\mathbb{R}^d)} < \varepsilon$ for all $k,l \geq K$. Now as $f_K \in L^p$ there exists $R > 0$ such that $\|f_K\|_{L^p(\mathbb{R}^d\setminus B_R)} < \varepsilon$. This yields

$$\|f_k\|_{L^p(\mathbb{R}^d\setminus B_R)} \leq \|f_K\|_{L^p(\mathbb{R}^d\setminus B_R} + \|f_k - f_K\|_{L^p(\mathbb{R}^d\setminus B_R)} < 2\varepsilon \tag{16}$$

for all $k \geq K$, as wanted. □

Thus, for $\varepsilon > 0$, we may apply Dave's Lemma 2 to our sequence to obtain $R > 0$ and $K \in \mathbb{N}$ such that for all $k,l \geq K$ we have

$$\|f_k\|_{L^p(\mathbb{R}^d\setminus B_R)} < \varepsilon \qquad \|f_k - f_l\|_{L^p(\mathbb{R}^d)} < \varepsilon \tag{17}$$

Using this we estimate that for all $l \geq k \geq N := \max(K, \lceil R \rceil)$ we have

$$\begin{aligned}
\|\widehat{f_k} - \widehat{f_l}\|_{L^p(\mathbb{R}^d)} &= \|\chi_k f_k - \chi_l f_l\|_{L^p(\mathbb{R}^d)} \leq \|\chi_k (f_k - f_l)\|_{L^p(\mathbb{R}^d)} + \|(\chi_k - \chi_l) f_l\|_{L^p(\mathbb{R}^d)} && (18)\\
&\leq \|f_k - f_l\|_{L^p(\mathbb{R}^d)} + \|f_k - f_l\|_{L^p(B_k)} + \|(\chi_k - \chi_l) f_l\|_{L^p(\mathbb{R}^d \setminus B_k)} && (19)\\
&\leq 2\varepsilon + \|\underbrace{(\chi_k - \chi_l) f_l}_{\leq |f_l|}\|_{L^p(\mathbb{R}^d \setminus B_R)} \leq 2\varepsilon + \|f_l\|_{L^p(\mathbb{R}^d \setminus B_R)} < 3\varepsilon && (20)
\end{aligned}$$

By using the inequality (20) (and some arbitrary $l \in \mathbb{N}$ large enough) and equation (17) we again estimate for all $k \geq N$ that

$$\begin{aligned}
\|\widehat{f_k} - f_k\|_{L^p(\mathbb{R}^d)} &\leq \|\widehat{f_k} - \widehat{f_l}\|_{L^p(\mathbb{R}^d)} + \|\widehat{f_l} - f_K\|_{L^p(\mathbb{R}^d)} + \|f_K - f_k\|_{L^p(\mathbb{R}^d)} && (21)\\
&< 4\varepsilon + \|\widehat{f_l} - f_K\|_{L^p(\mathbb{R}^d)} \leq 4\varepsilon + \|f_l - f_K\|_{L^p(B_R)} + \|\widehat{f_l} - f_K\|_{L^p(\mathbb{R}^d \setminus B_R)} && (22)\\
&\leq 5\varepsilon + \|f_l\|_{L^p(\mathbb{R}^d \setminus B_R)} + \|f_K\|_{L^p(\mathbb{R}^d \setminus B_R)} < 7\varepsilon && (23)
\end{aligned}$$

which concludes the argument.

4 EXERCISE 35:

We are given a sequence $(f_l) \subset \mathscr{C}^\infty$ which is Cauchy with respect to the H^1 norm. As in exercise 34 we set $\widehat{f_k} := \chi_k f_k$. We shall now prove that $(\widehat{f_k}) \subset \mathscr{C}^\infty$ is Cauchy in H^1. Note that exercise 9 also holds for $f \in \mathscr{C}^\infty \cap H^k$ in the measure theoretic framework, that is, we know that in our case for $k = 1$ we have

$$\|f\|_{H^1} \simeq \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_{L^2} \tag{24}$$

(the proof is the same as for exercise 9 with the difference being that the Fourier transform is an automorphism on the measure theoretic L^2-space). Before using this we collect some helpful side facts. First note that

$$\partial_j \widehat{f_k} = \frac{1}{k} \partial_j \chi_k f_k + \chi_k \partial_j f_k = \frac{1}{k} \partial_j \chi_k f_k + \widehat{\partial_j f_k} \tag{25}$$

Moreover, as $(f_k) \subset \mathscr{C}^\infty$ is Cauchy in H^1 we have by the equivalence of the norms in (24) that $(f_k) \subset \mathscr{C}^\infty$ and all $(\partial_j f_k) \subset \mathscr{C}^\infty$ for $1 \leq j \leq d$ are Cauchy in L^2. Hence by exercise 34 we have that $(\widehat{f_k}) \subset \mathscr{S}$ and all $(\widehat{\partial_j f_k}) \subset \mathscr{S}$ for $1 \leq j \leq d$ are Cauchy in L^2. By considering

$$\|\widehat{f_k} - \widehat{f_l}\|_{H^1} \simeq \|\widehat{f_k} - \widehat{f_l}\|_{L^2} + \sum_{j=0}^{d} \|\partial_j \widehat{f_k} - \partial_j \widehat{f_l}\|_{L^2} \tag{26}$$

we infer that it is thus sufficient to show that $\|\partial_j \widehat{f_k} - \partial_j \widehat{f_l}\|_{L^2} \to 0$ for every $1 \leq j \leq d$, since it was already noted that $\|\widehat{f_k} - \widehat{f_l}\|_{L^2}$ tends to 0. Unpacking this expression leads to

$$\begin{aligned}
\|\partial_j \widehat{f_k} - \partial_j \widehat{f_l}\|_{L^2} &= \left\|\frac{1}{k} \partial_j \chi_k f_k - \frac{1}{l} \partial_j \chi_l f_l + \widehat{\partial_j f_k} - \widehat{\partial_j f_l}\right\|_{L^2} && (27)\\
&\leq \left\|\frac{1}{k} \partial_j \chi_k f_k - \frac{1}{l} \partial_j \chi_l f_l\right\|_{L^2} + \|\widehat{\partial_j f_k} - \widehat{\partial_j f_l}\|_{L^2} && (28)
\end{aligned}$$

Again we have already noted that the second summand in the final inequality (28) tends to 0 and therefore it is only left to show that the second summand also tends to 0. In order to do that we first define

$$M := \max_{j=1}^{d} \max_{x \in \operatorname{supp}(\chi)} |\partial_j \chi(x)| < \infty \tag{29}$$

We finish the proof by estimating

$$\begin{aligned}
\left\|\frac{1}{k} \partial_j \chi_k f_k - \frac{1}{l} \partial_j \chi_l f_l\right\|_{L^2} &\leq \left\|\frac{1}{k} \partial_j \chi_k f_k - \frac{1}{k} \partial_j \chi_l f_l\right\|_{L^2} + \left|\frac{1}{k} - \frac{1}{l}\right| \|\partial_j \chi_l f_l\|_{L^2} && (30)\\
&\leq \frac{M}{k} (\|f_k\|_{L^2} + \|f_l\|_{L^2}) + M \left|\frac{1}{k} - \frac{1}{l}\right| \|f_l\|_{L^2} \to 0 && (31)
\end{aligned}$$