(35) X_i - число очков на i-той кости, $i=1,2$

Совместное распр. $Y=\min(X_1, X_2)$ и $Z=X_2$, их E, D и cov. $Y_{опт}$ -?

$X_1 \sim$

1	2	3	4	5	6
1/6	1/6	1/6	1/6	1/6	1/6

$X_2 \sim$

1	2	3	4	5	6
1/6	1/6	1/6	1/6	1/6	1/6

$Y \sim$

1	2	3	4	5	6
11/36	1/4	7/36	5/36	1/12	1/36

$Z = X_2$

$$Y, Z \sim \begin{array}{c|c} \begin{pmatrix} \frac{6}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\ 0 & \frac{5}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\ 0 & 0 & \frac{4}{36} & \frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\ 0 & 0 & 0 & \frac{3}{36} & \frac{1}{36} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & \frac{2}{36} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{36} \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \; Y \\ \hline \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} & \\ Z & \end{array}$$

(из форм. полн. вер.)

$$E(Y) = \frac{11}{36} + \frac{1}{2} + \frac{7}{12} + \frac{5}{9} + \frac{5}{12} + \frac{1}{6} = \frac{91}{36} \approx 2,53$$

$$E(Z) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{7}{2} = 3,5$$

$$E(Y^2) = \frac{11}{36} + \frac{4}{4} + \frac{9 \cdot 7}{36} + \frac{16 \cdot 5}{36} + \frac{25}{12} + \frac{36}{36} = \frac{301}{36} = 8,36$$

$$E(Z^2) = \frac{1+4+9+16+25+36}{6} = \frac{91}{6} = 15,16$$

$$D(Z) = \frac{91}{6} - \frac{49}{4} = 2,92$$

$$D(Y) = \frac{301}{36} - \left(\frac{91}{36}\right)^2 = 1,97$$

$$\operatorname{cov}(Y, Z) = E(YZ) - E(Y) \cdot E(Z)$$

$$E(YZ) = 1 \cdot \frac{6}{36} + \frac{1}{36}(2+3+4+5+6) + 4\,\frac{5}{36} + \frac{1}{36}(6+8+10+12) +$$

$$+9\cdot\frac{4}{36}+\frac{1}{36}(12+15+18)+16\cdot\frac{3}{36}+4\cdot\frac{1}{36}(5+6)+25\frac{2}{36}+\frac{5}{36}\cdot 6+1=$$

$$=\frac{371}{36}\approx 10,3$$

$$\mathrm{cov}(Y,Z)=\frac{371}{36}-\frac{91}{36}\cdot\frac{21}{6}=\frac{315}{216}\approx 1,46$$

Z - видим, Y - предск.

$$\hat{Y}=E(Y)+\sqrt{D(Y)}\cdot\rho(Y,Z)\left[\frac{Z-E(Z)}{\sqrt{D(Z)}}\right]$$

$$\rho(Y,Z)=\frac{\mathrm{cov}(Y,Z)}{\sqrt{D(Y)}\sqrt{D(Z)}}=\frac{9}{\sqrt{219}}$$

$$\hat{Y}=\frac{91}{36}+\frac{\sqrt{2555}}{36}\cdot\frac{9}{\sqrt{219}}\left(\frac{Z-\frac{21}{6}}{\sqrt{105/6}}\right)=\frac{91}{36}+\frac{3}{2}\sqrt{\frac{2553}{22995}}\cdot$$

$$\cdot\left(Z-\frac{21}{6}\right)=\frac{91}{36}+\frac{1}{2}\left(Z-\frac{21}{6}\right)$$

36 $P(\xi=-1)=P(\xi=1)=a$ $P(\xi=0)=1-2a$

$E(|\xi|)=1\cdot 2a+0\cdot(1-2a)=2a$

Нер-во Чебышева:

$$P(|\xi|\geqslant 1)\leqslant E(|\xi|)=2a$$

При этом точное значение

$$P(|\xi|\geqslant 1)=2a=E(|\xi|).$$

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а) $P(\xi = \eta)$ б) $P(\xi > \eta)$ в) $P(\xi + \eta = k)$

г) $P(\xi = l \mid \xi + \eta = k)$ д) $P(\xi = k \mid \xi = \eta)$

ξ и η независимые и имеют геометр. распр. с параметром p.

а) $P(\xi = \eta) = \sum\limits_{m=1}^{\infty} (pq^{m-1})^2 = \frac{p^2}{1-q^2} = \frac{p}{1+q}$

б) $P(\xi > \eta) = P(\eta > \xi)$ – т.к. они независимы и имеют одинаковое распр.

$P(\xi > \eta) + P(\xi = \eta) + P(\xi < \eta) = 1$

$P(\xi > \eta) = \frac{1 - P(\xi = \eta)}{2} = \frac{1 + q - p}{2(1+q)} = \frac{q}{1+q}$

в) $P(\xi + \eta = k) = \sum\limits_{i} P(\xi = i) \cdot P(\eta = k - i) =$

$= \sum\limits_{i=1}^{k-1} pq^{i-1} \cdot pq^{k-i-1} = p^2 \sum\limits_{i=1}^{k-1} q^{k-2} = p^2 q^{k-2}(k-1)$

г) при $l > k$: $P(\xi = l \mid \xi + \eta = k) = 0$

если $l \leq k$: $P(\xi = l \mid \xi + \eta = k) = \dfrac{P(\xi = l, \xi + \eta = k)}{P(\xi + \eta = k)} =$

$$= \frac{P(\xi = l)\ P(\eta = k - l)}{P(\xi + \eta = k} = \frac{p q^{l-1} \cdot p \cdot q^{k-l-1}}{p^2 q^{k-2} (k-1)} = \frac{1}{k-1}$$

Здесь $k \geq 2$, $1 \leq l \leq k-1$.

