The Knowledge [1]

Sample book subtitle [2]

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October 29, 2021

[1]This is a footnote.
[2]This is yet another footnote.
[3]cuppajoeman.com

Dedicated to Calvin and Hobbes.

Contents

List of Figures

List of Tables

Todo list

Preface

This book contains things that we know, have proven or have learned.

Un-numbered sample section

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Structure of book

The book is partitioned into different sections based on the area of mathematics it is involved with.

About the companion website

The website[1] for this file contains:

- A link to (freely downlodable) latest version of this document.
- Link to download LaTeX source for this document.
- Miscellaneous material (e.g. suggested readings etc).

Acknowledgements

- A special word of thanks to professors who wanted to make sure I understood and learned as much as possible Alfonso Gracia-Saz[2], Jean-Baptiste Campesato[3], Z-Module, riv, PlanckWalk, franciman, qergle from #math on https://libera.chat/.
- I'm deeply indebted my parents, colleagues and friends for their support and encouragement.

[1] https://github.com/cuppajoeman/stuff-I-know
[2] https://www.math.toronto.edu/cms/alfonso-memorial/
[3] https://math.univ-angers.fr/~campesato/

1

Introductory Chapter

"This is a quote and I don't know who said this."

– Author's name, *Source of this quote*

1.1 Section heading

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Table 1.1: Sample table

S. No.	Column#1	Column#2	Column#3
1	50	837	970
2	47	877	230
3	31	25	415
4	35	144	2356
5	45	300	556

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2

Linear Algebra

2.1 Vectors

Definition 1: Algebraic Dot Product

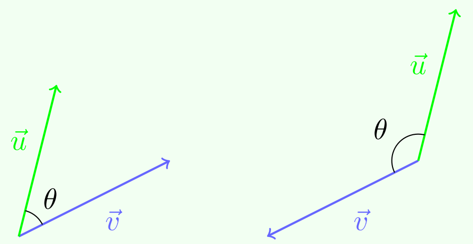
Let $u_1, u_2, \ldots, u_{n-1}, u_n$ and $v_1, v_2, \ldots, v_{n-1}, v_n$ denote the components of $\vec{u}$ and $\vec{v}$ respectively, then we have:

$$\vec{v} \cdot \vec{u} \overset{\mathtt{D}}{=} \sum_{i=1}^{n} v_i u_i$$

Definition 2: GeometricD ot Product

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and let θ be the angle between the two (the one in the range $[0, \pi]$), then we have the dot product:

$$\vec{v} \cdot \vec{u} \overset{\mathtt{D}}{=} \|v\| \, \|u\| \cos(\theta)$$



2.2 Matrices

Definition 3: Matrix Multiplication

If $\mathbf{A}$ *is an* $m \times n$ *matrix and* $\mathbf{B}$ *is an* $n \times p$ *matrix,*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

the matrix product $\mathbf{C} = \mathbf{AB}$ *(denoted without multiplication signs or dots) is defined to be the* $m \times p$ *matrix*

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

for $i = 1, \ldots, m$ *and* $j = 1, \ldots, p$*.*

```python
def multiply_matrices(matrix_A, matrix_B):
  """
  Given two matrices, we compute the matrix multiplication of the two of them

  Supposing that matrix_A is MxN and that matrix_B is NxK
  then the resulting matrix is M x K (1)
  """

  transpose_matrix_B = transpose_matrix(matrix_B)

  num_rows_of_matrix_A = len(matrix_A) # This is M
  num_columns_of_matrix_A = len(matrix_A[0]) #This is N

  num_rows_of_matrix_B = len(matrix_B) #this is N
  num_columns_of_matrix_B = len(matrix_B[0])# This is K

  resulting_matrix = []

  #initialize the matir xto be all zeros
  # then we will fill in the entries with the correct values
  for i in range(num_rows_of_matrix_A):
    blank_row = []
    for j in range(num_columns_of_matrix_B):
      blank_row.append(0)
    resulting_matrix.append(blank_row)
  # Notice that the matrix now has dimensions ‘num_rowsof_matrix_A‘ x ‘
   num_columns_of_matrix_B‘
  # Or equivalently M x K , as we noted by  (1)

  for row_index in range(num_rows_of_matrix_A):
    for column_index in range(num_rows_of_matrix_B):
      row_from_matrix_A = matrix_A[row_index]
      transposed_column_from_matrix_B = transpose_matrix_B[column_index]
      resulting_matrix[row_index][column_index] = dot_product(row_from_matrix_A,
   transposed_column_from_matrix_B)

  return resulting_matrix


def transpose_matrix(matrix):
```

```
  """Given a matrix turn it into a new matrix where it's rows are equal to it's columns"""
  rows = len(matrix)
  columns = len(matrix[0]) # assuming matrix is non-empty

  transpose_matrix = []
  for i in range(columns):
    temp_new_row = []
    for j in range(rows):
      temp_new_row.append(matrix[j][i])
    transpose_matrix.append(temp_new_row)

  return transpose_matrix



def dot_product(v1, v2):
  dot_product = 0
  for i in range(len(row)):
    dot_product += v1[i] * v2[i]
  return dot_product
```

3

First Order Logic

"This is a quote and I don't know who said this."

