

The Knowledge

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Dedicated to the proofs that were left as exercises to the readers

Contents

1	Analysis	4
1.1	Limits	5
1.2	Differentiation	7
1.3	Multi Variable	9
2	Linear Algebra	10
2.1	Vectors	10
2.2	Matrices	11
3	First Order Logic	13
3.1	Deductions	13
3.2	Completeness	14
4	Topology	15
4.1	Topological Spaces and Continuous Functions	15
4.1.1	Basis for a Topology	15
4.1.2	The Subspace Topology	15
4.1.3	The Product Topology	15
4.1.4	The Metric Topology	16
4.2	Connectedness and Compactness	18
4.2.1	Compact Spaces	18
5	Computer Science	19

List of Theorems

1.0.1 Triangle Inequality	4
1.2.1 Chain Rule	7
3.2.1 Completeness Theorem	14
4.1.1 Basis for the Box Topology	16
4.1.2 Basis for the Product Topology	16

List of Propositions

1.1.1 Limit of Constant	5
1.1.2 Constant in Limit	5
1.1.3 Sum of Limits	6
1.2.1 Little O and Differentiability Equivalence	8
2.1.1 Algebraic Def Implies Geometric Def of Dot Product	10
4.1.1 Epsilon Balls Form A Basis	17

List of Lemmas

1.0.1 Absolute Value is Equal to max	4
3.1.1 Universal connection to Variable Assignment Function	13
4.1.1 Epsilon Ball Contains Another	17
4.2.1 Covering Yields Finite Covering if and only if Compact	18

List of Definitions

1.1.1 Real Valued Limit	5
1.3.1 Directional Derivative	9
2.1.1 Algebraic Dot Product	10
2.1.2 Geometric Dot Product	10
2.2.1 Matrix Multiplication	11
4.1.1 Basis	15
4.1.2 Subspace Topology	15
4.1.3 Product Topology	15
4.1.4 Box Topology	16
4.1.5 \mathbb{R} Omega	16
4.1.6 A metric	16
4.1.7 Epsilon Ball	17
4.1.8 Metric Topology	17
4.1.9 Bounded Subset of a Metric Space	18
4.2.1 Covering	18
4.2.2 Compact Space	18
5.0.1 Big-O	19
5.0.2 Little-o	19

List of Examples

4.1.1 Discrete Metric	17
4.2.1 Closed and Bounded, not Compact	18

Preface

This book contains knowledge that that me or my peers have obtained, the purpose is to explain things fundamentally and in full detail so that someone who has never touched the subject may be able to understand it. It will focus on conveying the ideas that are involved in synthesizing the new knowledge with less of a focus on the results themselves.

It differs from a normal textbook in that it is open source and will fall under more continuous development rather than having editions that periodically come out.

Here is a link to the book: <https://github.com/cuppajoeman/knowledge-book/blob/main/build/book.pdf>

Structure of book

The book is partitioned into different sections based on the domain it is involved with. There may be shared definitions and theorems throughout the chapters, but in general it will start more elementary and get more advanced.

Knowledge

In this book you will find many results, will will characterize them as being one of the following

- Theorems – Results that are of importance and who's proof is not easily found (maybe using a novel idea)
- Propositions – Results of less importance who's proof could be constructed without a novel idea
- Lemmas – Results that are technical intermediate steps which has no standing as an independent result on first observation *
- Corollaries – Results which follow readily from an existing result of greater importance

Recommendations

By now you might know that in order to actually get better at mathematics you have to engage with it. This book may be used as a reference at times, but I highly recommend trying to re-prove statements or coming up with your own ideas before instantly looking at the solutions.

About the companion website

The website[†] for this file contains:

- A link to (freely downloadable) latest version of this document.
- Link to download \LaTeX source for this document.

