§12 **248**

$$\int_{-\infty}^{+\infty} \sin x \, dx$$

В смысле главного значения:

$$\lim_{A \to +\infty} \int_{-A}^{A} \sin x \, dx = \lim_{A \to +\infty} -\cos x \Big|_{-A}^{A} = \lim_{A \to +\infty} 0 = \underline{0}$$

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$$\int_{-0}^{+\infty} \frac{dx}{1-x^2}$$

Сделаем замену

$t = \ln x$

$$\int_{-\infty}^{+\infty} \frac{de^t}{1-e^{2t}}$$ в смысле главного значения:

$$\lim_{A\to+\infty} \int_{-A}^{A} \frac{de^t}{1-e^{2t}} = \lim_{A\to+\infty} -\frac{1}{2}\ln\left|\frac{e^{2t}-1}{e^{2t}+1}\right|\Bigg|_{-A}^{A} =$$

$$= \lim_{A\to+\infty}\left(-\frac{1}{2}\ln\left|\frac{e^{2A}-1}{e^{2A}+1}\right| + \frac{1}{2}\ln\left|\frac{e^{-2A}-1}{e^{-2A}+1}\right|\right) =$$

$$= \lim_{A\to+\infty} \frac{1}{2}\ln\left|\frac{e^{-2A}-1}{e^{-2A}+1}\cdot\frac{e^{2A}+1}{e^{2A}-1}\right| = \lim_{A\to+\infty} \frac{1}{2}\ln\left|\frac{e^{-2A}-e^{2A}}{e^{2A}-e^{-2A}}\right| = \lim_{A\to+\infty} 0 = 0$$

§17. 1(3)

$$f(x) = \operatorname{sign}(x-a) - \operatorname{sign}(x-b), \quad b>a$$

$$f(x) = \begin{cases} 0, & x<a \\ 2, & a<x<b \\ 0, & x>b \\ 1, & x=a, \text{ и } x=b \end{cases}$$

$$a(y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cos ty\,dt = \frac{1}{\pi}\left[\int_{-\infty}^{a} 0\cdot\cos ty\,dt + \int_{a}^{b} 2\cos ty\,dt + \int_{b}^{+\infty} 0\cos ty\,dt\right] = \frac{2}{\pi}\,\frac{\sin by - \sin ay}{y}$$

$$b(y) = \frac{1}{\pi}\int_{a}^{b} 2\sin ty\,dt = \frac{2}{\pi}\,\frac{\cos ay - \cos by}{y}$$

$$I_f(x) = \frac{2}{\pi}\int_0^{+\infty}\left[\frac{\sin by - \sin ay}{y}\cos xy + \frac{\cos ay - \cos by}{y}\cdot\sin xy\right]dy = \frac{2}{\pi}\int_0^{+\infty}\frac{\sin(y(x-a)) - \sin(y(x-b))}{y}\,dy.$$

§17 2(4)

$$f(x) = \begin{cases} \sin \omega x, & \text{если } |x| \leqslant 2\pi n/\omega \\ 0, & \text{если } |x| > \frac{2\pi n}{\omega} \end{cases} \qquad n \in N,\ \omega > 0$$

Эта ф-ция нечётная $\rightarrow$ $a(y) = 0$

$$b(y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin ty\,dt = \frac{1}{\pi}\int_{-\frac{2\pi n}{\omega}}^{\frac{2\pi n}{\omega}} \sin\omega t \sin ty\,dt =$$

$$= \frac{2}{\pi}\int_{0}^{\frac{2\pi n}{\omega}} \sin\omega t \sin ty\,dt = \frac{2}{\pi\cdot 2}\int_{0}^{\frac{2\pi n}{\omega}} \big(\cos(\omega t - ty) -$$

$$- \cos(\omega t + ty)\big)\,dt = \frac{1}{\pi}\left[\frac{\sin(\omega t - ty)}{\omega - y} - \frac{\sin(\omega t + ty)}{\omega + y}\right]\Bigg|_{0}^{\frac{2\pi n}{\omega}} =$$

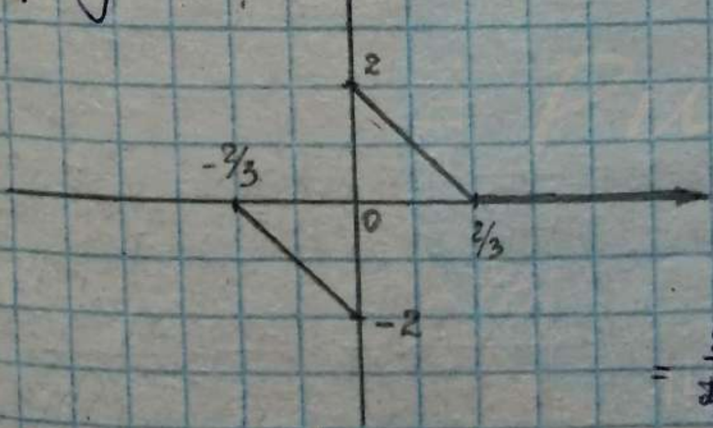
$$= \frac{1}{\pi}\,\frac{-\sin\frac{2\pi n}{\omega}y}{\omega - y} - \frac{1}{\pi}\,\frac{\sin\frac{2\pi n}{\omega}y}{\omega + y} = \frac{2\omega}{\pi}\,\frac{\sin\frac{2\pi n}{\omega}y}{y^2 - \omega^2}$$

$$I_f(x) = \frac{2\omega}{\pi}\int_{0}^{+\infty} \frac{\sin\frac{2\pi n}{\omega}y}{y^2-\omega^2}\sin xy\,dy$$

