The Knowledge

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Dedicated to the proofs that were left as exercises to the readers

Contents

List of Theorems

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Preface

This book contains knowledge that that me or my peers have obtained, the purpose is to explain things fundamentally and in full detail so that someone who has never touched the subject may be able to understand it. It will focus on conveying the ideas that are involved in synthesizing the new knowledge with less of a focus on the results themselves.

It differs from a normal textbook in that it is open source and will fall under more continuous development rather than having editions that periodically come out. It also differs in the sense that it welcomes other users to improve the book.

Here is a link to the book: `https://github.com/cuppajoeman/knowledge-book/blob/main/build/book.pdf`

Structure of book

The book is partitioned into different sections based on the domain it is involved with. There may be shared definitions and theorems throughout the chapters, but in general it will start more elementary and get more advanced.

Knowledge

In this book you will find many results, will will characterize them as being one of the following

- Theorems - Results that are of importantance and who's proof is not easily found (maybe using a novel idea)
- Propositions - Results of less importance who's proof could be constructed without a novel idea
- Lemmas - Results that are technical intermediate steps which has no standing as an independent result on first observation [*]
- Corollaries - Results which follow readily from an existing result of greater importance

Recommendations

By now you might know that in order to actually get better at mathematics you have to engage with it. This book may be used as a reference at times, but I highly recommend trying to re-prove statements or coming up with your own ideas before instantly looking at the solutions.

About the companion website

The website[†] for this file contains:

- A link to (freely downloadable) latest version of this document.
- Link to download LaTeXsource for this document.

[*]But sometimes they escape, as their usage becomes more than just an intermediate step, as Zorn's or Fatou's Lemmas did

[†]`https://github.com/cuppajoeman/knowledge-book`

Acknowledgements

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- Thanks to Z-Module, riv, PlanckWalk, franciman, qergle from #math on `https://libera.chat/`.

[‡] `https://www.math.toronto.edu/cms/alfonso-memorial/`

[§] `https://math.univ-angers.fr/~campesato/`

[¶] `https://www.galgr.com/`

Contributing

Contributions to the project are very welcome, let's delve into how to get started with this.

If you want to contribute to the project it's most likely that a contribution will fall into one of the following categories

- Content Based
  - Adding Definitions, Theorems, ...
  - Finishing TODO's
  - Formatting of the book
- Structural Layout of Project
  - Organization
  - Simplyfing the existing structure of the directories
  - Making scripts which set up new structures
- External
  - Adding explanatory content to help onboard new users
  - Getting others involved
  - Creating infrastructure to support users (Github discussions)

Communication

All communication will occur through github disccusions. You can access it here:

Content Based

If you want to add a new top level structure, the best thing to do is to verify with other members of the project if it warrants it's own top level structure, otherwise it can be added as a substructure of an existing one.

Supposing that you are on linux, the easiest way to make a new structure and start working on it:

```
user@machine:~# cp -r structure new_structure
user@machine:~# cd new_structure
user@machine:~# mv content.tex new_structure.tex
user@machine:~# nvim new_structure.tex
```

Otherwise if you're adding a new theorem, it could be:

```
user@machine:~# cd existing_structure/theorems
user@machine:~# cp theorem.tex my_new_theorem.tex
user@machine:~# nvim my_new_theorem.tex
user@machine:~# ...
user@machine:~# git add -A && git commit -m "add my new theorem" && git push
```

Creating Files

For example if want to create a new definition for topology we would go into the definition folder for topology and create a new file using pothole case and don't include any extraneous words, for example it is better not to append "the" to the front of your file names when not specifically required.

1

Analysis

Theorem 1.0.1: Triangle Inequality

Let $a, b \in \mathbb{R}$ then

$$|a+b| \leq |a| + |b|$$

Proof

- $a + b \leq |a| + b \leq |a| + |b| \Leftrightarrow a + b \leq |a| + |b|$
- $-\left(a+b\right) = -a - b \leq |a| - b \leq |a| + |b| \Leftrightarrow -\left(a+b\right) \leq |a| + |b|$
  - Which are both derived by the fact that $|x| = \max\left(x, -x\right) \geq x, -x$
- Finally we can see that $|a+b| = \max\left(a+b, -\left(a+b\right)\right)$, so no matter which one of $\left\{a+b, -\left(a+b\right)\right\}$ $|a+b|$ is equal to we have that

$$|a+b| \leq |a| + |b|$$

■

Note that the triangle equality is intuitively telling us that the shortest distance between two points is a straight line

Lemma 1.0.1: Absolute Value is Equal to max

$\forall x \in \mathbb{R}$

$$|x| = \max\left(x, -x\right)$$

Proof

- If $x \geq 0$ then $|x| = x$ and $\max\left(x, -x\right) = x$
- If $x < 0$ then $|x| = -x$ and $\max\left(x, -x\right) = -x$ since $x < 0 \leftrightarrow -x > 0$ and therefore $-x > x$

■

1.1 Limits

Proposition 1.1.1: Limit of Constant

Let $f(x) = \alpha \in \mathbb{R}$ be a constant function, then

$$\lim_{x \to a} f(x) = \alpha$$

Proof

- Let $\varepsilon \in \mathbb{R}^+$, let δ be fixed as any real number, then assume that $\forall x \in \operatorname{dom}(f)$ that $0 < |x - a| < \delta$, we will show that $|f(x) - \alpha| < \varepsilon$.
- Recall that $f(x) = \alpha$ for all $x \in \operatorname{dom}(f)$, therefore $|f(x) - \alpha| = |\alpha - \alpha| = 0 < \varepsilon$ as needed.

$\blacksquare$

Notice that in the above proof that we didn't use our hypothesis in the proof of the consequent, which makes sense because no matter which interval you look in the function is constant there, thus it doesn't depend on the hypothesis at all.

