(13) a) $\cancel{N}(a, \sigma^2)$

$$\rho_\xi(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

Для параметров (0,1)

$$\varphi_\xi(t) = \int_{-\infty}^{+\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \overset{y=x-it}{=\!=} e^{-\frac{t^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy =$$

$$= e^{-t^2/2} \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\pi} \cdot \sqrt{2} = e^{-\frac{t^2}{2}}$$

$\xi' = a + \sigma\xi \sim N(a, \sigma^2)$, где $\xi \sim N(0,1)$

$$\varphi_{\xi'}(t) = e^{ita} e^{-\frac{\sigma^2 t^2}{2}} = e^{ita - \frac{\sigma^2 t^2}{2}}$$

б) $U(0, a)$

~~Дл~~ Для параметров (0,1):

$$\varphi_\xi(t) = \int_0^1 e^{itx} dx = \begin{cases} 1, & t = 0 \\ \frac{e^{it} - 1}{it}, & t \neq 0 \end{cases}$$

$\xi' = a\xi$

$$\varphi_{\xi'}(t) = \begin{cases} 1, & t = 0 \\ \frac{e^{iat} - 1}{ita}, & t \neq 0 \end{cases}$$

c) $\rho_\xi(n) = \frac{\lambda^n}{n!} e^{-\lambda}$ $n = 0, 1, 2, \ldots$

$$\varphi_\xi(t) = \sum_{n=0}^{\infty} e^{itn} \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^{it})^n}{n!} =$$

$$= e^{-\lambda} e^{\lambda e^{it}} = e^{\lambda e^{it} - \lambda}$$

d) $\rho_\alpha(x) = C_\alpha x^{-2}\left(1 - \cos\frac{x}{\alpha}\right) \qquad x \in \mathbb{R}$

$$I = \int_{-\infty}^{+\infty} C_\alpha x^{-2}\left(1 - \cos\frac{x}{\alpha}\right) dx = 1$$ — усл-ие нормир.

$$f(x,\alpha) = \begin{cases} \dfrac{1-\cos\frac{x}{\alpha}}{x^2} & x \neq 0 \\ \dfrac{1}{2\alpha^2} & x = 0 \end{cases}$$

при $\alpha \to +\infty$ интеграл сх-ся

$$f'(x,\alpha) = \begin{cases} -\dfrac{1}{\alpha^2}\,\dfrac{\sin\frac{x}{\alpha}}{x}, & x \neq 0 \\ -\dfrac{1}{\alpha^3}, & x = 0 \end{cases}$$

$\int_{-\infty}^{+\infty} \frac{\sin\frac{x}{\alpha}}{x} dx$ - инт. Дирихле, сх-ся равном. по $\alpha \in [-\alpha_0, \alpha_0], \alpha_0 > 0$

Значит можем применить теорему о дифф. по параметру:

$$I'(\alpha) = -\frac{C_\alpha}{\alpha^2} \underbrace{\int_{-\infty}^{+\infty} \frac{\sin\frac{x}{\alpha}}{x} dx}_{\text{инт. Дирихле}} = -\frac{C_\alpha}{\alpha^2}\pi \operatorname{sign}\alpha$$

$$I(\alpha) = \frac{C_\alpha}{|\alpha|}\pi + \text{const} \qquad I(\alpha \to +\infty) = 0 \hookrightarrow \text{const} = 0$$

$$C_\alpha = \frac{|\alpha|}{\pi} \hookrightarrow \varphi_\alpha(t) = \frac{|\alpha|}{\pi}\int_{-\infty}^{+\infty} \frac{1-\cos\frac{x}{\alpha}}{x^2} e^{itx} dx$$

$$\varphi_\alpha(t) = \frac{2|\alpha|}{\pi}\int_0^{+\infty} \frac{1-\cos\frac{x}{\alpha}}{x^2}\cos tx\, dx = \frac{2|\alpha|}{\pi}\int_0^{+\infty} \frac{\cos tx - \frac{1}{2}\cos x\left(\frac{1}{\alpha} - t\right) - \frac{1}{2}\cos x\left(\frac{1}{\alpha}+t\right)}{x^2} dt.$$

Применим т. о дифф. по параметру к этому интегралу (сх-ся равном. по призн. Дирихле, как и продиф. интегр.)

