$\underline{\underline{\S 22}}$

1(5) $\quad \sin^8 x + \cos^8 x = \left(\frac{1-\cos 2x}{2}\right)^4 + \left(\frac{1+\cos 2x}{2}\right)^4 =$

$= \frac{1}{2^4}\left(1 - 2\cos 2x + \cos^2 2x\right)^2 + \frac{1}{2^4}\left(1 + 2\cos 2x + \cos^2 2x\right)^2 =$

$= \frac{1}{2^4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)^2 + \frac{1}{2^4}\left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)^2 =$

$= \frac{1}{2^4}\left(\frac{9}{4} - \cancel{6\cos 2x} + 4\cos^2 2x - \cancel{2\cos 2x \cos 4x} +\right.$

$\left. + \frac{\cos^2 4x}{4} + \frac{3}{2}\cos 4x\right) + \frac{1}{2^4}\left(\frac{9}{4} + \cancel{6\cos 2x} +\right.$

$\left. + 4\cos^2 2x + \cancel{2\cos 2x \cos 4x} + \frac{\cos^2 4x}{4} + \frac{3}{2}\cos 4x\right) =$

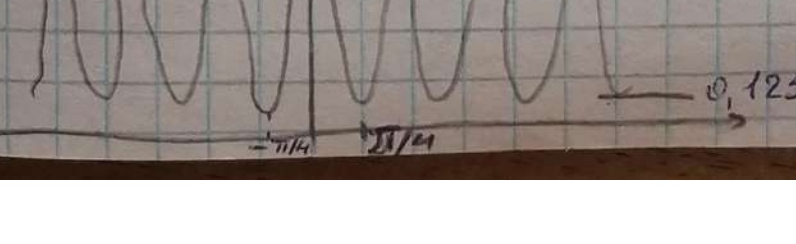
$= \frac{9}{4}\cdot\frac{1}{2^3} + \frac{\cos^2 2x}{2} + \frac{\cos^2 4x}{32} + \frac{3\cos 4x}{16} =$

$= \frac{9}{32} + \frac{1}{2}\,\frac{1+\cos 4x}{2} + \frac{1}{32}\,\frac{1+\cos 8x}{2} + \frac{3\cos 4x}{16} =$

$= \frac{9}{32} + \frac{1}{4} + \frac{7}{16}\cos 4x + \frac{1}{64} + \frac{\cos 8x}{64} =$

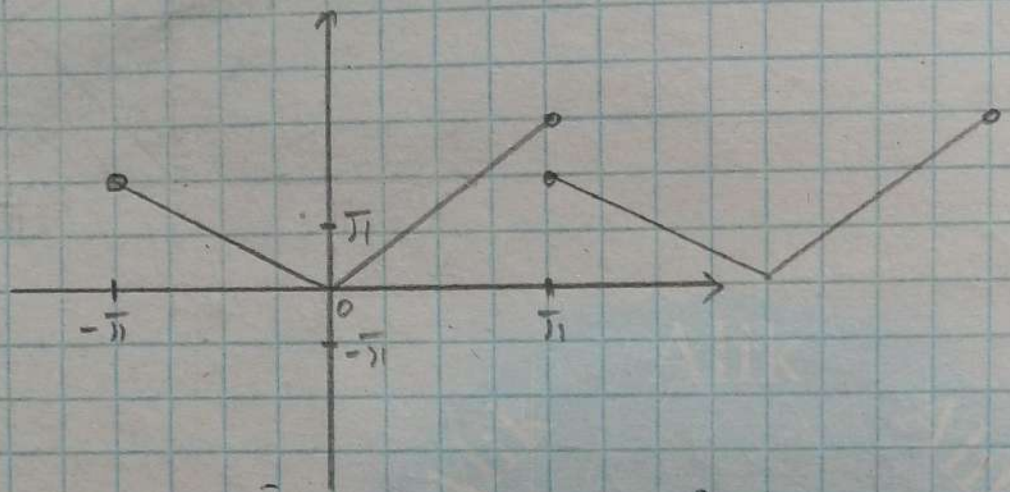
$= \frac{35}{64} + \frac{7}{16}\cos 4x + \frac{\cos 8x}{64}$ //

Т. к. ряд из конечного числа членов, то он сх-ся равномерно.