Definition 1.1.1: Real Valued Limit

Suppose f is a real valued function, then we say that the limit of f at a is L and write $\lim_{x \to a} f(x) = L$ when the following holds:

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ \text{ such that } \forall x \in \operatorname{dom}(f), 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Proposition 1.1.2: Constant in Limit

Assume that the following limit exists $\lim_{x \to a} f(x)$ and define L to be it's value, then for any $c \in \mathbb{R}$

$$\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$$

Proof

- If $c = 0$ then using the fact that the limit of a constant is that constant itself, we have:

$$\lim_{x \to a} [0f(x)] = \lim_{x \to a} 0 = 0 = 0 \lim_{x \to a} f(x)$$

- If $c \neq 0$, we must prove that for any $\varepsilon_c \in \mathbb{R}^{>0}$ there exists δ_c such that for all $x_c \in \operatorname{dom}(f)$

$$|x_c - a| \leq \delta_c \Rightarrow |cf(x_c) - cL| \leq \varepsilon_c$$

- Notice that if we were to let ε in the orignal definition be equal to $\frac{\varepsilon_c}{|c|}$ then we could multiply the equation after the implication on both sides by $|c|$ (so that we can absorb it into the absolute value).
- Let $\varepsilon_c \in \mathbb{R}^{>0}$ since the original limit holds for any epsilon, bind ε to $\frac{\varepsilon_c}{|c|}$ and we get δ such that for all $x \in \operatorname{dom}(f)$, the following holds:

$$|x - a| \leq \delta \Rightarrow |f(x) - L| \leq \frac{\varepsilon_c}{|c|} \tag{α}$$

- Take $\delta_c = \delta$, let $x_c \in \operatorname{dom}(f)$ and bind x in the original definition to x_c, and assume that $|x_c - a| \leq \delta_c$, because of our choice for δ_c we satisfy α's hypothesis with x replaced by x_c and we get

$$|f(x_c) - L| \leq \frac{\varepsilon_c}{c} \Leftrightarrow |c| |f(x) - L| \leq \varepsilon_c$$

- Since for any $a, b \in \mathbb{R}$ we have $|ab| = |a|\,|b|$ we can conclude with distributivity in $\mathbb{R}$ that

$$\left|cf\left(x_c\right) - cL\right| \leq \varepsilon_c$$

As required.

■

Proposition 1.1.3: Sum of Limits

If $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$ *both exist then*

$$\lim_{x\to a}[f(x) + g(x)] = L + M$$

Proof

- Suppose antecedent holds, so we have existsance of the limits $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, now we'd like to show that the limit $\lim_{x\to a}[f(x) + g(x)]$ equals $L + M$
- Let $\varepsilon \in \mathbb{R}^+$, now from the fact that the two prior limits exist we can substitute in $\frac{\varepsilon}{2}$ into them, to get δ_L and δ_M respectively, such that

$$\forall x \in \operatorname{dom}\left(f\right), 0 < |x - a| < \delta_L \Rightarrow \left|f\left(x\right) - L\right| < \frac{\varepsilon}{2}$$

and

$$\forall x \in \operatorname{dom}\left(g\right), 0 < |x - a| < \delta_M \Rightarrow \left|g\left(x\right) - M\right| < \frac{\varepsilon}{2}$$

- Take $\delta = \min\left(\delta_L, \delta_M\right)$, so that we may utilize the above inequalities as we assume that $\forall x \in \operatorname{dom}\left(f + g\right)$ that $0 < |x - a| < \delta$, and proceed to show that $\left|\left(f\left(x\right) + g\left(x\right)\right) - \left(L + M\right)\right| < \varepsilon$, since our x is bounded by the min of the two deltas from the other limits, we utilize the inequalities in tandem with the triangle equality to obtain:

$$\left|\left(f\left(x\right) + g\left(x\right)\right) - \left(M + L\right)\right| = \left|\left(f\left(x\right) - L\right) + \left(g\left(x\right) - M\right)\right| \leq \left|f\left(x\right) - L\right| + \left|g\left(x\right) - M\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

■

1.2 Differentiation

Theorem 1.2.1: Chain Rule

Given two functions f and g where g is differentiable at the point $\overline{x}$ and f differentiable at the point $\overline{y} = g(\overline{x})$ then

$$(f \circ g)'(\overline{x}) = f'(g(\overline{x})) \cdot g(\overline{x})$$

Proof

- We define the following two new functions

$$v(h) = \frac{g(\overline{x}+h) - g(\overline{x})}{h} - g'(\overline{x}) \text{ and } w(k) = \frac{f(\overline{y}+k) - f(\overline{y})}{k} - f'(\overline{y})$$

α : Note that $\lim_{h\to 0} v(h) = 0$ and $\lim_{k\to 0} w(k) = 0$, and that we can re-arrange the above to:

$$\left(v(h) + g'(\overline{x})\right) \cdot h + g(\overline{x}) = g(\overline{x}+h) \text{ and } \left(w(k) + f'(\overline{y})\right) \cdot k + f(\overline{y}) = f(\overline{y}+k)$$

- Now it follows that

$$\begin{aligned} f\left(g(\overline{x}+h)\right) &= f\left(\left(v(h) + g'(\overline{x})\right) \cdot h + g(\overline{x})\right) \\ &= f\left(g(\overline{x}) + \left(v(h) + g'(\overline{x})\right) \cdot h\right) \\ &= f\left(\overline{y} + \left(v(h) + g'(\overline{x})\right) \cdot h\right) \\ &= \left(w\left(\left(v(h) + g'(\overline{x})\right) \cdot h\right) + f'(\overline{y})\right) \cdot \left(v(h) + g'(\overline{x})\right) \cdot h + f(\overline{y}) \end{aligned}$$

- Now recall that we are interested in $(f \circ g)'(\overline{x})$, so we're going to have to look at:

$$\lim_{h\to 0} \frac{f\left(g(\overline{x}+h)\right) - f(g(\overline{x}))}{h}$$

- Notice that the term on the left of the sum in the numerator is something we already know, and therefore it becomes:

$$\lim_{h\to 0} \frac{\left(w\left(\left(v(h) + g'(\overline{x})\right) \cdot h\right) + f'(\overline{y})\right) \cdot \left(v(h) + g'(\overline{x})\right) \cdot h + f(\overline{y}) - f(g(\overline{x}))}{h}$$

- By noting that $f(\overline{y}) = f(g(\overline{x}))$, then cancelling out the h we can see that the above simplifies to

$$\lim_{h\to 0} \left(w\left(\left(v(h) + g'(\overline{x})\right) \cdot h\right) + f'(\overline{y})\right) \cdot \left(v(h) + g'(\overline{x})\right)$$

- By α on v and w with $k = g(\overline{x}) \cdot h$, we have that the above limit is:

$$\lim_{h\to 0} \left(w\left(\left(g'(\overline{x})\right) \cdot h\right) + f'(\overline{y})\right) \cdot \left(g'(\overline{x})\right) = \lim_{h\to 0} f'(\overline{y}) \cdot g'(\overline{x}) = \boxed{f'(g(\overline{x})) \cdot g'(\overline{x})}$$

