Mathematical Programs with Complementarity Constraints and Related Problems

Prof. Dr. Alexandra Schwartz
schwartz@gsc.tu-darmstadt.de

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Ressources

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You can find

- my slides
- and some more extensive lecture notes

at

github.com/alexandrabschwartz/Winterschool2018

If you have any questions, please come to me during the week or contact me at

schwartz@gsc.tu-darmstadt.de

Goals

At the end of the week you should

- know the definition and applications of MPCCs,
- be able to explain why MPCCs cannot be solved directly using standard theory,
- understand the origins and relations of different optimality conditions for MPCCs,
- know what to look for in a relaxation algorithm for MPCCs,
- be able to transfer the MPCC ideas to some related problem classes.

Literature

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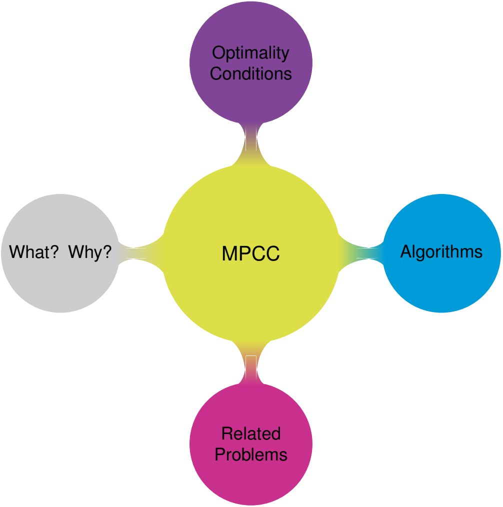
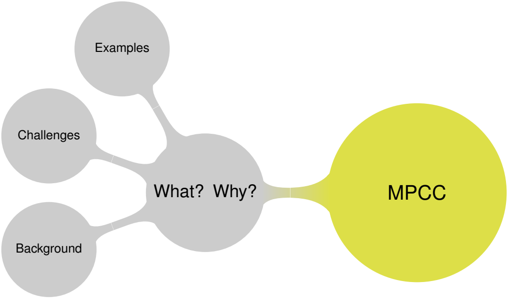
- Luo, Pang, Ralph: *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press
- Outrata, Kocvara, Zowe: *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*, Kluwer Academic Publishers
- Mordukhovich: *Variational Analysis and Generalized Differentiation I and II*, Springer
- Rockafellar, Wets: *Variational Analysis*, Springer
- Numerous articles by Jane Ye, Jiri Outrata, Christian Kanzow, and many others

Contents of the Course



What is the plan for today?

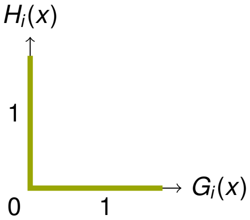


Problem Formulation and Examples

Mathematical Program with Complementarity Constraints (MPCC)

A mathematical program with complementarity constraints (MPCC) is of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$
$$0 \leq G(x) \perp H(x) \geq 0.$$



We assume that

- $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^p$, and $G, H : \mathbb{R}^n \to \mathbb{R}^q$ are continuously differentiable,
- the feasible set

$$X := \{x \in \mathbb{R}^n \mid g(x) \leq 0, \quad h(x) = 0, \quad 0 \leq G(x) \perp H(x) \geq 0\}$$

is nonempty.

Mathematical Program with Equilibrium Constraints

A mathematical program with equilibrium constraints (MPEC) is of the form

$$\min_{x,y} f(x,y) \quad \text{s.t.} \quad x \in X, y \in S(x),$$

where $x \rightrightarrows S(x)$ is the solution set of the equilibrium constraint, e.g. of

- a lower-level optimization problem,
- a lower-level Nash game,
- or a physical equilibrium.

Mathematical Program with Equilibrium Constraints

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- or a physical equilibrium.

The solution set $S(x)$ can often be described using

- complementarity constraints $0 \leq G(x, y) \perp H(x, y) \geq 0$,
- variational inequalities $F(y^*)^T(y - y^*) \geq 0 \quad \forall y \in Y$,
- generalized equations $0 \in F(y^*) + N_Y(y^*)$.

