Задача 2

Атом водорода помещён в однор. электрич. поле с напр. $\vec{E}$. Гамильтониан будет выглядеть:

$$\hat{H} = \hat{H}_0 + \hat{V} = \frac{\hat{P}^2}{2m} - \frac{e^2}{r} - \vec{d}\vec{E} = \frac{\hat{P}^2}{2m} - \frac{e^2}{r} \underbrace{- eEr\cos\theta}_{\hat{V}}$$

Оценим внутриатомное напряж.: $E_a = \frac{e}{a^2} \simeq 5\cdot 10^9$ В/см

Т.к. оно велико относительно внешнего, то восп-ся th возмущ.

$$\psi^{(0)} \equiv \psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}$$ - основн. сост. атома водорода

Поправка 1 порядка $\varepsilon^1 = \langle \psi^{(0)} | \hat{V} | \psi^{(0)} \rangle = -\frac{Ee}{\pi a^3} \int_0^\infty r^3 e^{-\frac{2r}{a}} dr \cdot$

$$\cdot \int_0^\pi \sin\theta \cos\theta d\theta \cdot \int_0^{2\pi} d\varphi = 0$$

Из теории поля доп. энергия за счёт поляризации

$$\varepsilon^{(2)} = -\frac{\bar{P}\bar{E}}{2} = -\frac{\alpha E^2}{2}$$

При этом $\varepsilon^{(2)} = \langle \psi^{(0)} | \hat{V} | \psi^{(1)} \rangle$

$$\hat{H}_0 |\psi^{(1)}\rangle + \hat{V} |\psi^{(0)}\rangle = \varepsilon^{(0)} |\psi^{(1)}\rangle + \underset{0}{\varepsilon^{(1)}} |\psi^{(0)}\rangle$$

$$(\hat{H}_0 - \varepsilon^{(0)}) |\psi^{(1)}\rangle = -\hat{V} |\psi^{(0)}\rangle$$

Переходя к волн. ф.: $(\hat{H}_0 - \varepsilon^{(0)}) \psi^{(1)} = -\hat{V} \psi^{(0)}$

$$\varepsilon^{(0)} = -\frac{1}{2}\frac{me^4}{\hbar^2}, \quad \hat{p} = -i\hbar\nabla$$

$$\left(-\frac{\hbar^2}{2m}\Delta + \frac{1}{2}\frac{me^2}{\hbar^2} - \frac{e^2}{r}\right)\psi^{(1)} = \frac{eEr\cos\theta \, e^{-\frac{r}{a}}}{\sqrt{\pi a^3}} \quad (1)$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \quad (2)$$

Лапласиан в сферич.: $\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{\hat{l}^2}{r^2}$ (3)

$$\hat{L}^2 Y_{lm}(\theta,\varphi) = l(l+1) Y_{lm}(\theta,\varphi) \quad (4)$$

Т.к. правая часть (1) имеет форму собств. ф-ции оператора момента, то решение ищем в виде:

$$\psi^{(1)}(r,\theta,\varphi) = g(r)\, e^{-\frac{r}{a}}\, Y_{10}(\theta,\varphi) \quad (5)$$

Подставим (2)-(5) в (1):

$$\left(-\frac{\hbar^2}{2m}\Delta_r + \frac{\hbar^2}{mr^2} + \frac{1}{2}\frac{me^4}{\hbar^2} - \frac{e^2}{r}\right)\cdot g(r)\, e^{-\frac{r}{a}}\, Y_{10} = \frac{2Ee}{\sqrt{3a^3}}\, r e^{-\frac{r}{a}} Y_{10}$$

$$\Delta_r\left(g(r)e^{-\frac{r}{a}}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial(ge^{-\frac{r}{a}})}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\left(r^2\frac{\partial g}{\partial r} - \frac{r^2}{a}g\right)e^{-\frac{r}{a}}\right) =$$

$$= e^{-\frac{r}{a}}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial g}{\partial r}\right) - \frac{1}{a}\frac{\partial g}{\partial r} + \frac{1}{a^2}g - \frac{2}{ra}g - \frac{1}{a}\frac{\partial g}{\partial r}\right) =$$

$$= e^{-\frac{r}{a}}\left(\Delta_r g - \frac{2}{a}\frac{\partial g}{\partial r} - \frac{2}{ra}g + \frac{1}{a^2}g\right)$$

$$-\frac{\hbar^2}{2m}\left(\Delta_r g - \frac{2}{a}\frac{\partial g}{\partial r}\right) + g\left(\frac{\hbar^2}{mr^2} - \frac{e^2}{r} + \frac{\hbar^2}{mra}\right) = \frac{2Eer}{\sqrt{3a^3}}$$