■

The chain rule is sometimes written as $\frac{df}{dx}(x) = \frac{df}{dg}(x)\frac{dg}{dx}(x)$, but notice that this extends liebniz notation as we only know that $\frac{df}{dx} = f'(x)$, and it is not defined for when we take the derivative with respect to another function. This motivates the definition

$$\frac{df}{dg}(x) = \frac{df}{dx}\left(g(x)\right)$$

Proposition 1.2.1: Little O and Differentiability Equivalence

A function f is differentiable at the point $\overline{x}$ if and only if there is a number $\alpha \in \mathbb{R}$ for any $h \in \mathbb{R}$ we have

$$f\left(\overline{x}+h\right) = f(\overline{x}) + h\alpha + E\left(h\right)$$

Where $E \in o\left(h\right)$

Proof

- $\Rightarrow$
  - Assume that f is differentiable, therefore we have

    $$f'(\overline{x}) = \lim_{h\to 0} \frac{f\left(\overline{x}+h\right) - f(x)}{h}$$

  - Equivalently:

    $$\lim_{h\to 0} \frac{f\left(\overline{x}+h\right) - f(x)}{h} - f'(\overline{x}) = 0$$

  - Since $\lim_{h\to 0} f'(\overline{x}) = f'(\overline{x})$ as it's a constant, we may use the limit law to deduce:

    $$\lim_{h\to 0}\left(\frac{f\left(\overline{x}+h\right) - f(x)}{h} - f'(\overline{x})\right) = 0 \Leftrightarrow \lim_{h\to 0} \frac{f\left(\overline{x}+h\right) - f(\overline{x}) - hf'(\overline{x})}{h} = 0$$

  - That is to say that $f\left(\overline{x}+h\right) - f(\overline{x}) - hf'(\overline{x}) \in o\left(h\right)$, set $E\left(h\right) = f\left(\overline{x}+h\right) - f(\overline{x}) - hf'(\overline{x})$ then we have that

    $$f\left(\overline{x}+h\right) = f(\overline{x}) + hf'(\overline{x}) + E\left(h\right)$$

    where we've taken $\alpha = f'(\overline{x})$
- $\Leftarrow$
  - Assume that there is a number $\alpha \in \mathbb{R}$ and $E \in o\left(h\right)$, so that for any $h \in \mathbb{R}$ we have:

    $$f\left(\overline{x}+h\right) = f(\overline{x}) + h\alpha + E\left(h\right) \Leftrightarrow l\left(h\right) = f\left(\overline{x}+h\right) - f(\overline{x}) - h\alpha$$

    thus $f\left(\overline{x}+h\right) - f(\overline{x}) - h\alpha \in o\left(h\right)$.
  - So then by definition of little-o, we get:

    $$\lim_{h\to 0} \frac{f\left(\overline{x}+h\right) - f(\overline{x}) - h\alpha}{h} = 0$$

  - Since $\lim_{h\to 0} \alpha = \alpha$ we may add it to both sides using the limit law on the left to obtain:

    $$\lim_{h\to 0}\left(\frac{f\left(\overline{x}+h\right) - f(\overline{x}) - h\alpha}{h} + \alpha\right) = \alpha \Leftrightarrow \lim_{h\to 0} \frac{f\left(\overline{x}+h\right) - f(\overline{x})}{h} = \alpha$$

    Meaning that $\alpha = f'(\overline{x})$ and that f is differentiable

■

Notice that the number α is equal to a limit and that limits are unique, therefore the solution $\alpha = f'(\overline{x})$ is the unique solution to the above proposition. Also since this it the unique solution, we know that there is no other real besides $f'(\overline{x})$ where this holds, therefore $f'(\overline{x})$ is the best linear approximation to f at the point $\overline{x}$.

Finally notice that if you solve for $E\left(h\right)$ and then recall that $E\left(h\right) \in o\left(h\right)$ we have:

$$\lim_{h\to 0} \frac{\overbrace{\overbrace{f\left(\overline{x}+h\right)}^{\text{actual value}} - \overbrace{\left(f(\overline{x}) + hf'(\overline{x})\right)}^{\text{linear estimation}}}^{\text{error}}}{h} = 0$$

the error is going to zero at a rate which is faster than linear, as if the error were decreasing at a linear rate the limit would evaluate to some non-zero constant.

1.3 Sequences

Definition 1.3.1: Bounded Sequence

A sequence $\{x_n\}$ *is said to be bounded if* $\exists M \in \mathbb{R}$ *such that* $|x_n| \leq M$ *for all* $n \in \mathbb{N}$

1.4 Multi Variable

Definition 1.4.1: Directional Derivative

Let $U \subseteq \mathbb{R}^n$ *be open,* $f : U \to \mathbb{R}$ *and* $\vec{a} \in U$ *and* $\vec{v} \in \mathbb{R}^n$

$$\partial_{\vec{v}} f\left(\vec{a}\right) \stackrel{\text{D}}{=} \lim_{t \to 0 \in \mathbb{R}} \frac{f\left(\vec{a} + t\vec{v}\right) - f\left(\vec{a}\right)}{t}$$

2

Linear Algebra

2.1 Vectors

Definition 2.1.1: Algebraic Dot Product

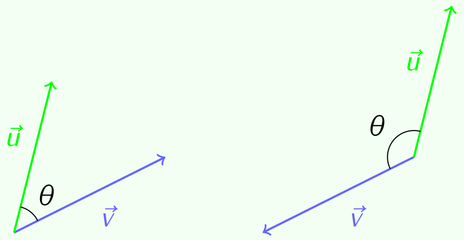
Let $u_1, u_2, \ldots, u_{n-1}, u_n$ and $v_1, v_2, \ldots, v_{n-1}, v_n$ denote the components of $\vec{u}$ and $\vec{v}$ respectively, then we have:

$$\vec{v} \cdot \vec{u} \overset{D}{=} \sum_{i=1}^{n} v_i u_i$$

Definition 2.1.2: Geometric Dot Product

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and let θ be the angle between the two (the one in the range $[0, \pi]$), then we have the dot product:

$$\vec{v} \cdot \vec{u} \overset{D}{=} ||v|| \left\|u\right\| \cos\left(\theta\right)$$



