The Knowledge

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October 30, 2021

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Dedicated to the proofs that were left as exercises to the readers

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Preface

This book contains knowledge that that me or my peers have obtained, the purpose is to explain things fundamentally and in full detail so that someone who has never touched the subject may be able to understand it. It will focus on conveying the ideas that are involved in synthesizing the new knowledge with less of a focus on the results themselves.

Structure of book

The book is partitioned into different sections based on the domain it is involved with. There may be shared definitions and theorems throughout the chapters, but in general it will start more elementary and get more advanced.

Knowledge

In this book you will find many results, will will characterize them as being one of the following

- Theorems - Results that are of importantance and who's proof is not easily found (maybe using a novel idea)
- Propositions - Results of less importance who's proof could be constructed without a novel idea
- Lemmas - Results that are technical intermediate steps which has no standing as an independent result on first observation [†]
- Corollaries - Results which follow readily from an existing result of greater importance

Recommendations

By now you might know that in order to actually get better at mathematics you have to engage with it. This book may be used as a reference at times, but I highly recommend trying to re-prove statements or coming up with your own ideas before instantly looking at the solutions.

[†] But sometimes they escape, as their usage becomes more than just an intermediate step, as Zorn's or Fatou's Lemmas did

About the companion website

The website[*] for this file contains:

- A link to (freely downlodable) latest version of this document.
- Link to download LaTeX source for this document.

Acknowledgements

- A special word of thanks to professors who wanted to make sure I understood and learned as much as possible Alfonso Gracia-Saz[†], Jean-Baptiste Campesato[‡], Z-Module, riv, PlanckWalk, franciman, qergle from #math on `https://libera.chat/`.

[*] `https://github.com/cuppajoeman/knowledge-book`
[†] `https://www.math.toronto.edu/cms/alfonso-memorial/`
[‡] `https://math.univ-angers.fr/~campesato/`

Contributing

Contributions to the project are very welcome, let's delve into how to get started with this.

If you want to contribute to the project it's most likely that a contribution will fall into one of the following categories

- Content Based
  - Adding Definitions, Theorems, ...
  - Finishing TODO's
  - Formatting of the book
- Structural Layout of Project
  - Organization
  - Simplyfing the existing structure of the directories
  - Making scripts which set up new structures
- External
  - Adding explanatory content to help onboard new users
  - Getting others involved
  - Creating infrastructure to support users (Github discussions)

Content Based

If you're looking to add content to the project finish this

1

Linear Algebra

1.1 Vectors

Definition 1: Algebraic Dot Product

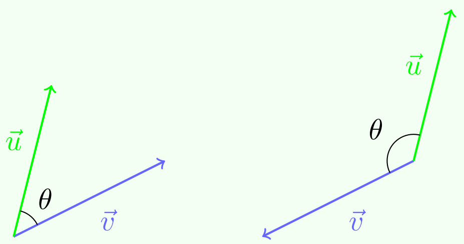
Let $u_1, u_2, \dots, u_{n-1}, u_n$ *and* $v_1, v_2, \dots, v_{n-1}, v_n$ *denote the components of* $\vec{u}$ *and* $\vec{v}$ *respectively, then we have:*

$$\vec{v} \cdot \vec{u} \overset{\mathtt{D}}{=} \sum_{i=1}^{n} v_i u_i$$

Definition 2: Geometric Dot Product

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ *and let* θ *be the angle between the two (the one in the range* $[0, \pi]$*), then we have the dot product:*

$$\vec{v} \cdot \vec{u} \overset{\mathtt{D}}{=} \|v\| \, \|u\| \cos(\theta)$$



Proposition 1: Algebraic Def Implies Geometric Def of Dot Product

$$\sum_{i=1}^{n} v_i u_i = \|v\| \, \|u\| \cos(\theta)$$

Proof

- Consider two vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$, we know that from the polar coordinate system we can represent them as

$$\vec{x} = (\|x\| \cos(\theta_x), \|x\| \sin(\theta_x)) \quad \text{and} \quad \vec{y} = (\|y\| \cos(\theta_y), \|y\| \sin(\theta_y))$$

- The the algebraic definition implies the following

$$\vec{x} \cdot \vec{y} = \|x\| \|y\| \cos(\theta_x) \cos(\theta_y) + \|x\| \|y\| \sin(\theta_x) \sin(\theta_y) = \|x\| \|y\| (\cos(\theta_x) \cos(\theta_y) + \sin(\theta_x) \sin(\theta_y))$$

- Then we can recall the sin angle difference formula to obtain:

$$\vec{x} \cdot \vec{y} = \|x\| \|y\| (\cos(\theta_x - \theta_y))$$

- And just noting that $\theta_x - \theta_y$ is the angle between the vectors $\vec{x}$ and $\vec{y}$ as needed.

