Mathematical Programs with Complementarity Constraints and Related Problems

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Ressources

You can find

- my slides
- and some more extensive lecture notes

at

github.com/alexandrabschwartz/Winterschool2018

If you have any questions, please come to me during the week or contact me at

schwartz@gsc.tu-darmstadt.de

Goals

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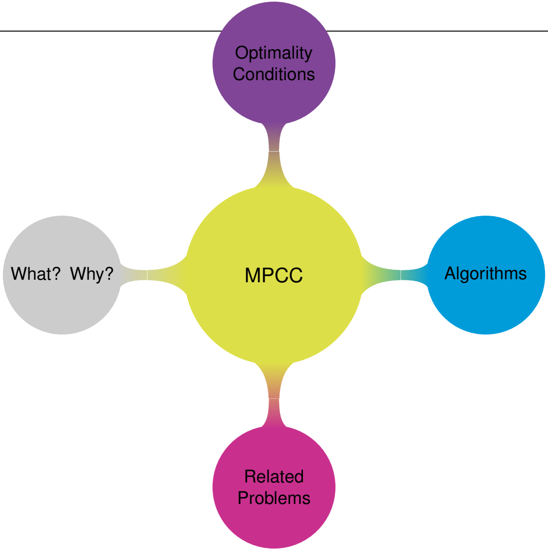
At the end of the week you should

- know the definition and applications of MPCCs,
- be able to explain why MPCCs cannot be solved directly using standard theory,
- understand the origins and relations of different optimality conditions for MPCCs,
- know what to look for in a relaxation algorithm for MPCCs,
- be able to transfer the MPCC ideas to some related problem classes.

Contents of the Course



Literature

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- Luo, Pang, Ralph: *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press
- Outrata, Kocvara, Zowe: *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*, Kluwer Academic Publishers
- Mordukhovich: *Variational Analysis and Generalized Differentiation I and II*, Springer
- Rockafellar, Wets: *Variational Analysis*, Springer
- Numerous articles by Jane Ye, Jiri Outrata, Christian Kanzow, and many others

What is the plan for today?

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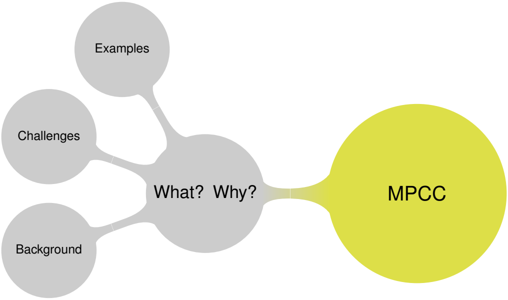
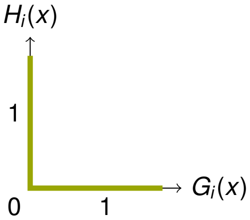


Problem Formulation and Examples

Mathematical Program with Complementarity Constraints (MPCC)

A mathematical program with complementarity constraints (MPCC) is of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$
$$0 \leq G(x) \perp H(x) \geq 0.$$



We assume that

- $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^p$, and $G, H : \mathbb{R}^n \to \mathbb{R}^q$ are continuously differentiable,
- the feasible set

$$X := \{x \in \mathbb{R}^n \mid g(x) \leq 0, \quad h(x) = 0, \quad 0 \leq G(x) \perp H(x) \geq 0\}$$

is nonempty.

Mathematical Program with Equilibrium Constraints

A mathematical program with equilibrium constraints (MPEC) is of the form

$$\min_{x,y} f(x, y) \quad \text{s.t.} \quad x \in X, y \in S(x),$$

where $x \rightrightarrows S(x)$ is the solution set of the equilibrium constraint, e.g. of

- a lower-level optimization problem,
- a lower-level Nash game,
- or a physical equilibrium.

Mathematical Program with Equilibrium Constraints

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- a lower-level optimization problem,
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- or a physical equilibrium.

The solution set $S(x)$ can often be described using

- complementarity constraints $0 \leq G(x,y) \perp H(x,y) \geq 0$,
- variational inequalities $F(y^*)^T(y - y^*) \geq 0 \quad \forall y \in Y$,
- generalized equations $0 \in F(y^*) + N_Y(y^*)$.