5(2)

$$f(x)=\begin{cases}2-3x, & 0\le x\le \frac{2}{3}\\ 0, & x>\frac{2}{3}\end{cases}$$

Продолжим её нечётным образом на $(-\infty,0)$



$a(y)=0$

$$b(y)=\frac{2}{\pi}\int_0^{2/3}(2-3t)\sin ty\,dt=$$

$$=\frac{2}{\pi}\left(2\,\frac{1-\cos\frac{2}{3}y}{y}+\right.$$

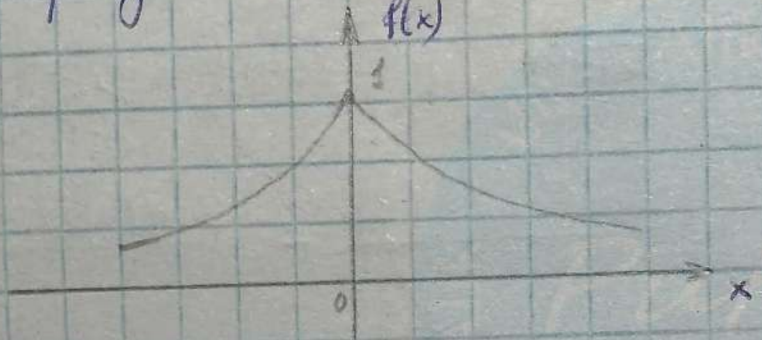
$$\left.+\frac{3}{y}\left(t\cos ty\Big|_0^{2/3}-\int_0^{2/3}\cos ty\,dt\right)\right)=\frac{2}{\pi}\left(\frac{2}{y}-3\frac{\sin\frac{2}{3}y}{y^2}\right)$$

$$I_f=\frac{2}{\pi}\int_0^{+\infty}\frac{2y-3\sin\frac{2}{3}y}{y^2}\sin xy\,dy$$

6(1)

$$f(x) = e^{-\alpha x}, \quad x \geq 0, \quad \alpha > 0$$

Продолжим её чётным образом на $(-\infty, 0)$:



$$b(y) = 0$$

$$a(y) = \frac{2}{\pi}\int_0^{+\infty} e^{-\alpha t}\cos ty\,dt = \frac{2}{\pi}\,\mathrm{Re}\left(\int_0^{+\infty} e^{t(iy-\alpha)}\,dt\right) =$$

$$= \frac{2}{\pi}\,\mathrm{Re}\left(\frac{1}{\alpha - iy}\right) = \frac{2}{\pi}\,\frac{\alpha}{\alpha^2+y^2}$$

$$I_f(x) = \frac{2\alpha}{\pi}\int_0^{+\infty}\frac{\cos xy}{\alpha^2+y^2}\,dy$$

7(4)

$$f(x) = \begin{cases} \sin x, & \text{если } |x| \le \pi \\ 0, & \text{если } |x| > \pi \end{cases}$$

$$F(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ixy} dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin x \, e^{-ixy} dx =$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\pi} (-i \sin xy \cdot \sin x) dx = \frac{-i}{\sqrt{2\pi}} \int_{0}^{\pi} [\cos(x - xy) -$$

$$- \cos(x + xy)] dx = -\frac{i}{\sqrt{2\pi}} \left(\frac{\sin(x-xy)}{1-y} - \frac{\sin(x+xy)}{1+y} \right) \Bigg|_0^{\pi} =$$

$$= -i \sqrt{\frac{2}{\pi}} \frac{\sin \pi y}{1-y^2}$$

8(1,5)

1) $f(x) = x e^{-\alpha|x|}$, $\alpha > 0$

$$F(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ixy} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x e^{-\alpha|x|} e^{-ixy} dx =$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} (-ix e^{-\alpha x} \sin xy) dx = -i\sqrt{\frac{2}{\pi}} \operatorname{Im} \int_{0}^{+\infty} x e^{-\alpha x + ixy} dx =$$

$$= -i\sqrt{\frac{2}{\pi}} \operatorname{Im}\left(\frac{x e^{-\alpha x+ixy}}{iy-\alpha} - \frac{e^{-\alpha x+ixy}}{(iy-\alpha)^2}\right)\Bigg|_{0}^{+\infty} = -i\sqrt{\frac{2}{\pi}} \operatorname{Im}\left(\frac{1}{(iy-\alpha)^2}\right) =$$

$$= -i\sqrt{\frac{2}{\pi}} \frac{2\alpha y}{(\alpha^2+y^2)^2} = -i\sqrt{\frac{8}{\pi}} \frac{\alpha y}{(y^2+\alpha^2)^2} \,//$$