■

1.2 Matrices

Definition 3: Matrix Multiplication

If $\mathbf{A}$ *is an* $m \times n$ *matrix and* $\mathbf{B}$ *is an* $n \times p$ *matrix,*

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

the matrix product $\mathbf{C} = \mathbf{AB}$ *(denoted without multiplication signs or dots) is defined to be the* $m \times p$ *matrix*

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

for $i = 1, \ldots, m$ *and* $j = 1, \ldots, p$.

```
def multiply_matrices(matrix_A, matrix_B):
```

```python
  """
  Given two matrices, we compute the matrix multiplication of the two of them

  Supposing that matrix_A is MxN and that matrix_B is NxK
  then the resulting matrix is M x K (1)
  """

  transpose_matrix_B = transpose_matrix(matrix_B)

  num_rows_of_matrix_A = len(matrix_A) # This is M
  num_columns_of_matrix_A = len(matrix_A[0]) #This is N

  num_rows_of_matrix_B = len(matrix_B) #this is N
  num_columns_of_matrix_B = len(matrix_B[0])# This is K

  resulting_matrix = []

  #initialize the matir xto be all zeros
  # then we will fill in the entries with the correct values
  for i in range(num_rows_of_matrix_A):
    blank_row = []
    for j in range(num_columns_of_matrix_B):
      blank_row.append(0)
    resulting_matrix.append(blank_row)
  # Notice that the matrix now has dimenisions `num_rowsof_matrix_A` x `num_columns_of_matrix_B`
  # Or equivalently M x K , as we noted by  (1)

  for row_index in range(num_rows_of_matrix_A):
    for column_index in range(num_rows_of_matrix_B):
      row_from_matrix_A = matrix_A[row_index]
      transposed_column_from_matrix_B = transpose_matrix_B[column_index]
      resulting_matrix[row_index][column_index] = dot_product(row_from_matrix_A, transposed_column_from_matrix_B)

  return resulting_matrix


def transpose_matrix(matrix):
  """Given a matrix turn it into a new matrix where it's rows are equal to it's columns"""
  rows = len(matrix)
  columns = len(matrix[0]) # assuming matrix is non-empty

  transpose_matrix = []
  for i in range(columns):
    temp_new_row = []
    for j in range(rows):
      temp_new_row.append(matrix[j][i])
    transpose_matrix.append(temp_new_row)

  return transpose_matrix
```

```
def dot_product(v1, v2):
  dot_product = 0
  for i in range(len(row)):
    dot_product += v1[i] * v2[i]
  return dot_product
```

2

First Order Logic

"This is a quote and I don't know who said this."

– Author's name, *Source of this quote*

2.1 Deductions

Lemma 1: Universal connection to Variable Assignment Function

$$\Sigma \vdash \theta \text{ if and only if } \Sigma \vdash \forall x \theta$$

Note this lemma might seem quite strange, but note it actually makes sense,

finish why

x ——— **Proof** ——— x

- $\Rightarrow$
  - Suppose that $\Sigma \vdash \theta$, therefore we have a deduction $\mathcal{D}$ of θ, then the proof

$$\begin{aligned}
&\mathcal{D} \\
&[(\forall y\,(y = y)) \land \neg\,(\forall y\,(y = y))] \to \theta && \text{(taut. PC)} \\
&[(\forall y\,(y = y)) \land \neg\,(\forall y\,(y = y))] \to (\forall x)\,\theta && \text{(QR)} \\
&(\forall x)\,\theta && \text{(PC)}
\end{aligned}$$

- $\Leftarrow$
  - Suppose that $\Sigma \vdash \forall x\theta$, so we have a deduction of it, call it $\mathcal{D}$, then the following deduction suffices

$$\begin{gathered}
\mathcal{D} \\
\forall x\theta \\
\forall x\theta \to \theta^x_x \\
\theta^x_x
\end{gathered}$$

2.2 Completeness

Theorem 1: Completeness Theorem

Suppose that Σ is a set of $\mathcal{L}$-formulas, where $\mathcal{L}$ is a countable langauge and ϕ is an $\mathcal{L}$-formula. If $\Sigma \models \phi$, then $\Sigma \vdash \phi$.