Proposition 2.1.1: Algebraic Def Implies Geometric Def of Dot Product

$$\sum_{i=1}^{n} v_i u_i = ||v|| \left\|u\right\| \cos\left(\theta\right)$$

Proof

- Consider two vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$, we know that from the polar coordinate system we can represent them as

$$\vec{x} = \left(||x|| \cos\left(\theta_x\right), ||x|| \sin\left(\theta_x\right)\right) \quad \text{and} \quad \vec{y} = \left(\left\|y\right\| \cos\left(\theta_y\right), \left\|y\right\| \sin\left(\theta_y\right)\right)$$

- The the algebraic definition implies the following

$$\vec{x} \cdot \vec{y} = ||x|| \left\|y\right\| \cos\left(\theta_x\right) \cos\left(\theta_y\right) + ||x|| \left\|y\right\| \sin\left(\theta_x\right) \sin\left(\theta_y\right) = ||x|| \left\|y\right\| \left(\cos\left(\theta_x\right) \cos\left(\theta_y\right) + \sin\left(\theta_x\right) \sin\left(\theta_y\right)\right)$$

- Then we can recall the sin angle difference formula to obtain:

$$\vec{x} \cdot \vec{y} = ||x|| \left\|y\right\| \left(\cos\left(\theta_x - \theta_y\right)\right)$$

- And just noting that $\theta_x - \theta_y$ is the angle between the vectors $\vec{x}$ and $\vec{y}$ as needed.

■

2.2 Matrices

Definition 2.2.1: Matrix Multiplication

If $\mathbf{A}$ *is an* $m \times n$ *matrix and* $\mathbf{B}$ *is an* $n \times p$ *matrix,*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

the matrix product $\mathbf{C} = \mathbf{AB}$ *(denoted without multiplication signs or dots) is defined to be the* $m \times p$ *matrix*

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

for $i = 1, \ldots, m$ *and* $j = 1, \ldots, p$.

```python
def multiply_matrices(matrix_A, matrix_B):
  """
  Given two matrices, we compute the matrix multiplication of the two of them

  Supposing that matrix_A is MxN and that matrix_B is NxK
  then the resulting matrix is M x K (1)
  """

  transpose_matrix_B = transpose_matrix(matrix_B)

  num_rows_of_matrix_A = len(matrix_A) # This is M
  num_columns_of_matrix_A = len(matrix_A[0]) #This is N

  num_rows_of_matrix_B = len(matrix_B) #this is N
  num_columns_of_matrix_B = len(matrix_B[0])# This is K

  resulting_matrix = []

  #initialize the matir xto be all zeros
  # then we will fill in the entries with the correct values
  for i in range(num_rows_of_matrix_A):
    blank_row = []
    for j in range(num_columns_of_matrix_B):
      blank_row.append(0)
    resulting_matrix.append(blank_row)
```

```
  # Notice that the matrix now has dimenisions ‘num_rowsof_matrix_A‘ x ‘num_columns_of_matrix_B‘
  # Or equivalently M x K , as we noted by  (1)

  for row_index in range(num_rows_of_matrix_A):
    for column_index in range(num_rows_of_matrix_B):
      row_from_matrix_A = matrix_A[row_index]
      transposed_column_from_matrix_B = transpose_matrix_B[column_index]
      resulting_matrix[row_index][column_index] = dot_product(row_from_matrix_A,
    transposed_column_from_matrix_B)

  return resulting_matrix


def transpose_matrix(matrix):
  """Given a matrix turn it into a new matrix where it’s rows are equal to it’s columns"""
  rows = len(matrix)
  columns = len(matrix[0]) # assuming matrix is non-empty

  transpose_matrix = []
  for i in range(columns):
    temp_new_row = []
    for j in range(rows):
      temp_new_row.append(matrix[j][i])
    transpose_matrix.append(temp_new_row)

  return transpose_matrix



def dot_product(v1, v2):
  dot_product = 0
  for i in range(len(row)):
    dot_product += v1[i] * v2[i]
  return dot_product
```

3

First Order Logic

3.1 Languages

Definition 3.1.1: First-Order Language

A first order language $\mathcal{L}$ is an infinite collection of distinct symbols, no one of which is properly contained in another, sparated into the following cateogories:

1. *Parentheses: (,).*
2. *Connectives: $\vee, \neg$.*
3. *Quantifier: $\forall$.*
4. *Variables, one for each positive integer n: $v_1, v_2, \ldots, v_n, \ldots$ The set of variable symbols will be denoted Vars.*
5. *Equality symbol: $=$.*
6. *Constant symbols: Some set of zero or more symbols.*
7. *Function symbols: For each positive integer n, some set of zero or more n-ary function symbols.*
8. *Relation symbols: For each positive integer n, some set of zero or more n-ary relation symbols.*

Definition 3.1.2: A Term's Variable Replaced by a Term

Suppose that u is a term, x is a variable, and t is a term. We define the term u_t^x (read "u with x replaced by t") as follows:

1. *If u is a variable not equal to x, then u_t^x is u.*
2. *If u is x, then u_t^x is t.*
3. *If u is a constant symbol, then u_t^x is u.*
4. *If $u :\equiv f u_1 u_2 \ldots u_n$, where f is an n-ary function symbol and the u_i are terms, then u_t^x is $f\,(u_1)_t^x\,(u_2)_t^x \ldots (u_n)_t^x$*

- In the fourth clause of the definition above and in the first two clauses of the next definition, the parentheses are not really there. Because ${u_1}_t^x$ is hard to read so the parentheses have been added in the interest of readability.
- If we let t be $g(c)$ and we let u be $f(x, y) + h(z, x, g(x))$, then u_t^x is

$$f(g(c), y) + h(z, g(c), g(g(c)))$$

Definition 3.1.3: A Term is Substitutable for a Variable

Suppose that ϕ is an $\mathcal{L}$-formula, t is a term, and x is a variable. We say that t is substitutable for x in ϕ if

1. *ϕ is atomic, or*
2. *$\phi :\equiv \neg(\alpha)$ and t is substitutable for x in α, or*
3. *$\phi :\equiv (\alpha \vee \beta)$ and t is substitutable for x in both α and β, or*
4. *$\phi :\equiv (\forall y)(\alpha)$ and either*
   - *x is not free in ϕ, or*
   - *y does not occur in t and t is substitutable for x in α.*