Ищем $g(r)$ в виде полинома: $g(r) = A + Br + Cr^2$

$$\Delta_r g = \frac{1}{r^2}\frac{\partial}{\partial r}\left(Br^2 + 2Cr^3\right) = 6C + \frac{2B}{r}$$

$$\frac{\partial g}{\partial r} = B + 2Cr$$

$$-\frac{\hbar^2}{2m}\left(6C + \frac{2B}{r} - \frac{2B}{a} - \frac{4C}{a}r\right) + (A + Br + Cr^2)\cdot$$

$$\cdot\left(\frac{\hbar^2}{mr^2} - \frac{e^2}{r} + \frac{\hbar^2}{mar}\right) = \frac{2Eer}{\sqrt{3a^3}}$$

Приравняем коэфф. при степенях r:

r^0: $\frac{A\hbar^2}{m} = 0$

r^{-1}: $-\frac{\hbar^2 B}{m} - Ae^2 + \frac{A\hbar^2}{ma} + \frac{B\hbar^2}{m} = 0$

r^0: $\frac{B\hbar^2}{ma} - \frac{3C\hbar^2}{m} - Be^2 + \frac{B\hbar^2}{ma} + \frac{C\hbar^2}{m} = 0$

r^1: $\frac{2C\hbar^2}{ma} - Ce^2 + \frac{C\hbar^2}{ma} = \frac{2Ee}{\sqrt{3a^3}}$

Подставим $a = \frac{\hbar^2}{me^2}$:

$$\begin{cases} A = 0 \\ 0 = 0 \\ Be^2 = \frac{2C\hbar^2}{m} \\ \frac{2C\hbar^2}{ma} = \frac{2Ee}{\sqrt{3a^3}} \end{cases} \hookrightarrow \quad \begin{aligned} &A = 0 \\ &B = 2a\cdot C = \frac{2aE}{e\sqrt{3a^3}} \\ &C = \frac{E}{e\sqrt{3a^3}} \end{aligned}$$

$$g(r) = \frac{2Er}{e\sqrt{3a^3}}\left(a + \frac{r}{2}\right)$$

$$\psi^{(1)}(r) = \frac{2Er}{e\sqrt{3a^3}}\left(a + \frac{r}{2}\right)e^{-\frac{r}{a}}Y_{10}(\theta,\varphi)$$

$$\mathcal{E}^{(2)} = \langle\psi^{(0)}|\hat{V}|\psi^{(1)}\rangle = \int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\frac{e^{-\frac{r}{a}}}{\sqrt{\pi a^3}}(-Eer\cos\theta)\frac{2Er}{e\sqrt{3a^3}}\cdot$$

$$\cdot\left(a + \frac{r}{2}\right)e^{-\frac{r}{a}}\sqrt{\frac{3}{4\pi}}\cos\theta\, r^2\sin\theta\, dr\, d\theta\, d\varphi = -\frac{E^2}{\pi a^3}\cdot\frac{e}{e}\int_0^{+\infty} r^4\cdot$$

$$\cdot\left(a + \frac{r}{2}\right)e^{-\frac{2r}{a}}dr\int_0^{\pi}\cos^2\theta\, d(-\cos\theta)\int_0^{2\pi}d\varphi = -\frac{4E^2}{3a^3}\int_0^{+\infty}r^4\left(a + \frac{r}{2}\right)e^{-\frac{2r}{a}}dr =$$

$$= -\frac{4E^2}{3a^3}\left(a\cdot\frac{a^5}{2^5}4! + \frac{1}{2}\frac{a^6}{2^6}5!\right) = -\frac{4E^2a^3}{3}\left(\frac{2^3\cdot 3}{2^5} + \frac{2^3\cdot 5\cdot 3}{2^7}\right) = -\frac{9}{4}E^2a^3$$