– Author's name, *Source of this quote*

3.1 Deductions

Lemma 1: Universal connection to Variable Assignment Function

$\Sigma \vdash \theta$ *if and only if* $\Sigma \vdash \forall x \theta$

Note this lemma might seem quite strange, but note it actually makes sense,

finish why

Proof

- $\Rightarrow$
  - Suppose that $\Sigma \vdash \theta$, therefore we have a deduction $\mathcal{D}$ of θ, then the proof

$$\begin{aligned} &\mathcal{D} \\ &[(\forall y\,(y = y)) \land \neg\,(\forall y\,(y = y))] \to \theta && \text{(taut. PC)} \\ &[(\forall y\,(y = y)) \land \neg\,(\forall y\,(y = y))] \to (\forall x)\,\theta && \text{(QR)} \\ &(\forall x)\,\theta && \text{(PC)} \end{aligned}$$

- $\Leftarrow$
  - Suppose that $\Sigma \vdash \forall x \theta$, so we have a deduction of it, call it $\mathcal{D}$, then the following deduction suffices

$$\begin{gathered} \mathcal{D} \\ \forall x \theta \\ \forall x \theta \to \theta^x_x \\ \theta^x_x \end{gathered}$$

3.2 Completeness

Theorem 1: Completeness Theorem

Suppose that Σ is a set of $\mathcal{L}$-formulas, where $\mathcal{L}$ is a countable langauge and ϕ is an $\mathcal{L}$-formula. If $\Sigma \models \phi$, then $\Sigma \vdash \phi$.

Setup

- *We start by assuming that $\Sigma \models \phi$, we must show that $\Sigma \vdash \phi$.*
- *If ϕ is not a sentence then we can always prove ϕ' which is the same as ϕ with all of it's variables bound*
  - *We can do that by appending $(\forall x_f)$ where each x_f is a free varaible of ϕ to the front of ϕ*
- *Therefore we will prove it for all sentences ϕ*

  justify why this is equivalent

4

Topology

4.1 Topological Spaces and Continuous Functions

4.1.1 Basis for a Topology

Definition 4: Basis

If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

1. *For each $x \in X$, there is at least one basis element B containing x.*
2. *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for, each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

4.1.2 The Subspace Topology

Definition 5: Subspace Topology

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology. With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.

4.1.3 The Product Topology

Definition 6: Product Topology

Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \left\{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\right\}$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called the product topology. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a product space.

Theorem 2: Basis for the Box Topology

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in \mathcal{J}} X_\alpha$.

Theorem 3: Basis for the Product Topology

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in \mathcal{J}} X_\alpha$.

Definition 7: R Omega

$\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Recall that

$$\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} X_n$$

with $X_n = \mathbb{R}$ for each n

4.1.4 The Metric Topology

Definition 8: A metric

A metric on a set X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

having the following properties:

1. *$d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. *Triangle Inequality: $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.*

Example 1: Discrete Metric

$d : X \times X \to \mathbb{R}$ given by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & \text{otherwise} \end{cases}$$

Definition 9: Epsilon Ball

Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the ϵ-ball centered at $\boldsymbol{x}$. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

Definition 10: Metric Topology

If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the metric topology induced by d.

Definition 11: Bounded Subset of a Metric Space

Let X be a metric space with metric d. A subset A of X is said to be bounded if there is some number $M \in \mathbb{R}$ such that

$$d\left(a_1, a_2\right) \leq M$$

for every pair of points $a_1, a_2 \in A$

4.2 Connectedness and Compactness

Example 2: Closed and Bounded, not Compact

A metric space X and a closed and bounded subspace Y of X that is not compact.

- Consider the set $X = \left\{\frac{1}{n} : n \in \mathbb{N}^{+}\right\}$, with the discrete metric, it is bounded because the for any two points $a, b \in X, d\left(a, b\right) \leq 1$
- Let X be an infinite set and let consider the discrete metric on that set, the metric topology which it induces (call it $\mathcal{T}$) is the discrete topology of X. Therefore if we consider any subset Y of X it is closed, as $X \setminus Y \in \mathcal{T}$ (remember it's the discrete topology). But the open covering $\{\{x\} : x \in X\}$ has no finite subcollection which also covers X .

closed and bounded proof

4.2.1 Compact Spaces

Definition 12: Covering

A collection A of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of A is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition 13: Compact Space

A space X is said to be compact if every open covering A of X contains a finite subcollection that also covers X.

Lemma 2: Covering Yields Finite Covering if and only if Compact

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.