Mathematical Programs with Complementarity Constraints and Related Problems

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Ressources

You can find

- my slides
- and some more extensive lecture notes

at

github.com/alexandrabschwartz/Winterschool2018

If you have any questions, please come to me during the week or contact me at

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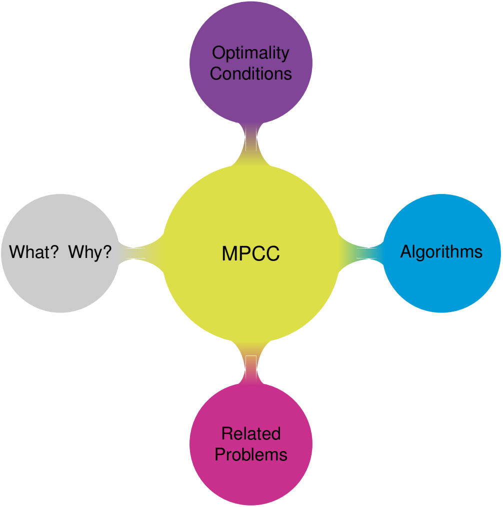
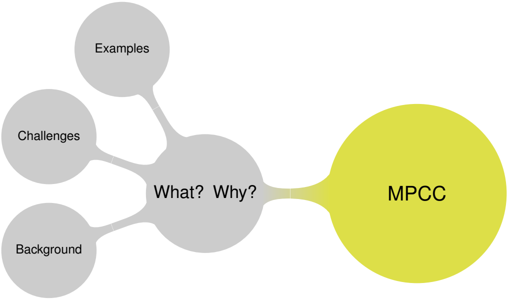
Contents of the Course



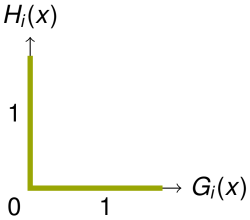
What did we do yesterday?



Mathematical Program with Complementarity Constraints (MPCC)

A mathematical program with complementarity constraints (MPCC) is of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \le 0, \quad h(x) = 0,$$
$$0 \le G(x) \perp H(x) \ge 0.$$



We assume that

- $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^p$, and $G, H : \mathbb{R}^n \to \mathbb{R}^q$ are continuously differentiable,
- the feasible set

$$X := \{x \in \mathbb{R}^n \mid g(x) \le 0, \quad h(x) = 0, \quad 0 \le G(x) \perp H(x) \ge 0\}$$

is nonempty.

Challenges of the NLP Reformulation of MPCC

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- Most standard constraint qualifications for the NLP reformulation of MPCC are violated.
- Without constraint qualification the KKT conditions are not necessary optimality conditions.
- The Fritz-John conditions are satisfied at every feasible point of the NLP reformulation.

Some Important Cones

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Let $X \subseteq \mathbb{R}^n$ be nonempty and $x^* \in X$.

- The (Bouligand) tangent cone is defined as

$$T_X(x^*) := \{d \in \mathbb{R}^n \mid \exists (x^k)_k \to_X x^*, (t_k)_k \geq 0 \ : \ t_k(x^k - x^*) \to d\}.$$

- The polar cone to a set $K \subseteq \mathbb{R}^n$ is defined as

$$K^\circ := \{w \in \mathbb{R}^n \mid w^T d \leq 0 \quad \forall d \in K\}.$$

- For a polyhedron $K = \{x \mid A^T x \leq 0, B^T x = 0\}$ one knows

$$K^\circ = \{w = A\lambda + B\mu \mid \lambda \geq 0\}.$$

- The Fréchet normal cone of X at x^* is defined as

$$N_X^F(x^*) := T_X(x^*)^\circ = \{w \in \mathbb{R}^n \mid \limsup_{x \to x^*, x \in X \setminus \{x^*\}} w^T \frac{x - x^*}{\|x - x^*\|} \leq 0\}.$$

- The limiting or Mordukhovich normal cone of X at x^* is defined as

$$N_X^M(x^*) := \{w \in \mathbb{R}^n \mid \exists (x^k)_k \to x^*, w^k \in N_X^F(x^k) \ : \ w^k \to w\}.$$

The Bouligand and Clarke Subdifferential

Consider a locally Lipschitz continuous map $F : \mathbb{R}^n \to \mathbb{R}^m$.

- Rademacher's Theorem: F is differentiable almost everywhere, i.e. the complement of

$$D_F := \{x \in \mathbb{R}^n \mid F \text{ is differentiable in } x\}.$$

  has Lebesgue measure zero.