Ответ: $\alpha = \frac{9}{2}a^3$

Упражнение 2

$$\hat{H} = \hat{H}_0 + \hat{V}(t)$$

$$\hat{H}_0 = \frac{\hat{\vec{p}}^2}{2m} + \frac{m\omega^2 \hat{\vec{r}}^2}{2}$$

$\hat{V}(t) = -\hat{\vec{d}} \cdot \vec{E} = -e\hat{\vec{r}} \cdot \vec{E}$ – энергия вз-ия с полем

$$|\psi_{вз}(t)\rangle \equiv e^{\frac{i}{\hbar}\hat{H}_0 t} |\psi_ш(t)\rangle$$

Подставим в нестац. ур-ие Шрёд.:

$$i\hbar \frac{\partial |\psi_ш(t)\rangle}{\partial t} = (\hat{H}_0 + \hat{V}) |\psi_ш(t)\rangle$$

$$\cancel{H_0 e^{-\frac{i}{\hbar}\hat{H}_0 t} |\psi_{вз}(t)\rangle} + e^{-\frac{i}{\hbar}\hat{H}_0 (t)} i\hbar \frac{\partial}{\partial t} |\psi_{вз}(t)\rangle =$$

$$= (\cancel{\hat{H}_0} + \hat{V}(t)) e^{-\frac{i}{\hbar}\hat{H}_0 (t)} |\psi_{вз}(t)\rangle$$

$$i\hbar \frac{\partial |\psi_{вз}(t)\rangle}{\partial t} = e^{\frac{i}{\hbar}\hat{H}_0 t} \hat{V}(t) e^{-\frac{i}{\hbar}\hat{H}_0 t} |\psi_{вз}(t)\rangle$$

Откуда $\hat{V}_{вз}(t) \equiv e^{\frac{i}{\hbar}\hat{H}_0 t} \hat{V}(t) e^{-\frac{i}{\hbar}\hat{H}_0 t}$ – только возмущ.

В нашем случае $\hat{V}_{вз}(t) = -e \underbrace{e^{\frac{i}{\hbar}\hat{H}_0 t} \hat{\vec{r}} e^{-\frac{i}{\hbar}\hat{H}_0 t}}_{\text{представление Гейзенберга для } \hat{H}_0} \vec{E} =$

$$= -e \vec{r}_{вз} \cdot \vec{E}$$

Справедлива задача Коши для ур-ия Гейзенберга:

$$\begin{cases} \frac{d\hat{\vec{r}}_{вз}}{dt} = \frac{i}{\hbar} [\hat{H}_0, \hat{\vec{r}}_{вз}] = \frac{i}{\hbar} [\hat{H}_0, \hat{\vec{r}}]_{вз} \\ \hat{\vec{r}}_{вз}(0) = \hat{\vec{r}}_ш \end{cases}$$

$$[\hat{H}_0, \hat{\vec{r}}]_{вз} = \frac{1}{2m} [\hat{\vec{p}}^2, \hat{\vec{r}}]_{вз}$$

$$[\hat{\vec{p}}^2, \hat{r}_\beta] = [\hat{p}_\alpha \hat{p}_\alpha, \hat{r}_\beta] = \hat{p}_\alpha [\hat{p}_\alpha, \hat{r}_\beta] + [\hat{p}_\alpha, \hat{r}_\beta] \hat{p}_\alpha =$$

$$= -2i\hbar \delta_{\alpha\beta} \hat{p}_\alpha = -2i\hbar \hat{p}_\beta$$

Откуда $[\hat{\vec{p}}^2, \hat{\vec{r}}]_{вз} = -2i\hbar\hat{\vec{p}}_{вз}$

$$\begin{cases} \dfrac{d\hat{\vec{r}}_{вз}}{dt} = \dfrac{\hat{\vec{p}}_{вз}}{m} \\ \hat{\vec{r}}_{вз}(0) = \hat{\vec{r}}_{ш} \end{cases}$$

