Mathematical Programs with Complementarity Constraints and Related Problems

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Ressources

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You can find

- my slides
- and some more extensive lecture notes

at

github.com/alexandrabschwartz/Winterschool2018

If you have any questions, please come to me during the week or contact me at

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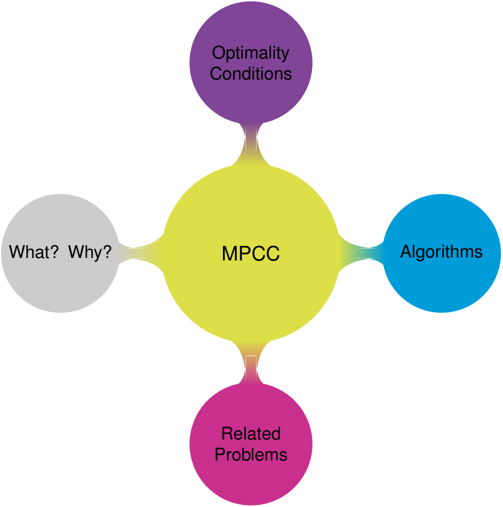
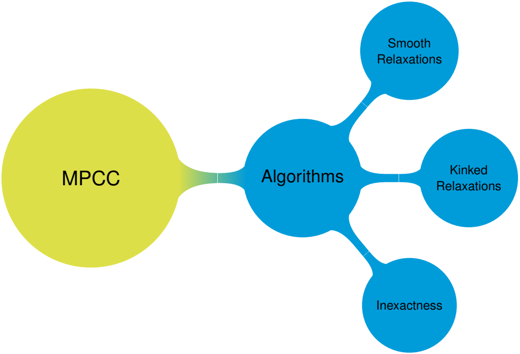
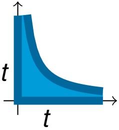
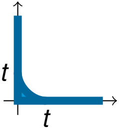
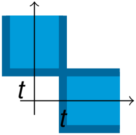
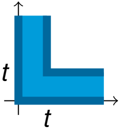
Contents of the Course



What did we do yesterday?



Relaxation Methods: Exact KKT Points

relaxation	global	local	nonsmooth	kinked
authors	Scholtes	Steff./Ulbr.	Kadrani et al.	Kanzow, S.
geometry				
stationarity under CQ	C-stat. MPEC-MFCQ	C-stat. MPEC-CPLD	M-stat. MPEC-CPLD	M-stat. MPEC-CPLD

Relaxation Methods: Inexact KKT Points

Let $t_k \downarrow 0$ and $\varepsilon_k = o(t_k)$. Let x^k be ε_k-KKT points of NLP(t_k) and assume $x^k \to x^*$. Then the limit x^* satisfies:

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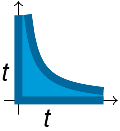
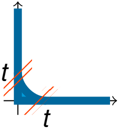
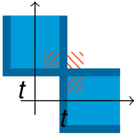
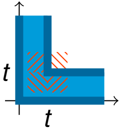
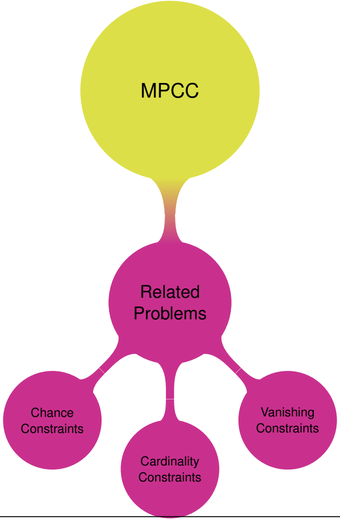
geometry				
stationarity	C-stat.	W-stat.	W-stat.	W-stat.
under CQ	MPEC-MFCQ	MPEC-MFCQ	MPEC-MFCQ	MPEC-MFCQ

What is the plan for today?



