(Н1) Несимметр. монета

мат.ожидание кол-ва бросаний до выпадения первого герба

p - вер-сть выпад. герба

ξ - кол-во бросаний до выпад. 1-го герба.

Аналог. 32, так как это геометр. распр. с параметром p.

$E(\xi) = \frac{1}{p}$

(1)

$$\rho_L(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

$L = 2\pi r$, $r = \frac{L}{2\pi}$, $S = \pi r^2 = \frac{L^2}{4\pi}$

$$E(S) = \int_0^1 \rho_L(x) \cdot \frac{x^2}{4\pi}\, dx = \frac{x^3}{12\pi}\Big|_0^1 = \frac{1}{12\pi}$$

$$E(S^2) = \int_0^1 \rho_L(x) \cdot \frac{x^4}{16\pi^2}\, dx = \frac{x^5}{5 \cdot 16\pi^2}\Big|_0^1 = \frac{1}{80\pi^2}$$

$$D(S) = E(S^2) - E^2(S) = \frac{1}{80\pi^2} - \frac{1}{144\pi^2} = \frac{1}{\pi^2 \cdot 180}$$

Ответ: $E(S) = \frac{1}{12\pi}$, $D(S) = \frac{1}{180\pi^2}$

② $\rho_\xi(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$ $\qquad \rho_\eta(y) = \begin{cases} 1, & y \in [0,1] \\ 0, & y \in [0,1] \end{cases}$

$l = |\xi - \eta|$ - расст. между точками

$$E(l) = \iint \rho_\eta(y) \cdot \rho_\xi(x) \cdot |x-y| \, dx\, dy =$$

$$= \int_0^1 \rho_\eta(y)\, dy \int_0^1 \rho_\xi(x)\, |x-y|\, dx = \int_0^1 dy \left(\int_0^y (y-x)\, dx + \right.$$

$$\left. + \int_y^1 (x-y)\, dx \right) = \int_0^1 dy \left(\left(yx - \frac{x^2}{2}\right)\Big|_0^y + \left(\frac{x^2}{2} - yx\right)\Big|_y^1 \right) =$$

$$= \int_0^1 dy \left(y^2 - \frac{y^2}{2} + \frac{1}{2} - y - \frac{y^2}{2} + y^2 \right) = \int_0^1 \left(y^2 - y + \frac{1}{2} \right) dy =$$

$$= \left(\frac{y^3}{3} - \frac{y^2}{2} + \frac{y}{2}\right)\Big|_0^1 = \frac{1}{3}$$

$$E(l^2) = \int_0^1 \rho_2(y)\,dy \int_0^1 \rho_1(x)\,(y-x)^2\,dx =$$

$$= \int_0^1 dy \int_0^1 (y^2 - 2xy + x^2)\,dx = \int_0^1 \left(y^2 - y + \frac{1}{3}\right) dy =$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} = \frac{1}{6}.$$

$$D(l) = E(l^2) - E^2(l) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

Ответ: $E(l) = \frac{1}{3}$, $D(l) = \frac{1}{18}$

③

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a) $\rho_1(x) = 1 \qquad 0 \le x \le 1 \qquad g(x) = -\ln(1-x)$

$$\rho_2(y) = \rho_1(g^{-1}(y)) \cdot |(g^{-1}(y))'|$$

$$y = -\ln(1-x)$$

$$e^{-y} = 1 - x \longrightarrow x = 1 - e^{-y}$$

$$g^{-1}(y) = h(y) = 1 - e^{-y}$$

$$h'(y) = e^{-y}$$

$$\rho_2(y) = 1 \cdot e^{-y} \qquad 0 \le y < +\infty$$

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б) $\rho_1(x) = e^{-x} \qquad x \ge 0 \qquad g(x) = \ln(x)$

$$\rho_2(y) = \rho_1(g^{-1}(y)) \cdot |(g^{-1}(y))'|$$

$$y = \ln x \qquad x = e^{y} \longrightarrow g^{-1}(y) = e^{y}$$

$$\rho_\eta(y) = e^{-e^y} \cdot e^{y} = e^{y-e^y} \qquad y \in (-\infty, +\infty)$$

c) $\rho(x) = e^{-x}, \quad x \geq 0 \qquad g(x) = \{x\}$

$$F_\eta(y) = P(g(\xi) \leq y) = \begin{cases} 0, & y < 0 \\ 1, & y \geq 1 \\ ? & y \in [0,1) \end{cases}$$

Когда $y \in [0,1)$:

$$F_\eta(y) = P\Big(\xi \in \bigcup_{k\geq 0}^{\infty} [k, k+y]\Big) = \sum_{k\geq 0}^{\infty} P(\xi \in [k, k+y]) =$$

дробн часть $\leq y$ на этом [ ]

$$= \sum_{k\geq 0}^{\infty} \int_k^{k+y} e^{-x}\,dx = \sum_{k\geq 0}^{\infty} \left(e^{-k} - e^{-(k+y)}\right) = (1 - e^{-y}) \sum_{k\geq 0}^{\infty} e^{-k} =$$

$$= \frac{1-e^{-y}}{1-e^{-1}}$$ из геом. прогр.