Example: Bilevel Optimization

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A bilevel optimization problem (in optimistic formulation) is of the form

$$\min_{x,y} f(x,y) \quad \text{s.t.} \quad x \in X, y \in \operatorname*{argmin}_y \{F(x,y) \mid y \in Y(x)\},$$

where

$$Y(x) \quad = \quad \{y \mid G(x,y) \leq 0, H(x,y) = 0\}.$$

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where

$$Y(x) \quad = \quad \{y \mid G(x,y) \leq 0, H(x,y) = 0\}.$$

Exercise

How can you rewrite the lower level $y \in \operatorname{argmin}_y\{F(x,y) \mid y \in Y(x)\}$ *using*

- *complementarity constraints,*
- *a variational inequality,*
- *a generalized equation?*

Example: Reformulations of a Bilevel Problem

Consider the lower level $y \in \text{argmin}_y\{F(x,y) \mid y \in Y(x)\}$.

- Under a suitable constraint qualification, we can replace the lower level by the KKT conditions

$$\begin{aligned}
&\nabla_y F(x,y) + \nabla_y G(x,y)\lambda + \nabla_y H(x,y)\mu = 0,\\
&0 \leq \lambda \perp G(x,y) \leq 0,\\
&H(x,y) = 0.
\end{aligned}$$

Example: Reformulations of a Bilevel Problem

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- If $Y(x)$ is convex, we can use the variational inequality

$$\nabla_y F(x,y)^T(z-y) \geq 0 \quad \forall z \in Y(x).$$

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\end{aligned}$$

- If $Y(x)$ is convex, we can use the variational inequality

$$\nabla_y F(x,y)^T(z-y) \ge 0 \quad \forall z \in Y(x).$$

- Or we rewrite B-stationarity as a generalized equation

$$\nabla_y F(x,y)^T d \ge 0 \; \forall d \in T_{Y(x)}(y) \quad \Longleftrightarrow \quad 0 \in \nabla_y F(x,y) + N_{Y(x)}(y).$$

Challenges

Nonlinear Program (NLP)

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A nonlinear program (NLP) is of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0.$$

Denote the feasible set by

$$Z := \{x \in \mathbb{R}^n \mid g(x) \leq 0,\ h(x) = 0\}$$

and for a point $x^* \in Z$ define the set of active inequalities

$$I_g(x^*) \quad = \quad \{i \mid g_i(x^*) = 0\}.$$

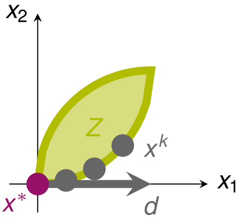
Optimality Conditions for NLPs: B-Stationarity

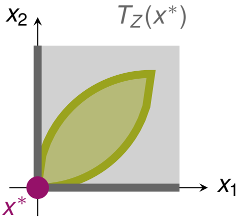
Let x^* be a local minimum of NLP. Then x^* is B-stationary, i.e.

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in T_Z(x^*),$$

where the (Bouligand) tangent cone is defined as

$$T_X(x^*) := \{d \in \mathbb{R}^n \mid \exists (x^k)_k \to_X x^*, (t_k)_k \geq 0 \ : \ t_k(x^k - x^*) \to d\}.$$





Constraint Qualifications for NLPs

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Under a constraint qualification in x^* such as

- LICQ: the gradients

$$\nabla g_i(x^*)\ (i \in I_g(x^*)), \quad \nabla h_i(x^*)\ (i = 1, \dots, p)$$

  are linearly independent;
- MFCQ: the gradients

$$\nabla g_i(x^*)\ (i \in I_g(x^*)) \quad \text{and} \quad \nabla h_i(x^*)\ (i = 1, \dots, p)$$

  are **positively** linearly independent;
- Abadie CQ: $T_Z(x^*) = L_Z(x^*)$;
- Guignard CQ: $T_Z(x^*)^\circ = L_Z(x^*)^\circ$

we can replace the tangent cone by the simpler linearized tangent cone

$$\begin{aligned} L_Z(x^*) \quad = \quad & \{d \in \mathbb{R}^n \mid \nabla g_i(x^*)^T d \leq 0 \quad \forall i \in I_g(x^*), \\ & \nabla h_i(x^*)^T d = 0 \quad \forall i = 1, \dots, p\} \end{aligned}$$

Optimality Conditions for NLPs: KKT Conditions

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Let x^* be a local minimum of NLP, where a CQ holds. Then x^* is a KKT point, i.e.