To understand the motivation behind the fourth clause, consider the formula:

$$\phi :\equiv \left(\forall x\right) \left(\forall y\right) x = y$$

Then one might want to say that $\left(\left(\forall y\right) x = y\right)_t^x$ where t is any term

Definition 3.1.4: Free Variable in a Formula

Suppose that v is a variable and ϕ is a formula. We will say that v is free in ϕ if

1. *ϕ is atomic and v occurs in (is a symbol in) ϕ, or*
2. *$\phi :\equiv (\neg\alpha)$ and v is free in α, or*
3. *$\phi :\equiv (\alpha \vee \beta)$ and v is free in at least one of α or β, or*
4. *$\phi :\equiv (\forall u)(\alpha)$ and v is not u and v is free in α.*

Examples

- Thus, if we look at the formula
$$\forall v_2 \neg \left(\forall v_3\right) \left(v_1 = S\left(v_2\right) \vee v_3 = v_2\right)$$
the variable v_1 is free whereas the variables v_2 and v_3 are not free.

- A slightly more complicated example is
$$\left(\forall v_1 \forall v_2 \left(v_1 + v_2 = 0\right)\right) \vee v_1 = S(0)$$
  - In this formula, v_1 is free whereas v_2 is not free. Especially when a formula is presented informally, you must be careful about the scope of the quantifiers and the placement of parentheses.

Notes

- We will have occasion to use the informal notation $\forall x \phi(x)$. This will mean that ϕ is a formula and x is among the free variables of ϕ.
- If we then write $\phi(t)$, where t is an $\mathcal{L}$-term, that will denote the formula obtained by taking ϕ and replacing each occurrence of the variable x with the term t.

3.2 Deductions

Definition 3.2.1: Deduction of a Formula

Suppose that Σ is a collection of $\mathcal{L}$-formulas and D is a finite sequence $(\phi_1, \phi_2, \ldots, \phi_n)$ of $\mathcal{L}$-formulas. We will say that D is a deduction from Σ if for each $i, 1 \leq i \leq n$, either

1. *$\phi_i \in \Lambda\,(\phi_i$ is a logical axiom), or*
2. *$\phi_i \in \Sigma\,(\phi_i$ is a nonlogical axiom), or*
3. *There is a rule of inference (Γ, ϕ_i) such that $\Gamma \subseteq \{\phi_1, \phi_2, \ldots, \phi_{i-1}\}$.*

If there is a deduction from Σ, the last line of which is the formula ϕ, we will call this a deduction from Σ of ϕ, and write $\Sigma \vdash \phi$.

Definition 3.2.2: Logical Axioms

Given a first order language $\mathcal{L}$ the set Λ of logical axioms is the collection of all $\mathcal{L}-$formulas that fall into one of the following categories:

$$x = x \text{ for each variable } x$$
$$\left[(x_1 = y_1) \land (x_2 = y_2) \land \cdots \land (x_n = y_n)\right] \to \left(f\,(x_1, x_2, \ldots, x_n) = f\,(y_1, y_2, \ldots, y_n)\right)$$
$$\left[(x_1 = y_1) \land (x_2 = y_2) \land \cdots \land (x_n = y_n)\right] \to \left(R\,(x_1, x_2, \ldots, x_n) \to R\,(y_1, y_2, \ldots, y_n)\right)$$
$$(\forall x \phi) \to \phi_t^x \text{ if } t \text{ is substitutable for } x \text{ in } \phi$$
$$\phi_t^x \to (\exists x \phi) \text{ if } t \text{ is substitutable for } x \text{ in } \phi$$

To refer to them easily we label them by moving down the above list E1, E2, E3, Q1, Q2

Lemma 3.2.1: Universal connection to Variable Assignment Function

$$\Sigma \vdash \theta \text{ if and only if } \Sigma \vdash \forall x \theta$$

Proof

- $\Rightarrow$
  - Suppose that $\Sigma \vdash \theta$, therefore we have a deduction $\mathcal{D}$ of θ, then the proof

$$\mathcal{D}$$
$$\left[\left(\forall y\,(y = y)\right) \land \neg\left(\forall y\,(y = y)\right)\right] \to \theta \qquad \text{(taut. PC)}$$
$$\left[\left(\forall y\,(y = y)\right) \land \neg\left(\forall y\,(y = y)\right)\right] \to (\forall x)\,\theta \qquad \text{(QR)}$$
$$(\forall x)\,\theta \qquad \text{(PC)}$$

- $\Leftarrow$
  - Suppose that $\Sigma \vdash \forall x \theta$, so we have a deduction of it, call it $\mathcal{D}$, then the following deduction suffices

$$\mathcal{D}$$
$$\forall x \theta$$
$$\forall x \theta \to \theta_x^x$$
$$\theta_x^x$$

■

3.3 Completeness

Definition 3.3.1: Consistent Set of L Formulas

Let Σ be a set of $\mathcal{L}$-formulas. We will say that Σ is inconsistent if there is a deduction from Σ of $[(\forall x)x = x] \land \neg[(\forall x)x = x]$. We say that Σ is consistent if it is not inconsistent.

Proposition 3.3.1: Contradiction has no Model

There is no $\mathcal{L}$-Structure $\mathfrak{A}$ such that $\mathfrak{A} \models \bot$

Proof

Suppose there was a model of $\bot$, that would mean that we have an $\mathcal{L}$-Structure $\mathfrak{A}$ such that $\mathfrak{A} \models \bot$. Recall that $\bot :\equiv \left[\left(\forall x\right) x = x\right] \land \neg \left[\left(\forall x\right) x = x\right]$, so then we would have to have $\mathfrak{A} \models \left(\forall x\right) x = x$ and also not have $\mathfrak{A} \models \left(\forall x\right) x = x$ which is a contradiction. ■

Lemma 3.3.1: Constant Extension still Consistent

If Σ is a consistent set of $\mathcal{L}$-sentences and $\mathcal{L}'$ is an extension by constants of $\mathcal{L}$, then Σ is consistent when viewed as a set of $\mathcal{L}'$-sentences.

Proof

- Suppose for the sake of contradiction that Σ is not consistent when viewed as a set of $\mathcal{L}'$-sentences, so $\Sigma \vdash \bot$, considering the set of all deductions of $\bot$ from Σ we may find a deduction $\mathcal{D}$ which uses the least number of the newly added constants by the well ordering principle, let this number be $n \in \mathbb{N}$ and that $n > 0$ or else we would have a deduction from $\mathcal{L}$ of $\bot$ which would be a contradiction.

