

# Stuff I Know<sup>1</sup>

## Sample book subtitle<sup>2</sup>

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October 28, 2021

<sup>1</sup>This is a footnote.

<sup>2</sup>This is yet another footnote.

<sup>3</sup>[cuppajoeman.com](https://cuppajoeman.com)

Dedicated to Calvin and Hobbes.

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# Preface

This book contains things that I know, have proven or have learned.

## Un-numbered sample section

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## Another sample section

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## Structure of book

The book is partitioned into different sections based on the area of mathematics it is involved with.

## About the companion website

The website<sup>1</sup> for this file contains:

- A link to (freely downloadable) latest version of this document.
- Link to download LaTeX source for this document.
- Miscellaneous material (e.g. suggested readings etc).

## Acknowledgements

- A special word of thanks to professors who wanted to make sure I understood and learned as much as possible Alfonso Gracia-Saz<sup>2</sup>, Jean-Baptiste Campesato<sup>3</sup>, Z-Module, riv, PlanckWalk, franciman, qergle from #math on <https://libera.chat/>.
- I'm deeply indebted my parents, colleagues and friends for their support and encouragement.

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<sup>1</sup><https://github.com/cuppajoeman/stuff-I-know>

<sup>2</sup><https://www.math.toronto.edu/cms/alfonso-memorial/>

<sup>3</sup><https://math.univ-angers.fr/~campesato/>

# Introductory Chapter

*“This is a quote and I don’t know who said this.”*

– Author’s name, *Source of this quote*

## 1.1 Section heading

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Table 1.1: Sample table

S. No.	Column#1	Column#2	Column#3
1	50	837	970
2	47	877	230
3	31	25	415
4	35	144	2356
5	45	300	556

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<sup>2</sup>[www.example.com](http://www.example.com)

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## 2

# First Order Logic

*“This is a quote and I don’t know who said this.”*

– Author’s name, *Source of this quote*

## 2.1 Deductions

### Lemma 1: Universal connection to Variable Assignment Function

$\Sigma \vdash \theta$  if and only if  $\Sigma \vdash \forall x\theta$

Note this lemma might seem quite strange, but note it actually makes sense, \_\_\_\_\_

finish  
why

**Proof**

•  $\Rightarrow$

– Suppose that  $\Sigma \vdash \theta$ , therefore we have a deduction  $\mathcal{D}$  of  $\theta$ , then the proof

$$\begin{array}{l} \mathcal{D} \\ [(\forall y (y = y)) \wedge \neg (\forall y (y = y))] \rightarrow \theta \quad \text{(taut. PC)} \\ [(\forall y (y = y)) \wedge \neg (\forall y (y = y))] \rightarrow (\forall x) \theta \quad \text{(QR)} \\ (\forall x) \theta \quad \text{(PC)} \end{array}$$

•  $\Leftarrow$

– Suppose that  $\Sigma \vdash \forall x\theta$ , so we have a deduction of it, call it  $\mathcal{D}$ , then the following deduction suffices

$$\begin{array}{l} \mathcal{D} \\ \forall x\theta \\ \forall x\theta \rightarrow \theta_x^x \\ \theta_x^x \end{array}$$

## 2.2 Completeness

### Theorem 1: Completeness Theorem

Suppose that  $\Sigma$  is a set of  $\mathcal{L}$ -formulas, where  $\mathcal{L}$  is a countable language and  $\phi$  is an  $\mathcal{L}$ -formula. If  $\Sigma \models \phi$ , then  $\Sigma \vdash \phi$ .

#### Setup

- We start by assuming that  $\Sigma \models \phi$ , we must show that  $\Sigma \vdash \phi$ .
- If  $\phi$  is not a sentence then we can always prove  $\phi'$  which is the same as  $\phi$  with all of its variables bound
  - We can do that by appending  $(\forall x_f)$  where each  $x_f$  is a free variable of  $\phi$  to the front of  $\phi$
- Therefore we will prove it for all sentences  $\phi$

justify why this is equivalent

# 3

## Topology

### 3.1 Topological Spaces and Continuous Functions

#### 3.1.1 Basis for a Topology

##### Definition 1: Basis

If  $X$  is a set, a basis for a topology on  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$  (called basis elements) such that

1. For each  $x \in X$ , there is at least one basis element  $B$  containing  $x$ .
2. If  $x$  belongs to the intersection of two basis elements  $B_1$  and  $B_2$ , then there is a basis element  $B_3$  containing  $x$  such that  $B_3 \subset B_1 \cap B_2$ .

If  $\mathcal{B}$  satisfies these two conditions, then we define the topology  $\mathcal{T}$  generated by  $\mathcal{B}$  as follows: A subset  $U$  of  $X$  is said to be open in  $X$  (that is, to be an element of  $\mathcal{T}$ ) if for, each  $x \in U$ , there is a basis element  $B \in \mathcal{B}$  such that  $x \in B$  and  $B \subset U$ . Note that each basis element is itself an element of  $\mathcal{T}$ .