Example: Bilevel Optimization

A bilevel optimization problem (in optimistic formulation) is of the form

$$\min_{x,y} f(x,y) \quad \text{s.t.} \quad x \in X, y \in \operatorname*{argmin}_y \{F(x,y) \mid y \in Y(x)\},$$

where

$$Y(x) \quad = \quad \{y \mid G(x,y) \leq 0, H(x,y) = 0\}.$$

Example: Bilevel Optimization

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A bilevel optimization problem (in optimistic formulation) is of the form

$$\min_{x,y} f(x,y) \quad \text{s.t.} \quad x \in X, y \in \operatorname*{argmin}_{y} \{F(x,y) \mid y \in Y(x)\},$$

where

$$Y(x) \quad = \quad \{y \mid G(x,y) \leq 0, H(x,y) = 0\}.$$

Exercise

How can you rewrite the lower level $y \in \operatorname{argmin}_y \{F(x,y) \mid y \in Y(x)\}$ *using*

- *complementarity constraints,*
- *a variational inequality,*
- *a generalized equation?*

Example: Reformulations of a Bilevel Problem

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Consider the lower level $y \in \operatorname{argmin}_y \{F(x, y) \mid y \in Y(x)\}$.

- Under a suitable constraint qualification, we can replace the lower level by the KKT conditions

$$
\begin{aligned}
&\nabla_y F(x, y) + \nabla_y G(x, y)\lambda + \nabla_y H(x, y)\mu = 0,\\
&0 \leq \lambda \perp G(x, y) \leq 0,\\
&H(x, y) = 0.
\end{aligned}
$$

Example: Reformulations of a Bilevel Problem

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- If $Y(x)$ is convex, we can use the variational inequality

$$\nabla_y F(x,y)^T(z-y) \geq 0 \quad \forall z \in Y(x).$$

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\end{aligned}$$

- If $Y(x)$ is convex, we can use the variational inequality

$$\nabla_y F(x,y)^T(z-y) \geq 0 \quad \forall z \in Y(x).$$

- Or we rewrite B-stationarity as a generalized equation

$$\nabla_y F(x,y)^T d \geq 0 \; \forall d \in T_{Y(x)}(y) \quad \Longleftrightarrow \quad 0 \in \nabla_y F(x,y) + N_{Y(x)}(y).$$

Challenges

Nonlinear Program (NLP)

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A nonlinear program (NLP) is of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0.$$

Denote the feasible set by

$$Z := \{x \in \mathbb{R}^n \mid g(x) \leq 0,\ h(x) = 0\}$$

and for a point $x^* \in Z$ define the set of active inequalities

$$I_g(x^*) \quad = \quad \{i \mid g_i(x^*) = 0\}.$$

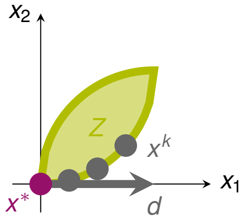
Optimality Conditions for NLPs: B-Stationarity

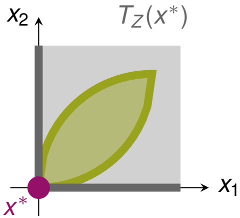
Let x^* be a local minimum of NLP. Then x^* is B-stationary, i.e.

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in T_Z(x^*),$$

where the (Bouligand) tangent cone is defined as

$$T_X(x^*) := \{d \in \mathbb{R}^n \mid \exists (x^k)_k \to_X x^*, (t_k)_k \geq 0 \;:\; t_k(x^k - x^*) \to d\}.$$