$$I(\alpha) = \int_0^{+\infty} \frac{\cos x - \frac{1}{2}\cos x\left(\frac{1}{\alpha} - t\right) - \frac{1}{2}\cos x\left(\frac{1}{\alpha} + t\right)}{x^2}\, dx$$

$$I'(\alpha) = -\frac{1}{2\alpha^2}\int_0^{+\infty} \frac{\sin x\left(\frac{1}{\alpha} - t\right) + \sin x\left(\frac{1}{\alpha} + t\right)}{x}\, dx =$$

$$= -\frac{\pi}{4\alpha^2}\left(\operatorname{sign}\left(\frac{1}{\alpha} - t\right) + \operatorname{sign}\left(\frac{1}{\alpha} + t\right)\right)$$

$$I'(\alpha) = \begin{cases} -\frac{\pi}{4\alpha^2}\operatorname{sign}\alpha \cdot 2, & |t| < \left|\frac{1}{\alpha}\right| \\ -\frac{\pi}{4\alpha^2}\operatorname{sign}\alpha, & |t| = \left|\frac{1}{\alpha}\right| \\ 0, & |t| > \left|\frac{1}{\alpha}\right| \end{cases}$$

Откуда

$$I(\alpha) = \begin{cases} \frac{\pi}{2|\alpha|} + const, & |t| < \left|\frac{1}{\alpha}\right| \\ 0, & |t| \geq \left|\frac{1}{\alpha}\right| \end{cases}$$

(доопр. в т. разрыва 0 и в т. 0 $const = 0$, т. к. $I(\alpha \to +\infty) = 0$, $|t| \geq \frac{1}{\alpha}$

$$I\left(\alpha = \frac{1}{t}\right) = \int_0^{+\infty} \frac{\cos xt - 1}{x^2}\, dt = -\frac{\pi t}{2}$$

$$\varphi_\alpha(t) = \begin{cases} 1 - |\alpha t|, & |t| < \left|\frac{1}{\alpha}\right| \\ 0, & |t| \geq \left|\frac{1}{\alpha}\right| \end{cases}$$

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a) $\varphi(t) = \cos t = \dfrac{e^{it} + e^{-it}}{2}$ - дискр. случ. велич.

$$F_\xi(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{2}, & -1 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

в) $\varphi(t) = e^{it} \cos t = \dfrac{e^{2it} + 1}{2}$

$$F_\xi(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

в ряд Тейлора

c) $\varphi(t) = (2 - e^{it})^{-1} = \dfrac{1}{2} \dfrac{1}{1 - \frac{1}{2} e^{it}} = \dfrac{1}{2} \cdot \sum\limits_{k=0}^{\infty} \dfrac{e^{itk}}{2^k}$

$P(\xi = k) = \dfrac{1}{2^{k+1}}$ - откуда ясна ф-ция распред.

d) $\varphi(t) = \dfrac{1}{2} + \dfrac{\cos t}{2} + i \dfrac{\sin t}{6} = \dfrac{1}{2} + e^{it} \cdot \dfrac{1}{3} + \dfrac{1}{6} e^{-it}$

$$F_\xi(x) = \begin{cases} 0, & x < -1 \\ \frac{1}{6}, & -1 \le x < 0 \\ \frac{2}{3}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

(15) a) $\varphi(t) = 4t^{-2}\cos t \cdot \sin^2\frac{t}{2} = \frac{2\cos t}{t^2} - \frac{2\cos^2 t}{t^2}$

$$E\xi = \frac{1}{i}\frac{d\varphi}{dt}\Big|_{t=0} = \frac{1}{i}\frac{d}{dt}\left(\frac{2}{t^2}(\cos t - \cos^2 t)\right)\Big|_{t=0} =$$

$$= \left[-\frac{4}{t^3 i}(\cos t - \cos^2 t) + \frac{2}{it^2}(\sin 2t - \sin t)\right]\Big|_{t=0}$$

$$\lim_{t\to 0} -\frac{4}{t^3 i}\left(\cancel{1} - \frac{t^2}{2} + \frac{t^4}{24} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{4t^2}{2\cdot 2} - \frac{16t^4}{24\cdot 2} + o(t^5)\right) +$$

$$+ \frac{2}{it^2}\left(2t - \frac{8t^3}{6} - t + \frac{t^3}{6} + o(t^4)\right) =$$

$$= \lim_{t\to 0} -\frac{2}{t\cdot i} + \frac{2}{it} + o(t^2) = \underline{0}$$

$$E\xi^2 = -\frac{d^2\varphi}{dt^2}\Big|_{t=0} = -\Big[+\frac{12}{t^4}(\cos t - \cos^2 t) +$$

