16.1(3)

$\Delta u = 0, \quad r < 1 \qquad u(r=1) = \cos^4\varphi$

Шаг 1:

$$\cos^4\varphi = \frac{3}{8} + \frac{1}{2}\cos 2\varphi + \frac{1}{8}\cos 4\varphi$$

Шаг 2:

$$u = A + Br^2\cos 2\varphi + Cr^4\cos 4\varphi$$

$$u|_{r=1} = A + B\cos 2\varphi + C\cos 4\varphi = \frac{3}{8} + \frac{1}{2}\cos 2\varphi + \frac{1}{8}\cos 4\varphi$$

Откуда $A = \frac{3}{8}$, $B = \frac{1}{2}$, $C = \frac{1}{8}$

Ответ: $u = \frac{3}{8} + \frac{1}{2}r^2\cos 2\varphi + \frac{1}{8}r^4\cos 4\varphi$

16.2(3)

$\left.\frac{\partial u}{\partial r}\right|_{r=R} = \sin^3\varphi \qquad \Delta u = 0, \ r < R$

Шаг 1:

$$\sin^3\varphi = \frac{3}{4}\sin\varphi - \frac{1}{4}\sin 3\varphi$$

Шаг 2: $u = A + Br\sin\varphi + Cr^3\sin 3\varphi$

$u|_{r=R} = A + BR\sin\varphi + CR^3\sin 3\varphi = \frac{3}{4}\sin\varphi - \frac{1}{4}\sin 3\varphi$

Откуда $A = 0$, $B = \frac{3}{4R}$, $C = -\frac{1}{4R^3}$

Ответ: $u = \frac{3}{4R} r\sin\varphi - \frac{r^3}{4R^3}\sin 3\varphi$

Т2

(1) a) $\Delta u = 12x$, $r = \sqrt{x^2+y^2}$, $1 < r < 2$

(2) $u|_{r=1} = 2\cos^3\varphi + 1 - \sin\varphi\cos\varphi$

(3) $u|_{r=2} = 16\cos^3\varphi - 4\sin\varphi\cos\varphi$

Шаг 0: частн. реш. ур-ия (1)

$\tilde{u} = 2x^3 = 2r^3\cos^3\varphi$

(4) сделаем замену $u = v + \tilde{u}$

Подставим (4) в (1), (2), (3), попутно сделаем Шаг 1.

$(1)'$ $\Delta v = 0$, $1 < r < 2$

$(2)'$ $v|_{r=1} = 1 - \frac{1}{2}\sin 2\varphi$

$(3)'$ $v|_{r=2} = -2\sin 2\varphi$

Шаг 2:

$v = A + B\ln r + \left(Cr^2 + \frac{D}{r^2}\right)\sin 2\varphi \hookrightarrow$ в $(2)'$, $(3)'$

$A + B\cdot 0 + (C + D)\sin 2\varphi \equiv 1 - \frac{1}{2}\sin 2\varphi$

$A + B\ln 2 + \left(4C + \frac{D}{4}\right)\sin 2\varphi \equiv -2\sin 2\varphi$

$\begin{cases} A + B\cdot 0 = 1 \\ A + B\ln 2 = 0 \end{cases}$ $\begin{cases} C + D = -\frac{1}{2} \\ 4C + \frac{D}{4} = -2 \end{cases}$

$A = 1,\ B = -\frac{1}{\ln 2},\ C = -\frac{1}{2},\ D = 0$

Ответ: $u = 2r^3\cos^3\varphi + 1 - \log_2 r - \frac{1}{2}r^2\sin 2\varphi =$

$= 2x^3 + 1 - \frac{1}{2}\log_2(x^2+y^2) - xy$

б) $\Delta u = 4\frac{(x-y)^2}{x^2+y^2}$, $1 < r < 2$, $r = \sqrt{x^2+y^2}$

$(2u - u_r)\big|_{r=1} = \frac{2xy}{x^2+y^2}$, $u_r\big|_{r=2} = \frac{8x^2}{x^2+y^2}$

Шаг 0.1:

$4\frac{r^2 - r^2\sin 2\varphi}{r^2} = 4(1 - \sin 2\varphi)$

Шаг 0.2:

$\Delta u = 4 \hookrightarrow$ частн. реш. $\tilde{u} = r^2$

$\Delta u = -4\sin 2\varphi \hookrightarrow$ частн. реш. $\tilde{\tilde{u}} = \sin 2\varphi \cdot g(r)$

$r^2 g''(r) + rg'(r) - 4g = -4r^2$

Замена $r = e^t$ $\quad g(r) = y(\ln r)$

$\cancel{r^2}\left(\ddot{y}(\ln r)\frac{1}{\cancel{r^2}} - \dot{y}(\ln r)\frac{1}{\cancel{r^2}}\right) + \cancel{r}\,\dot{y}(\ln r)\frac{1}{\cancel{r}} - 4y(\ln r) = -4r^2$

$\ddot{y}(t) - \dot{y}(t) + \dot{y}(t) - 4y(t) = -4e^{2t}$

$\ddot{y}(t) - 4y(t) = -4e^{2t}$

$\tilde{y}_r = Ate^{2t}$

$(Ae^{2t} + 2Ate^{2t})'_t = 4Ae^{2t} + 4Ate^{2t}$

$4Ae^{2t} + 4Ate^{2t} - 4Ate^{2t} = -4e^{2t} \hookrightarrow \tilde{y}_r = -te^{2t}$

$\tilde{g}_r(r) = r^2\ln r \hookrightarrow \tilde{\tilde{u}} = -r^2\ln r \cdot \sin 2\varphi$

Замена $u = v + r^2 - r^2 \ln r \sin 2\varphi$

$$\frac{2xy}{x^2+y^2} = \frac{r^2 \sin 2\varphi}{r^2} = \sin 2\varphi$$

$$\frac{8x^2}{x^2+y^2} = 8\cos^2\varphi = 4 + 4\cos 2\varphi$$

Шаг 1:

$$u_r = v_r + 2r - 2r\ln r \sin 2\varphi - r\sin 2\varphi$$

$$2v + \cancel{2} - 0 - v_r - \cancel{2} + \cancel{\sin 2\varphi} = \cancel{\sin 2\varphi} \hookrightarrow (2v - v_r)\big|_{r=1} = 0$$

$$v_r + \cancel{4} - 4\ln 2 \sin 2\varphi - 2\sin 2\varphi = \cancel{4} + 4\cos 2\varphi$$

$$\hookrightarrow v_r\big|_{r=2} = 4\cos 2\varphi + (2 + 4\ln 2)\sin 2\varphi$$

Шаг 2.

$$\begin{cases} v = A + B\ln r + \left(Cr^2 + \frac{D}{r^2}\right)\cos 2\varphi + \left(Er^2 + \frac{F}{r^2}\right)\sin 2\varphi \\ 2(A + B\cdot 0 + (C+D)\cos 2\varphi + (E+F)\sin 2\varphi) - B - \\ - (2C - 2D)\cos 2\varphi - (2E - 2F)\sin 2\varphi) = 0 \\ \frac{B}{2} + \left(4C - \frac{2D}{8}\right)\cos 2\varphi + \left(4E - \frac{2F}{8}\right)\sin 2\varphi = \\ = 4\cos 2\varphi + (2 + \ln 2 \cdot 4)\sin 2\varphi \end{cases}$$

$2A - B = 0$ $\qquad$ $B/2 = 0$

$2C + 2D - 2C + 2D = 0$ $\qquad$ $4C - \frac{2D}{8} = 4$

$2E + 2F - 2E + 2F = 0$ $\qquad$ $4E - \frac{2F}{8} = 2 + 4\ln 2$

$B = 0,\ A = 0,\ D = 0,\ C = 1,\ F = 0,\ E = \frac{1}{2} + \ln 2$

Ответ: $u = r^2 \cos 2\varphi + \left(\frac{1}{2} + \ln 2\right) r^2 \sin 2\varphi + r^2 -$

$- r^2 \ln r \sin 2\varphi$