#### 3.1.2 The Subspace Topology

##### Definition 2: Subspace Topology

Let  $X$  be a topological space with topology  $\mathcal{T}$ . If  $Y$  is a subset of  $X$ , the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on  $Y$ , called the subspace topology. With this topology,  $Y$  is called a subspace of  $X$ ; its open sets consist of all intersections of open sets of  $X$  with  $Y$ .

#### 3.1.3 The Product Topology

##### Definition 3: Product Topology

Let  $\mathcal{S}_\beta$  denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$$

and let  $\mathcal{S}$  denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$

The topology generated by the subbasis  $\mathcal{S}$  is called the product topology. In this topology  $\prod_{\alpha \in J} X_\alpha$  is called a product space.

**Theorem 2: Basis for the Box Topology**

Suppose the topology on each space  $X_\alpha$  is given by a basis  $\mathcal{B}_\alpha$ . The collection of all sets of the form

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where  $B_\alpha \in \mathcal{B}_\alpha$  for each  $\alpha$ , will serve as a basis for the box topology on  $\prod_{\alpha \in \mathcal{J}} X_\alpha$ .

**Theorem 3: Basis for the Product Topology**

Suppose the topology on each space  $X_\alpha$  is given by a basis  $\mathcal{B}_\alpha$ . The collection of all sets of the form

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where  $B_\alpha \in \mathcal{B}_\alpha$  for finitely many indices  $\alpha$  and  $B_\alpha = X_\alpha$  for all the remaining indices, will serve as a basis for the product topology  $\prod_{\alpha \in \mathcal{J}} X_\alpha$ .

**Definition 4:  $\mathbb{R}$  Omega**

$\mathbb{R}^\omega$ , the countably infinite product of  $\mathbb{R}$  with itself. Recall that

$$\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} X_n$$

with  $X_n = \mathbb{R}$  for each  $n$

**3.1.4 The Metric Topology****Definition 5: A metric**

A metric on a set  $X$  is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

having the following properties:

1.  $d(x, y) \geq 0$  for all  $x, y \in X$ ; equality holds if and only if  $x = y$ .
2.  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .
3. Triangle Inequality:  $d(x, y) + d(y, z) \geq d(x, z)$ , for all  $x, y, z \in X$ .

**Example 1: Discrete Metric**

$d : X \times X \rightarrow \mathbb{R}$  given by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & \text{otherwise} \end{cases}$$

**Definition 6: Epsilon Ball**

Given  $\epsilon > 0$ , consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points  $y$  whose distance from  $x$  is less than  $\epsilon$ . It is called the  $\epsilon$ -ball centered at  $x$ . Sometimes we omit the metric  $d$  from the notation and write this ball simply as  $B(x, \epsilon)$ , when no confusion will arise.

**Definition 7: Metric Topology**

If  $d$  is a metric on the set  $X$ , then the collection of all  $\epsilon$ -balls  $B_d(x, \epsilon)$ , for  $x \in X$  and  $\epsilon > 0$ , is a basis for a topology on  $X$ , called the metric topology induced by  $d$ .

**Definition 8: Bounded Subset of a Metric Space**

Let  $X$  be a metric space with metric  $d$ . A subset  $A$  of  $X$  is said to be bounded if there is some number  $M \in \mathbb{R}$  such that

$$d(a_1, a_2) \leq M$$

for every pair of points  $a_1, a_2 \in A$

## 3.2 Connectedness and Compactness

**Example 2: Closed and Bounded, not Compact**

A metric space  $X$  and a closed and bounded subspace  $Y$  of  $X$  that is not compact.

- Consider the set  $X = \left\{ \frac{1}{n} : n \in \mathbb{N}^+ \right\}$ , with the **discrete metric**, it is bounded because for any two points  $a, b \in X$ ,  $d(a, b) \leq 1$ .
- Let  $X$  be an infinite set and let consider the discrete metric on that set, the metric topology which it induces (call it  $\mathcal{T}$ ) is the discrete topology of  $X$ . Therefore if we consider any subset  $Y$  of  $X$  it is closed, as  $X \setminus Y \in \mathcal{T}$  (remember it's the discrete topology). But the open covering  $\{\{x\} : x \in X\}$  has no finite subcollection which also covers  $X$ .

closed  
and  
bounded  
proof

### 3.2.1 Compact Spaces

**Definition 9: Covering**

A collection  $A$  of subsets of a space  $X$  is said to cover  $X$ , or to be a covering of  $X$ , if the union of the elements of  $A$  is equal to  $X$ . It is called an open covering of  $X$  if its elements are open subsets of  $X$ .

**Definition 10: Compact Space**

A space  $X$  is said to be compact if every open covering  $A$  of  $X$  contains a finite subcollection that also covers  $X$ .

**Lemma 2: Covering Yields Finite Covering if and only if Compact**

Let  $Y$  be a subspace of  $X$ . Then  $Y$  is compact if and only if every covering of  $Y$  by sets open in  $X$  contains a finite subcollection covering  $Y$ .