XII. 7.6

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$ — ур-ие теплопроводности

Схема Кранка-Николсона:

$$\frac{u_m^{n+1} - u_m^n}{\tau} = \frac{D}{2}\left(\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right)$$

$$u_m^{n+1}\left(1 + \frac{\tau D}{h^2}\right) = u_m^n\left(1 - \frac{D\tau}{h^2}\right) + \frac{D\tau}{2h^2}\left(u_{m+1}^n + u_{m-1}^n + u_{m+1}^{n+1} + u_{m-1}^{n+1}\right)$$

Все коэфф. должны быть неотрицательны для монотонности по Фридриху

$$1 - \frac{D\tau}{h^2} \geqslant 0 \longrightarrow D \leqslant \frac{h^2}{\tau}$$

XIII. 7.3

$$\frac{y_m^{n+1} - y_m^n}{\tau} = D\frac{y_{m-1}^n - 2y_m^{n+1} + y_{m+1}^n}{h^2}$$

$$y_m^{n+1}\left(\frac{1}{\tau} + \frac{2D}{h^2}\right) = \frac{y_m^n}{\tau} + D\frac{y_{m-1}^n + y_{m+1}^n}{h^2}$$

Схема эквивалентна явной с уменьшенным шагом по времени $\tau^* = \dfrac{1}{\dfrac{1}{\tau} + \dfrac{2D}{h^2}}$

XIII. 9.3

$$\frac{y_m^{n+1}-y_m^n}{\tau}=\frac{y_{m-1}^n-2y_m^n+y_{m+1}^n}{h^2}$$

$$y_m^{n+1}=y_m^n+\tau y'_t+\frac{\tau^2}{2}y''_{tt}+O(\tau^3)$$

$$y_{m\pm1}^n=y_m^n\pm hy'_x+\frac{h^2}{2}y''_{xx}\pm\frac{h^3}{6}y'''_{xxx}+\frac{h^4}{24}y^{(IV)}_{xxxx}+\frac{h^5}{120}y^{(V)}_x+O(h^6)$$

$$y'_t+\frac{\tau}{2}y''_{tt}=y''_{xx}+\frac{h^2}{12}y^{(IV)}_{xxxx}+O(h^4)+O(\tau^2)$$

Из ур-ия $y'_t=Dy''_{xx}$

$$y''_{tt}=Dy'''_{xxt}=D(y'_t)''_{xx}=D^2y^{(IV)}_x$$

$$D^2\frac{\tau}{2}-\frac{h^2}{12}=0\qquad \tau=\frac{h^2}{6D^2}$$

Ответ: при $\frac{\tau}{h^2}=\frac{1}{6D^2}$

XIII. 9.8

$$\frac{y_m^{n+1}-y_m^{n-1}}{2\tau}=\frac{y_{m+1}^n-y_m^{n+1}-y_m^{n-1}+y_{m-1}^n}{h^2}$$

$$y_m^n\sim\lambda^n e^{ikmh}$$

$$\frac{\lambda^2-1}{2\tau}=\frac{\lambda e^{ikh}-\lambda^2-1+\lambda e^{-ikh}}{h^2}$$

$$\lambda^2-1=\frac{4\tau}{h^2}\lambda\cos kh-\frac{\lambda^2+1}{h^2}2\tau$$

$$\lambda^2\left(1+\frac{2\tau}{h^2}\right)-\frac{4\tau}{h^2}\lambda\cos kh-1=0$$

$$\frac{\Delta}{4}=\frac{4\tau^2}{h^4}\cos^2kh+1+\frac{2\tau}{h^2}>0 \hookrightarrow \lambda=\frac{\frac{2\tau}{h^2}\cos kh\pm\sqrt{\frac{\Delta}{4}}}{1+\frac{2\tau}{h^2}}$$

$$|\lambda|\le1$$

Если $\frac{\Delta}{4} = \left(1 + \frac{2\tau}{h^2} - \frac{2\tau}{h^2}\cos kh\right)^2$

$$\frac{4\tau^2}{h^4}\cos^2 kh + 1 + \frac{2\tau}{h^2} \leq 1 + \frac{4\tau}{h^2} + \frac{4\tau^2}{h^4} + \frac{4\tau^2}{h^4}\cos^2 kh -$$

$$- \frac{2\tau}{h^2}\cos kh - \frac{4\tau^2}{h^4}\cos kh$$

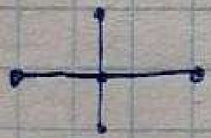
$$\frac{2\tau}{h^2} + \frac{4\tau^2}{h^4} - \frac{2\tau}{h^2}\cos kh - \frac{4\tau^2}{h^4}\cos kh \geq 0$$

$\frac{2\tau}{h^2} 2\sin^2 kh + \frac{4\tau^2}{h^4} 2\sin^2 kh \geq 0$ – верно $\forall \tau, h$

Если $\frac{\Delta}{4} = \left(1 + \frac{2\tau}{h^2} + \frac{2\tau}{h^2}\cos kh\right)^2$

