Stuff I Know [1]

Sample book subtitle [2]

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October 28, 2021

[1]This is a footnote.
[2]This is yet another footnote.
[3]cuppajoeman.com

Dedicated to Calvin and Hobbes.

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Preface

This book contains things that I know, have proven or have learned.

Un-numbered sample section

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Structure of book

The book is partitioned into different sections based on the area of mathematics it is involved with.

About the companion website

The website[1] for this file contains:

- A link to (freely downlodable) latest version of this document.
- Link to download LaTeX source for this document.
- Miscellaneous material (e.g. suggested readings etc).

Acknowledgements

- A special word of thanks to professors who wanted to make sure I understood and learned as much as possible Alfonso Gracia-Saz[2], Jean-Baptiste Campesato[3], Z-Module, riv, PlanckWalk, franciman, qergle from #math on `https://libera.chat/`.
- I'm deeply indebted my parents, colleagues and friends for their support and encouragement.

[1] https://github.com/cuppajoeman/stuff-I-know
[2] https://www.math.toronto.edu/cms/alfonso-memorial/
[3] https://math.univ-angers.fr/~campesato/

1

Introductory Chapter

"This is a quote and I don't know who said this."

– Author's name, *Source of this quote*

1.1 Section heading

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Table 1.1: Sample table

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1	50	837	970
2	47	877	230
3	31	25	415
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5	45	300	556

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2

First Order Logic

"This is a quote and I don't know who said this."

– Author's name, *Source of this quote*

2.1 Deductions

Lemma 1: Universal connection to Variable Assignment Function

$\Sigma \vdash \theta$ *if and only if* $\Sigma \vdash \forall x \theta$

Note this lemma might seem quite strange, but note it actually makes sense,

finish why

x — Proof — x

- $\Rightarrow$
  - Suppose that $\Sigma \vdash \theta$, therefore we have a deduction $\mathcal{D}$ of θ, then the proof

$$\begin{aligned}
&\mathcal{D} \\
&[(\forall y\,(y = y)) \land \neg\,(\forall y\,(y = y))] \to \theta && \text{(taut. PC)} \\
&[(\forall y\,(y = y)) \land \neg\,(\forall y\,(y = y))] \to (\forall x)\,\theta && \text{(QR)} \\
&(\forall x)\,\theta && \text{(PC)}
\end{aligned}$$

- $\Leftarrow$
  - Suppose that $\Sigma \vdash \forall x \theta$, so we have a deduction of it, call it $\mathcal{D}$, then the following deduction suffices

$$\begin{gathered}
\mathcal{D} \\
\forall x \theta \\
\forall x \theta \to \theta^x_x \\
\theta^x_x
\end{gathered}$$

2.2 Completeness

Theorem 1: Completeness Theorem

Suppose that Σ is a set of $\mathcal{L}$-formulas, where $\mathcal{L}$ is a countable langauge and ϕ is an $\mathcal{L}$-formula. If $\Sigma \models \phi$, then $\Sigma \vdash \phi$.

Setup

- *We start by assuming that $\Sigma \models \phi$, we must show that $\Sigma \vdash \phi$.*
- *If ϕ is not a sentence then we can always prove ϕ' which is the same as ϕ with all of it's variables bound*
  - *We can do that by appending $(\forall x_f)$ where each x_f is a free varaible of ϕ to the front of ϕ*
- *Therefore we will prove it for all sentences ϕ*

  justify why this is equivalent

3

Topology

3.1 Topological Spaces and Continuous Functions

3.1.1 Basis for a Topology

Definition 1: Basis

If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

1. *For each $x \in X$, there is at least one basis element B containing x.*
2. *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for, each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

3.1.2 The Subspace Topology

Definition 2: Subspace Topology

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology. With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.

3.1.3 The Product Topology

Definition 3: Product Topology

Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \left\{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\right\}$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called the product topology. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a product space.

Theorem 2: Basis for the Box Topology

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in \mathcal{J}} X_\alpha$.

Theorem 3: Basis for the Product Topology

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in \mathcal{J}} X_\alpha$.

Definition 4: R Omega

$\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Recall that

$$\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} X_n$$

with $X_n = \mathbb{R}$ for each n

3.1.4 The Metric Topology

Definition 5: A metric

A metric on a set X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

having the following properties:

1. *$d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. *Triangle Inequality: $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.*

Example 1: Discrete Metric

$d : X \times X \to \mathbb{R}$ given by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & \text{otherwise} \end{cases}$$

Definition 6: Epsilon Ball

Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the ϵ-ball centered at $\boldsymbol{x}$. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

Definition 7: Metric Topology

If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the metric topology induced by d.

Definition 8: Bounded Subset of a Metric Space

Let X be a metric space with metric d. A subset A of X is said to be bounded if there is some number $M \in \mathbb{R}$ such that

$$d(a_1, a_2) \leq M$$

for every pair of points $a_1, a_2 \in A$

3.2 Connectedness and Compactness

Example 2: Closed and Bounded, not Compact

A metric space X and a closed and bounded subspace Y of X that is not compact.

- Consider the set $X = \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$, with the discrete metric, it is bounded because the for any two points $a, b \in X, d(a, b) \leq 1$

closed and bounded proof

- Let X be an infinite set and let consider the discrete metric on that set, the metric topology which it induces (call it $\mathcal{T}$) is the discrete topology of X. Therefore if we consider any subset Y of X it is closed, as $X \setminus Y \in \mathcal{T}$ (remember it's the discrete topology). But the open covering $\{\{x\} : x \in X\}$ has no finite subcollection which also covers X .

3.2.1 Compact Spaces

Definition 9: Covering

A collection A of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of A is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition 10: Compact Space

A space X is said to be compact if every open covering A of X contains a finite subcollection that also covers X.

Lemma 2: Covering Yields Finite Covering if and only if Compact

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.