Constraint Qualifications for NLPs

Under a constraint qualification in x^* such as

- LICQ: the gradients

  $$\nabla g_i(x^*)\ (i \in I_g(x^*)), \quad \nabla h_i(x^*)\ (i = 1, \dots, p)$$

  are linearly independent;
- MFCQ: the gradients

  $$\nabla g_i(x^*)\ (i \in I_g(x^*)) \quad \text{and} \quad \nabla h_i(x^*)\ (i = 1, \dots, p)$$

  are **positively** linearly independent;
- Abadie CQ: $T_Z(x^*) = L_Z(x^*)$;
- Guignard CQ: $T_Z(x^*)^\circ = L_Z(x^*)^\circ$

we can replace the tangent cone by the simpler linearized tangent cone

$$\begin{aligned} L_Z(x^*) \quad = \quad & \{d \in \mathbb{R}^n \mid \nabla g_i(x^*)^T d \leq 0 \quad \forall i \in I_g(x^*), \\ & \nabla h_i(x^*)^T d = 0 \quad \forall i = 1, \dots, p\} \end{aligned}$$

Optimality Conditions for NLPs: KKT Conditions

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Let x^* be a local minimum of NLP, where a CQ holds. Then x^* is a KKT point, i.e.

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in L_Z(x^*),$$

which is equivalent to the existence of (Lagrange) multipliers $\lambda \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ such that

$$\begin{aligned}
&\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu = 0,\\
&0 \leq \lambda \perp g(x^*) \leq 0,\\
&h(x^*) = 0.
\end{aligned}$$

NLP Reformulation of MPCC

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Exercise

Show that the following conditions are equivalent for $a, b \in \mathbb{R}^n$*:*

- $0 \leq a \perp b \geq 0$
- $a \geq 0, b \geq 0, a \circ b = (a_i b_i)_i = 0$
- $a \geq 0, b \geq 0, a \circ b \leq 0$
- $a \geq 0, b \geq 0, a^T b \leq 0$

NLP Reformulation of MPCC

Exercise

Show that the following conditions are equivalent for $a, b \in \mathbb{R}^n$:

- $0 \leq a \perp b \geq 0$
- $a \geq 0, b \geq 0, a \circ b = (a_i b_i)_i = 0$
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Thus, we can reformulate MPCC as the equivalent NLP

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$
$$G(x) \geq 0, \quad H(x) \geq 0, \quad G(x) \circ H(x) = 0.$$

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Show that the following conditions are equivalent for $a, b \in \mathbb{R}^n$:

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$$G(x) \geq 0, \quad H(x) \geq 0, \quad G(x) \circ H(x) = 0.$$

Why are we not done here?

NLP Reformulation versus Constraint Qualifications

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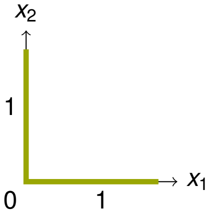
Exercise

Consider the simple MPCC

$$\min_{x \in \mathbb{R}^2} f(x) \quad s.t. \quad 0 \leq x_1 \perp x_2 \geq 0.$$

In which feasible points is LICQ / MFCQ / Abadie CQ / Guignard CQ for the NLP reformulation satisfied?



NLP Reformulation versus Constraint Qualifications

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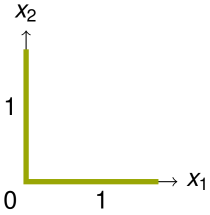

In which feasible points is LICQ / MFCQ / Abadie CQ / Guignard CQ for the NLP reformulation satisfied?

For general MPCCs one knows that

- LICQ and MFCQ are violated in all feasible points,
- Abadie CQ is usually violated,
- only Guignard CQ has a chance to hold.

A “simple” MPCC by S. Scholtes

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Consider the linear MPCC

$$\min_{x \in \mathbb{R}^3} x_1 + x_2 - x_3 \quad \text{s.t.} \quad \begin{aligned} -4x_1 + x_3 &\leq 0, \\ -4x_2 + x_3 &\leq 0, \\ 0 \leq x_1 \perp x_2 &\geq 0. \end{aligned}$$

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Consider the linear MPCC

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Its global minimum $x^* = (0, 0, 0)$ is not a KKT point:

$$\begin{aligned} \lambda_3 &= 1 - 4\lambda_1 \geq 0, \\ \lambda_4 &= 1 - 4\lambda_2 \geq 0, \\ \lambda_1 + \lambda_2 &= 1, \text{ where } \lambda_1 \geq 0, \lambda_2 \geq 0, \end{aligned}$$

Consequently Guignard CQ cannot be satisfied.