- Let v be a variable that isn't used in $\mathcal{D}$ and let c be one of the newly added constants which is used in $\mathcal{D}$ and let $\mathcal{D}_v$ be created where for each line $\phi \in \mathcal{D}$ we create ϕ_v by replacing all occurances of c in ϕ with v, and note that the last line of $\mathcal{D}_v$ is still $\bot$, at this point we don't know if $\mathcal{D}_v$ is a deduction, so we have to check that it is.

- If ϕ is an equality axiom or an element of Σ then $\phi_v :\equiv \phi$ because equality axioms only contain variables and not constants, also Σ is a set of $\mathcal{L}$ sentences and so it can't contain any of the new constants, and so ϕ_v is a valid step in the deduction since it is still an equality axiom or an element of Σ

- If ϕ is $\left(\forall x\right) \theta \to \theta^x_t$ then ϕ_v is $\left(\forall x\right) \theta_v \to \left(\theta_v\right)^x_{t_v}$, to see why we use t_v as well as θ_v try $\theta :\equiv c = x$ and $t :\equiv c + 3$

■

Notice that if we write $\Sigma \models \bot$ it means that for any $\mathcal{L}$-Structure $\mathfrak{A}$ if $\mathfrak{A} \models \Sigma$ then $\mathfrak{A} \models \bot$ by the above discussion that forces $\mathfrak{A} \models \Sigma$ to be false, and therefore Σ has no model.

Theorem 3.3.1: Completeness Theorem

Suppose that Σ is a set of $\mathcal{L}$-formulas, where $\mathcal{L}$ is a countable langauge and ϕ is an $\mathcal{L}$-formula. If $\Sigma \models \phi$, then $\Sigma \vdash \phi$.

Setup

- *We start by assuming that $\Sigma \models \phi$, we must show that $\Sigma \vdash \phi$.*
- *If ϕ is not a sentence then we can always prove ϕ' which is the same as ϕ with all of it's variables bound*
  - *We can do that by appending $(\forall x_f)$ where each x_f is a free varaible of ϕ to the front of ϕ*
- *Therefore we will prove it for all sentences ϕ*

3.4 Incompleteness

Definition 3.4.1: Representable Set

A set $A \subseteq \mathbb{N}^k$ is said to be representable (in N) if there is an $\mathcal{L}_{NT}$-formula $\phi(\underline{x})$ such that

$$\begin{array}{ll} \forall a \in A & N \vdash \phi(\bar{a}) \\ \forall \underline{b} \notin A & N \vdash \neg\phi(\bar{b}) \end{array}$$

In this case we will say that the formula ϕ represents the set A.

Definition 3.4.2: Weakly Representable Set

A set $A \subseteq \mathbb{N}^k$ is said to be weakly representable (in N) if there is an $\mathcal{L}_{NT}$-formula $\phi(x)$ such that

$$\begin{array}{ll} \forall a \in A & N \vdash \phi(\bar{a}) \\ \forall b \notin A & N \nvdash \phi(\bar{b}) \end{array}$$

In this case we will say that the formula ϕ weakly represents the set A.

Definition 3.4.3: Total Function

Suppose that $A \subseteq \mathbb{N}^k$ and suppose that $f : A \to \mathbb{N}$. If $A = \mathbb{N}^k$ we will say that f is a total function

Definition 3.4.4: Partial Function

Suppose that $A \subsetneq \mathbb{N}^k$ and suppose that $f : A \to \mathbb{N}$, then we will call f a partial function.

Definition 3.4.5: Representable Function

Suppose that $f : \mathbb{N}^k \to \mathbb{N}$ is a total function. We will say that f is a representable function (in N) if there is an $\mathcal{L}_{NT}$ formula $\phi\left(x_1, \ldots, x_{k+1}\right)$ such that, for all $a_1, a_2, \ldots a_{k+1} \in \mathbb{N}$

- *If $f\left(a_1, \ldots, a_k\right) = a_{k+1}$, then $N \vdash \phi\left(\overline{a_1}, \ldots, \overline{a_{k+1}}\right)$*
- *If $f\left(a_1, \ldots, a_k\right) \neq a_{k+1}$, then $N \vdash \neg\phi\left(\overline{a_1}, \ldots, \overline{a_{k+1}}\right)$*

Definition 3.4.6: Definable Set

We will say that a set $A \subseteq \mathbb{N}^k$ is definable if there is a formula $\phi(x)$ such that

$$\begin{array}{ll} \forall a \in A & \mathfrak{N} \models \phi(\bar{a}) \\ \forall \underline{\downarrow b} \notin A & \mathfrak{N} \models \neg\phi(\bar{b}) \end{array}$$

In this case, we will say that ϕ defines the set A.

Corollary 3.4.1: I

$A \subseteq \mathbb{N}^k$ is definable by a Δ formula, then it is representable

4

Topology

4.1 Topological Spaces and Continuous Functions

4.1.1 Basis for a Topology

Definition 4.1.1: Basis

If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

1. *For each $x \in X$, there is at least one basis element B containing x.*
2. *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for, each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

4.1.2 The Subspace Topology

Definition 4.1.2: Subspace Topology

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology. With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.

4.1.3 The Product Topology

Definition 4.1.3: Product Topology

Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \left\{\pi_\beta^{-1}\left(U_\beta\right) \mid U_\beta \text{ open in } X_\beta\right\}$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called the product topology. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a product space.

Definition 4.1.4: Box Topology

Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha\in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha\in J} U_\alpha$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the box topology.

Theorem 4.1.1: Basis for the Box Topology

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha\in\mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha\in\mathcal{J}} X_\alpha$.

Theorem 4.1.2: Basis for the Product Topology

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha\in\mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha\in\mathcal{J}} X_\alpha$.

Definition 4.1.5: R Omega

$\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Recall that

$$\mathbb{R}^\omega = \prod_{n\in\mathbb{N}} X_n$$

with $X_n = \mathbb{R}$ for each n

4.1.4 The Metric Topology

Definition 4.1.6: A metric

A metric on a set X is a function

$$d : X \times X \to \mathbb{R}$$

having the following properties:

1. *$d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. *Triangle Inequality: $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.*

Example 4.1.1: Discrete Metric

$d : X \times X \to \mathbb{R}$ given by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & \text{otherwise} \end{cases}$$

Definition 4.1.7: Epsilon Ball

Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the ϵ-ball centered at $\mathbf{x}$. Sometimes we omit the metric d from the notation and write this ball simply as $B(x, \epsilon)$, when no confusion will arise.