*But sometimes they escape, as their usage becomes more than just an intermediate step, as Zorn's or Fatou's Lemmas did

[†]<https://github.com/cuppajoeman/knowledge-book>

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[‡]<https://www.math.toronto.edu/cms/alfonso-memorial/>

[§]<https://math.univ-angers.fr/~campesato/>

[¶]<https://www.galgr.com/>

Contributing

Contributions to the project are very welcome, let's delve into how to get started with this.

If you want to contribute to the project it's most likely that a contribution will fall into one of the following categories

- Content Based
 - Adding Definitions, Theorems, ...
 - Finishing TODO's
 - Formatting of the book
- Structural Layout of Project
 - Organization
 - Simplifying the existing structure of the directories
 - Making scripts which set up new structures
- External
 - Adding explanatory content to help onboard new users
 - Getting others involved
 - Creating infrastructure to support users (Github discussions)

Content Based

If you want to add a new top level structure, the best thing to do is to verify with other members of the project if it warrants it's own top level structure, otherwise it can be added as a substructure of an existing one.

Supposing that you are on linux, the easiest way to make a new structure and start working on it:

```
user@machine:~# cp -r structure new_structure
user@machine:~# cd new_structure
user@machine:~# mv content.tex new_structure.tex
user@machine:~# nvim new_structure.tex
```

Otherwise if you're adding a new theorem, it could be:

```
user@machine:~# cd existing_structure/theorems
user@machine:~# cp theorem.tex my_new_theorem.tex
user@machine:~# nvim my_new_theorem.tex
user@machine:~# ...
user@machine:~# git add -A && git commit -m "add my new theorem" && git push
```

Creating Files

For example if want to create a new definition for topology we would go into the definition folder for topology and create a new file using pothole case and don't include any extraneous words, for example it is better not to append "the" to the front of your file names when not specifically required.

1

Analysis

Theorem 1.0.1: Triangle Inequality

Let $a, b \in \mathbb{R}$ then

$$|a + b| \leq |a| + |b|$$

Proof

- $a + b \leq |a| + b \leq |a| + |b| \Leftrightarrow a + b \leq |a| + |b|$
- $-(a + b) = -a - b \leq |a| - b \leq |a| + |b| \Leftrightarrow -(a + b) \leq |a| + |b|$
 - Which are both derived by the fact that $|x| = \max(x, -x) \geq x, -x$
- Finally we can see that $|a + b| = \max(a + b, -(a + b))$, so no matter which one of $\{a + b, -(a + b)\}$ $|a + b|$ is equal to we have that

$$|a + b| \leq |a| + |b|$$

■

Note that the triangle equality is intuitively telling us that the shortest distance between two points is a straight line

Lemma 1.0.1: Absolute Value is Equal to max

$\forall x \in \mathbb{R}$

$$|x| = \max(x, -x)$$

Proof

- If $x \geq 0$ then $|x| = x$ and $\max(x, -x) = x$
- If $x < 0$ then $|x| = -x$ and $\max(x, -x) = -x$ since $x < 0 \Leftrightarrow -x > 0$ and therefore $-x > x$

■

1.1 Limits

Proposition 1.1.1: Limit of Constant

Let $f(x) = \alpha \in \mathbb{R}$ be a constant function, then

$$\lim_{x \rightarrow a} f(x) = \alpha$$

Proof

- Let $\varepsilon \in \mathbb{R}^+$, let δ be fixed as any real number, then assume that $\forall x \in \text{dom}(f)$ that $0 < |x - a| < \delta$, we will show that $|f(x) - \alpha| < \varepsilon$.
- Recall that $f(x) = \alpha$ for all $x \in \text{dom}(f)$, therefore $|f(x) - \alpha| = |\alpha - \alpha| = 0 < \varepsilon$ as needed. ■

Notice that in the above proof that we didn't use our hypothesis in the proof of the consequent, which makes sense because no matter which interval you look in the function is constant there, thus it doesn't depend on the hypothesis at all.