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in L_Z(x^*),$$

which is equivalent to the existence of (Lagrange) multipliers $\lambda \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ such that

$$\begin{aligned}
&\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu = 0, \\
&0 \leq \lambda \perp g(x^*) \leq 0, \\
&h(x^*) = 0.
\end{aligned}$$

NLP Reformulation of MPCC

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Exercise

Show that the following conditions are equivalent for $a, b \in \mathbb{R}^n$*:*

- $0 \leq a \perp b \geq 0$
- $a \geq 0, b \geq 0, a \circ b = (a_i b_i)_i = 0$
- $a \geq 0, b \geq 0, a \circ b \leq 0$
- $a \geq 0, b \geq 0, a^T b \leq 0$

NLP Reformulation of MPCC

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Exercise

Show that the following conditions are equivalent for $a, b \in \mathbb{R}^n$:

- $0 \leq a \perp b \geq 0$
- $a \geq 0, b \geq 0, a \circ b = (a_i b_i)_i = 0$
- $a \geq 0, b \geq 0, a \circ b \leq 0$
- $a \geq 0, b \geq 0, a^T b \leq 0$

Thus, we can reformulate MPCC as the equivalent NLP

$$\begin{aligned} \min_x f(x) \quad \text{s.t.} \quad & g(x) \leq 0, \quad h(x) = 0, \\ & G(x) \geq 0, \quad H(x) \geq 0, \quad G(x) \circ H(x) = 0. \end{aligned}$$

NLP Reformulation of MPCC

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Exercise

Show that the following conditions are equivalent for $a, b \in \mathbb{R}^n$:

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$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$
$$G(x) \geq 0, \quad H(x) \geq 0, \quad G(x) \circ H(x) = 0.$$

Why are we not done here?

NLP Reformulation versus Constraint Qualifications

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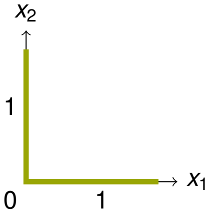
Exercise

Consider the simple MPCC

$$\min_{x \in \mathbb{R}^2} f(x) \quad s.t. \quad 0 \leq x_1 \perp x_2 \geq 0.$$

In which feasible points is LICQ / MFCQ / Abadie CQ / Guignard CQ for the NLP reformulation satisfied?



NLP Reformulation versus Constraint Qualifications

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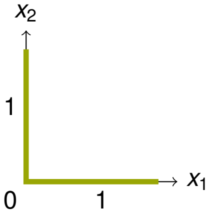

In which feasible points is LICQ / MFCQ / Abadie CQ / Guignard CQ for the NLP reformulation satisfied?

For general MPCCs one knows that

- LICQ and MFCQ are violated in all feasible points,
- Abadie CQ is usually violated,
- only Guignard CQ has a chance to hold.

A “simple” MPCC by Scholtes

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Consider the linear MPCC

$$\begin{aligned} \min_{x \in \mathbb{R}^3} x_1 + x_2 - x_3 \quad \text{s.t.} \quad & -4x_1 + x_3 \leq 0, \\ & -4x_2 + x_3 \leq 0, \\ & 0 \leq x_1 \perp x_2 \geq 0. \end{aligned}$$

A "simple" MPCC by Scholtes

Consider the linear MPCC

$$\min_{x \in \mathbb{R}^3} x_1 + x_2 - x_3 \quad \text{s.t.} \quad \begin{aligned} -4x_1 + x_3 &\leq 0, \\ -4x_2 + x_3 &\leq 0, \\ 0 \leq x_1 \perp x_2 &\geq 0. \end{aligned}$$

Its global minimum $x^* = (0, 0, 0)$ is not a KKT point:

$$\begin{aligned} \lambda_3 &= 1 - 4\lambda_1 \geq 0, \\ \lambda_4 &= 1 - 4\lambda_2 \geq 0, \\ \lambda_1 + \lambda_2 &= 1, \text{ where } \lambda_1 \geq 0, \lambda_2 \geq 0, \end{aligned}$$

Consequently Guignard CQ cannot be satisfied.

Challenges of the NLP Reformulation of MPCC

- Most standard CQs for the NLP reformulation of MPCC are violated.