$\frac{2\tau}{h^2} 2\cos^2 kh + \frac{4\tau^2}{h^4} 2\cos^2 kh \geq 0 \rightarrow$ аналогично

Значит схема устойчива при любых шагах



Разложим отн. центральной точки

$$y_m^{n\pm1} = y_m^n \pm \tau y_t' + \frac{\tau^2}{2} y_{tt}'' + O(\tau^3)$$

$$y_{m\pm1}^n = y_m^n \pm h y_x' + \frac{h^2}{2} y_{xx}'' \pm \frac{h^3}{6} y_{xxx}''' + O(h^4)$$

$$y_t' + O(\tau^2) = -\frac{\tau^2}{h^2} y_{tt}'' + y_{xx}'' + O(h^2) + O\left(\frac{\tau^3}{h^2}\right)$$

$$y_t' + c^2 y_{tt}'' = y_{xx}'' + O(h^2) + O(\tau)$$

Схема approx. ур-ие при $c = \text{const}$ с порядком $O(\tau, h^2)$.

## XIII. 9.9

$$\frac{1}{12}\frac{y_{m+1}^{n+1}-y_{m+1}^{n}}{\tau}+\frac{5}{6}\frac{y_m^{n+1}-y_m^n}{\tau}+\frac{1}{12}\frac{y_{m-1}^{n+1}-y_{m-1}^{n}}{\tau}=$$

$$=\frac{1}{2}\frac{y_{m+1}^{n}-2y_m^n+y_{m-1}^n}{h^2}+\frac{1}{2}\frac{y_{m+1}^{n+1}-2y_m^{n+1}+y_{m-1}^{n+1}}{h^2}$$

Порядок есть $O(\tau, h^2)$

На устойчивость: $y_m^n \sim \lambda^n e^{ikmh}$

$$\lambda\left[\frac{1}{12\tau}\left(e^{ikh}+e^{-ikh}\right)\right]+\frac{5}{6\tau}(\lambda-1)-\frac{1}{12\tau}\left(e^{ikh}-e^{-ikh}\right)=$$

$$=\frac{1}{2}\frac{e^{ikh}+e^{-ikh}-2}{h^2}+\frac{1}{2\lambda}\frac{e^{ikh}+e^{-ikh}-2}{h^2}$$

$$\lambda\left(\frac{\cos kh}{6\tau}+\frac{5}{6\tau}+\frac{2\sin^2\frac{kh}{2}}{h^2}\right)=\frac{\cos kh}{6\tau}+\frac{5}{6\tau}-\frac{2\sin^2\frac{kh}{2}}{h^2}$$

$$\lambda=\frac{\frac{1}{6}\cos kh+\frac{5}{6}-\frac{2\tau}{h^2}\sin^2\frac{kh}{2}}{\frac{1}{6}\cos kh+\frac{5}{6}+\frac{2\tau}{h^2}\sin^2\frac{kh}{2}}$$

$$|\lambda|\leq 1$$

## XIII. 9.17

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f(x,t)$$

а) $$\begin{cases}\dfrac{u^{n+\frac12}-u^n}{\tau/2}=\Lambda_{xx}u^{n+\frac12}+\Lambda_{yy}u^n+f\\[2mm] \dfrac{u^{n+1}-u^{n+\frac12}}{\tau/2}=\Lambda_{xx}u^{n+\frac12}+\Lambda_{yy}u^{n+1}+f\end{cases}$$

$$\begin{cases} \left(1-\frac{\tau}{2}\Lambda_{xx}\right)u^{n+\frac{1}{2}} - \left(1+\frac{\tau}{2}\Lambda_{yy}\right)u^n = \frac{\tau}{2}f \\ \left(1-\frac{\tau}{2}\Lambda_{yy}\right)u^{n+1} - \left(1+\frac{\tau}{2}\Lambda_{xx}\right)u^{n+\frac{1}{2}} = \frac{\tau}{2}f \end{cases}$$

Обозначим $A = 1-\frac{\tau}{2}\Lambda$ $\quad B = \frac{\tau}{2}\Lambda + 1$

$$\begin{cases} A_x u^{n+\frac{1}{2}} - B_y u^n = \frac{\tau}{2}f & \Big| \cdot B_x \\ A_y u^{n+1} - B_x u^{n+\frac{1}{2}} = \frac{\tau}{2}f & \Big| \cdot A_x \end{cases}$$

$$\begin{cases} B_x A_x u^{n+\frac{1}{2}} - B_x B_y u^n = \frac{\tau}{2}B_x f \\ A_x A_y u^{n+1} - A_x B_x u^{n+\frac{1}{2}} = \frac{\tau}{2}A_x f \end{cases}$$

$$A_x A_y u^{n+1} - B_x B_y u^n - \cancelto{0}{(A_x B_x - B_x A_x)}\, u^{n+\frac{1}{2}} = \frac{\tau}{2}(A_x + B_x) f$$

$$\left(1-\frac{\tau}{2}\Lambda_{xx}\right)\left(1-\frac{\tau}{2}\Lambda_{yy}\right)u^{n+1} - \left(1+\frac{\tau}{2}\Lambda_{xx}\right)\left(1+\frac{\tau}{2}\Lambda_{yy}\right)u^n =$$
$$= \tau f$$