Definition 1.1.1: Real Valued Limit

Suppose f is a real valued function, then we say that the limit of f at a is L and write $\lim_{x \rightarrow a} f(x) = L$ when the following holds:

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ \text{ such that } \forall x \in \text{dom}(f), 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Proposition 1.1.2: Constant in Limit

Assume that the following limit exists $\lim_{x \rightarrow a} f(x)$ and define L to be it's value, then for any $c \in \mathbb{R}$

$$\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

Proof

- If $c = 0$ then using the fact that the limit of a constant is that constant itself, we have:

$$\lim_{x \rightarrow a} [0f(x)] = \lim_{x \rightarrow a} 0 = 0 = 0 \lim_{x \rightarrow a} f(x)$$

- If $c \neq 0$, we must prove that for any $\varepsilon_c \in \mathbb{R}^{>0}$ there exists δ_c such that for all $x_c \in \text{dom}(f)$

$$|x_c - a| \leq \delta_c \Rightarrow |cf(x_c) - cL| \leq \varepsilon_c$$

- Notice that if we were to let ε in the original definition be equal to $\frac{\varepsilon_c}{|c|}$ then we could multiply the equation after the implication on both sides by $|c|$ (so that we can absorb it into the absolute value).
- Let $\varepsilon_c \in \mathbb{R}^{>0}$ since the original limit holds for any epsilon, bind ε to $\frac{\varepsilon_c}{|c|}$ and we get δ such that for all $x \in \text{dom}(f)$, the following holds:

$$|x - a| \leq \delta \Rightarrow |f(x) - L| \leq \frac{\varepsilon_c}{|c|} \quad (\alpha)$$

- Take $\delta_c = \delta$, let $x_c \in \text{dom}(f)$ and bind x in the original definition to x_c , and assume that $|x_c - a| \leq \delta_c$, because of our choice for δ_c we satisfy α 's hypothesis with x replaced by x_c and we get

$$|f(x_c) - L| \leq \frac{\varepsilon_c}{|c|} \Leftrightarrow |c| |f(x_c) - L| \leq \varepsilon_c$$

- Since for any $a, b \in \mathbb{R}$ we have $|ab| = |a| |b|$ we can conclude with distributivity in \mathbb{R} that

$$|cf(x_c) - cL| \leq \varepsilon_c$$

As required. ■

Proposition 1.1.3: Sum of Limits

If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ both exist then

$$\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$$

Proof

- Suppose antecedent holds, so we have existence of the limits $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, now we'd like to show that the limit $\lim_{x \rightarrow a} [f(x) + g(x)]$ equals $L + M$
- Let $\varepsilon \in \mathbb{R}^+$, now from the fact that the two prior limits exist we can substitute in $\frac{\varepsilon}{2}$ into them, to get δ_L and δ_M respectively, such that

$$\forall x \in \text{dom}(f), 0 < |x - a| < \delta_L \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

and

$$\forall x \in \text{dom}(g), 0 < |x - a| < \delta_M \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$$

- Take $\delta = \min(\delta_L, \delta_M)$, so that we may utilize the above inequalities as we assume that $\forall x \in \text{dom}(f + g)$ that $0 < |x - a| < \delta$, and proceed to show that $|(f(x) + g(x)) - (L + M)| < \varepsilon$, since our x is bounded by the min of the two deltas from the other limits, we utilize the inequalities in tandem with the triangle equality to obtain:

$$|(f(x) + g(x)) - (L + M)| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
■

1.2 Differentiation

Theorem 1.2.1: Chain Rule

Given two functions f and g where g is differentiable at the point \bar{x} and f differentiable at the point $\bar{y} = g(\bar{x})$ then

$$(f \circ g)'(\bar{x}) = f'(g(\bar{x})) \cdot g'(\bar{x})$$

Proof

- We define the following two new functions

$$v(h) = \frac{g(\bar{x} + h) - g(\bar{x})}{h} - g'(\bar{x}) \text{ and } w(k) = \frac{f(\bar{y} + k) - f(\bar{y})}{k} - f'(\bar{y})$$

