(7) $\xi_1$ и $\xi_2$ имеют плотн. распр. $\rho(x)$. Найти совместн. плотн. распр. $(r, \varphi)$ т. $(\xi_1, \xi_2)$ $\quad 0 \le \varphi < 2\pi$

$\xi_1 = r \cdot \cos\varphi$

$\xi_2 = r \cdot \sin\varphi$

$$A = \begin{cases} x^2 + y^2 \le r_0^2 \\ \operatorname{arctg} \frac{y}{x} \le \varphi_0 \end{cases}$$

$$P(r \le r_0, \varphi \le \varphi_0) = \iint\limits_A \rho_{\xi_1}(x)\, \rho_{\xi_2}(y)\, dx\, dy =$$

$$= \int_0^{r_0} \int_0^{\varphi_0} \rho(r\cos\varphi)\, \rho(r\sin\varphi)\, r\, dr\, d\varphi$$

Откуда получим $\rho_{r,\varphi}(r_0, \varphi_0) = \rho(r_0 \cos\varphi_0) \cdot \rho(r_0 \sin\varphi_0)\, \underline{r_0}$

где $0 \le r_0 \le +\infty \qquad 0 \le \varphi_0 < 2\pi$

(8) С.в. $\xi$ и $\eta$ нез. с равном. р. на $[0, a]$

$\rho(\chi_1), \rho(\chi_2), \rho(\chi_3), \rho(\chi_4)$ - ? $\quad \chi_1 = \xi + \eta \quad \chi_2 = \xi - \eta$

$\chi_3 = \xi \cdot \eta \quad \chi_4 = \dfrac{\xi}{\eta}$.

1) $\rho_{\xi+\eta}(x) = \int\limits_0^x \rho_\xi(y) \cdot \rho_\eta(x-y)\, dy = \dfrac{x}{a^2}, \quad x \in [0, a]$

$$\rho_{\xi+\eta}(x) = \int\limits_{x-a}^{a} \rho_\xi(y) \cdot \rho_\eta(x-y)\, dy = \frac{2}{a} - \frac{x}{a^2}, \quad x \in [a, 2a]$$

$$\rho_{\chi_1}(x) = \begin{cases} 0, & x \in (-\infty, 0) \cup (2a, +\infty) \\ \dfrac{x}{a^2}, & x \in [0, a) \\ \dfrac{2}{a} - \dfrac{x}{a^2}, & x \in [a, 2a] \end{cases}$$

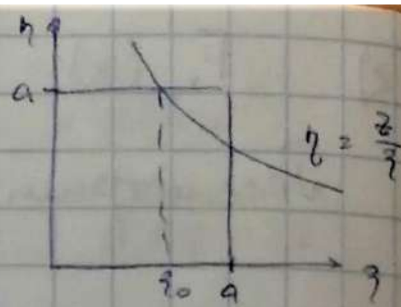
2) $F_{x_2}(x) = P(\xi - \eta \le x)$

Рассмотрим $\zeta = -\eta$ : $\rho_{\zeta}(x) = \begin{cases} 0, & x \notin [-a, 0] \\ \frac{1}{a}, & x \in [-a, 0] \end{cases}$

При $-a \le x \le 0$, $0 < x \le a$ ан-но п. 1 получим:

$$\rho_{x_2}(x) = \begin{cases} 0, & x \notin [-a, a] \\ \frac{x+a}{a^2}, & x \in [-a, 0] \quad \leftarrow \int_{-a}^{x} \rho_{\xi}(y) \cdot \rho_{\zeta}(x-y)\,dy = \frac{x+a}{a^2} \\ \frac{2}{a} - \frac{x+a}{a^2}, & x \in (0, a] \end{cases}$$

3) $F_{X_3}(z) = P(\xi\eta \le z) =$




$$= \int_0^{\xi_0} \rho_\xi(x)\,dx \int_0^a \rho_\eta(y)\,dy + \int_{\xi_0}^a \rho_\xi(x)\,dx \int_0^{z/x} \rho_\eta(y)\,dy =$$

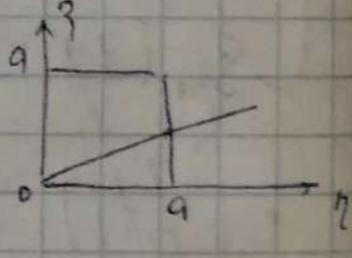
$$= \frac{1}{a^2}\left(\xi_0 a + \int_{\xi_0}^a \frac{z}{x}\,dx\right) = \frac{\xi_0}{a} + \cancel{\xi_0}\,\frac{z}{a^2} \ln x \Big|_{\xi_0}^a =$$

$$= \frac{\xi_0}{a} + \frac{z}{a^2} \ln \frac{a}{\xi_0}$$ , а $\underline{\underline{\xi_0 = \frac{z}{a}}}$

$$P(\xi\eta \le z) = \frac{z}{a^2} + \frac{z}{a^2} \ln\left(\frac{a^2}{z}\right)$$

$$\rho_{X_3}(z) = \begin{cases} 0, & z \notin [0, a^2] \\ \frac{1}{a^2} \ln \frac{a^2}{z}, & z \in [0, a^2] \end{cases}$$

4) $z \in [0,1]$:




$$P(\xi \le z \cdot \eta) = \frac{1}{a^2} \cdot \frac{a \cdot za}{2} = \frac{z a^2}{a^2 2} = \frac{z}{2}$$