§22 11)

$$f(x)=\begin{cases}-2x, & -\pi < x \le 0 \\ 3x, & 0 < x \le \pi\end{cases} \qquad x_0=\pi$$



$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{0}(-2x)\,dx+\frac{1}{\pi}\int_{0}^{\pi}3x\,dx=$$

$$=-\frac{1}{\pi}\,x^2\Big|_{-\pi}^{0}+\frac{1}{\pi}\,\frac{3x^2}{2}\Big|_{0}^{\pi}=\frac{3}{2}\pi+\pi=\frac{\pi}{2}\cdot 5 \longrightarrow \frac{a_0}{2}=\frac{\pi}{4}\cdot 5$$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{0}(-2x)\cos kx\,dx+\frac{1}{\pi}\int_{0}^{\pi}3x\cos kx\,dx=$$

$$=\frac{1}{\pi}\left[-\int_{-\pi}^{0}2x\,d\frac{\sin kx}{k}+\int_{0}^{\pi}3x\,d\frac{\sin kx}{k}\right]=$$

$$=\frac{1}{\pi}\left[-2x\,\frac{\sin kx}{k}\Big|_{-\pi}^{0}+\int_{-\pi}^{0}\frac{\sin kx}{k}2\,dx+3x\frac{\sin kx}{k}\Big|_{0}^{\pi}-\right.$$

$$\left.-\int_{0}^{\pi}\frac{\sin kx}{k}3\,dx\right]=\frac{1}{\pi}\left(0-\frac{2}{k}\,\frac{\cos kx}{k}\Big|_{-\pi}^{0}+0+\right.$$

$$\left.+\frac{3}{k^2}\cos kx\Big|_{0}^{\pi}\right)=\frac{1}{\pi k^2}\left(-2(1-(-1)^k)+3((-1)^k-1)\right)=$$

$$=\frac{1}{\pi k^2}5\left((-1)^k-1\right)$$

$$a_{2l} = 0 \qquad a_{2l+1} = -\frac{10}{\pi(2l+1)^2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{0}(-2x)\sin kx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 3x \sin kx\,dx =$$

$$= \frac{1}{\pi}\Big[\, 2x\frac{\cos kx}{k}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\cos kx}{k}2\,dx - 3x\frac{\cos kx}{k}\Big|_{0}^{\pi} +$$

$$+ \int_{0}^{\pi} 3\frac{\cos kx}{k}\,dx\Big] = \frac{1}{\pi}\left(0 + 2\pi\frac{(-1)^{k+1}}{k} - 0 + 3\pi\frac{(-1)^{k+1}}{k} + 0 + 0\right) =$$