д) $P(\xi = k \mid \xi = \eta) = P(\eta = k) = p q^{k-1}$

(38) $\xi \geq 0,\ \xi \in \mathbb{Z}$ Док-ть $\overset{E(\xi)}{\overset{|||}{M\xi}} = \sum_{k\geq 1}^{\infty} P(\xi \geq k)$

$$E(\xi) = \sum_{k=0}^{\infty} k\,P(\xi = k) = \sum_{k\geq 1}^{\infty} P(\xi = k) + \sum_{k\geq 2}^{\infty} P(\xi = k) + \ldots = P(\xi \geq 1) + P(\xi \geq 2) + \ldots = \sum_{k=1}^{\infty} P(\xi \geq k)$$

(39) В N ячеек случайно размещ. n неразл. шаров. ξ - число пустых ячеек.

Мы можем разложить $C_{N+n-1}^{n} = \frac{(N+n-1)!}{(N-1)!\,n!}$

$$\alpha_k = \begin{cases} 1, & \text{если } k\text{-тый пуст} \\ 0, & \text{если } k\text{-тый не пуст} \end{cases}$$

$$\xi = \sum_{k=1}^{N} \alpha_k$$

$$E(\xi) = \sum_{k=1}^{N} E(\alpha_k) = NE(\alpha_1)$$

$$E(\alpha_1) = P(\{1\text{-я ячейка пуста}\})$$

$$P(\{1\text{-я пуста}\}) = \frac{C_{(N-1)+n-1}^{n}}{C_{N+n-1}^{n}} = \frac{N-1}{N+n-1}$$

$$E(\xi) = \frac{N(N-1)}{N+n-1} . \qquad \xi^2 = \sum_{k,j=1}^{N} \alpha_k \alpha_j$$

$$E(\xi^2) = \sum_{k,j=1}^{N} E(\alpha_k, \alpha_j) = N E(\alpha_1^2) + (N^2 - N) E(\alpha_1 \alpha_2)$$

$$E(\alpha_1^2) = E(\alpha_1)$$

$E(\alpha_1 \cdot \alpha_2) = P(\{\text{первые две пусты}\})$

$$P(\{\text{первые две пусты}\}) = \frac{C^n_{(N-2)+n-1}}{C^n_{N+n-1}} = \frac{(N-2)(N-1)}{(N+n-2)(N+n-1)}$$

$$E(\xi^2) = \frac{N(N-1)}{N+n-1} + \frac{N(N-1)(N-2)(N-1)}{(N+n-2)(N+n-1)}$$

$$D(\xi) = E(\xi^2) - E^2(\xi) = \frac{N(N-1)}{N+n-1} + \frac{N(N-1)^2(N-2)}{(N+n-2)(N+n-1)} - \frac{N^2(N-1)^2}{(N+n-1)^2}$$

(40) ξ и η – числа появлений единицы и шестёрки при n брос. игральной кости

$$\alpha_k = \begin{cases} 1, & \text{на } k\text{-том выпала „1"} \\ 0, & \text{на } k\text{-том не выпала „1"} \end{cases}$$

Аналогично β_k для „6"

$$\xi = \sum_{k=1}^{n} \alpha_k, \quad \eta = \sum_{k=1}^{n} \beta_k$$

$$E(\xi) = \sum_{k=1}^{n} E(\alpha_k) = n \cdot \frac{1}{6} = \frac{n}{6} = E(\eta)$$

$$E(\xi^2) = E\left(\sum_{k,j=1}^{n} \alpha_k \cdot \alpha_j\right) = E\left(\sum_{k=1}^{n} \alpha_k^2\right) + E\left(\sum_{k,j:\ k\neq j}^{n} \alpha_k \alpha_j\right) =$$

$$= \frac{n}{6} + (n^2 - n)\left(\frac{1}{6}\right)^2 = \frac{n^2}{36} + \frac{5n}{36}$$

$$D(\xi) = E(\xi^2) - E^2(\xi) = \frac{n^2}{36} + \frac{5n}{36} - \frac{n^2}{36} = \frac{5n}{36} = D(\eta)$$

$$E(\xi\eta) = E\left(\sum_{k,j=1}^{n} \alpha_k \beta_j\right) = \sum_{k=1}^{n} \underbrace{E(\alpha_k \beta_k)}_{0} + \sum_{k\neq j}' E(\alpha_k \beta_j) =$$

$$= (n^2 - n) \cdot \frac{1}{36}$$

$$\operatorname{cov}(\xi, \eta) = E(\xi\eta) - E(\xi) \cdot E(\eta) = \frac{n^2 - n}{36} - \frac{n^2}{36} = -\frac{n}{36}$$

$$\rho(\xi, \eta) = \frac{\operatorname{cov}(\xi, \eta)}{\sqrt{D(\xi) D(\eta)}} = -\frac{n\, 36}{36 \cdot 5n} = -\frac{1}{5}$$