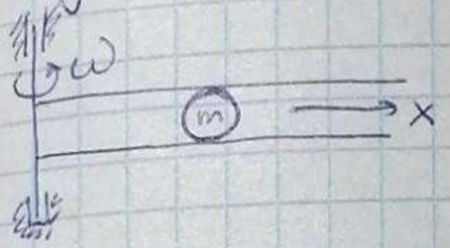
Задача 19.1 Составить и решить каноническ. ур-ния его относит. движ.



Рассмотрим обобщ. коорд. $q = x$.

$$T = m\frac{\dot{x}^2}{2} \qquad \Pi = \Pi^{пер} = \frac{\omega^2 x^2 m}{2}$$

$$L = T - \Pi = \frac{m\dot{x}^2}{2} - \frac{\omega^2 x^2 m}{2}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \dot{x} = \frac{p_x}{m}$$

$$H = \frac{p_x^2}{m} - \frac{m p_x^2}{2m^2} - \frac{\omega^2 x^2 m}{2}$$

$$\begin{cases} \dot{x} = \dfrac{p_x}{m} \\ \dot{p}_x = -\dfrac{\partial H}{\partial x} = \omega^2 x m \end{cases}$$

$$\ddot{x} = \frac{\partial p_x}{\partial x}\dot{x}\frac{1}{m} + \frac{\dot{p}_x}{m} = \frac{\dot{p}_x}{m} = \omega^2 x$$

$$\ddot{x} = \omega^2 x \hookrightarrow \lambda^2 = \omega^2 \qquad \lambda = \pm\omega$$

$$\begin{cases} x = C_1 e^{\omega t} + C_2 e^{-\omega t} \\ p_x = \omega m C_1 e^{\omega t} - \omega m C_2 e^{-\omega t} \end{cases}$$

Найдём константы через нач. значения

$$x_0 = C_1 + C_2 \qquad P_0 = \omega m C_1 - \omega m C_2$$

$$C_1 = x_0 - C_2$$

$$p_0 = \omega m x_0 - 2\omega m C_2$$

Откуда $C_2 = \dfrac{p_0 - \omega m x_0}{-2\omega m} = \dfrac{x_0}{2} - \dfrac{p_0}{2\omega m}$

$$C_1 = x_0 - C_2 = \frac{x_0}{2} + \frac{p_0}{2\omega m}.$$

$$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 1/m \\ \omega \end{pmatrix} \left(\frac{m x_0}{2} + \frac{p_0}{2\omega} \right) e^{\omega t} + \begin{pmatrix} 1/m \\ -\omega \end{pmatrix} \left(\frac{m x_0}{2} - \frac{p_0}{2\omega} \right) \cdot e^{-\omega t}$$

Задача 19.9

$$L = \frac{(\dot{q}_1 - \dot{q}_2)^2 + a\dot{q}_1^2 t^2}{2} - a\cos q_2$$

Найти гамильтониан и составить канонич. ур-ие движения

$$P_{q_1} = \frac{\partial L}{\partial \dot{q}_1} = (\dot{q}_1 - \dot{q}_2) + a\dot{q}_1 t^2$$

$$P_{q_2} = -(\dot{q}_1 - \dot{q}_2)$$

$$H = P_{q_1} \cdot \hat{\dot{q}}_1 + P_{q_2} \cdot \hat{\dot{q}}_2 - \frac{(\hat{\dot{q}}_1 - \hat{\dot{q}}_2)^2 + a\hat{\dot{q}}_1^2 t^2}{2} + a\cos q_2$$

$$\hat{\dot{q}}_1 = \frac{P_{q_1} + P_{q_2}}{a t^2} \qquad \hat{\dot{q}}_2 = P_{q_2} + \frac{P_{q_1} + P_{q_2}}{a t^2}$$

$$H = p_{q_1} \cdot \frac{p_{q_1} + p_{q_2}}{a t^2} + p_{q_2} \cdot \left(p_{q_2} + \frac{p_{q_1} + p_{q_2}}{a t^2} \right) -$$

$$- \frac{p_{q_2}^2 + a t^2 \left(\frac{p_{q_1} + p_{q_2}}{a t^2} \right)^2}{2} + a \cos q_2 = \frac{p_{q_2}^2}{2} + \frac{(p_{q_1} + p_{q_2})^2}{2 a t^2} + a \cos q_2$$

Канонич. ур-ия движения:

$$\begin{cases} \dot{q}_1 = \frac{\partial H}{\partial p_{q_1}} = \frac{p_{q_1} + p_{q_2}}{a t^2} \\ \dot{q}_2 = \frac{\partial H}{\partial p_{q_2}} = p_{q_2} + \frac{p_{q_1} + p_{q_2}}{a t^2} \\ \dot{p}_{q_1} = -\frac{\partial H}{\partial q_1} = 0 \longrightarrow p_{q_1} = const = p_{q_{1_0}} \\ \dot{p}_{q_2} = -\frac{\partial H}{\partial q_2} = a \sin q_2 \end{cases}$$