$$\rho_2(y) = F_\eta'(y) = \begin{cases} 0, & y \notin [0,1) \\ \dfrac{e^{-y}}{1-e^{-1}}, & y \in [0,1) \end{cases}$$

d) $\rho_\xi(x) = \pi^{-1}(1+x^2)^{-1} \qquad -\infty < x < +\infty \qquad g(x) = \frac{1}{x}$

$$\rho_\eta(y) = \rho_\xi(g^{-1}(y)) \cdot |(g^{-1}(y))'|$$

$$y = \frac{1}{x} \qquad x = \frac{1}{y}$$

$$\rho_\eta(y) = \pi^{-1}\left(1 + \frac{1}{y^2}\right)^{-1} \cdot \left|-\frac{1}{y^2}\right| = \frac{1}{\pi(y^2+1)} \qquad -\infty < y < +\infty$$

e) $\rho_\xi(x) = \pi^{-1}(1+x^2)^{-1}$ $\quad -\infty < x < +\infty \quad$ $g(x) = \dfrac{2x}{1-x^2}$

$$y = \frac{2x}{1-x^2} \qquad y - yx^2 - 2x = 0 \qquad x = \frac{-1 \pm \sqrt{1+y^2}}{y}$$

1) $x_1 = -\dfrac{1+\sqrt{1+y^2}}{y} \qquad x \in (-\infty, -1) \cup (1, +\infty)$

$$x_1' = \frac{1+\sqrt{1+y^2}}{y^2\sqrt{1+y^2}}$$

$$\rho_1 = \frac{1}{\pi}\,\frac{1}{1+x_1^2}\cdot|x_1'| = \frac{1+\sqrt{1+y^2}}{\pi\cdot 2(y^2+1)(\sqrt{1+y^2}+1)}$$

2) $x_2 = -\dfrac{1-\sqrt{1+y^2}}{y} \qquad x \in (-1, 1)$

$$x_2' = \frac{\sqrt{1+y^2}-1}{y^2\sqrt{1+y^2}}$$

$$\rho_2 = \frac{1}{\pi}\,\frac{1}{1+x_2^2}\,|x_2'| = \frac{\sqrt{1+y^2}-1}{\pi\cdot 2(y^2+1)(\sqrt{1+y^2}-1)}$$

$$\rho_\eta(y) = \rho_1 + \rho_2 = \frac{1+\sqrt{1+y^2}}{2\pi(y^2+1)(\sqrt{1+y^2}+1)} + \frac{\sqrt{1+y^2}-1}{2\pi(y^2+1)(\sqrt{1+y^2}-1)} =$$

$$= \frac{1+y^2-1+1+y^2-1}{2\pi(y^2+1)(1+y^2-1)} = \frac{2y^2}{2\pi(y^2+1)y^2} = \frac{1}{\pi(y^2+1)}$$

$$-\infty < y < +\infty$$

(4) $F(x)$ - ф-ция распр. с.в. ξ непр. и стр. возр.

$F(\eta)$, $E(\eta)$ - ? $\quad \eta = F(\xi)$

$$F_\eta(z) = P(F(\xi) \leq z) = \begin{cases} 0, & z \leq 0 \\ 1, & z \geq 1 \\ ?, & z \in (0,1) \end{cases}$$

Когда $z \in (0, 1)$:

$$F_\eta(z) = P(\xi \leq F^{-1}(z)) = F_\xi(F^{-1}(z)) = z$$

$$F_\eta(z) = \begin{cases} 0, & z \leq 0 \\ z, & z \in (0,1) \\ 1, & z \geq 1 \end{cases} \qquad E_\eta = \int_0^1 z \cdot 1 \, dz = \frac{1}{2}$$

⑤ с. в. $\xi_1, \ldots, \xi_n$ нез. и один. распр. с плотн. $\rho(x)$.

Найти распр. с. в. $\alpha = \min\limits_{1 \le k \le n} (\xi_k)$ $\beta = \max\limits_{1 \le k \le n} \xi_k$

$$F_\beta(x) = P(\beta \le x) = P(\xi_1 \le x, \xi_2 \le x, \ldots, \xi_n \le x) = (F_\xi(x))^n$$

$$F_\alpha(x) = P(\alpha \le x) = 1 - P(\alpha > x) = 1 - P(\xi_1 > x, \xi_2 > x, \ldots, \xi_n > x) =$$

$$= 1 - P(\xi_1 > x) \cdot \ldots \cdot P(\xi_n > x) = 1 - (1 - F_\xi(x))^n$$

так как $P(\xi_i > x) = 1 - P(\xi_i \le x) = 1 - F_{\xi_i}(x)$

~~Ответ.~~ $F_\beta(x) = (F_\xi(x))^n$ $F_\alpha(x) = 1 - (1 - F_\xi(x))^n$.

а так как $\rho(x) = F'(x)$, то $F_\beta(x) = \left(\int \rho(x)\,dx\right)^n$

$F_\alpha(x) = 1 - \left(1 - \int \rho(x)\,dx\right)^n$.