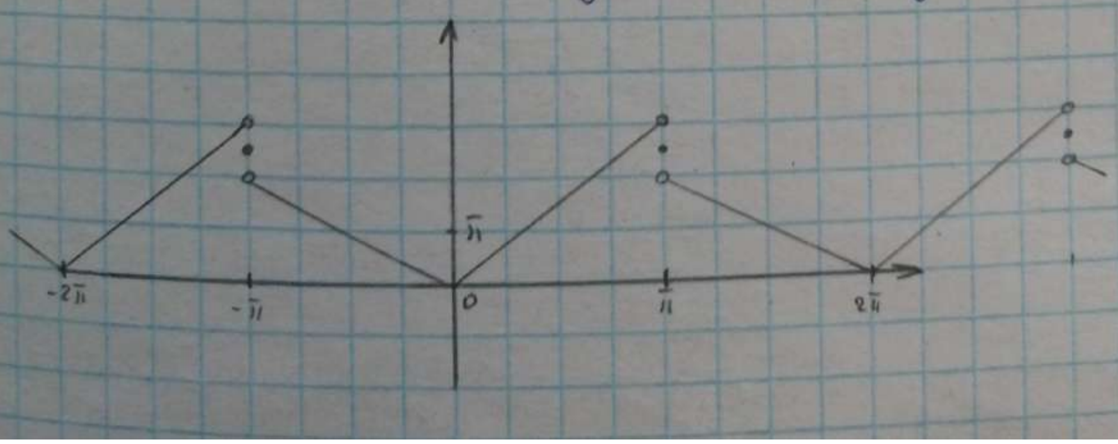
$$= \frac{(-1)^{k+1}}{k}5$$

$$b_{2l} = -\frac{5}{2l} \qquad b_{2l+1} = \frac{5}{2l+1}$$

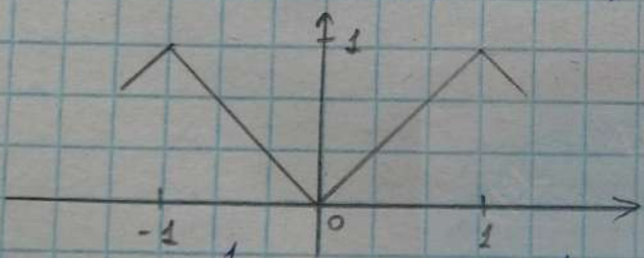
$$f(x) \sim \frac{\pi}{4}5 + \sum_{k=1}^{\infty}\left(\frac{5((-1)^k - 1)}{\pi k^2}\cos kx + 5\frac{(-1)^{k+1}}{k}\sin kx\right)$$

$$f(x) \sim \frac{5\pi}{4} - 5\sum_{k=1}^{\infty}\left((1-(-1)^k)\frac{\cos kx}{\pi k^2} + (-1)^k\frac{\sin kx}{k}\right)$$

Т.к. есть точки разрыва, то ряд не сх-ся равномерно. График суммы ряда:



§22 14) $f(x) = |x|, \quad x \in [-1, 1]$



ф-ция чётная, значит $b_k = 0$

$$a_0 = \frac{1}{1}\int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = x^2\Big|_0^1 = 1 \qquad \frac{a_0}{2} = \frac{1}{2}$$

$$a_k = \frac{1}{1}\int_{-1}^{1} |x| \cos k\pi x\, dx = 2\int_0^1 x \cos k\pi x\, dx =$$

$$= 2x\, \frac{\sin k\pi x}{k\pi}\Big|_0^1 - 2\int_0^1 \frac{\sin(k\pi x)}{k\pi}\, dx =$$

$$= 2\,\frac{(-1)^k - 1}{k^2 \pi^2}$$

$$a_{2l} = 0 \qquad a_{2l+1} = -\frac{4}{\pi^2 (2l+1)^2}$$

$$f(x) \sim \frac{1}{2} - \frac{4}{\pi^2} \sum_{l=0}^{\infty} \frac{\cos \pi (2l+1) x}{(2l+1)^2}$$

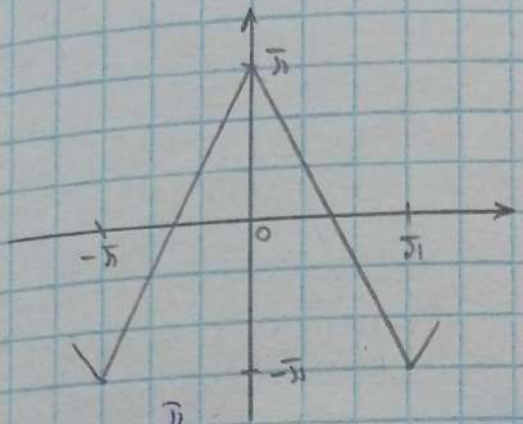
Ряд сх-ся равномерно по признаку Вейерштрасса, т.к. $\left| \frac{\cos \pi (2l+1) x}{(2l+1)^2} \right| \leq \frac{1}{(2l+1)^2}$ - сх-ся.

График суммы ряда совпадает с графиком ф-ции.

§22 30) $f(x) = \pi - 2x$ $\quad 0 < x \le \pi$

1) чётная $\quad$ 2) нечётная

①



т. к. чётная, то $b_k = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi}(\pi - 2x)\,dx = 0$$

$$a_k = \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)\cos kx\,dx = \frac{2}{\pi}\left[\int_0^{\pi}(\pi-2x)\,d\,\frac{\sin kx}{k}\right] =$$