Аналогично для оператора импульса

$$\begin{cases} \dfrac{d\hat{\vec{p}}_{вз}}{dt} = \dfrac{i}{\hbar}[\hat{H}_0, \hat{\vec{p}}]_{вз} \\ \hat{\vec{p}}_{вз}(0) = \hat{\vec{p}}_{ш} \end{cases}$$

$$[\hat{H}_0, \hat{\vec{p}}]_{вз} = \frac{m\omega^2}{2}[\hat{\vec{r}}^2, \hat{\vec{p}}]_{вз}$$

$$[\hat{\vec{r}}^2, \hat{p}_\beta] = [\hat{r}_\alpha \hat{r}_\alpha, \hat{p}_\beta] = \hat{r}_\alpha[\hat{r}_\alpha, \hat{p}_\beta] + [\hat{r}_\alpha, \hat{p}_\beta]\hat{r}_\alpha =$$

$$= 2i\hbar\delta_{\alpha\beta}\vec{r}_\alpha = 2i\hbar\hat{r}_\beta$$

Откуда $[\hat{\vec{r}}^2, \hat{\vec{p}}]_{вз} = 2i\hbar\hat{\vec{r}}_{вз}$

$$\begin{cases} \dfrac{d\hat{\vec{p}}_{вз}}{dt} = -m\omega^2\vec{r}_{вз} \\ \hat{\vec{p}}_{вз}(0) = \hat{\vec{p}} \end{cases}$$

$$\frac{d^2\hat{\vec{r}}_{вз}}{dt^2} = \frac{1}{m}\frac{d\hat{\vec{p}}_{вз}}{dt} = -\omega^2\hat{\vec{r}}_{вз}$$

$$\hat{\vec{r}}_{вз} = \hat{\vec{A}}\cos\omega t + \hat{\vec{B}}\sin\omega t$$

$$\hat{\vec{r}}_{вз}(0) = \hat{\vec{A}} = \hat{\vec{r}}$$

$$\frac{d\hat{\vec{r}}_{вз}}{dt}(0) = \frac{\hat{\vec{p}}_{вз}(0)}{m} = \frac{\hat{\vec{p}}}{m} = \hat{\vec{B}}\omega \hookrightarrow \hat{\vec{B}} = \frac{1}{\omega}\frac{\hat{\vec{p}}}{m}$$

Ответ: $\hat{V}_{вз}(t) = -e\left(\hat{\vec{r}}\cos\omega t + \dfrac{\hat{\vec{p}}}{m\omega}\sin\omega t\right)\vec{E}(t)$

Упражнение 3.

Гамильтониан свободной частицы:

$$\hat{H} = \frac{\hat{\bar{p}}^2}{2m}$$

Задача Коши для ур-ия Гейзенберга:

$$\begin{cases} \dfrac{d\hat{\bar{r}}_r}{dt} = \dfrac{i}{\hbar}[\hat{H}, \hat{\bar{r}}]_r = \dfrac{\hat{\bar{p}}_r}{m} \\ \hat{\bar{r}}_r(0) = \hat{\bar{r}} \end{cases}$$

$$\begin{cases} \dfrac{d\hat{\bar{p}}_r}{dt} = \dfrac{i}{\hbar}[\hat{H}, \hat{\bar{p}}]_r = 0 \\ \hat{\bar{p}}_r(0) = \hat{\bar{p}} \end{cases}$$

$$\frac{d^2\hat{\bar{r}}_r}{dt^2} = 0 \hookrightarrow \hat{\bar{r}}_r = \hat{\bar{A}}t + \hat{\bar{B}}$$

$$\hat{\bar{r}}_r(0) = \hat{\bar{B}} = \hat{\bar{r}}$$

$$\frac{d\hat{\bar{r}}_r(0)}{dt} = \frac{\hat{\bar{p}}_r(0)}{m} = \frac{\hat{\bar{p}}}{m} = \hat{\bar{A}}$$

Ответ:

$$\begin{cases} \hat{\bar{r}}_r = \hat{\bar{r}} + \dfrac{\hat{\bar{p}}t}{m} \\ \hat{\bar{p}}_r = \bar{p} \end{cases}$$

Задача 3.

$$n = 2 \hookrightarrow \quad l = \left[\begin{matrix} 0 \\ 1 \end{matrix}\right. \hookrightarrow m = \left[\begin{matrix} 0 \\ 0, \pm 1 \end{matrix}\right.$$

