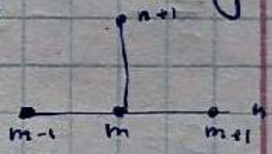
XIV. 9.11 a)

$$\begin{cases} \dfrac{u_m^{n+1} - u_m^n}{\tau} + \dfrac{-3u_{m+1}^n + 4u_m^n - u_{m-1}^n}{2h} + \dfrac{-3v_{m+1}^n + 3v_{m-1}^n}{2h} = f_m^n \\ \dfrac{v_m^{n+1} - v_m^n}{\tau} + \dfrac{-u_{m+1}^n + u_{m-1}^n}{2h} + \dfrac{-v_{m+1}^n + 4v_m^n - 3v_{m-1}^n}{2h} = g_m^n \end{cases}$$



Иссл. на спектр. устойчивость:

$$u_m^n \sim \alpha\lambda^n e^{ikhm}$$

$$v_m^n \sim \beta\lambda^n e^{ikhm}$$

$$\begin{cases} \alpha\dfrac{\lambda-1}{\tau} + \alpha\dfrac{-3e^{ikh}+4-e^{-ikh}}{2h} + \beta\dfrac{-3e^{ikh}+3e^{-ikh}}{2h} = 0 \\ \beta\dfrac{\lambda-1}{\tau} + \alpha\dfrac{-e^{-ikh}+e^{-ikh}}{2h} + \beta\dfrac{-e^{ikh}+4-3e^{-ikh}}{2h} = 0 \end{cases}$$

Смотрим, чтобы детерм. системы обращ. в 0

$$D = \begin{vmatrix} \dfrac{\lambda-1}{\tau} + \dfrac{-3e^{ikh}+4-e^{-ikh}}{2h} & -\dfrac{3}{2h}\left(e^{ikh}-e^{-ikh}\right) \\ \dfrac{-e^{-ikh}+e^{-ikh}}{2h} & \dfrac{\lambda-1}{\tau} + \dfrac{-3e^{-ikh}-e^{ikh}+4}{2h} \end{vmatrix} = 0$$

$$a = \frac{\lambda-1}{\tau} \qquad b = \frac{-3e^{ikh}+4-e^{-ikh}}{2h}$$

$$D = \begin{vmatrix} a+b & -\frac{3i}{h}\sin kh \\ -\frac{i}{h}\sin kh & a+\bar{b} \end{vmatrix} = a^2 + |b|^2 + ab + a\bar{b} + \frac{3}{h^2}\sin^2 kh = 0$$

$$4h^2|b|^2 = (-4\cos kh + 4)^2 + (2\sin kh)^2 = 64\sin^4\frac{kh}{2} + 4\sin^2 kh \quad \text{-def}$$

$$a(b+\bar{b}) = 2a\,\mathrm{Re}(b) = \frac{a}{h}(4-4\cos kh) = \frac{8a}{h}\sin^2\frac{kh}{2}$$

$$a^2 + \frac{16}{h^2}\sin^4\frac{kh}{2} + \frac{1}{h^2}\sin^2 kh + \frac{8a}{h}\sin^2\frac{kh}{2} + \frac{3}{h^2}\sin^2 kh = 0$$

$$a^2 + \frac{8a}{h}\sin^2\frac{kh}{2} + \frac{16}{h^2}\sin^4\frac{kh}{2} + \frac{4}{h^2}\sin^2 kh = 0$$

$$\frac{D}{4} = \cancel{\left(\frac{4}{h}\sin^2\frac{kh}{2}\right)^2} - \cancel{\frac{16}{h^2}\sin^4\frac{kh}{2}} - \frac{4}{h^2}\sin^2 kh = -\frac{4}{h^2}\sin^2 kh$$

$$a = -\frac{4}{h}\sin^2 kh \pm \frac{2i}{h}\sin kh$$

$$\lambda = 1 - \frac{4\tau}{h}\sin^2\frac{kh}{2} \pm \frac{2i\tau}{h}\sin kh$$

$$|\lambda|^2 = \left(1 - \frac{4\tau}{h}\sin^2\frac{kh}{2}\right)^2 + \frac{4\tau^2}{h^2}\sin^2 kh = 1 - \frac{8\tau}{h}\sin^2\frac{kh}{2} +$$

$$+ \frac{16\tau^2}{h^2}\sin^4\frac{kh}{2} + \frac{4\tau}{h^2}\sin^2 kh \leq 1$$

$$-\frac{8\tau}{h} + \frac{16\tau^2}{h^2}\sin^2\frac{kh}{2} + \frac{16\tau^2}{h^2}\cos^2\frac{kh}{2} \leq 0$$

$$\frac{16\tau^2}{h^2} - \frac{8\tau}{h} \leq 0 \hookrightarrow \frac{\tau}{h} \leq \frac{1}{2}$$ – усл-ие устойчивости схемы