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- Most standard CQs for the NLP reformulation of MPCC are violated.
- Without CQ the KKT conditions are not necessary optimality conditions.
- The Fritz-John conditions, i.e. the existence of $\alpha \geq 0$, $\lambda \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ such that

$$\begin{aligned}
&\alpha \nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu = 0,\\
&0 \leq \lambda \perp g(x^*) \leq 0,\\
&h(x^*) = 0,\\
&(\alpha, \lambda, \mu) \neq 0
\end{aligned}$$

  are necessary optimality conditions without a CQ, but they are satisfied at every feasible point of the NLP reformulation.

Background: Normal Cones and Subdifferentials

Polar Cone and Fréchet Normal Cone

Let $X \subseteq \mathbb{R}^n$ be nonempty and $x^* \in X$.

- The (Bouligand) tangent cone is defined as

$$T_X(x^*) := \{d \in \mathbb{R}^n \mid \exists (x^k)_k \to_X x^*, (t_k)_k \geq 0 \ : \ t_k(x^k - x^*) \to d\}.$$

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- The polar cone to a set $K \subseteq \mathbb{R}^n$ is defined as

$$K^\circ := \{w \in \mathbb{R}^n \mid w^T d \leq 0 \quad \forall d \in K\}.$$

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- For a polyhedral cone $P = \{d \mid A^T d \leq 0, B^T d = 0\}$ one knows

$$P^\circ = \{w = A\lambda + B\mu \mid \lambda \geq 0\}.$$

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- The Fréchet normal cone of X at x^* is defined as

$$N_X^F(x^*) := T_X(x^*)^\circ = \{w \in \mathbb{R}^n \mid \limsup_{x \to x^*, x \in X \setminus \{x^*\}} w^T \frac{x - x^*}{\|x - x^*\|} \leq 0\}.$$

For $x^* \notin X$ one defines $N_X^F(x^*) := \emptyset$.

Examples for the Fréchet Normal Cone

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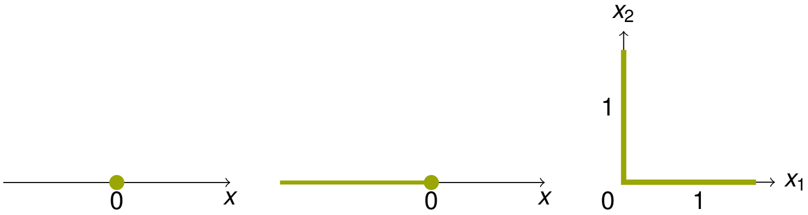
Exercise

Consider the feasible sets

- $X = \{0\} \subset \mathbb{R}$,
- $X = (-\infty, 0] \subset \mathbb{R}$,
- $X = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \perp x_2 \geq 0\} \subset \mathbb{R}^2$

and compute the Fréchet normal cone at all feasible points.



Limiting Normal Cone

The limiting or Mordukhovich normal cone of X at x^* is defined as

$$N_X^M(x^*) := \{w \in \mathbb{R}^n \mid \exists (x^k)_k \to x^*, w^k \in N_X^F(x^k) \ : \ w^k \to w\}.$$

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Exercise

Consider the feasible set $X = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \perp x_2 \geq 0\}$ *and compute the limiting normal cone at* $x^* = (0, 0)^T$*.*

A Calculus Rule for the Limiting Normal Cone

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$$N^M_{A\cap B}(x^*) = ???$$

A Calculus Rule for the Limiting Normal Cone

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$$N^M_{A\cap B}(x^*) = ???$$

- Let $A, B \subseteq \mathbb{R}^n$ be nonempty and closed, $x^* \in A \cap B$ and assume that the set-valued function

  $$M(y) := \{x \in A \mid x + y \in B\}$$

  is calm at $(0, x^*)$. Then

  $$N^M_{A\cap B}(x^*) \subseteq N^M_A(x^*) + N^M_B(x^*).$$

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is calm at $(0, x^*)$. Then
$$N^M_{A\cap B}(x^*) \subseteq N^M_A(x^*) + N^M_B(x^*).$$
- A set-valued map $M : Y \subseteq \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is calm at $(y^*, x^*) \in \mathrm{graph}(M)$ if there exist $\delta > 0$, $\varepsilon > 0$, $L > 0$ such that
$$M(y) \cap B_\varepsilon(x^*) \subseteq M(y^*) + L\|y - y^*\|B_1(0) \quad \forall y \in B_\delta(y^*).$$

A Calculus Rule for the Limiting Normal Cone

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$$M(y) \cap B_\varepsilon(x^*) \subseteq M(y^*) + L\|y - y^*\| B_1(0) \quad \forall y \in B_\delta(y^*).$$
- Polyhedral multifunctions M, i.e. set-valued maps whose graph can be written as the union of finitely many polyhedra, are calm at all $(y^*, x^*) \in \text{graph}(M)$.