$\psi_1 = \psi_{200} = R_{20}(r)\, Y_{00}(\theta,\varphi)$

$\psi_2 = \psi_{210} = R_{21}(r)\, Y_{10}(\theta,\varphi)$

$\psi_3 = \psi_{21-1} = R_{21}(r)\, Y_{1-1}(\theta,\varphi)$

$\psi_4 = \psi_{211} = R_{21}(r)\, Y_{11}(\theta,\varphi)$

кратность вырождения есть $4\ (=n^2)$ $\hookrightarrow$ чтобы найти волновые ф-ции 0-го приближ., необх. решить ур-ие:

$$\det(V_{\alpha\beta} - E_n^{(1)}\delta_{\alpha\beta}) = 0 \qquad (*)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} - \frac{e^2}{r} - er\cos\theta\, \mathcal{E} = \hat{H}_0 + \hat{V}$$

$$R_{20}(r) = \left(\frac{1}{2a}\right)^{3/2}\left(2 - \frac{r}{a}\right) e^{-\frac{r}{2a}}$$

$$R_{21}(r) = \left(\frac{1}{2a}\right)^{3/2}\left(\frac{r}{\sqrt{3}a}\right) e^{-\frac{r}{2a}}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1\pm1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$

$$V_{\alpha\beta} = \begin{pmatrix} \langle\psi_1|\hat{V}|\psi_1\rangle & \langle\psi_1|\hat{V}|\psi_2\rangle & \dots & \langle\psi_1|\hat{V}|\psi_4\rangle \\ \dots & \dots & & \dots \\ \langle\psi_4|\hat{V}|\psi_1\rangle & \dots & \dots & \langle\psi_4|\hat{V}|\psi_4\rangle \end{pmatrix}$$

$$\langle\psi_{nlm}|\hat{V}|\psi_{nl'm'}\rangle = \int_0^{+\infty} e\mathcal{E}\, R(r)\, R'(r)\, r^3 dr \cdot \int_0^{\pi} f(\theta)\, d\theta \cdot$$

$$\cdot \int_0^{2\pi} e^{-im\varphi} e^{im'\varphi} d\varphi = I_1 \cdot I_2 \cdot I_3$$

$I_3 = 2\pi\,\delta_{mm'}$ $\hookrightarrow$ откуда занулится 10 интегралов:

$\langle\psi_1|\hat{V}|\psi_{3,4}\rangle = 0 \qquad \langle\psi_3|\hat{V}|\psi_{1,2,4}\rangle = 0$

$\langle\psi_2|\hat{V}|\psi_{3,4}\rangle = 0 \qquad \langle\psi_4|\hat{V}|\psi_{1,2,3}\rangle = 0$

$$I_2 = \int_0^{\pi} \mathcal{Y}_{lm}(\theta)\, \mathcal{Y}_{l'm'}(\theta) \cos\theta \sin\theta\, d\theta$$

Здесь $\mathcal{Y}_{lm}(\theta) = \mathcal{Y}_{lm}(\theta, \varphi) \cdot \frac{1}{e^{im\varphi}}$.

Св-во ф-ций:

$$\underbrace{\hat{I}}_{\text{опер. инверсии}} \mathcal{Y}_{lm}(\theta, \varphi) = \mathcal{Y}_{lm}(\pi - \theta, \varphi + \pi) = (-1)^l \mathcal{Y}_{lm}(\theta, \varphi)$$

$$I_2 \underset{\theta \to \pi - \theta}{=\!=} \int_{\pi}^{0} \mathcal{Y}_{lm}(\pi - \theta)\, \mathcal{Y}_{l'm'}(\pi - \theta) \cos\theta \sin\theta\, d\theta =$$

$$= -\int_0^{\pi} (-1)^l \cdot (-1)^{l'} \mathcal{Y}_{lm}(\theta)\, \mathcal{Y}_{l'm'}(\theta)\, d\theta = (-1)^{l+l'+1} I_2$$

т.е. если $l + l' = 0$, или 2, то $I_2 = 0$ $\hookrightarrow$

$\hookrightarrow$ $\langle \psi_i | \hat{V} | \psi_i \rangle = 0, \ i = \overline{1, 4}$ (занулили ещё 4 инт.)