XIV. 9.14

б)

$$\begin{cases} \dfrac{\partial u}{\partial t} + 2\dfrac{\partial u}{\partial x} - 2\dfrac{\partial v}{\partial x} + 3\dfrac{\partial w}{\partial x} = f(t,x) \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x} = g(t,x) \\ \dfrac{\partial w}{\partial t} + \dfrac{\partial u}{\partial x} + 3\dfrac{\partial v}{\partial x} - \dfrac{\partial w}{\partial x} = h(t,x) \end{cases}$$

$u(0,x) = \varphi_1(x)$, $v(0,x) = \varphi_2(x)$, $w(0,x) = \varphi_3(x)$, $0 \leq x \leq 1$

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

$$|A - \lambda E| = \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 1 - 3) +$$

$$+ 2(-1-\lambda-1) + 3(3-1+\lambda) = (2-\lambda)(\lambda^2-4) + 2(-2-\lambda) +$$

$$+ 3(2+\lambda) = (\lambda+2)\left((2-\lambda)(\lambda-2) - 2 + 3\right) = 0$$

$\lambda = -2, 1, 3$ $\hookrightarrow$ система гиперболическая

$\underline{\lambda = -2}$:

$$(\omega_1^1 \ \omega_2^1 \ \omega_3^1) \begin{pmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix} = (0 \ 0 \ 0)$$

$$\begin{pmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 4 & -2 & 3 \\ 0 & 14 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\omega_2^1 = -\frac{1}{14}\omega_3^1$

$\omega_1^1 = -\frac{11}{14}\omega_3^1$ $\quad (\omega_1^1 \ \omega_2^1 \ \omega_3^1) = \begin{pmatrix} -11/14 \\ -1/14 \\ 1 \end{pmatrix}^T$

$\underline{\lambda = 1}$:

$$\begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 0 & 3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$\omega_2^2 = \omega_3^2 \quad \omega_1^2 = -\omega_3^2 \quad \hookrightarrow (\omega_1^2 \ \omega_2^2 \ \omega_3^2) =$

$-\omega_3^2 - 2\omega_3^2 + 3\omega_3^2 = 0 \hookrightarrow \omega_3^2 = 1 \quad = (-1 \ 1 \ 1)$

$\lambda = 3:$

$$\begin{pmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 0 & 5 & -5 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} -1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \omega_1^3 & \omega_2^3 & \omega_3^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$\Omega = \begin{pmatrix} -\frac{11}{14} & -\frac{1}{14} & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad R = \Omega \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\frac{11}{14}u & -\frac{1}{14}v + w \\ -u & +v + w \\ u & +v + w \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}$$

$$\begin{cases} \dfrac{\partial R_1}{\partial t} - 2\dfrac{\partial R_1}{\partial x} = -\dfrac{11}{14} f - \dfrac{1}{14} g + h \\ \dfrac{\partial R_2}{\partial t} + \dfrac{\partial R_2}{\partial x} = -f + g + h \\ \dfrac{\partial R_3}{\partial t} + 3\dfrac{\partial R_3}{\partial x} = f + g + h \end{cases}$$

1) Здесь 3 ур-ия на левой границе, 3 на правой. подходит

2) 4 на правой (ЛНЗ) $\Rightarrow$ не подходит

3) Аналогично пункту 1

4) Аналогично пункту 1

(2) Схема с аппроксим. выше 1 пор. по т. Годунова не может быть монотонной.

Рассм. обычный правый уголок

$$\frac{R_{1m}^{n+1}-R_{1m}^{n}}{\tau}-2\,\frac{R_{1m+1}^{n}-R_{1m}^{n+1}}{h}=0$$

Схема устойчива при $\left|\frac{2\tau}{h}\right|<1$

XIV. 9.6

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0 \quad a=const>0$$

$$\frac{y_m^{n+1}-y_m^n}{\tau}+a\,\frac{y_{m+1}^n-y_{m-1}^n}{2h}=\xi\tau\,\frac{y_{m+1}^n-2y_m^n+y_{m-1}^n}{h^2}$$