α : Note that $\lim_{h \rightarrow 0} v(h) = 0$ and $\lim_{k \rightarrow 0} w(k) = 0$, and that we can re-arrange the above to:

$$(v(h) + g'(\bar{x})) \cdot h + g(\bar{x}) = g(\bar{x} + h) \text{ and } (w(k) + f'(\bar{y})) \cdot k + f(\bar{y}) = f(\bar{y} + k)$$

- Now it follows that

$$\begin{aligned} f(g(\bar{x} + h)) &= f\left((v(h) + g'(\bar{x})) \cdot h + g(\bar{x})\right) \\ &= f\left(g(\bar{x}) + (v(h) + g'(\bar{x})) \cdot h\right) \\ &= f\left(\bar{y} + (v(h) + g'(\bar{x})) \cdot h\right) \\ &= \left(w\left((v(h) + g'(\bar{x})) \cdot h\right) + f'(\bar{y})\right) \cdot (v(h) + g'(\bar{x})) \cdot h + f(\bar{y}) \end{aligned}$$

- Now recall that we are interested in $(f \circ g)'(\bar{x})$, so we're going to have to look at:

$$\lim_{h \rightarrow 0} \frac{f(g(\bar{x} + h)) - f(g(\bar{x}))}{h}$$

- Notice that the term on the left of the sum in the numerator is something we already know, and therefore it becomes:

$$\lim_{h \rightarrow 0} \frac{\left(w\left((v(h) + g'(\bar{x})) \cdot h\right) + f'(\bar{y})\right) \cdot (v(h) + g'(\bar{x})) \cdot h + f(\bar{y}) - f(g(\bar{x}))}{h}$$

- By noting that $f(\bar{y}) = f(g(\bar{x}))$, then cancelling out the h we can see that the above simplifies to

$$\lim_{h \rightarrow 0} \left(w\left((v(h) + g'(\bar{x})) \cdot h\right) + f'(\bar{y})\right) \cdot (v(h) + g'(\bar{x}))$$

- By α on v and w with $k = g'(\bar{x}) \cdot h$, we have that the above limit is:

$$\lim_{h \rightarrow 0} \left(w\left(g'(\bar{x}) \cdot h\right) + f'(\bar{y})\right) \cdot g'(\bar{x}) = \lim_{h \rightarrow 0} f'(\bar{y}) \cdot g'(\bar{x}) = \boxed{f'(g(\bar{x})) \cdot g'(\bar{x})}$$

■

The chain rule is sometimes written as $\frac{df}{dx}(x) = \frac{df}{dg}(x) \frac{dg}{dx}(x)$, but notice that this extends liebniz notation as we only know that $\frac{df}{dx} = f'(x)$, and it is not defined for when we take the derivative with respect to another function. This motivates the definition

$$\frac{df}{dg}(x) = \frac{df}{dx}(g(x))$$

Proposition 1.2.1: Little O and Differentiability Equivalence

A function f is differentiable at the point \bar{x} if and only if there is a number $\alpha \in \mathbb{R}$ for any $h \in \mathbb{R}$ we have

$$f(\bar{x} + h) = f(\bar{x}) + h\alpha + E(h)$$

Where $E \in o(h)$

Proof

• \Rightarrow

– Assume that f is differentiable, therefore we have

$$f'(\bar{x}) = \lim_{h \rightarrow 0} \frac{f(\bar{x} + h) - f(\bar{x})}{h}$$

– Equivalently:

$$\lim_{h \rightarrow 0} \frac{f(\bar{x} + h) - f(\bar{x})}{h} - f'(\bar{x}) = 0$$

– Since $\lim_{h \rightarrow 0} f'(\bar{x}) = f'(\bar{x})$ as it's a constant, we may use the limit law to deduce:

$$\lim_{h \rightarrow 0} \left(\frac{f(\bar{x} + h) - f(\bar{x})}{h} - f'(\bar{x}) \right) = 0 \Leftrightarrow \lim_{h \rightarrow 0} \frac{f(\bar{x} + h) - f(\bar{x}) - hf'(\bar{x})}{h} = 0$$