если $z > 1$:

$$P(\xi \le z \cdot \eta) = \frac{1}{a^2} \cdot \left(a^2 - \frac{1}{2} a \cdot \frac{a}{z}\right) = 1 - \frac{1}{2z}$$

$$\rho_{X_4}(z) = \begin{cases} 0, & z < 0 \\ 1/2, & z \in [0, 1] \\ \frac{1}{2z^2}, & z \ge 1 \end{cases}$$

(9) $\xi \sim U[-1,1]$

| $\eta$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $P(\eta)$ | $1/3$ | $1/3$ | $1/3$ |

$$F_{\xi\eta}(z) = P(\xi\eta \le z) = \sum_{k=-1}^{1} P(\xi k \le z) \cdot P(\eta = k) =$$

$$= \frac{1}{3}\left( P(-\xi \le z) + P(\xi \cdot 0 \le z) + P(\xi \le z)\right) =$$

$$= \frac{1}{3}\left( F_{-\xi}(z) + F_0(z) + F_\xi(z)\right)$$

$$F_{\xi\eta}(z) = \begin{cases} 0, & z \le -1 \\ \frac{x}{3} + \frac{1}{3}, & z \in (-1, 0) \\ \frac{x}{3} + \frac{2}{3}, & z \in [0, 1) \\ 1, & z \ge 1 \end{cases}$$

$$F_{\xi+\eta}(z) = P(\xi+\eta \le z) = \sum_{k=-1}^{1} P(\xi + k \le z) \cdot P(\eta = k) =$$

$$= \frac{1}{3}\left( P(\xi - 1 \le z) + P(\xi \le z) + P(\xi + 1 \le z)\right) =$$

$$= \frac{1}{3}\left( F_\xi(z+1) + F_\xi(z) + F_\xi(z-1)\right)$$

$$F_{\xi+\eta}(z) = \begin{cases} 0, & z \le -2 \\ \frac{x}{6} + \frac{1}{3}, & z \in (-2, -1] \\ \frac{x}{3} + \frac{1}{2}, & z \in [-1, 1] \\ \frac{x}{6} + \frac{2}{3}, & z \in (1, 2) \\ 1, & z \ge 2 \end{cases}$$

(10) $\xi \sim \exp(\lambda)$ $\qquad$ $P(\xi > k \mid \xi > l)$ - ?

$$F_\xi(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

если $k \leq l$, то $\qquad P(\xi > k \mid \xi > l) = 1$

если $k > l$:

$$P(\xi > k \mid \xi > l) = \overset{1}{\overset{\shortparallel}{P(\xi > l \mid \xi > k)}} \cdot \frac{P(\xi > k)}{P(\xi > l)} =$$

$$= \frac{1 - P(\xi \leq k)}{1 - P(\xi \leq l)} = \frac{1 - F_\xi(k)}{1 - F_\xi(l)} = e^{-\lambda(k-l)}.$$

(11) C найдём из того, что

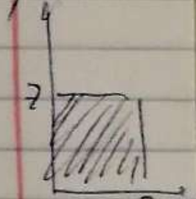
$$\iint\limits_{[0,1]\times[0,1]} \rho_{\xi,\eta}(x,y)\,dx\,dy = 1$$

$$C\int_0^1 dx \int_0^1 (x+y)\,dy = C\int_0^1 \left(x+\frac{1}{2}\right)dx = C\cdot\left(\frac{1}{2}+\frac{1}{2}\right) = C$$

Откуда $C = 1$.

$$\rho_\eta(y) = \int_{-\infty}^{+\infty} \rho_{\xi,\eta}(x,y)\,dx = \int_0^1 \rho_{\xi\eta}(x,y)\,dx = y + \frac{1}{2}$$

$$\rho_\xi(x) = \int_{-\infty}^{+\infty} \rho_{\xi,\eta}(x,y)\,dy = \int_0^1 \rho_{\xi,\eta}(x,y)\,dy = x + \frac{1}{2}$$




$$F_{\underset{\max(\xi,\eta)}{\gamma}}(z) = P(\max(\xi,\eta) \le z) = P(\xi \le z,\ \eta \le z) =$$

$$= \iint_A \rho_{\xi,\eta}(x,y)\,dx\,dy = \int_0^z dx \int_0^z (x+y)\,dy =$$

$$= \int_0^z xz + \frac{z^2}{2}\,dx = \frac{z^3}{2} + \frac{z^3}{2} = z^3.$$

$$F_\gamma(z) = \begin{cases} 0, & z \le 0 \\ z^3, & z \in (0,1) \\ 1, & z \ge 1 \end{cases} \longrightarrow \rho_\gamma(z) = \begin{cases} 0, & z \notin [0,1] \\ 3z^2, & z \in [0,1] \end{cases}$$

(12) $\xi_1, \ldots, \xi_n, \ldots \sim U[0,1]$

$$E\left(\frac{1}{n}\sum_{i=1}^{n} f(\xi_i)\right) = \frac{1}{n}\, n\, E\left(f(\xi_1)\right) = \int_{-\infty}^{+\infty} f(x)\, \rho_{\xi_1}(x)\, dx = \int_0^1 f(x)\, dx$$

По нер-ву Чебышева:

$$P\left(\left|\underbrace{\frac{1}{n}\sum_{i=1}^{n} f(\xi_i)}_{\eta} - \underbrace{\int_0^1 f(x)\, dx}_{E\eta}\right| > \varepsilon\right) \le \frac{D(\eta)}{\varepsilon^2} =$$

$$= \frac{1}{n\varepsilon^2} D(f(\xi_1)) = \frac{1}{n\varepsilon^2}\left(\underbrace{\int_0^1 f^2(x)\, dx}_{E(\eta^2)} - \underbrace{\left[\int_0^1 f(x)\, dx\right]^2}_{(E(\eta))^2}\right)$$