An even “simpler” MPCC by A. Schiela

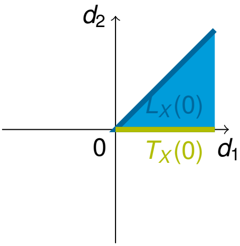
Consider the linear MPCC

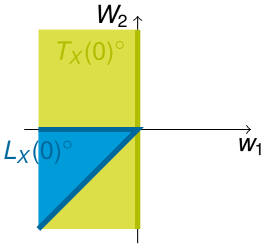
$$\min_{x \in \mathbb{R}^2} -x_2 \quad \text{s.t.} \quad x_2 - x_1 \leq 0, \quad 0 \leq x_1 \perp x_2 \geq 0.$$

Then $X = [0, \infty) \times \{0\}$ and $x^* = (0, 0)$ is a global minimum with

$$T_X(x^*) = [0, \infty) \times \{0\} \quad \text{and} \quad L_X(x^*) = \{d \in \mathbb{R}^2 \mid d_1 \geq 0, d_2 \geq 0, d_2 - d_1 \leq 0\},$$

$$T_X(x^*)^\circ = (-\infty, 0] \times \mathbb{R} \quad \text{and} \quad L_X(x^*)^\circ = \{w \in \mathbb{R}^2 \mid w_1 \leq 0, w_1 + w_2 \leq 0\}.$$





Challenges of the NLP Reformulation of MPCC

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- Most standard CQs for the NLP reformulation of MPCC are violated.

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- Without CQ the KKT conditions are not necessary optimality conditions.

Challenges of the NLP Reformulation of MPCC

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- Most standard CQs for the NLP reformulation of MPCC are violated.
- Without CQ the KKT conditions are not necessary optimality conditions.
- The Fritz-John conditions, i.e. the existence of $\alpha \geq 0$, $\lambda \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ such that

$$
\begin{aligned}
&\alpha \nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu = 0, \\
&0 \leq \lambda \perp g(x^*) \leq 0, \\
&h(x^*) = 0, \\
&(\alpha, \lambda, \mu) \neq 0
\end{aligned}
$$

are necessary optimality conditions without a CQ, but they are satisfied at every feasible point of the NLP reformulation.

Background: Normal Cones and Subdifferentials

Polar Cone and Fréchet Normal Cone

Let $X \subseteq \mathbb{R}^n$ be nonempty and $x^* \in X$.

- The (Bouligand) tangent cone is defined as

$$T_X(x^*) := \{d \in \mathbb{R}^n \mid \exists (x^k)_k \to_X x^*, (t_k)_k \geq 0 \ : \ t_k(x^k - x^*) \to d\}.$$

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- The polar cone to a set $K \subseteq \mathbb{R}^n$ is defined as

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- For a polyhedral cone $P = \{d \mid A^T d \leq 0, B^T d = 0\}$ one knows

$$P^\circ = \{w = A\lambda + B\mu \mid \lambda \geq 0\}.$$

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- For a polyhedral cone $P = \{d \mid A^T d \leq 0, B^T d = 0\}$ one knows

$$P^\circ = \{w = A\lambda + B\mu \mid \lambda \geq 0\}.$$

- The Fréchet normal cone of X at x^* is defined as

$$N_X^F(x^*) := T_X(x^*)^\circ = \{w \in \mathbb{R}^n \mid \limsup_{x \to x^*, x \in X \setminus \{x^*\}} w^T \frac{x - x^*}{\|x - x^*\|} \leq 0\}.$$

For $x^* \notin X$ one defines $N_X^F(x^*) := \emptyset$.