– That is to say that $f(\bar{x} + h) - f(\bar{x}) - hf'(\bar{x}) \in o(h)$, set $E(h) = f(\bar{x} + h) - f(\bar{x}) - hf'(\bar{x})$ then we have that

$$f(\bar{x} + h) = f(\bar{x}) + hf'(\bar{x}) + E(h)$$

where we've taken $\alpha = f'(\bar{x})$

• \Leftarrow

– Assume that there is a number $\alpha \in \mathbb{R}$ and $E \in o(h)$, so that for any $h \in \mathbb{R}$ we have:

$$f(\bar{x} + h) = f(\bar{x}) + h\alpha + E(h) \Leftrightarrow l(h) = f(\bar{x} + h) - f(\bar{x}) - h\alpha$$

thus $f(\bar{x} + h) - f(\bar{x}) - h\alpha \in o(h)$.

– So then by definition of little-o, we get:

$$\lim_{h \rightarrow 0} \frac{f(\bar{x} + h) - f(\bar{x}) - h\alpha}{h} = 0$$

– Since $\lim_{h \rightarrow 0} \alpha = \alpha$ we may add it to both sides using the limit law on the left to obtain:

$$\lim_{h \rightarrow 0} \left(\frac{f(\bar{x} + h) - f(\bar{x}) - h\alpha}{h} + \alpha \right) = \alpha \Leftrightarrow \lim_{h \rightarrow 0} \frac{f(\bar{x} + h) - f(\bar{x})}{h} = \alpha$$

Meaning that $\alpha = f'(\bar{x})$ and that f is differentiable



Notice that the number α is equal to a limit and that limits are unique, therefore the solution $\alpha = f'(\bar{x})$ is the unique solution to the above proposition. Also since this is the unique solution, we know that there is no other real besides $f'(\bar{x})$ where this holds, therefore $f'(\bar{x})$ is the best linear approximation to f at the point \bar{x} .

Finally notice that if you solve for $E(h)$ and then recall that $E(h) \in o(h)$ we have:

$$\lim_{h \rightarrow 0} \frac{\overbrace{f(\bar{x} + h) - (f(\bar{x}) + hf'(\bar{x}))}^{\text{error}}}{h} = 0$$

the error is going to zero at a rate which is faster than linear, as if the error were decreasing at a linear rate the limit would evaluate to some non-zero constant.

1.3 Multi Variable

Definition 1.3.1: Directional Derivative

Let $U \subseteq \mathbb{R}^n$ be open, $f : U \rightarrow \mathbb{R}$ and $\vec{d} \in U$ and $\vec{v} \in \mathbb{R}^n$

$$\partial_{\vec{v}} f(\vec{d}) \stackrel{\text{D}}{=} \lim_{t \rightarrow 0 \in \mathbb{R}} \frac{f(\vec{d} + t\vec{v}) - f(\vec{d})}{t}$$

2

Linear Algebra

2.1 Vectors

Definition 2.1.1: Algebraic Dot Product

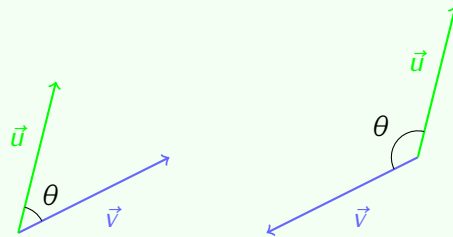
Let $u_1, u_2, \dots, u_{n-1}, u_n$ and $v_1, v_2, \dots, v_{n-1}, v_n$ denote the components of \vec{u} and \vec{v} respectively, then we have:

$$\vec{v} \cdot \vec{u} \stackrel{D}{=} \sum_{i=1}^n v_i u_i$$

Definition 2.1.2: Geometric Dot Product

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and let θ be the angle between the two (the one in the range $[0, \pi]$), then we have the dot product:

$$\vec{v} \cdot \vec{u} \stackrel{D}{=} \|\vec{v}\| \|\vec{u}\| \cos(\theta)$$



Proposition 2.1.1: Algebraic Def Implies Geometric Def of Dot Product

$$\sum_{i=1}^n v_i u_i = \|\vec{v}\| \|\vec{u}\| \cos(\theta)$$

Proof

- Consider two vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$, we know that from the polar coordinate system we can represent them as

$$\vec{x} = (\|\vec{x}\| \cos(\theta_x), \|\vec{x}\| \sin(\theta_x)) \quad \text{and} \quad \vec{y} = (\|\vec{y}\| \cos(\theta_y), \|\vec{y}\| \sin(\theta_y))$$

- The algebraic definition implies the following

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos(\theta_x) \cos(\theta_y) + \|\vec{x}\| \|\vec{y}\| \sin(\theta_x) \sin(\theta_y) = \|\vec{x}\| \|\vec{y}\| (\cos(\theta_x) \cos(\theta_y) + \sin(\theta_x) \sin(\theta_y))$$

- Then we can recall the sin angle difference formula to obtain:

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| (\cos(\theta_x - \theta_y))$$

- And just noting that $\theta_x - \theta_y$ is the angle between the vectors \vec{x} and \vec{y} as needed.



2.2 Matrices

Definition 2.2.1: Matrix Multiplication

If \mathbf{A} is an $m \times n$ matrix and \mathbf{B} is an $n \times p$ matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

the matrix product $\mathbf{C} = \mathbf{AB}$ (denoted without multiplication signs or dots) is defined to be the $m \times p$ matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj}$$

for $i = 1, \dots, m$ and $j = 1, \dots, p$.

```

1 def multiply_matrices(matrix_A, matrix_B):
2     """
3     Given two matrices, we compute the matrix multiplication of the two of them
4
5     Supposing that matrix_A is MxN and that matrix_B is NxK
6     then the resulting matrix is M x K (1)
7     """
8
9     transpose_matrix_B = transpose_matrix(matrix_B)
10
11     num_rows_of_matrix_A = len(matrix_A) # This is M
12     num_columns_of_matrix_A = len(matrix_A[0]) #This is N
13
14     num_rows_of_matrix_B = len(matrix_B) #this is N
15     num_columns_of_matrix_B = len(matrix_B[0])# This is K
16
17     resulting_matrix = []
18
19     #initialize the matrix to be all zeros
20     # then we will fill in the entries with the correct values
21     for i in range(num_rows_of_matrix_A):
22         blank_row = []
23         for j in range(num_columns_of_matrix_B):
24             blank_row.append(0)
25         resulting_matrix.append(blank_row)