Получаем 2 ненулевых элемента:

$$\langle \psi_1 | \hat{V} | \psi_2 \rangle = \langle \psi_2 | \hat{V} | \psi_1 \rangle = \int_0^{\infty} \frac{e\mathcal{E}}{8a^3} \left(2 - \frac{r}{a}\right) \frac{r}{\sqrt{3}a} e^{-\frac{r}{a}} dr \cdot$$

$$\cdot r^3 \int_0^{\pi} \frac{1}{\sqrt{4\pi}} \sqrt{\frac{3}{4\pi}} \cos^2\theta \sin\theta\, d\theta \int_0^{2\pi} d\varphi =$$

$$= \frac{e\mathcal{E}}{8a^3} \frac{1}{\sqrt{3}a} \frac{\sqrt{3}}{4\pi} \frac{2}{3} 2\pi \int_0^{\infty} r^4 \left(2 - \frac{r}{a}\right) e^{-\frac{r}{a}} dr =$$

$$= \frac{e\mathcal{E}}{24a^4} \left(a^5 \cdot 4! \cdot 2 - a^5 5!\right) = -3ea\mathcal{E}$$

$$V_{\alpha\beta} = \left(\begin{array}{cc|c} 0 & -3ea\mathcal{E} & \\ -3ea\mathcal{E} & 0 & 0 \\ \hline & 0 & 0 \end{array}\right)$$

Решаем (*):

$$\begin{vmatrix} -E^{(1)} & -3ea\varepsilon & 0 & 0 \\ -3ea\varepsilon & -E^{(1)} & 0 & 0 \\ 0 & 0 & -E^{(1)} & 0 \\ 0 & 0 & 0 & -E^{(1)} \end{vmatrix} = 0$$

$$(E^{(1)})^2 (E^{(1)} - 9e^2a^2\varepsilon^2) = 0$$

Откуда решения:

$$\begin{cases} E_1^{(1)} = E_2^{(1)} = 0 \\ E_3^{(1)} = -3ea\varepsilon \\ E_4^{(1)} = 3ea\varepsilon \end{cases}$$

– Получим 3 уровня энергии. (вырождение снялось не полностью)

$$E^{(0)} \longrightarrow \begin{aligned} &E = E^{(0)} + 3ea\varepsilon \\ &E = E^{(0)} \\ &E = E^{(0)} - 3ea\varepsilon \end{aligned}$$

Ищем правильные ВФ:

$$\begin{pmatrix} -E^{(1)} & -3ea\varepsilon & 0 & 0 \\ -3ea\varepsilon & -E^{(1)} & 0 & 0 \\ 0 & 0 & -E^{(1)} & 0 \\ 0 & 0 & 0 & -E^{(1)} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Усл-ие нормировки: $|C_1|^2 + |C_2|^2 + |C_3|^2 + |C_4|^2 = 1$

1) $E_{1,2}^{(1)} = 0 \hookrightarrow \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \hookrightarrow \tilde{\psi}_1^{(0)} = \psi_3 = \psi_{21-1}$

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \hookrightarrow \tilde{\psi}_2^{(0)} = \psi_4 = \psi_{211}$$

$$2)\ E_3^{(1)} = -3ea\varepsilon \quad \hookrightarrow \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \hookrightarrow \tilde{\psi}_3^{(0)} = \frac{\psi_1 + \psi_2}{\sqrt{2}} = \frac{\psi_{200} + \psi_{210}}{\sqrt{2}}$$

$$3)\ E_4^{(1)} = 3ea\varepsilon$$

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \hookrightarrow \tilde{\psi}_4^{(0)} = \frac{\psi_1 - \psi_2}{\sqrt{2}} = \frac{\psi_{200} - \psi_{210}}{\sqrt{2}}$$

Ответ: $E_{1,2}^{(1)} = 0$, $\tilde{\psi}_1^{(0)} = \psi_{21-1}$, $\tilde{\psi}_2^{(0)} = \psi_{211}$

$E_3^{(1)} = -3ae\varepsilon$, $\tilde{\psi}_3^{(0)} = \dfrac{\psi_{200} + \psi_{210}}{\sqrt{2}}$

$E_4^{(1)} = 3ae\varepsilon$, $\tilde{\psi}_4^{(0)} = \dfrac{\psi_{200} - \psi_{210}}{\sqrt{2}}$