The Bouligand and Clarke Subdifferential

Consider a locally Lipschitz continuous map $F : \mathbb{R}^n \to \mathbb{R}^m$.

- Rademacher's Theorem: F is differentiable almost everywhere, i.e. the complement of

$$D_F := \{x \in \mathbb{R}^n \mid F \text{ is differentiable in } x\}.$$

has Lebesgue measure zero.

The Bouligand and Clarke Subdifferential

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  has Lebesgue measure zero.
- The Bouligand subdifferential of F at $x^* \in \mathbb{R}^n$ is defined as

$$\partial^B F(x^*) := \{M \in \mathbb{R}^{m \times n} \mid \exists (x^k)_k \to_{D_F} x^* \ : \ F'(x^k) \to M\}.$$

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- The Clarke subdifferential of F at $x^* \in \mathbb{R}^n$ is defined as

$$\partial^C F(x^*) := \operatorname{conv} \partial^B F(x^*).$$

Examples for the Bouligand and Clarke Subdifferential

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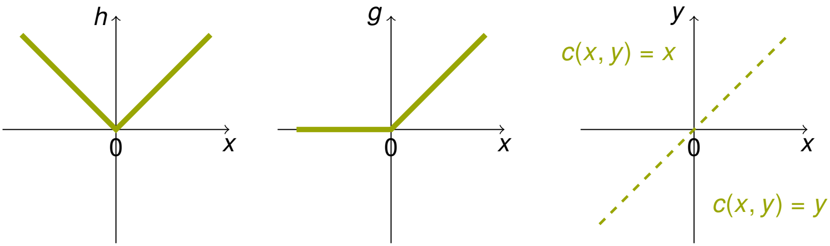
Exercise

Consider the functions

- $h(x) = |x|$,
- $g(x) = \max\{0, x\}$,
- $c(x, y) = \min\{x, y\}$

and compute the Bouligand and Clarke subdifferentials.



Properties of the Clarke Subdifferential

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz continuous and $x^* \in \mathbb{R}^n$.

- $\partial F(x^*)$ is nonempty, compact, and convex.
- If F is continuously differentiable around x^*, then $\partial^C F(x^*) = \{F'(x^*)\}$.

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- Let $f, f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^m \to \mathbb{R}^n$ be locally Lipschitz continuous and $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Then the following holds:
  - $\partial^C(cf)(x) = c\partial^C f(x)$ for all $c \in \mathbb{R}$.
  - $\partial^C(f_1 + f_2)(x) \subseteq \partial^C f_1(x) + \partial^C f_2(x)$.
  - $\partial^C(f_1 \cdot f_2)(x) \subseteq f_2(x)\partial^C f_1(x) + f_1(x)\partial^C f_2(x)$.
  - $\partial(f \circ G)(y) \subseteq \mathrm{conv}(\partial^C f(G(y)) \cdot \partial^C G(y))$ with equality if f is continuously differentiable or f convex and G continuously differentiable.

Fritz-John Conditions of Clarke

Let f, g, h be locally Lipschitz continuous and x^* be a local minimum of

$$\min_x f(x) \text{s.t.} x \in Z := \{x \in \mathbb{R}^n \mid g(x) \leq 0, h(x) = 0\}.$$

Then there exist multipliers $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$ such that

- $(\alpha, \lambda, \mu) \neq (0, 0, 0)$,
- $\alpha \geq 0$,
- $\lambda \geq 0$ and $\lambda_i = 0$ for all $i \notin I_g(x^*)$
- and

$$0 \quad \in \quad \alpha \partial^C f(x^*) + \sum_{i=1}^{m} \lambda_i \partial^C g_i(x^*) + \sum_{i=1}^{p} \mu_i \partial^C h_i(x^*)$$

What should you remember?

- The definition of MPCC.
- Why MPCC cannot be solved using standard NLP theory.
- The definition and calculus rule for the Fréchet and limiting normal cone.
- The definition and basic properties of the Clarke subdifferential.

What is the plan for tomorrow?