```

```
26 # Notice that the matrix now has dimenisions 'num_rowsof_matrix_A' x 'num_columns_of_matrix_B'
27 # Or equivalently M x K , as we noted by (1)
28
29 for row_index in range(num_rows_of_matrix_A):
30     for column_index in range(num_rows_of_matrix_B):
31         row_from_matrix_A = matrix_A[row_index]
32         transposed_column_from_matrix_B = transpose_matrix_B[column_index]
33         resulting_matrix[row_index][column_index] = dot_product(row_from_matrix_A,
34                             transposed_column_from_matrix_B)
35
36
37
38 return resulting_matrix
39
40
41
42
43
44 def transpose_matrix(matrix):
45     """Given a matrix turn it into a new matrix where it's rows are equal to it's columns"""
46     rows = len(matrix)
47     columns = len(matrix[0]) # assuming matrix is non-empty
48
49
50     transpose_matrix = []
51     for i in range(columns):
52         temp_new_row = []
53         for j in range(rows):
54             temp_new_row.append(matrix[j][i])
55         transpose_matrix.append(temp_new_row)
56
57
58 return transpose_matrix
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3

First Order Logic

3.1 Deductions

Lemma 3.1.1: Universal connection to Variable Assignment Function

$\Sigma \vdash \theta$ if and only if $\Sigma \vdash \forall x\theta$

Proof

• \Rightarrow

– Suppose that $\Sigma \vdash \theta$, therefore we have a deduction \mathcal{D} of θ , then the proof

$$\begin{array}{rcl} \mathcal{D} & & \\ \left[\left(\forall y (y = y) \right) \wedge \neg \left(\forall y (y = y) \right) \right] \rightarrow \theta & & \text{(taut. PC)} \\ \left[\left(\forall y (y = y) \right) \wedge \neg \left(\forall y (y = y) \right) \right] \rightarrow (\forall x) \theta & & \text{(QR)} \\ (\forall x) \theta & & \text{(PC)} \end{array}$$

• \Leftarrow

– Suppose that $\Sigma \vdash \forall x\theta$, so we have a deduction of it, call it \mathcal{D} , then the following deduction suffices

$$\begin{array}{c} \mathcal{D} \\ \forall x\theta \\ \forall x\theta \rightarrow \theta_x^x \\ \theta_x^x \end{array}$$

■

3.2 Completeness

Theorem 3.2.1: Completeness Theorem

Suppose that Σ is a set of \mathcal{L} -formulas, where \mathcal{L} is a countable language and ϕ is an \mathcal{L} -formula. If $\Sigma \models \phi$, then $\Sigma \vdash \phi$.

Setup

- We start by assuming that $\Sigma \models \phi$, we must show that $\Sigma \vdash \phi$.
- If ϕ is not a sentence then we can always prove ϕ' which is the same as ϕ with all of its variables bound
 - We can do that by appending $(\forall x_f)$ where each x_f is a free variable of ϕ to the front of ϕ
- Therefore we will prove it for all sentences ϕ

4

Topology

4.1 Topological Spaces and Continuous Functions

4.1.1 Basis for a Topology

Definition 4.1.1: Basis

If X is a set, a basis for a topology on X is a collection \mathcal{B} of subsets of X (called basis elements) such that

1. For each $x \in X$, there is at least one basis element B containing x .
2. If x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If \mathcal{B} satisfies these two conditions, then we define the topology \mathcal{T} generated by \mathcal{B} as follows: A subset U of X is said to be open in X (that is, to be an element of \mathcal{T}) if for, each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of \mathcal{T} .

4.1.2 The Subspace Topology

Definition 4.1.2: Subspace Topology

Let X be a topological space with topology \mathcal{T} . If Y is a subset of X , the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y , called the subspace topology. With this topology, Y is called a subspace of X ; its open sets consist of all intersections of open sets of X with Y .

4.1.3 The Product Topology

Definition 4.1.3: Product Topology

Let \mathcal{S}_β denote the collection

$$\mathcal{S}_\beta = \left\{ \pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta \right\}$$

and let \mathcal{S} denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in I} \mathcal{S}_\beta$$

The topology generated by the subbasis \mathcal{S} is called the product topology. In this topology $\prod_{\alpha \in I} X_\alpha$ is called a product space.

Definition 4.1.4: Box Topology

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha \in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha$$

where U_α is open in X_α , for each $\alpha \in J$. The topology generated by this basis is called the box topology.

Theorem 4.1.1: Basis for the Box Topology

Suppose the topology on each space X_α is given by a basis \mathcal{B}_α . The collection of all sets of the form