Задача 1

$$U = -\frac{U_0}{\operatorname{ch}^2 \frac{x}{a}}$$

По правилу квантования Бора - Зоммерфельда:

$$\int p(x)\,dx = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

Точки поворота: $p(x_0) = 0$, $p(-x_0) = 0$, $x_0 > 0$

$$p = \sqrt{2m(E-U)} = 0$$

$$\sqrt{2m\left(E + \frac{U_0}{\operatorname{ch}^2 \frac{x_0}{2}}\right)} = 0 \quad \hookrightarrow \quad \operatorname{ch}^2 \frac{x_0}{a} = -\frac{U_0}{E}$$

$$I = \int_{-x_0}^{x_0} \sqrt{2m\left(E_n + \frac{U_0}{ch^2\frac{x}{a}}\right)}\,dx = \pi\hbar\left(n + \frac{1}{2}\right)$$

Замена: $sh\frac{x}{a} = \sqrt{\frac{U_0}{|E_n|} - 1}\,\sin t$

$$dx = \frac{a}{ch\frac{x}{a}}\sqrt{\frac{U_0}{|E_n|} - 1}\,\cos t\,dt$$

$$ch^2\frac{x}{a} = 1 + sh^2\frac{x}{a} = 1 + \sin^2 t\left(\frac{U_0}{|E_n|} - 1\right)$$

$$I = \int_{-t_0}^{t_0} \sqrt{2m\left(-|E_n| - \sin^2 t \cdot U_0 + |E_n|\sin^2 t + U_0\right)} \cdot$$

$$\cdot \frac{a\cos t}{\cos^2 t + \sin^2 t\frac{U_0}{|E_n|}} \cdot \sqrt{\frac{U_0}{|E_n|} - 1}\,dt = \int_{-t_0}^{t_0} \sqrt{2m(U_0 - |E_n|)} \cdot$$

$$\cdot \frac{a\sqrt{\frac{U_0}{|E_n|} - 1}}{1 + tg^2 t\,\frac{U_0}{|E_n|}}\,dt = \sqrt{\frac{2m}{|E_n|}}\left(U_0 - |E_n|\right) a \int_{-t_0}^{t_0} \frac{dt}{1 + tg^2 t \cdot \frac{U_0}{|E_n|}}$$

$$y = \int \frac{1}{1 + z\,tg^2 t}\,dt, \quad z > 0$$

Замена $u = tg\,t$, $du = \frac{1}{\cos^2 t}\,dt$

$$y = \int \frac{\frac{1}{1+u^2}}{1 + zu^2}\,du = \int \frac{1}{(1+u^2)(1+zu^2)}\,du = \int \left(\frac{z}{z-1} \cdot \frac{1}{1+zu^2} -\right.$$

$$\left. - \frac{1}{z-1} \cdot \frac{1}{1+u^2}\right)du = \frac{z}{z-1}\int \frac{1}{1+zu^2}\,du - \frac{1}{z-1}\int \frac{1}{1+u^2}\,du =$$

$$= \frac{z}{z-1}\,\frac{arctg(\sqrt{z}\,u)}{\sqrt{z}} - \frac{1}{z-1} \cdot arctg\,u + C =$$

$$= \frac{\sqrt{z}\,arctg(\sqrt{z}\,tg\,t) - t}{z-1} + C, \quad C \in \mathbb{R}$$

$$I = \sqrt{\frac{2m}{|E_n|}}\left(U_0 - |E_n|\right) 2a \, \frac{\frac{\pi}{2}\sqrt{\frac{U_0}{|E_n|}} - \frac{\pi}{2}}{\frac{U_0}{|E_n|} - 1}$$

$$I = a\pi\sqrt{2m}\left(\sqrt{\frac{U_0}{|E_n|}} - 1\right)\sqrt{|E_n|} = \pi\hbar\left(n + \frac{1}{2}\right)$$

$$\sqrt{U_0} - \sqrt{|E_n|} = \frac{\hbar}{a\sqrt{2m}}\left(n + \frac{1}{2}\right)$$

$$E_n = -\left(\sqrt{U_0} - \frac{\hbar}{a\sqrt{2m}}\left(n + \frac{1}{2}\right)\right)^2$$