$$+ \frac{4}{t^3}(\sin t - \sin 2t) - \frac{4}{t^3}(\sin 2t - \sin t) +$$

$$+ \frac{2}{t^2}(2\cos 2t - \cos t)\Big]\Big|_{t=0}$$

$$-\lim_{t\to 0}\left[\frac{12}{t^4}\left(\frac{t^2}{2} - \frac{7}{24}t^4 + o(t^4)\right) - \frac{8}{t^3}\left(t - \frac{7}{6}t^3 + o(t^3)\right) +\right.$$

$$\left.+ \frac{2}{t^2}\left(1 - \frac{7}{2}t^2 + o(t^2)\right)\right] = -\left(-\frac{7}{2} + \frac{28}{3} - 7\right) = \frac{7}{6}$$

$$D\xi = E(\xi^2) - (E\xi)^2 = \frac{7}{6}$$

в) $\varphi_\xi(t) = (1-it)^{-p}\,(1+it)^{-q}$ $\qquad p, q > 0$

$$E\xi = \frac{1}{i}\frac{d\varphi}{dt}\Big|_{t=0} = \Big[\frac{1}{i}\big(pi(1-it)^{-p-1}(1+it)^{-q} - qi(1+it)^{-q-1}\cdot$$

$$\cdot(1-it)^{-p}\big)\Big]\Big|_{t=0} = p-q$$

$$E\xi^2 = -\frac{d^2\varphi}{dt^2}\Big|_{t=0} = -\Big(-p(+p+1)(1-it)^{-p-2}(1+it)^{-q} +$$

$$+ p\cdot 1(1-it)^{-p-1}\cdot q(1+it)^{-q-1} - q(q+1)(1+it)^{-q-2}\cdot(1-it)^{-p} +$$

$$+ qp(1+it)^{-q-1}(1-it)^{-p-1}\Big)\Big|_{t=0} = p(p+1) - pq +$$

$$+ q(q+1) - pq = (p-q)^2 + p + q$$

$$D\xi = E\xi^2 - (E\xi)^2 = p+q$$

c) $\varphi_\xi(t) = (\arcsin\theta)^{-1}\arcsin(\theta e^{it})$, $0 < \theta < 1$

$$E\xi = \frac{1}{i}\frac{d\varphi}{dt}\Big|_{t=0} = \frac{1}{i\arcsin\theta}\;\frac{i\theta e^{it}}{\sqrt{1-\theta^2 e^{2it}}}\Big|_{t=0} = \frac{\theta}{\arcsin\theta\sqrt{1-\theta^2}}$$

$$E\xi^2 = -\frac{d^2\varphi}{dt^2}\Big|_{t=0} = \Big[\frac{\theta e^{it}}{\arcsin\theta\sqrt{1-\theta^2 e^{2it}}} - \frac{1}{2}\,\frac{1}{\arcsin\theta}\,\frac{2i^2\theta^3 e^{3it}}{(1-\theta^2 e^{2it})^{3/2}}\Big]\Big|_{t=0} =$$

$$= \frac{\theta}{\arcsin\theta\sqrt{1-\theta^2}} + \frac{1}{\arcsin\theta}\,\frac{\theta^3}{(1-\theta^2)^{3/2}} = \frac{\theta}{\arcsin\theta\sqrt{(1-\theta^2)^3}}$$

$$D\xi = E\xi^2 - (E\xi)^2 = \frac{\theta}{\arcsin\theta\,(1-\theta^2)}\left(\frac{1}{\sqrt{1-\theta^2}} - \frac{\theta}{\arcsin\theta}\right)$$

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$$\xi_1, \ldots, \xi_n \sim N(0,1)$$

$$\eta = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$$

а) $f(x) = x^2 \hookrightarrow f^{-1}(x) = \pm\sqrt{x}$

$$\rho_\eta(x) = \rho(f_1^{-1}(x)) \, \underset{\text{"+"}}{\left|(f_1^{-1}(x))'\right|} + \rho(f_2^{-1}(x)) \, \underset{\text{"−"}}{\left|(f_2^{-1}(x))'\right|} =$$

$$= \begin{cases} 0, & x < 0 \\ \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}}, & x > 0 \end{cases}$$

Для 2х степеней с.в. восп-ся формулой свёртки.