5) $f(x) = \frac{d}{dx}\left(x^2 e^{-|x|}\right)$.

$$F(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d}{dx}\left(x^2 e^{-|x|}\right) e^{-ixy} dx = \frac{1}{\sqrt{2\pi}} e^{-ixy}\left(x^2 e^{-|x|}\right)\Bigg|_{-\infty}^{+\infty} +$$

$$+ \int_{-\infty}^{+\infty} x^2 e^{-|x|} \cdot e^{-ixy} \cdot iy \, dx = \frac{1}{\sqrt{2\pi}} iy \int_{-\infty}^{+\infty} x^2 e^{-|x|} e^{-ixy} dx =$$

$$= \sqrt{\frac{2}{\pi}} iy \int_{0}^{+\infty} x^2 e^{-x} \cos xy \, dx = \sqrt{\frac{2}{\pi}} iy \operatorname{Re} \int_{0}^{+\infty} x^2 e^{-x(1+iy)} dx =$$

$$= \sqrt{\frac{2}{\pi}} iy \operatorname{Re} \left[-\frac{1}{1+iy}\left[\left(e^{-x(1+iy)} x^2\right)\Big|_{0}^{+\infty} - 2\int_{0}^{+\infty} x e^{-x(1+iy)} dx\right]\right] =$$

$$= \sqrt{\frac{2}{\pi}} iy \cdot 2 \operatorname{Re}\left(\frac{1}{1+iy} \int_{0}^{+\infty} x e^{-x(1+iy)} dx\right) =$$

$$= \sqrt{\frac{2}{\pi}} \cdot 2iy \; Re\left(\frac{1}{(1+iy)^2}\left[\int_0^{+\infty} e^{-x(1+iy)}\, dx - x e^{-x(1+iy)}\Big|_0^{+\infty}\right)\right. =$$

$$= \sqrt{\frac{8}{\pi}}\, iy \; Re\left(\frac{1}{(1+iy)^3}\right) = \sqrt{\frac{8}{\pi}}\, iy \; Re\left(\frac{1}{1+3iy-3y^2-iy^2}\right) =$$

$$= \sqrt{\frac{8}{\pi}}\, iy \, \frac{1-3y^2}{(1-3y^2)^2+(3y-y^2)^2} = \sqrt{\frac{8}{\pi}}\, iy \, \frac{1-3y^2}{(1+y^2)^3}$$

14 (1, 3) $\quad \alpha = \frac{1}{1+|x|^5}$

1) $\hat{f}(y)$ - пр-ие ф-ции α.

Док-ть $\hat{f}(y)$ имеет непрер. на $\mathbb{R}$ произв. 3-го пор.

□ Пусть $k=0,1,2,3$.

$$\int_{-\infty}^{+\infty}\left|\frac{x^k}{1+|x|^5}\right|dx = 2\int_0^{+\infty}\frac{x^k}{1+x^5}dx = \underbrace{2\int_0^1\frac{x^k}{1+x^5}dx}_{\text{собств.}} +$$

$$+ 2\int_1^{+\infty}\frac{x^k}{1+x^5}dx$$

$$\int_1^{+\infty}\frac{x^k}{1+x^5}dx \le \int_1^{+\infty}x^{k-5}dx \qquad k-5<-1 \hookrightarrow$$

интегр. сх-ся по пр. сравн.

Значит $\alpha(x)$, $x\alpha(x)$, $x^2\alpha(x)$ и $x^3\alpha(x)$ абс. интегрируемы.

По т. о производной пр-ия Фурье

$$\exists\ (\hat{f}(y))''' = (-i)^3\,\widehat{x^3\alpha}(y)$$

то есть $F'''[\mathcal{L}](y) = i\,F[x^3\mathcal{L}](y)$

$x^3\mathcal{L}(x)$ абс. интегрир. $\hookrightarrow$ $F'''[\mathcal{L}](y)$ непрер.

3) $\hat{f}(y) = o\left(\frac{1}{y^5}\right)$ при $y \to \infty$.

Разложим в окрестн. 0:

$\mathcal{L}(x) = 1 - |x|^5 + o(x^5)$ $\hookrightarrow$ первые 4 произв.

$\mathcal{L}(x)$ непрерывны (и $\exists$)

Пятая произв. $\exists$ и явл-ся кусочно-непр.

По т. о пр-ии Фурье произв.

$F[\mathcal{L}^{(5)}(x)] = (iy)^5\,F[\mathcal{L}(x)]$

По лемме Римана об осцилляции

$F[\mathcal{L}^{(5)}(x)] \to 0,\ y \to +\infty$ $\hookrightarrow$ $F[\mathcal{L}(x)] = \hat{f}(y) = o\left(\frac{1}{y^5}\right)$