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α , will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$.

Theorem 4.1.2: Basis for the Product Topology

Suppose the topology on each space X_α is given by a basis \mathcal{B}_α . The collection of all sets of the form

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$.

Definition 4.1.5: \mathbb{R}^ω

\mathbb{R}^ω , the countably infinite product of \mathbb{R} with itself. Recall that

$$\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} X_n$$

with $X_n = \mathbb{R}$ for each n

4.1.4 The Metric Topology**Definition 4.1.6: A metric**

A metric on a set X is a function

$$d : X \times X \rightarrow \mathbb{R}$$

having the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. Triangle Inequality: $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.

Example 4.1.1: Discrete Metric

$d : X \times X \rightarrow \mathbb{R}$ given by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & \text{otherwise} \end{cases}$$

Definition 4.1.7: Epsilon Ball

Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ . It is called the ϵ -ball centered at x . Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

Lemma 4.1.1: Epsilon Ball Contains Another

Let $x \in X$ then for every $B(x, \epsilon)$ there is $y \in B(x, \epsilon)$ and $\delta \in \mathbb{R}^+$ such that

$$B(y, \delta) \subseteq B(x, \epsilon)$$

Proof

- Let $y \in B(x, \epsilon)$ we claim $B(y, \delta)$ where $\delta = \epsilon - d(x, y)$ works.
- Note $B(y, \delta) \subseteq B(x, \epsilon)$, since given any $z \in B(y, \delta)$ we have that $d(y, z) < \epsilon - d(x, y)$, rearranging gives us $d(x, y) + d(y, z) < \epsilon$ and by the triangle inequality we get $d(x, z) < \epsilon$ therefore $z \in B(x, \epsilon)$

■

Proposition 4.1.1: Epsilon Balls Form A Basis

The collection \mathcal{B} of all ϵ -balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X

Proof

- Let $x \in X$ then clearly $x \in B_d(x, \epsilon)$ works for any $\epsilon \in \mathbb{R}^+$
- Let $B_1, B_2 \in \mathcal{B}$ and let $y \in B_1 \cap B_2$, for each of B_1 and B_2 we have δ_1 and δ_2 respectively with $B(y, \delta_1) \subseteq B_1$ and $B(y, \delta_2) \subseteq B_2$, by taking $\delta = \min(\delta_1, \delta_2)$ then we have $B(y, \delta) \subseteq B_1 \cap B_2$ as needed.

■

Definition 4.1.8: Metric Topology

If d is a metric on the set X , then the collection of all ϵ -balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X , called the metric topology induced by d .

Definition 4.1.9: Bounded Subset of a Metric Space

Let X be a metric space with metric d . A subset A of X is said to be bounded if there is some number $M \in \mathbb{R}$ such that

$$d(a_1, a_2) \leq M$$

for every pair of points $a_1, a_2 \in A$

4.2 Connectedness and Compactness**Example 4.2.1: Closed and Bounded, not Compact**

A metric space X and a closed and bounded subspace Y of X that is not compact.

- Consider the set $X = \left\{ \frac{1}{n} : n \in \mathbb{N}^+ \right\}$, with the **discrete metric**, it is bounded because for any two points $a, b \in X$, $d(a, b) \leq 1$
- Let X be an infinite set and let consider the discrete metric on that set, the metric topology which it induces (call it \mathcal{T}) is the discrete topology of X . Therefore if we consider any subset Y of X it is closed, as $X \setminus Y \in \mathcal{T}$ (remember it's the discrete topology). But the open covering $\{\{x\} : x \in X\}$ has no finite subcollection which also covers X .

4.2.1 Compact Spaces**Definition 4.2.1: Covering**

A collection A of subsets of a space X is said to cover X , or to be a covering of X , if the union of the elements of A is equal to X . It is called an open covering of X if its elements are open subsets of X .

Definition 4.2.2: Compact Space

A space X is said to be compact if every open covering A of X contains a finite subcollection that also covers X .

Lemma 4.2.1: Covering Yields Finite Covering if and only if Compact

Let Y be a subspace of X . Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y .

Computer Science

Definition 5.0.1: Big-O

Let $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, then we define the set

$$\mathcal{O}(g) \stackrel{\text{D}}{=} \left\{ j : \exists c, B \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq B \rightarrow j(n) \leq cg(n) \right\}$$

And we say that f is in the big-O of g when $f \in \mathcal{O}(g)$

Definition 5.0.2: Little-o

We define the following set:

$$o(g) = \left\{ j : \lim_{x \rightarrow 0} \frac{j(x)}{g(x)} = 0 \right\}$$

We say that f is in the little-o of g when $f \in o(g)$