Setup

- *We start by assuming that $\Sigma \models \phi$, we must show that $\Sigma \vdash \phi$.*
- *If ϕ is not a sentence then we can always prove ϕ' which is the same as ϕ with all of it's variables bound*
  - *We can do that by appending $(\forall x_f)$ where each x_f is a free varaible of ϕ to the front of ϕ*
- *Therefore we will prove it for all sentences ϕ*

  justify why this is equivalent

3

Topology

3.1 Topological Spaces and Continuous Functions

3.1.1 Basis for a Topology

Definition 4: Basis

If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that

1. *For each $x \in X$, there is at least one basis element B containing x.*
2. *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows: A subset U of X is said to be open in X (that is, to be an element of $\mathcal{T}$) if for, each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

3.1.2 The Subspace Topology

Definition 5: Subspace Topology

Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology. With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.

3.1.3 The Product Topology

Definition 6: Product Topology

Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \left\{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\right\}$$

and let $\mathcal{S}$ denote the union of these collections,

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called the product topology. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a product space.

Theorem 2: Basis for the Box Topology

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in \mathcal{J}} X_\alpha$.

Theorem 3: Basis for the Product Topology

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form

$$\prod_{\alpha \in \mathcal{J}} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in \mathcal{J}} X_\alpha$.

Definition 7: R Omega

$\mathbb{R}^\omega$, the countably infinite product of $\mathbb{R}$ with itself. Recall that

$$\mathbb{R}^\omega = \prod_{n \in \mathbb{N}} X_n$$

with $X_n = \mathbb{R}$ for each n

3.1.4 The Metric Topology

Definition 8: A metric

A metric on a set X is a function

$$d : X \times X \to \mathbb{R}$$

having the following properties:

1. $d(x, y) \geq 0$ *for all* $x, y \in X$*; equality holds if and only if* $x = y$.
2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
3. *Triangle Inequality:* $d(x, y) + d(y, z) \geq d(x, z)$, *for all* $x, y, z \in X$.

Example 1: Discrete Metric

$d : X \times X \to \mathbb{R}$ *given by*

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & \textit{otherwise} \end{cases}$$

Definition 9: Epsilon Ball

Given $\epsilon > 0$, *consider the set*

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

of all points y *whose distance from* x *is less than* ϵ. *It is called the* ϵ*-ball centered at* $\boldsymbol{x}$. *Sometimes we omit the metric* d *from the notation and write this ball simply as* $B(x, \epsilon)$, *when no confusion will arise.*

Definition 10: Metric Topology

If d *is a metric on the set* X, *then the collection of all* ϵ*-balls* $B_d(x, \epsilon)$, *for* $x \in X$ *and* $\epsilon > 0$, *is a basis for a topology on* X, *called the metric topology induced by* d.

Definition 11: Bounded Subset of a Metric Space

Let X *be a metric space with metric* d. *A subset* A *of* X *is said to be bounded if there is some number* $M \in \mathbb{R}$ *such that*

$$d\left(a_1, a_2\right) \leq M$$

for every pair of points $a_1, a_2 \in A$

3.2 Connectedness and Compactness

Example 2: Closed and Bounded, not Compact

A metric space X and a closed and bounded subspace Y of X that is not compact.

- Consider the set $X = \left\{\frac{1}{n} : n \in \mathbb{N}^+\right\}$, with the discrete metric, it is bounded because the for any two points $a, b \in X, d(a, b) \leq 1$

closed and bounded proof

- Let X be an infinite set and let consider the discrete metric on that set, the metric topology which it induces (call it $\mathcal{T}$) is the discrete topology of X. Therefore if we consider any subset Y of X it is closed, as $X \setminus Y \in \mathcal{T}$ (remember it's the discrete topology). But the open covering $\{\{x\} : x \in X\}$ has no finite subcollection which also covers X.

3.2.1 Compact Spaces

Definition 12: Covering

A collection A of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of A is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition 13: Compact Space

A space X is said to be compact if every open covering A of X contains a finite subcollection that also covers X.

Lemma 2: Covering Yields Finite Covering if and only if Compact

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.