Examples for the Fréchet Normal Cone

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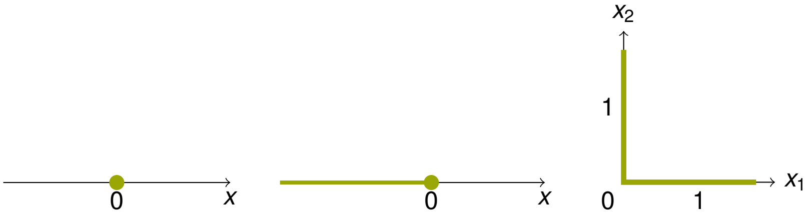
Exercise

Consider the feasible sets

- $X = \{0\} \subset \mathbb{R}$,
- $X = (-\infty, 0] \subset \mathbb{R}$,
- $X = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \perp x_2 \geq 0\} \subset \mathbb{R}^2$

and compute the Fréchet normal cone at all feasible points.



Limiting Normal Cone

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The limiting or Mordukhovich normal cone of X at x^* is defined as

$$N_X^M(x^*) := \{w \in \mathbb{R}^n \mid \exists (x^k)_k \to x^*, w^k \in N_X^F(x^k) \ : \ w^k \to w\}.$$

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Exercise

Consider the feasible set $X = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \perp x_2 \geq 0\}$ *and compute the limiting normal cone at* $x^* = (0,0)^T$*.*

A Calculus Rule for the Limiting Normal Cone

$$N^M_{A\cap B}(x^*) = ???$$

A Calculus Rule for the Limiting Normal Cone

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$$N^M_{A\cap B}(x^*) = ???$$

- Let $A, B \subseteq \mathbb{R}^n$ be nonempty and closed, $x^* \in A \cap B$ and assume that the set-valued function

  $$M(y) := \{x \in A \mid x + y \in B\}$$

  is calm at $(0, x^*)$. Then

  $$N^M_{A\cap B}(x^*) \subseteq N^M_A(x^*) + N^M_B(x^*).$$

A Calculus Rule for the Limiting Normal Cone

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is calm at $(0, x^*)$. Then
$$N^M_{A\cap B}(x^*) \subseteq N^M_A(x^*) + N^M_B(x^*).$$
- A set-valued map $M : Y \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is calm at $(y^*, x^*) \in \text{graph}(M)$ if there exist $\delta > 0$, $\varepsilon > 0$, $L > 0$ such that
$$M(y) \cap B_\varepsilon(x^*) \subseteq M(y^*) + L\|y - y^*\|B_1(0) \quad \forall y \in B_\delta(y^*).$$

A Calculus Rule for the Limiting Normal Cone

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- Let $A, B \subseteq \mathbb{R}^n$ be nonempty and closed, $x^* \in A \cap B$ and assume that the set-valued function
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$$N^M_{A\cap B}(x^*) \subseteq N^M_A(x^*) + N^M_B(x^*).$$
- A set-valued map $M : Y \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is calm at $(y^*, x^*) \in \text{graph}(M)$ if there exist $\delta > 0$, $\varepsilon > 0$, $L > 0$ such that
$$M(y) \cap B_\varepsilon(x^*) \subseteq M(y^*) + L\|y - y^*\| B_1(0) \quad \forall y \in B_\delta(y^*).$$
- Polyhedral multifunctions M, i.e. set-valued maps whose graph can be written as the union of finitely many polyhedra, are calm at all $(y^*, x^*) \in \text{graph}(M)$.

The Bouligand and Clarke Subdifferential

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Consider a locally Lipschitz continuous map $F : \mathbb{R}^n \to \mathbb{R}^m$.

- Rademacher's Theorem: F is differentiable almost everywhere, i.e. the complement of

$$D_F := \{x \in \mathbb{R}^n \mid F \text{ is differentiable in } x\}.$$

has Lebesgue measure zero.

The Bouligand and Clarke Subdifferential

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- Rademacher's Theorem: F is differentiable almost everywhere, i.e. the complement of

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  has Lebesgue measure zero.
- The Bouligand subdifferential of F at $x^* \in \mathbb{R}^n$ is defined as

$$\partial^B F(x^*) := \{M \in \mathbb{R}^{m\times n} \mid \exists (x^k)_k \to_{D_F} x^* \;:\; F'(x^k) \to M\}.$$

The Bouligand and Clarke Subdifferential

Consider a locally Lipschitz continuous map $F : \mathbb{R}^n \to \mathbb{R}^m$.