$$\rho_\eta(x) = \begin{cases} 0, & x \le 0 \\ \int_0^x \frac{1}{\sqrt{2\pi t}} e^{-\frac{t}{2}} \frac{1}{\sqrt{x-t}\sqrt{2\pi}} e^{-\frac{x-t}{2}} dt, & x > 0 \end{cases} \longrightarrow \begin{cases} 0, & x \le 0 \\ \frac{e^{-\frac{x}{2}}}{2\pi\sqrt{x}} \int_0^x t^{-\frac{1}{2}} \left(1 - \frac{t}{x}\right)^{-\frac{1}{2}} dt, & x > 0 \end{cases}$$

$$\frac{e^{-\frac{x}{2}}}{2\pi}\int_0^x u^{-\frac{1}{2}}(1-u)^{-\frac{1}{2}}du = \frac{e^{-\frac{x}{2}}}{2\pi} B\left(\frac{1}{2},\frac{1}{2}\right) =$$

$$= \frac{1}{2} e^{-\frac{x}{2}}$$

Значит $F_\eta(x) = \begin{cases} 1 - e^{-\frac{x}{2}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$ //

Характ. ф-ция:

$$\varphi_2(t) = \int_0^{+\infty} \frac{1}{2} e^{-\frac{x}{2}} e^{ixt} dx = \frac{1}{2} \frac{e^{x(it-\frac{1}{2})}}{it-\frac{1}{2}}\Bigg|_0^{+\infty} =$$

$$= \frac{1}{1-2it}$$ //

в) $\varphi_{\eta_2}(t) = \varphi_{\xi_1^2+\xi_2^2}(t) \hookrightarrow \varphi_{\xi_1^2}(t) = \frac{1}{\sqrt{1-2it}}$

$$\hookrightarrow \varphi_{\eta_n}(t) = \left(\frac{1}{\sqrt{1-2it}}\right)^n = \frac{1}{(1-2it)^{n/2}}$$

$$E(\eta_n^k) = \frac{1}{i^k} \frac{d^k \varphi_{\eta_n}(t)}{dt^k}\Bigg|_{t=0}$$

$$\frac{d\varphi_{\eta_n}(t)}{dt} = \frac{n}{-2}\frac{-2i}{(1-2it)^{n/2+1}} = \frac{n}{(1-2it)^{n/2+1}}$$

$$\frac{d^k\varphi_{\eta_n}(t)}{dt^k} = \frac{i^k\, n\,(n+2)\cdot\ldots\cdot(n+(k-1)\cdot 2)}{(1-2it)^{n/2+k}}$$

$$E(\eta_n^k) = \prod_{i=0}^{k-1}(n+2i)$$

(17) $\xi_{m,n}$ $(m=1,2,\dots,n)$ – н.с.в.

$$F_n(x) = P(\xi_{m,n} \leq x) = \begin{cases} 1-e^{-\alpha_n x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \alpha_n = \lambda n, \ \lambda > 0$$

$$\xi_n = \xi_{1,n} + \xi_{2,n} + \dots + \xi_{n,n} \longrightarrow ? \quad n \to +\infty$$

$$\varphi_{\xi_n}(t) = \prod_{m=1}^{n} \varphi_{\xi_{m,n}}(t) = \left(\varphi_{\xi_{m,n}}(t)\right)^n$$

$$\varphi_{\xi_{m,n}}(t) = \int_0^{+\infty} e^{itx} \alpha_n e^{-\alpha_n x} dx = \alpha_n \int_0^{+\infty} e^{x(it-\alpha_n)} dx = \frac{\alpha_n}{\alpha_n - it}$$

$$\varphi_{\xi_n}(t) = \left(\frac{\lambda n}{\lambda n - it}\right)^n = \left(\frac{1}{1-\frac{it}{\lambda n}}\right)^n = \left(1 - \frac{it}{\lambda n}\right)^{-n} \longrightarrow e^{\frac{it}{\lambda}}, \ n \to +\infty$$

Откуда $$F_{n\to\infty}(x) = \begin{cases} 0, & x < \frac{1}{\lambda} \\ 1, & x \geq \frac{1}{\lambda} \end{cases}$$

(18)

$$P(\pi_\lambda = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$E\pi_\lambda = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} =$$

$$= \lambda \cdot 1 = \lambda$$

$$E\pi_\lambda^2 = \sum_{k=0}^{\infty} \frac{k^2 \lambda^k}{k!} e^{-\lambda} = \lambda^2 + \lambda$$

$$D(\pi_\lambda) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Исходя из центр. предельн. теоремы:

$$\lim_{n \to \infty} P\left(\frac{\pi_\lambda - \overset{E\pi_\lambda}{\lambda}}{\underbrace{\sqrt{\lambda}}_{D(\pi_\lambda)}} \le x \right) = P(\eta \le x) = , \quad \eta \sim N(0,1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

Ответ: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$