Vanishing Constraints

Mathematical Program with Vanishing Constraints (MPVC)

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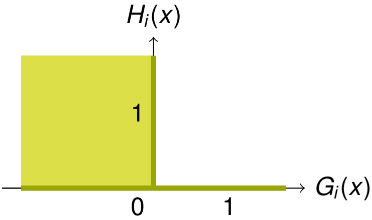
A mathematical program with vanishing constraints (MPVC) is of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$
$$H(x) \geq 0, \quad G(x) \circ H(x) \geq 0.$$



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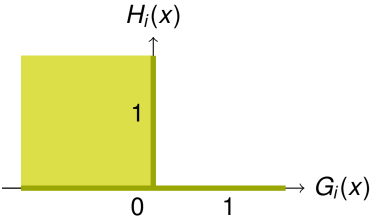
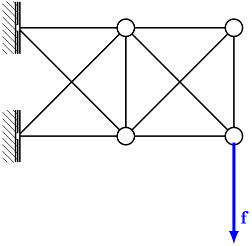


Two cases are possible:

- $H_i(x^*) > 0$ and thus $G_i(x^*) \le 0$
- $H_i(x^*) = 0$ and then $G_i(x^*) \in \mathbb{R}$ arbitrarily.

Example: Truss Design



Given some anchor points and a load point, how should you construct a load bearing truss?

Example: Truss Design (continued)

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$\min_{a,u} \quad \sum_i l_i a_i$	minimize volume of the truss
s.t. $\quad 0 \leq a \leq a_{\max}$	bounds on cross sections
$K(a)u = f$	force equilibrium
$f^T u \leq c$	bound on the deformation
$(\sigma_i(u)^2 - \sigma_{\max}^2) a_i \leq 0$	bound on the stress on bar i

Variables

- a_i cross sectional area of bar i
- u_i nodal displacements at the hinges

Constants

- f applied force
- l_i length of bar i
- $K(a)$ stiffness matrix of the truss
- $\sigma_i(u)$ stress on bar i

Relation between Vanishing and Complementarity Constraints

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One can reformulate an MPVC as an MPCC using slack variables as follows:

$$\begin{aligned} \min_{x,y} f(x) \quad \text{s.t.} \quad & g(x) \leq 0, h(x) = 0, \\ & G(x) - y \leq 0, \\ & 0 \leq H(x) \perp y \geq 0. \end{aligned}$$

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Exercise

How is the relation between the MPCC reformulation and the original MPVC?

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Then x^* is a solution of the MPVC if and only if (x^*, y^*) with

$$y_i^* \begin{cases} = 0 & \text{if } H_i(x^*) > 0, \\ \ge \max\{0, G_i(x^*)\} & \text{if } H_i(x^*) = 0 \end{cases}$$

if a solution of the corresponding MPCC. But y^* is not uniquely determined.

MPVCs versus Constraint Qualifications

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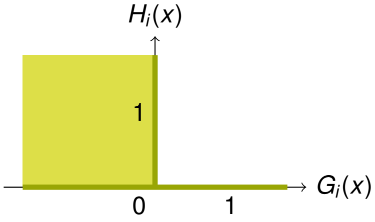
Vanishing Constraints:

$$H(x)_i \geq 0, \quad G(x)_i H(x)_i \geq 0 \qquad \forall i = 1, \dots, q$$

Let x^* be feasible for MPVC.

- If $H_i(x^*) = 0$ for at least one index $i \in \{1, \dots, q\}$, then LICQ is violated in x^*.
- If there exists an index i such that $H_i(x^*) = 0$ and $G_i(x^*) \geq 0$, then MFCQ is violated in x^*.
- The vanishing constraints i with $H_i(x^*) > 0$ are “well-behaved”.



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- One can define $V := \{(a, b) \in \mathbb{R}^2 \mid b \geq 0, ab \leq 0\}$ and use the condition

$$(\gamma_i, \nu_i) \in N_V^F(G_i(x^*), H_i(x^*)) \subseteq N_V^M(G_i(x^*), H_i(x^*))$$

  to obtain possible optimality conditions.

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- One can locally decompose the MPVC into TNLP(x^*, I) for all $I \subseteq \{i \mid G_i(x^*) = H_i(x^*) = 0\}$

$$\begin{aligned} \min_x f(x) \quad \text{s.t.} \quad & g(x) \leq 0, \quad h(x) = 0, \\ & G_i(x) \leq 0, \quad H_i(x) \geq 0 \quad \forall i \in \{i \mid G_i(x^*) < 0\} \cup I, \\ & H_i(x) = 0 \quad \forall i \in \{i \mid G_i(x^*) > 0\} \cup I^c \end{aligned}$$

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- One can use a nonsmooth reformulation

$$\varphi(a, b) = \max\{-b, \min\{a, b\}\} \leq 0.$$

Cardinality Constraints

Cardinality Constrained Optimization Problems

- A cardinality constrained problem is of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0, \quad \|x\|_0 \leq \kappa$$

with the "sparsity"-term

$$\|x\|_0 := \text{number of nonzero components of } x.$$

- These problems appear e.g. in
  - portfolio optimization,
  - compressed sensing,
  - feature selection,
  - communications engineering,
  - truss design.