Lemma 4.1.1: Epsilon Ball Contains Another

Let $x \in X$ then for every $B\left(x, \varepsilon\right)$ there is $y \in B\left(x, \varepsilon\right)$ and $\delta \in \mathbb{R}^+$ such that

$$B\left(y, \delta\right) \subseteq B\left(x, \varepsilon\right)$$

Proof

- Let $y \in B\left(x, \varepsilon\right)$ we claim $B\left(y, \delta\right)$ where $\delta = \varepsilon - d\left(x, y\right)$ works.
- Note $B\left(y, \delta\right) \subseteq B\left(x, \varepsilon\right)$, since given any $z \in B\left(y, \delta\right)$ we have that $d\left(y, z\right) < \varepsilon - d\left(x, y\right)$, rearranging gives us $d\left(x, y\right) + d\left(y, z\right) < \varepsilon$ and by the triangle inequality we get $d\left(x, z\right) < \varepsilon$ therefore $z \in B\left(x, \varepsilon\right)$

■

Proposition 4.1.1: Epsilon Balls Form A Basis

The collection $\mathcal{B}$ of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X

Proof

- Let $x \in X$ then clearly $x \in B_d\left(x, \varepsilon\right)$ works for any $\varepsilon \in \mathbb{R}^+$
- Let $B_1, B_2 \in \mathcal{B}$ and and let $y \in B_1 \cap B_2$, for each of B_1 and B_2 we have δ_1 and δ_2 respectively with $B\left(y, \delta_1\right) \subseteq B_1$ and $B\left(y, \delta_2\right) \subseteq B_2$, by taking $\delta = \min\left(\delta_1, \delta_2\right)$ then we have $B\left(y, \delta\right) \subseteq B_1 \cap B_2$ as needed.

■

Definition 4.1.8: Metric Topology

If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the metric topology induced by d.

Definition 4.1.9: Bounded Subset of a Metric Space

Let X be a metric space with metric d. A subset A of X is said to be bounded if there is some number $M \in \mathbb{R}$ such that

$$d\left(a_1, a_2\right) \leq M$$

for every pair of points $a_1, a_2 \in A$

Example 4.1.2: Function on R omega

Consider a function $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined by

$$h\left(x_1, x_2, \ldots\right) = \left(\alpha_1 x_1, \alpha_2 x_2, \ldots\right)$$

with $\alpha_1, \alpha_2, \ldots \in \mathbb{R} \setminus \{0\}$

1. *Is h continuous, when $\mathbb{R}^\omega$ is given the product topology?*
2. *Is h continuous, when $\mathbb{R}^\omega$ is given the box topology?*
3. *Is h continuous, when $\mathbb{R}^\omega$ is given the uniform topology?*

Proof

1. • Take an arbitrary basis element from the product topology, that is:

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_{\mathbb{R}}$ for finitely many indices α and $B_\alpha = \mathbb{R}$ for all the remaining indices.

α : Now note the following

– The inverse image of each of these intervals is a scaled version of itself therefore it is still an interval of $\mathbb{R}$, and it is therefore still open with respect to $\mathbb{R}$

– The inverse image of $\mathbb{R}$ under any scaling is still $\mathbb{R}$

• Therefore $h^{-1}\left(\prod_{\alpha \in \mathcal{J}} B_\alpha\right)$ is a product of finitely many intervals, and infinitely many $\mathbb{R}$'s and so it is open with respect to $\mathbb{R}^\omega$ as needed.

2. • Given an arbitrary basis element of the box topology we have

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_{\mathbb{R}}$ for each α

• Due to α we know that each of the intervals become new scaled intervals and so $h^{-1}\left(\prod_{\alpha \in \mathcal{J} B_\alpha}\right)$ is a product of open intervals and is therefore open in with respect $\mathbb{R}^\omega$ equipped with the box topology

3. Recall that a basis for the uniform topology are epsilon balls with radius less then 1 if that's the case, then so long as every α_i is greater than 1 then the inverse image scales them to be even smaller, resulting in an open set, otherwise the function h wouldn't be continuous, since all the α_i's are greater than one, it is continouous.

■

4.2 Connectedness and Compactness

Definition 4.2.1: Connected Space

Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

Notice that U, V are actually clopen, as $X \setminus U = V$ and $X \setminus V = U$ stating that V and U are closed as well.

Example 4.2.1: Closed and Bounded, not Compact

A metric space X and a closed and bounded subspace Y of X that is not compact.

- Consider the set $X = \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$, with the discrete metric, it is bounded because the for any two points $a, b \in X, d(a, b) \leq 1$
- Let X be an infinite set and let consider the discrete metric on that set, the metric topology which it induces (call it $\mathcal{T}$) is the discrete topology of X. Therefore if we consider any subset Y of X it is closed, as $X \setminus Y \in \mathcal{T}$ (remember it's the discrete topology). But the open covering $\{\{x\} : x \in X\}$ has no finite subcollection which also covers X.

Example 4.2.2: R omega Connected?

Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$. In which topologies is $\mathbb{R}^\omega$ connected?