- Rademacher's Theorem: F is differentiable almost everywhere, i.e. the complement of

$$D_F := \{x \in \mathbb{R}^n \mid F \text{ is differentiable in } x\}.$$

  has Lebesgue measure zero.
- The Bouligand subdifferential of F at $x^* \in \mathbb{R}^n$ is defined as

$$\partial^B F(x^*) := \{M \in \mathbb{R}^{m \times n} \mid \exists (x^k)_k \to_{D_F} x^* \;:\; F'(x^k) \to M\}.$$

- The Clarke subdifferential of F at $x^* \in \mathbb{R}^n$ is defined as

$$\partial^C F(x^*) := \operatorname{conv} \partial^B F(x^*).$$

Examples for the Bouliqand and Clarke Subdifferential

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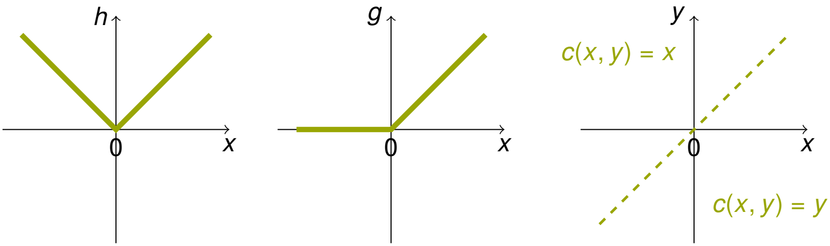
Exercise

Consider the functions

- $h(x) = |x|$,
- $g(x) = \max\{0, x\}$,
- $c(x, y) = \min\{x, y\}$

and compute the Bouligand and Clarke subdifferentials.



Properties of the Clarke Subdifferential

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz continuous and $x^* \in \mathbb{R}^n$.

- $\partial F(x^*)$ is nonempty, compact, and convex.
- If F is continuously differentiable around x^*, then $\partial^C F(x^*) = \{F'(x^*)\}$.

Properties of the Clarke Subdifferential

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz continuous and $x^* \in \mathbb{R}^n$.

- $\partial F(x^*)$ is nonempty, compact, and convex.
- If F is continuously differentiable around x^*, then $\partial^C F(x^*) = \{F'(x^*)\}$.
- Let $f, f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz continuous and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Then the following holds:
  - $\partial^C (cf)(x) = c\partial^C f(x)$ for all $c \in \mathbb{R}$.
  - $\partial^C (f_1 + f_2)(x) \subseteq \partial^C f_1(x) + \partial^C f_2(x)$.
  - $\partial^C (f_1 \cdot f_2)(x) \subseteq f_2(x)\partial^C f_1(x) + f_1(x)\partial^C f_2(x)$.
  - $\partial(f \circ G)(y) \subseteq \mathrm{conv}(\partial^C f(G(y)) \cdot \partial^C G(y))$ with equality if f is continuously differentiable or f convex and G continuously differentiable.

Fritz-John Conditions of Clarke

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Let f, g, h be locally Lipschitz continuous and x^* be a local minimum of

$$\min_x f(x) \text{s.t.} x \in Z := \{x \in \mathbb{R}^n \mid g(x) \leq 0, h(x) = 0\}.$$

Then there exist multipliers $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$ such that

- $(\alpha, \lambda, \mu) \neq (0, 0, 0)$,
- $\alpha \geq 0$,
- $\lambda \geq 0$ and $\lambda_i = 0$ for all $i \notin I_g(x^*)$
- and

$$0 \in \alpha \partial^C f(x^*) + \sum_{i=1}^{m} \lambda_i \partial^C g_i(x^*) + \sum_{i=1}^{p} \mu_i \partial^C h_i(x^*)$$

What should you remember?

- The definition of MPCC.
- Why MPCC cannot be solved using standard NLP theory.
- The definition and calculus rule for the Fréchet and limiting normal cone.
- The definition and basic properties of the Clarke subdifferential.

What is the plan for tomorrow?