Задача 19.15

$$H = \frac{1}{2}\,\frac{p_1^2 + p_2^2}{q_1^2 + q_2^2} + a(q_1^2 + q_2^2)$$

Найти ф-цию Лагранжа

$$L = p^T\dot{q} - H(q, \hat{p}, t)$$

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{q_1^2 + q_2^2}$$

$$p_1 = \dot{q}_1(q_1^2 + q_2^2)$$

$$p_2 = \dot{q}_2(q_1^2 + q_2^2)$$

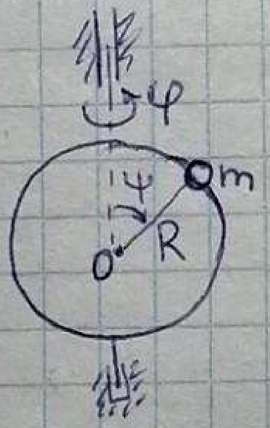
$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = \frac{p_2}{q_1^2 + q_2^2}$$

$$L = \dot{q}_1^2(q_1^2 + q_2^2) + \dot{q}_2^2(q_1^2 + q_2^2) - \frac{1}{2}\,\frac{\dot{q}_1^2(q_1^2 + q_2^2)^2 + \dot{q}_2^2(q_1^2 + q_2^2)^2}{q_1^2 + q_2^2} -$$

$$-a\left(q_1^2+q_2^2\right)=\left(q_1^2+q_2^2\right)\left(\dot{q}_1^2+\dot{q}_2^2\right)-\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right)\left(q_1^2+q_2^2\right)-$$

$$-a\left(q_1^2+q_2^2\right)=\left(q_1^2+q_2^2\right)\cdot\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2-2a\right)$$

Задача 19.19



Окружность массы M.
H – ? Составить канонич. ур-ия движ.

Запишем кинетич. и потенц. энергии системы:

$$T = \frac{J\dot{\varphi}^2}{2} + \frac{m}{2}\left((\dot{\psi}R)^2 + (\dot{\varphi}R\sin\psi)^2\right), \text{ где } J = MR^2/2$$

$$\Pi = mgR(\cos\psi + 1)$$

$$L = T - \Pi = \frac{MR^2\dot{\varphi}^2}{4} + \frac{m}{2}\left((\dot{\psi}R)^2 + (\dot{\varphi}R\sin\psi)^2\right) -$$

$$- mgR(\cos\psi + 1)$$

$$P_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = MR^2\frac{\dot{\varphi}}{2} + mR^2\sin^2\psi\,\dot{\varphi}$$

$$P_\psi = \frac{\partial L}{\partial\dot{\psi}} = mR^2\dot{\psi}$$

$$H = P_\varphi\frac{P_\varphi}{MR^2/2 + mR^2\sin^2\psi} + P_\psi\cdot\frac{P_\psi}{mR^2} + mgR\cdot(\cos\psi+1) -$$

$$- \frac{mR^2}{4}\frac{P_\varphi^2}{(MR^2/2 + mR^2\sin^2\psi)^2} - \frac{m}{2}\left(\frac{P_\psi^2}{(mR^2)^2}R^2 + R^2\sin^2\psi\,.\right.$$

$$\cdot \frac{p_\varphi^2}{\left(\frac{MR^2}{2}+mR^2\sin^2\psi\right)^2}\Big) = \frac{p_\varphi^2}{\frac{MR^2}{2}+mR^2\sin^2\psi} + \frac{p_\psi^2}{2mR^2} -$$

$$- \frac{p_\varphi^2}{2\left(\frac{MR^2}{2}+mR^2\sin^2\psi\right)} + mRg(\cos\psi+1) =$$

$$= \frac{p_\varphi^2}{MR^2+2mR^2\sin^2\psi} + \frac{p_\psi^2}{2mR^2} + mgR(\cos\psi+1).$$

Канонич. ур-ия движения:

$$\begin{cases} \dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{2p_\varphi}{MR^2+2mR^2\sin^2\psi} \\ \dot\psi = \frac{\partial H}{\partial p_\psi} = \frac{p_\psi}{mR^2} \\ \dot p_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \hookrightarrow p_\varphi = const \\ \dot p_\psi = -\frac{\partial H}{\partial \psi} = mgR\sin\psi + \frac{p_\varphi^2 \cdot 4mR^2\sin\psi\cos\psi}{(MR^2+2mR^2\sin^2\psi)^2} \end{cases}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial q}\underbrace{\dot q}_{\frac{\partial H}{\partial p}} + \frac{\partial H}{\partial p}\underbrace{\dot p}_{-\frac{\partial H}{\partial q}} = \frac{\partial H}{\partial t} = 0$$ - консерв. система

Значит $H = h = const$

$$h - mgR(\cos\psi+1) = \frac{p_\psi^2}{2mR^2} + \frac{p_\varphi^2}{MR^2+2mR^2\sin^2\psi}$$

$$p_\psi^2 = \frac{(h-mgR(\cos\psi+1))(M+2m\sin^2\psi)R^2 - p_\varphi^2}{(M+2m\sin^2\psi)R^2} 2mR^2$$