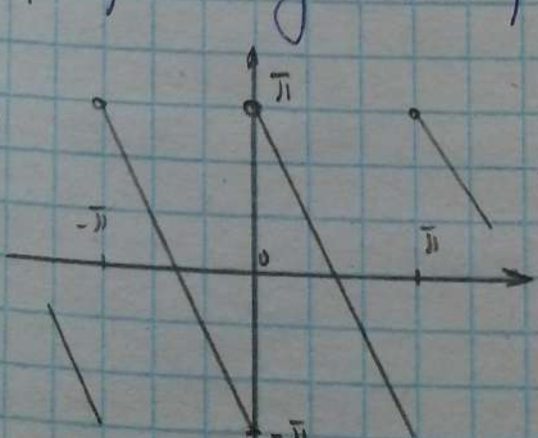
$$= \frac{2}{\pi}\left[\cancelto{0}{\frac{(\pi - 2x)\sin kx}{k}\Big|_0^{\pi}} + \frac{2}{k}\int_0^{\pi}\sin kx\,dx\right] =$$

$$= -\frac{4}{\pi k^2}\cos kx\Big|_0^{\pi} = \frac{4}{\pi k^2}\left(1 - (-1)^k\right)$$

$$f(x) \sim \frac{8}{\pi}\sum_{l=0}^{\infty}\frac{\cos(2l+1)x}{(2l+1)^2}$$ — сх-ся равномерно по призн. Вейерштрасса

График суммы ряда совп. с графиком ф-ции.

②



т. к. нечётная, то $a_k = 0$

$$b_k = \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)\sin kx\,dx = \frac{2}{\pi}\left[(\pi-2x)\frac{-\cos kx}{k}\Big|_0^{\pi} + \right.$$

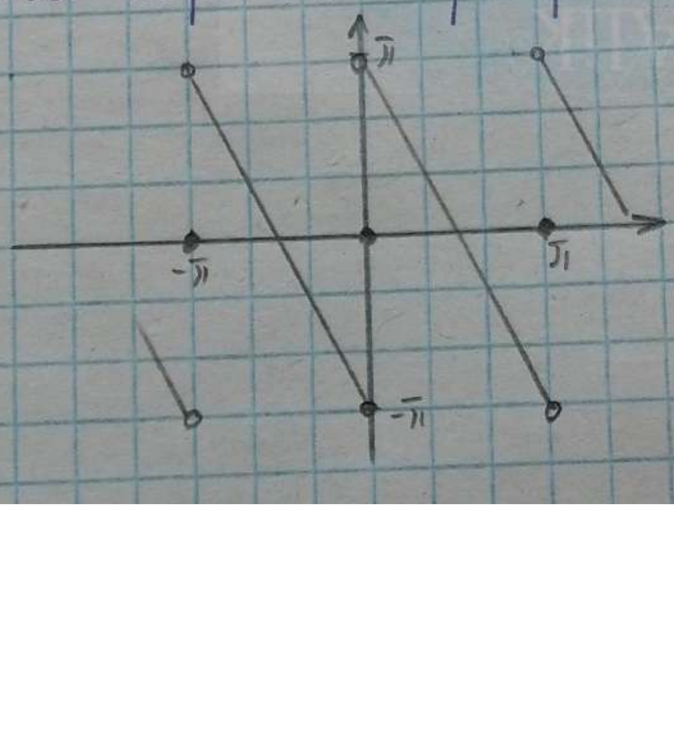
$$\left. + (-2)\cdot\int_0^{\pi}\cancel{\frac{\cos kx}{k}}\,dx\right] = \frac{2}{\pi}\left(-\pi\cdot\frac{(-1)^{k+1}}{k} + \frac{\pi}{k}\right) =$$

$$= \frac{2}{k}\left(1+(-1)^k\right)$$

$$b_{2l} = \frac{2\cdot 2}{2l} = \frac{2}{l} \qquad b_{2l+1} = 0$$

$$f(x) \sim 2\sum_{l=1}^{\infty}\frac{\sin 2lx}{l}$$

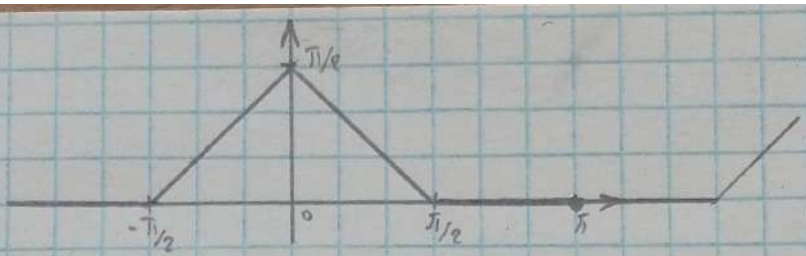
Т.к. есть точки разрыва, то ряд не сх-ся равномерно. График суммы ряда:



§22 42)

$$f(x) = \begin{cases} \pi/2 - x, & 0 < x < \pi/2 \\ 0, & \pi/2 \le x < \pi \end{cases}$$

Разложить по косинусам. Для этого продолжим её чётным образом:



Т.к. ф-ция чётная, то $b_k = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2} - x\right)dx = \frac{2}{\pi}\left(\frac{\pi}{2}x - \frac{x^2}{2}\right)\Bigg|_0^{\pi/2} =$$

$$= \frac{2}{\pi}\cdot\frac{\pi^2}{8} = \frac{\pi}{4} \qquad \frac{a_0}{2} = \frac{\pi}{8}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2} - x\right)\cos kx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{\pi}{2} - x\right)d\,\frac{\sin kx}{k}\right] =$$

$$= \frac{2}{\pi}\left[\cancel{\left(\frac{\pi}{2} - x\right)\frac{\sin kx}{k}\Bigg|_0^{\pi/2}}^{\,0} + \int_0^{\pi/2}\frac{\sin kx}{k}\,dx\right] =$$

$$= -\frac{2}{\pi}\;\frac{\cos kx}{k^2}\Bigg|_0^{\pi/2} = -\frac{2}{\pi k^2}\left(\cos k\frac{\pi}{2} - 1\right) =$$

$$= -\frac{2}{\pi k^2}\left(1 - 2\sin^2\frac{k\pi}{4} - 1\right) = \frac{4}{\pi k^2}\sin^2\frac{k\pi}{4}$$

$$f(x) \sim \frac{\pi}{8} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin^2\frac{k\pi}{4}}{k^2}\cos kx$$ – сх-ся равномерно по призн. Вейерштрасса

Т.к. точек разрыва нет, то график суммы ряда совпадает с графиком ф-ции.

§22 45) $f(x) = x^2$

1) на $[-\pi, \pi]$ по косинусам

2) на $(0, \pi)$ по синусам

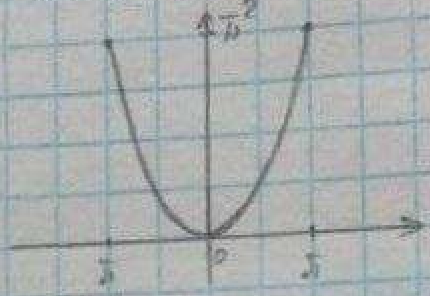
3) на $(0, 2\pi)$ по синусам и косинусам

пользуясь этими разложениями, найти

$$S_1 = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad S_2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \qquad S_3 = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

① на $[-\pi, \pi]$ по косинусам

Т.к четная, то $b_k = 0$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\,\frac{x^3}{3}\Big|_0^{\pi} = \frac{\pi^2 \cdot 2}{3}$$

$$\frac{a_0}{2} = \frac{\pi^2}{3}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} x^2 \cos kx\, dx = \frac{2}{\pi}\left(x^2 \frac{\sin kx}{k}\Big|_0^{\pi} - \int_0^{\pi} \frac{\sin kx}{k} dx^2\right) =$$

$$= \frac{2}{\pi}\left(0 - \int_0^{\pi} \frac{\sin kx}{k} 2x\, dx\right) = \frac{2}{\pi}\left(2x \frac{\cos kx}{k^2}\Big|_0^{\pi} - \right.$$

$$\left. - \int_0^{\pi} \frac{\cos kx}{k^2} 2\, dx\right) = \frac{2}{\pi}\left(2\pi \cdot \frac{\cos \pi k}{k^2} - 0\right) =$$

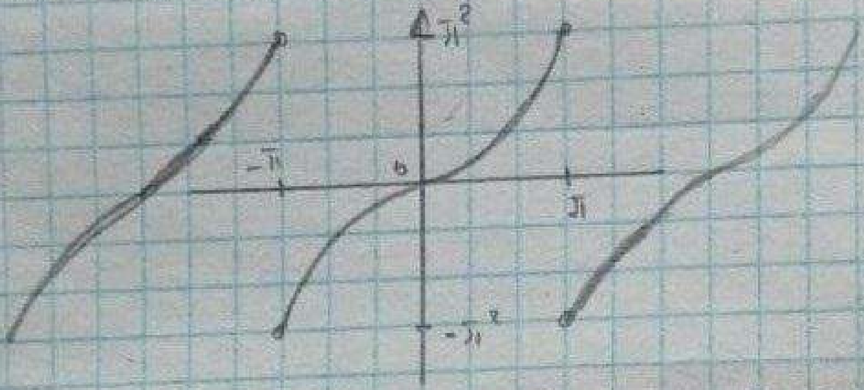
$$= 4\,\frac{(-1)^k}{k^2}$$

$$f(x) \sim \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k \cos kx}{k^2}$$ – сх-ся равномерно по призн. Вейерш. График суммы совп. с граф. $f(x)$