Example: Portfolio Optimization

Consider a basic portfolio selection problem

$$\min_x x^T Q x \quad \text{s.t.} \quad \begin{aligned} &\mu^T x \geq \rho, \\ &e^T x = 1, \\ &0 \leq x_i \leq u_i \quad \forall i = 1, \dots, n, \\ &\|x\|_0 \leq \kappa, \end{aligned}$$

where

- Q is the covariance matrix of the n possible assets,
- μ is the expected revenue,
- $e = (1, \dots, 1)^T$.

Relation to Sparse Optimization Problems

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For some weight $\rho > 0$ consider

$$\min_x f(x) + \rho \|x\|_0 \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0.$$

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- Sparse $\rightarrow$ Cardinality: A solution x^* of the sparse problem is always a solution of the cardinality constrained problem with $\kappa := \|x^*\|_0$.

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- Sparse $\rightarrow$ Cardinality: A solution x^* of the sparse problem is always a solution of the cardinality constrained problem with $\kappa := \|x^*\|_0$.
- Cardinality $\not\rightarrow$ Sparse: Consider the following example by O. Burdakov

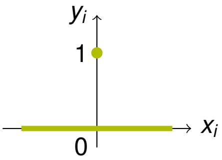
$$f(x) = \|Ax - b\|_2^2, \quad \text{where} \quad A = \begin{pmatrix} 0 & 3 & -3 \\ 3 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}.$$

There is no $\rho \geq 0$ for which one obtains the same solution as for $\kappa = 2$.

Continuous Reformulation using Complementarity Constraints

- **Idea**: Introduce binary variables y counting the zeros of x:

$$x \circ y = 0, \quad \|x\|_0 \leq n - e^T y \leq \kappa.$$



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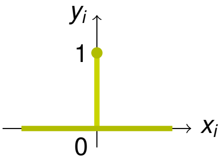

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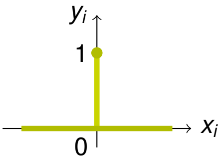
Relax the binary variables to $y \in [0, e]$.

- **Advantages**: No bounds on x needed; relaxation possible; solves the "right" problem.
- **Disadvantages**: Introduces a complementarity-type condition.



Relation between the Original Problem and the Continuous Reformulation

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Exercise

What is the relation between continuous reformulation

$$\min_{x,y} f(x) \quad s.t. \quad g(x) \leq 0, \quad h(x) = 0,$$
$$0 \leq y \leq e, \quad e^T y \geq n - \kappa,$$
$$x \circ y = 0.$$

and the original problem

$$\min_{x} f(x) \quad s.t. \quad g(x) \leq 0, \quad h(x) = 0, \quad \|x\|_0 \leq \kappa?$$

Relation between the Original Problem and the Continuous Reformulation (continued)

Global Solutions:

- x^* is a global solution of the original problem if and only if there exists y^* such that (x^*, y^*) is a global solution of the continuous reformulation.

Relation between the Original Problem and the Continuous Reformulation (continued)

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- If x^* is a local solution of the original problem, then for every y such that (x^*, y) is feasible, it is a local solution of the continuous reformulation.

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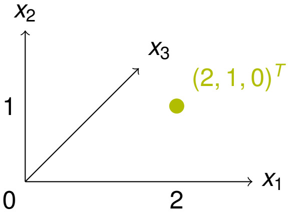
Local Solutions:

- If x^* is a local solution of the original problem, then for every y such that (x^*, y) is feasible, it is a local solution of the continuous reformulation.
- If (x^*, y^*) is a local solution of the continuous reformulation and $\|x^*\|_0 = \kappa$, then x^* is a local solution of the original problem.