Можем подставить во второе ур-ие системы:

$$\dot{\psi} = \frac{\sqrt{2\left[(h - mgR(\cos\psi + 1))(M + 2m\sin^2\psi)R^2 - p_\varphi^2\right]}}{R^2\sqrt{m(M + 2m\sin^2\psi)}}$$

После интегрирования получим

$$t = \int_{\psi_0}^{\psi} \frac{R^2\sqrt{m(M + 2m\sin^2\psi)}\, d\psi}{\sqrt{2\left[(h - mgR(\cos\psi + 1))(M + 2m\sin^2\psi)R^2 - p_\varphi^2\right]}}$$

И для φ подставим dt в первое:

$$\varphi - \varphi_0 = \int_{\psi_0}^{\psi} \frac{\sqrt{2m}\, p_\varphi\, d\psi}{\sqrt{(M + 2m\sin^2\psi)}\sqrt{\left((h - mgR(\cos\psi + 1))(M + 2m\sin^2\psi)R^2 - p_\varphi^2\right]}}$$

Задача 19.35

$$\dot{q} = Aq + Bp$$
$$\dot{p} = Cq + Dp$$

A, B, C, D: – гамильт. система – ?

Для того, чтобы эта система была Гамильтоновой, должна $\exists H$:

$\dot{q} = \frac{\partial H}{\partial p}$ $\quad \dot{p} = -\frac{\partial H}{\partial q}$ такая, что

$$\frac{\partial^2 H}{\partial p \partial q} = \frac{\partial^2 H}{\partial q \partial p}$$

$$\frac{\partial \dot q}{\partial q^T} = \frac{\partial^2 H}{\partial q^T \partial p} \qquad \frac{\partial \dot p}{\partial p^T} = -\frac{\partial^2 H}{\partial p^T \partial q}$$

Т.к. H - дважды непрер. дифф, то

$$\frac{\partial^2 H}{\partial q^T \partial p} = + \frac{\partial^2 H}{\partial p^T \partial q} \hookrightarrow \frac{\partial \dot q}{\partial q^T} = -\frac{\partial \dot p}{\partial p^T}$$

Отсюда получаем $A = -D = A^T$

т.к. $\frac{\partial^2 H}{\partial q^T \partial p}$ - симм. матр.

Продифф. по другим координатам:

$\frac{\partial \dot q}{\partial p^T} = \frac{\partial^2 H}{\partial p^T \partial p}$ - симм. матр $\hookrightarrow B = B^T$

И аналогично $C = C^T$.

Ответ: $A = -D = A^T$, $B = B^T$, $C = C^T$.

Задача 19.77

$$H=\sqrt{m^2c^4+c^2\left(\vec{p}-\frac{e}{c}A\right)^2}+e\varphi$$

$\vec{p}$ - импульс φ и A - скалярн. и вект. потенц. поля. L - ?

$$\dot{\vec{q}}=\frac{\partial H}{\partial \vec{p}}=\frac{c^2\left(\vec{p}-\frac{e}{c}A\right)}{\sqrt{m^2c^4+c^2\left(\vec{p}-\frac{e}{c}A\right)^2}}$$

$$\dot{\vec{q}}^2\left(m^2c^4+c^2\left(\vec{p}-\frac{e}{c}A\right)^2\right)=c^4\left(\vec{p}-\frac{e}{c}A\right)^2$$

$$\dot{\vec{q}}^2m^2c^4+\left(\vec{p}-\frac{e}{c}A\right)^2\left(\dot{\vec{q}}^2c^2-c^4\right)=0$$

Виразим $\vec{p}$:

$$\left(\vec{p} - \frac{e}{c}A\right)^2 = \frac{\dot{\vec{q}}^2 m^2 c^4}{c^2(c^2 - \dot{\vec{q}}^2)} = \frac{\dot{\vec{q}}^2 m^2 c^2}{c^2 - \dot{\vec{q}}^2}$$

$$\vec{p} = \dot{\vec{q}} \frac{mc}{\sqrt{c^2 - \dot{\vec{q}}^2}} + \frac{e}{c}A$$

$$L = \hat{\vec{p}}^T \cdot \dot{\vec{q}} - H(\vec{q}, \hat{\vec{p}}, t) = \dot{\vec{q}} \cdot \dot{\vec{q}} \frac{mc}{\sqrt{c^2 - \dot{\vec{q}}^2}} + \frac{e}{c}(A, \dot{\vec{q}}) -$$

$$- H = \frac{mc\,\dot{\vec{q}}^2}{\sqrt{c^2 - \dot{\vec{q}}^2}} + \frac{e}{c}(A, \dot{\vec{q}}) - \frac{mc^3}{\sqrt{c^2 - \dot{\vec{q}}^2}} - e\varphi =$$

$$\| \; H - e\varphi$$

$$= -mc^2\sqrt{1 - \frac{\dot{\vec{q}}^2}{c^2}} + \frac{e}{c}(A, \dot{\vec{q}}) - e\varphi = \underline{L}$$