Proof

1. - Consider the product topology, we will show that $\mathbb{R}^\omega$ is not connected. Consider the set A of bounded sequences in $\mathbb{R}^\omega$, and the complement of A, namely the unbounded sequence of $\mathbb{R}^\omega$ let's label this set as $B = \mathbb{R}^\omega \setminus A$.
     - Note that A is open in the box topology as if we fix any $\varepsilon > 0$ we may define for any $\vec{x} \in \mathbb{R}^\omega$
       $$U_{\vec{x}} \overset{\text{D}}{=} (x_1 - \varepsilon, x_1 + \varepsilon) \times (x_2 - \varepsilon, x_2 + \varepsilon) \times \dots$$
     - If $\vec{a}$ is a bounded sequence then U is a set of bounded sequences, as it only ever adds constants to the terms of $\vec{a}$, therefore $\vec{a} \in U_{\vec{a}}$ and $U_{\vec{a}} \subseteq A$, therefore A is open.
     - If $\vec{b}$ is an unbounded sequence then U is a set of unbounded sequences, as adding a constant to every element to an unbounded sequence yields an unbounded sequence. So for every $\vec{b} \in B$ there we have $\vec{b} \in U_{\vec{b}}$ and $U_{\vec{b}} \subseteq B$ so B is open.
   - Therefore A non-trivial set which is both open and closed in $\mathbb{R}^\omega$ and so it is disconnected.
2. Note that $\mathbb{R}^\omega$ is closed with the uniform topology by following the same proof with $\varepsilon = 1$
3. - Now if we consider $\mathbb{R}^\omega$ with the product topology we will see that $\mathbb{R}^\omega$ is connected.
   - Let $\mathbb{R}^n_0 \overset{\text{D}}{=} \{(x_1, x_2, \dots) : \text{ where for } i > n \text{ we have that } x_i = 0\}$
   - $\mathbb{R}^n_0$ is homeomorphic to $\mathbb{R}^n$ and since $\mathbb{R}$ is connected, and the finite cartesian product of connected spaces is connected, we get that $\mathbb{R}^n_0$ is connected.
   - We now will show that the closure of $\mathbb{R}^\infty$ is equal to $\mathbb{R}^\omega$ and then by the fact that the closure of a connected set is closed we obtain that $\mathbb{R}^\omega$ is connected.
   - To show that R^ω is equal to the closure of $\mathbb{R}^\infty$ we proceed by using the fact that a point is part of the closure of a set if and only if every basis element intersects it:
     - Let $\vec{a} \in \mathbb{R}^\omega$ and let $B = \prod_{i \in \mathbb{N}} B_i$ be some basis element for the product topology with $\vec{a} \in U$, now we have to show this set intersects $\mathbb{R}^\infty$. Since only finitely many of the B_i are equal to basis elements and the rest are equal to $\mathbb{R}$ that means there is some $N \in \mathbb{N}$ such that $\forall j \in \mathbb{N}^{\geq N}, B_j = \mathbb{R}$, therefore the point $\vec{b} = (a_1, a_2, \dots, a_{N-1}, a_N, 0, 0, 0, \dots) \in B \cap \mathbb{R}^\infty$

■

Proposition 4.2.1: Connected Implies Closure Connected

Let $A \subseteq X$ be a connected subspace of X then $\overline{A}$ is also connected.

Proof

- Suppose for the sake of contradiction that there is a separation B, C of $\overline{A}$, then $B \cup C = \overline{A}$ and note that that means that $A \subseteq B \cup C$ since $A \subseteq \overline{A}$ so $\left(B \cup C\right) \cap A = \overline{A} \cap A = A$.
- Therefore $\left(B \cap A\right) \cup \left(C \cap A\right) = A$ is a separation of A noting that $B \cap A$ and $C \cap A$ are non-empty because ...

■

Definition 4.2.2: Totally Disconnected

A topological space is totally disconnected if it's only connected subspaces are one-point sets.

Consider $\mathbb{R}_\ell$ if we have $\{a\}$ and $\{b\}$ then the open sets $\left(-\infty, b\right)$ and $[b, \infty)$ is a separation of $\mathbb{R}_\ell$, therefore it's disconnected. Similarly for any two points a, b in $\mathbb{Q}$ we have some irrational number r between the two, and thus $\left(-\infty, r\right)_{\mathbb{Q}}, \left(r, \infty\right)_{\mathbb{Q}}$ is a separation thus $\mathbb{Q}$ is totally disconnected under this topology.

Let's look at $\mathbb{R}$ with the fite complement topology. Right off the bat, we note that if a set is finite in $\mathbb{R}$ it's complement must be infinite therefore if $\mathbb{R}$ was completely disconnected it would mean for any singleton sets, we have a separation U, V, but that means that U and V must be infinite, but we then get a contradiction as $\mathbb{R} = U \cup V$ so $R \setminus U = V$, now since U was open this implies that V is finite, which is a contradiction. This idea may be extended to $\mathbb{R}^2$.

4.2.1 Compact Spaces

Definition 4.2.3: Covering

A collection A of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of A is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition 4.2.4: Compact Space

A space X is said to be compact if every open covering A of X contains a finite subcollection that also covers X.

Lemma 4.2.1: Covering Yields Finite Covering if and only if Compact

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

5

Computer Science

Definition 5.0.1: Big-O

Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, *then we define the set*

$$\mathcal{O}\left(g\right) \stackrel{\mathrm{D}}{=} \left\{j : \exists c, B \in R^{+}, \forall n \in \mathbb{N}, n \geq B \to j\left(n\right) \leq cg\left(n\right)\right\}$$

And we say that f *is in the big-O of* g *when* $f \in \mathcal{O}\left(g\right)$

Definition 5.0.2: Little-o

We define the following set:

$$o\left(g\right) = \left\{j : \lim_{x \to 0} \frac{j\left(x\right)}{g\left(x\right)} = 0\right\}$$

We say that f *is in the little-o of* g *when* $f \in o\left(g\right)$

6

Probability

Definition 6.0.1: Conditional Independence

Suppose A and B are independent events, then we would want $P\left(A \mid C\right) = P\left(A \mid B \cap C\right)$:

$$P\left(A \mid C\right) = \frac{P\left(A \cap C\right)}{P\left(C\right)} \text{ and } P\left(A \mid B \cap C\right) = \frac{P\left(A \cap B \cap C\right)}{P\left(B \cap C\right)}$$

So then we want:

$$\begin{gathered} P\left(A \mid C\right) = P\left(A \mid B \cap C\right) \\ \Updownarrow \\ \frac{P\left(A \cap C\right)}{P\left(C\right)} = \frac{P\left(A \cap B \cap C\right)}{P\left(B \cap C\right)} \\ \Updownarrow \\ \frac{P\left(A \cap B \cap C\right)}{P\left(C\right)} = \frac{P\left(A \cap C\right) P\left(B \cap C\right)}{P\left(C\right) P\left(C\right)} \\ \Updownarrow \\ P\left(A \cap B \mid C\right) = P\left(A \mid C\right) P\left(B \mid C\right) \end{gathered}$$

6.1 Bayesian Interference

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Programming

Working in the Shell

Convert all files names to lowercase

```
user@machine:~# for f in ‘find‘; do mv -v "$f" "‘echo $f  tr ’[A-Z]’ ’[a-z]’‘"; done
```