Example for Spurious Local Solutions

Consider the 3-dimensional cardinality constrained problem

$$\min_x \|x - (2,1,0)^T\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \leq 1.$$



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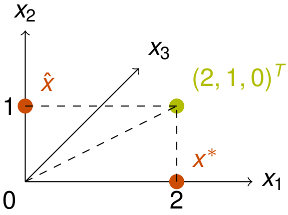

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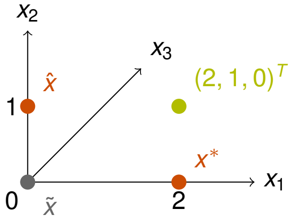

However, the continuous relaxation has additional local minimizer such as $(\tilde{x}, \tilde{y})$ with $\tilde{x} = (0,0,0)^T$ and $\tilde{y} = (1,1,0)$.

Half and Full Complementarity Formulation

- The continuous reformulation results in

$$\min_{x,y} f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$
$$0 \leq y \leq e, \quad e^T y \geq n - \kappa,$$
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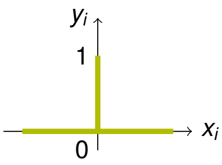

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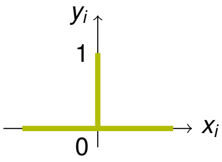

- The full complementarity formulation is given by

$$\min_{x^+ - x^-, y} f(x^+ - x^-) \quad \text{s.t.} \quad g(x^+ - x^-) \leq 0, \quad h(x^+ - x^-) = 0,$$
$$y \leq e, \quad e^T y \geq n - \kappa,$$
$$x^+, x^- \geq 0, \quad y \geq 0, \quad (x^+ + x^-) \circ y = 0.$$

Why should you not treat the continuous reformulation as an MPCC?

- In all points of interest MPCC-LICQ and MPCC-MFCQ are violated.

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- The analogues of M-, C- and W-stationarity coincide.
- All local minima of the continuous reformulation are S-stationary (= KKT points) under a Guignard-type CQ.
- The Scholtes relaxation converges to S-stationary points under an MFCQ-type CQ.

Chance Constraints

Optimization Problems with Chance Constraints

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An optimization problems with chance constraints is of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$
$$P(G(x,\xi) \leq 0) \geq \eta,$$

where

- $\eta \in [0,1]$ is the level of confidence, e.g. $\eta = 0.9,\ 0.95,\ 0.99,$
- ξ is a random variable with finitely many possible realization $\xi_1 \dots, \xi_N$ and known probabilities $p_1, \dots, p_N$.

Example: Portfolio Optimization

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Assume that

- we can invest in n assets,
- N possible scenarios i can occur with probabilities p_i and prognosed returns $\xi_i \in \mathbb{R}^n$,
- we want to invest a budget of 1 to maximize the expected return
- but limit the probability of having losses to $1 - \eta \in (0, 1)$.

We have to solve

$$\max_x \sum_{i=1}^{N} p_i \xi_i^T x \quad \text{s.t.} \quad x \geq 0, \quad \sum_{i=1}^{n} x_i = 1,$$
$$P(\xi^T x \geq 0) \geq \eta$$

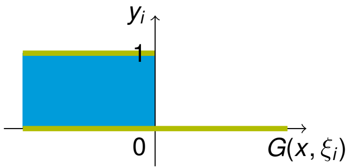
Reformulation of Chance Constraints

The chance constrained problem

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0, \quad P(G(x,\xi) \leq 0) \geq \eta,$$

can equivalently be rewritten as

$$\begin{aligned} \min_{x,y} f(x) \quad \text{s.t.} \quad & g(x) \leq 0, \quad h(x) = 0, \\ & y \in \{0,1\}^N, \\ & p^T y \geq \eta, \\ & y_i G(x,\xi_i) \leq 0. \end{aligned}$$



Similarly to cardinality constraints, we can relax the binary variables to $y \in [0,1]^N$.

What should you remember?

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- There are many classes of optimization problems with similar structures/problems as MPCCs.
- Reformulating them as an MPCC is not always the best approach.
- But one can use ideas from MPCCs to handle these classes such as
  - decomposition of the feasible set into easier pieces,
  - nonsmooth reformulation,
  - using normal cones to obtain conditions on multipliers.

What did we do this week?