② на $(0, \pi)$ по синусам

Для этого ф-цию продлим так, чтобы

она была нечётной:



т.к. нечётная, то $a_k = 0$

$$b_k = \frac{2}{\pi}\int_0^{\pi} x^2 \sin kx\,dx = -\frac{2}{\pi}\int_0^{\pi} x^2\, d\frac{\cos kx}{k} = -\frac{2}{\pi}\cdot$$

$$\cdot\left(x^2\frac{\cos kx}{k}\Big|_0^{\pi} - \int_0^{\pi}\frac{\cos kx}{k}2x\,dx\right) = -\frac{2}{\pi}\left(\frac{\pi^2\cos\pi k}{k} - \right.$$

$$\left.-\left(\frac{2x}{k}\frac{\sin kx}{k}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin kx}{k^2}2\,dx\right)\right) = -\frac{2}{\pi}\left(\frac{\pi^2(-1)^k}{k} - \right.$$

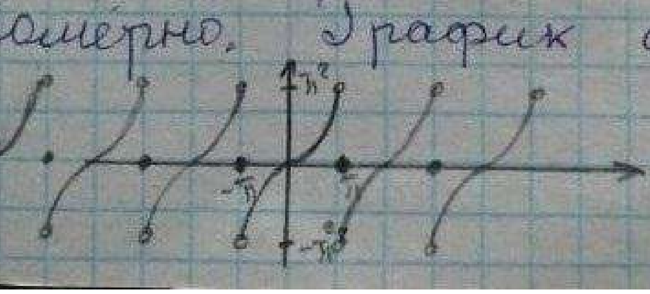
$$\left.-0 - \frac{\cos kx}{k^3}2\Big|_0^{\pi}\right) = \frac{4}{\pi}\frac{\cos kx}{k^3}\Big|_0^{\pi} - \frac{2\pi^2}{\pi}\frac{(-1)^k}{k} =$$

$$= \frac{4}{\pi}\frac{(-1)^k}{k^3} - \frac{4}{\pi}\frac{1}{k^3} - \frac{2\pi^2}{\pi}\frac{(-1)^k}{k} = \frac{2}{\pi}\left(\frac{2(-1)^k}{k^3} + \frac{2}{k^3}(-1)^k(-1)^{k+1} - \right.$$

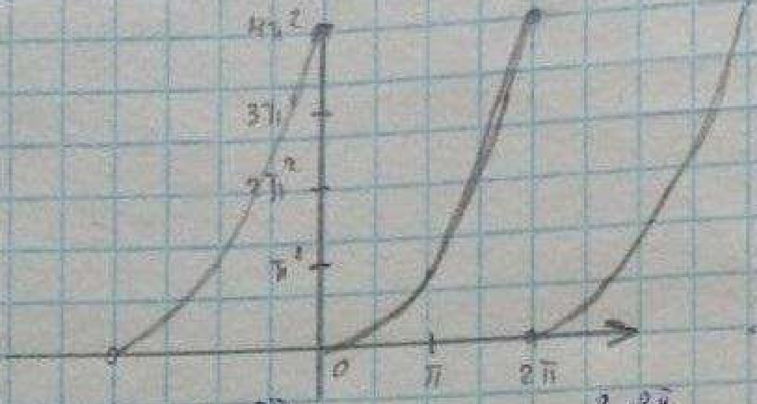
$$\left.-\frac{1\pi^2(-1)^k}{k}\right) = \frac{2}{\pi}(-1)^{k+1}\left(\frac{\pi^2}{k} - \frac{2(1-(-1)^k)}{k^3}\right)$$

$$f(x) \sim \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{\pi^2}{k} - \frac{2(1-(-1)^k)}{k^3}\right)\sin kx$$

Т.к. есть точки разрыва, то ряд не сх-ся равномерно. График суммы ряда:



③ на $(0, 2\pi)$ по синусам и косинусам



$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{1}{\pi}\frac{x^3}{3}\Big|_0^{2\pi} = \frac{8\pi^3}{3\pi} = \frac{8}{3}\pi^2 \qquad \frac{a_0}{2} = \frac{4}{3}\pi^2$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos kx\, dx = \frac{1}{\pi}\left(\cancel{\frac{x^2 \sin kx}{k}\Big|_0^{2\pi}}^{\to 0} - \int_0^{2\pi} \frac{\sin kx}{k} 2x\, dx\right) =$$

$$= \frac{1}{\pi}\left(\frac{\cos kx}{k^2} 2x \Big|_0^{2\pi} - \int_0^{2\pi} \frac{\cos kx}{k^2} 2\, dx\right) = \frac{1}{\pi}\left(\frac{4\pi}{k^2} - 0 - 0\right) =$$

$$= \frac{4}{k^2}$$

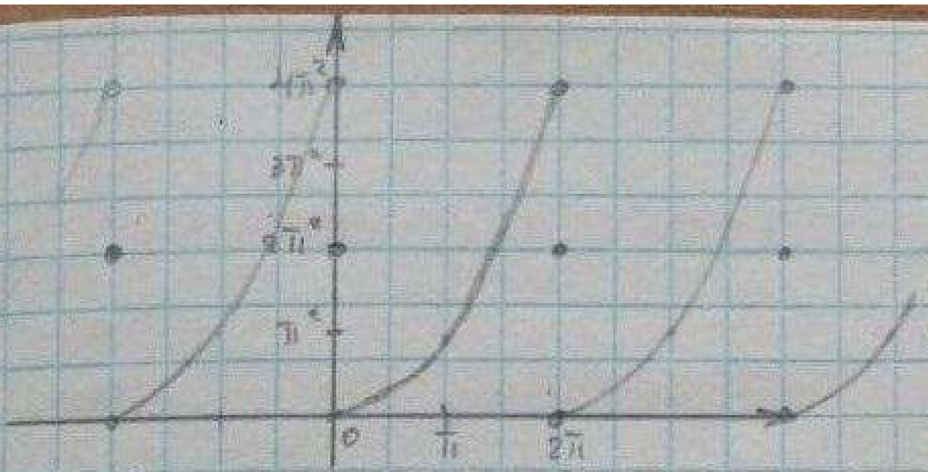
$$b_k = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin kx\, dx = \frac{1}{\pi}\left(-x^2 \frac{\cos kx}{k}\Big|_0^{2\pi} + \int_0^{2\pi} \frac{\cos kx}{k} 2x\, dx\right) =$$

$$= \frac{1}{\pi}\left(-\frac{4\pi^2}{k} + \cancel{\frac{\sin kx}{k^2}}^{\to 0} 2x\Big|_0^{2\pi} - \int_0^{2\pi} \cancel{\frac{\sin kx}{k^2}}^{\to 0} 2\, dx\right) = -\frac{4\pi}{k}$$

$$f(x) \sim \frac{4}{3}\pi^2 + 4\sum_{k=1}^{\infty}\left(\frac{\cos kx}{k^2} - \frac{\pi \sin kx}{k}\right)$$

Т. к. есть точки разрыва, то ряд не сх-ся равномерно.

График суммы ряда:



Для S_1:

Из 3 в т. $x=0$ сумма ряда равна $2\pi^2$

$$2\pi^2 = \frac{4}{3}\pi^2 + 4\sum_{k=1}^{\infty}\left(\frac{1}{k^2} - \frac{0}{k}\right)$$

$$\frac{6-4}{3}\pi^2 = 4\sum_{k=1}^{\infty}\frac{1}{k^2} \longrightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

Для S_2:

Из 3 в т. $x=\pi$ сумма ряда равна π^2

$$\pi^2 = \frac{4}{3}\pi^2 + 4\sum_{k=1}^{\infty}\left(\frac{(-1)^k}{k^2} - 0\right)$$

$$\pi^2 - \frac{4}{3}\pi^2 = -4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} \longrightarrow \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}$$

Для S_3:

$$S_3 = \frac{S_1+S_2}{2} = \frac{\frac{\pi^2}{6}+\frac{\pi^2}{12}}{2} = \frac{\pi^2}{8}$$

пр-ся непосредственно.