$\xi=\frac{h}{2}$ - схема Лакса

$\xi=1$ - Лакса-Вендрофора

$\xi=0$ - Куранта-Изаксона-Риса

Иссл. на устойчивость:

$$y_m^n\sim\lambda^n e^{ikmh}$$

$$\frac{\lambda-1}{\tau}+a\,\frac{e^{ikh}-e^{-ikh}}{2h}=\xi\tau\,\frac{e^{ikh}-2+e^{-ikh}}{h^2}$$

$$\lambda=1-i\frac{a\tau}{h}\sin kh-\frac{\xi\tau^2}{h^2}4\sin^2\frac{kh}{2}$$

$$|\lambda|^2=\left(1-\frac{\xi\tau^2}{h^2}4\sin^2\frac{kh}{2}\right)^2+\frac{a^2\tau^2}{h^2}\sin^2 kh=1-8\,\frac{\xi\tau^2}{h^2}\sin^2\frac{kh}{2}+$$

$$+16\,\frac{\xi^2\tau^4}{h^4}\sin^4\frac{kh}{2}+4\,\frac{a^2\tau^2}{h^2}\sin^2\frac{kh}{2}\left(1-\sin^2\frac{kh}{2}\right)=$$

$$=1+4\frac{\tau^2}{h^2}\sin^2\frac{kh}{2}\left(a^2-2\xi\right)+\frac{4\tau^2}{h^2}\sin^4\frac{kh}{2}\left(4\,\frac{\xi^2\tau^2}{h^2}-a^2\right)\leq 1$$

$$a^2-2\xi+\sin^2\frac{kh}{2}\left(4\,\frac{\xi^2\tau^2}{h^2}-a^2\right)\leq 0$$

$$\sin^2\frac{kh}{2} \geqslant \frac{2\xi - a^2}{a^2 - \frac{4\xi^2\tau^2}{h^2}} \qquad \forall k\in\mathbb{Z}$$

При этом $\forall k \in \mathbb{Z}$ $\quad \sin^2\frac{kh}{2}\le 1$

$$\hookrightarrow \quad \frac{2\xi-a^2}{a^2-\frac{4\xi^2\tau^2}{h^2}} \le 1$$

$$2\xi - a^2 \le a^2 - \frac{4\xi^2\tau^2}{h^2}$$

$$a^2 \geqslant \xi + \frac{2\xi^2\tau^2}{h^2}$$

$$\xi^2 + \frac{h^2}{2\tau^2}\xi - \frac{h^2}{2\tau^2}a^2 \le 0$$

$$\xi = \left[-\frac{h^2}{2\tau^2} \pm \sqrt{\frac{h^4}{4\tau^4}+a^2\frac{2h^2}{\tau^2}}\right]\frac{1}{2}$$

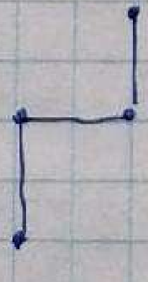
Нер-во вып-но при:

$$-\frac{h^2}{2\tau^2} - \sqrt{\frac{h^4}{4\tau^4}+\frac{2h^2}{\tau^2}a^2} \le \xi \le -\frac{h^2}{2\tau^2} + \sqrt{\frac{h^4}{4\tau^4}+\frac{2h^2}{\tau^2}a^2}$$

Сл-но так схема будет устойчивой

XIV.9.8

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \quad a = const > 0$$



Метод неопр. коэфф.

$$\alpha u_m^{n-1} + \beta u_m^n + \gamma u_{m+1}^n + \delta u_{m+1}^{n+1} = 0$$

Alik Putin

$$u_m^{n-1} = u_m^n - \tau u'_t + \frac{\tau^2}{2} u''_{tt} + O(\tau^3)$$

$$u_{m+1}^n = u_m^n + h u'_x + \frac{h^2}{2} u''_{xx} + O(h^3)$$

$$u_{m+1}^{n+1} = u_m^n + \tau u'_t + h u'_x + \frac{\tau^2}{2} u''_{tt} + \frac{h^2}{2} u''_{xx} + \tau h u''_{tx} + O(h^3, \tau^3)$$

$$\begin{cases} \alpha + \beta + \gamma + \delta = 0 \\ -\alpha + \delta = \frac{1}{\tau} \\ \gamma + \delta = \frac{a}{h} \\ \alpha + \delta = 0 \end{cases} \qquad \delta = \frac{1}{2\tau}, \quad \alpha = -\frac{1}{2\tau}$$

$$\gamma = -\frac{1}{2\tau} + \frac{a}{h}, \quad \beta = \frac{1}{2\tau} - \frac{a}{h}$$

$$\frac{u_{m+1}^{n+1} + u_m^n}{2} - \frac{u_{m+1}^n + u_m^{n-1}}{2} + \frac{a\tau}{h}\left(u_{m+1}^n - u_m^n\right) = 0$$

При $\sigma = \frac{a\tau}{h} = \frac{1}{2}$, $\frac{a}{h} = \frac{1}{2\tau} \hookrightarrow \gamma = 0, \beta = 0$

$$u_{m+1}^{n+1} = u_m^{n-1}$$

$$\tau u'_t + h u'_x = -\tau u'_t$$

$u'_t + \frac{h}{2\tau} u'_x = u'_t + a u'_x = 0 \hookrightarrow$ схема точна при $\sigma = \frac{a\tau}{h}$