Оставляем линейные по $\tau$ слагаемые:

$$u^{n+1} - \left(\frac{\tau}{2}\Lambda_{xx} + \frac{\tau}{2}\Lambda_{yy}\right)u^{n+1} - u^n - \left(\frac{\tau}{2}\Lambda_{xx} + \frac{\tau}{2}\Lambda_{yy}\right)u^n = \tau f$$

$$\frac{u^{n+1} - u^n}{\tau} = (\Lambda_{xx} + \Lambda_{yy})\frac{u^{n+1} + u^n}{2} + f$$

$$u_t = u_{xx} + u_{yy} + f + O(\tau) + O(h_x^2) + O(h_y^2)$$ – аппроксим.

$$u^n \sim \lambda^n e^{i(kh_x + mh_y)}$$

$$\Lambda_{xx}u^n = -\frac{4}{h_x^2}\sin^2\frac{kh_x}{2} \qquad \Lambda_{yy}u^n = -\frac{4}{h_y^2}\sin^2\frac{mh_y}{2}$$

$$\frac{\lambda_1^{1/2} - 1}{\tau/2} = -\frac{4}{h_x^2}\underbrace{\sin^2\frac{kh_x}{2}}_{a}\lambda_1^{1/2} - \frac{4}{h_y^2}\underbrace{\sin^2\frac{mh_y}{2}}_{b}$$

$$\lambda_1^{1/2} = \frac{1 - \frac{\sigma_y b}{2}}{1 + \frac{\sigma_x a}{2}}, \text{ где } \sigma_x = \frac{\tau}{h_x^2}, \quad \sigma_y = \frac{\tau}{h_y^2}$$

Аналогично $\lambda_2^{1/2} = \dfrac{1-\sigma_x a\cdot\frac{1}{2}}{1+\frac{\sigma_y b}{2}}$

$$\lambda = \lambda_1^{1/2}\lambda_2^{1/2} = \frac{1-\frac{\sigma_y b}{2}}{1+\frac{\sigma_x a}{2}}\cdot\frac{1-\frac{\sigma_x a}{2}}{1+\frac{\sigma_y b}{2}}$$

$|\lambda| \le 1$, т.к. числит. меньше знаменателя

Схема безусловно устойчива

б) $$\begin{cases} \dfrac{u^{n+\frac{1}{2}}-u^n}{\tau} = \dfrac{1}{2}\left(\Lambda_{xx}u^{n+\frac{1}{2}} + \Lambda_{yy}u^n\right) + f \\ \dfrac{u^{n+1}-u^{n+\frac{1}{2}}}{\tau} = \dfrac{1}{2}\left(\Lambda_{xx}u^{n+1} + \Lambda_{yy}u^n\right) \end{cases}$$

Обозначим $A = 1-\frac{\tau}{2}\Lambda_{xx}$, $B = 1+\frac{\tau}{2}\Lambda_{yy}$

$$\begin{cases} Au^{n+\frac{1}{2}} - Bu^n = f\tau & |\cdot B \\ Au^{n+1} - Bu^{n+\frac{1}{2}} = 0 & |\cdot A \end{cases}$$

$$AAu^{n+1} - BBu^n = Bf\tau$$

Оставим линейные по $\tau$ слагаемые:

$$u^{n+1} - \tau\Lambda_{xx}u^{n+1} - u^n - \tau\Lambda_{yy}u^n = f\tau$$

$$\frac{u^{n+1}-u^n}{\tau} = \Lambda_{xx}u^{n+1} + \Lambda_{yy}u^n + f$$

$u_t = u_{xx} + u_{yy} + O(\tau) + O(h_x^2) + O(h_y^2) + f$ - аппрокс

$$u^n \sim \lambda^n e^{i(kh_x + mh_y)}$$

$$\frac{\lambda_1^{1/2} - 1}{\tau} = \frac{1}{2}\left( -\underbrace{\frac{4}{h_x^2}\sin^2\frac{kh_x}{2}}_{a}\lambda_1^{1/2} - \underbrace{\frac{4}{h_y^2}\sin^2\frac{mh_y}{2}}_{b}\right)$$

$$\lambda_1^{1/2} = \frac{1 - \frac{\sigma_y b}{2}}{1 + \frac{\sigma_x a}{2}}$$

$$\lambda_2^{1/2} = \frac{1 - \frac{\sigma_y b}{2}}{1 + \frac{\sigma_x a}{2}} \longrightarrow \lambda = \lambda_1^{1/2}\lambda_2^{1/2} = \left(\frac{1 - \frac{\sigma_y b}{2}}{1 + \frac{\sigma_x a}{2}}\right)^2$$

$|\lambda| < 1 \hookrightarrow$ схема безусловно устойчива