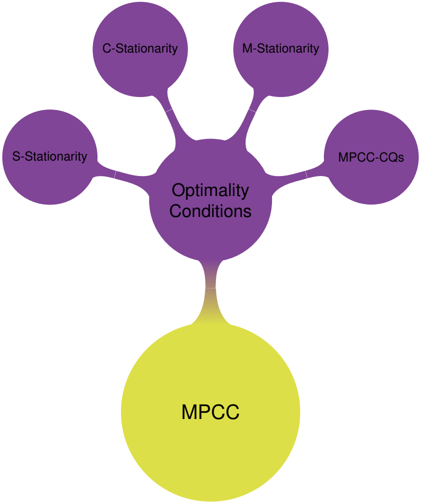
- The Bouligand subdifferential of F at $x^* \in \mathbb{R}^n$ is defined as

$$\partial^B F(x^*) := \{M \in \mathbb{R}^{m \times n} \mid \exists (x^k)_k \to_{D_F} x^* \ : \ F'(x^k) \to M\}.$$

- The Clarke subdifferential of F at $x^* \in \mathbb{R}^n$ is defined as

$$\partial^C F(x^*) := \operatorname{conv} \partial^B F(x^*).$$

What is the plan for today?



KKT based Optimality Conditions

KKT Conditions for the NLP Reformulation

Recall the NLP reformulation of the MPCC:

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$
$$G(x) \geq 0, \quad H(x) \geq 0, \quad G(x) \circ H(x) = 0.$$

Exercise

Write down the KKT conditions for the NLP reformulation and simplify them.

The following index sets may be helpful:

$$\begin{aligned} I_{0+}(x^*) &= \{i \mid G_i(x^*) = 0, \quad H_i(x^*) > 0\}, \\ I_{+0}(x^*) &= \{i \mid G_i(x^*) > 0, \quad H_i(x^*) = 0\}, \\ I_{00}(x^*) &= \{i \mid G_i(x^*) = 0, \quad H_i(x^*) = 0\}. \end{aligned}$$

S-Stationarity for MPCC

A feasible point $x^* \in X$ of MPCC is called S-stationary,

- if it is a KKT point of the NLP reformulation,
- or equivalently there exist $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$, $\gamma, \nu \in \mathbb{R}^p$ with

$$
\begin{aligned}
&\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)\gamma - \nabla H(x^*)\nu = 0,\\
&\lambda \geq 0 \text{ and } \lambda_i = 0 \quad \forall i \notin I_g(x^*),\\
&\gamma_i = 0 \quad \forall i \in I_{+0}(x^*),\\
&\nu_i = 0 \quad \forall i \in I_{0+}(x^*),\\
&\gamma_i \geq 0, \nu \geq 0 \quad \forall i \in I_{00}(x^*).
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&\gamma_i \geq 0, \nu \geq 0 \quad \forall i \in I_{00}(x^*).
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When is S-stationarity a necessary optimality condition?

Sufficient Condition for Guignard CQ

- Guignard CQ: $T_X(x^*)^\circ = L_X(x^*)^\circ$
- Under Guignard CQ the KKT conditions hold at a local minimum x^* of MPCC.
- The implication $L_X(x^*)^\circ \subseteq T_X(x^*)^\circ$ is always satisfied.
- How can we ensure

$$T_X(x^*)^\circ \subseteq L_X(x^*)^\circ?$$

Exercise

Consider the NLP reformulation of MPCC and compute

$$L_X(x^*) \quad \textit{and} \quad L_X(x^*)^\circ.$$

Linearized Cone and Polar COne of the NLP Reformulation

Let $x^* \in X$ be feasible for MPCC. Then

$$
\begin{aligned}
L_X(x^*) = \{d \in \mathbb{R}^n \quad | \quad & \nabla g_i(x^*) \leq 0 \quad \forall i \in I_g(x^*), \\
& \nabla h(x^*)^T d = 0, \\
& \nabla G_i(x^*)^T d = 0 \quad \forall i \in I_{0+}(x^*), \\
& \nabla H_i(x^*)^T d = 0 \quad \forall i \in I_{+0}(x^*), \\
& \nabla G_i(x^*)^T d \geq 0, \nabla H_i(x^*)^T d \geq 0 \quad \forall i \in I_{00}(x^*)\}
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Linearized Cone and Polar COne of the NLP Reformulation

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L_X(x^*)^\circ = \{w \in \mathbb{R}^n \quad | \quad & w = \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)\gamma - \nabla H(x^*)\nu, \\
& \lambda \geq 0 \text{ if } i \in I_g(x^*), \\
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& \gamma_i \geq 0, \nu_i \geq 0 \text{ if } i \in I_{00}(x^*)\}
\end{aligned}
$$

Approximation of the Tangent Cone of MPCC

- The reason why Abadie CQ for MPCC is likely to fail is the nonconvexity of X.
- For every $I \subseteq I_{00}(x^*)$ define the tightened program TNLP(x^*, I) as

$$\begin{aligned}\min_x f(x) \quad \text{s.t.} \quad & g(x) \leq 0, h(x) = 0, \\ & G_i(x) = 0, H_i(x) \geq 0 \quad \forall i \in I_{0+}(x^*) \cup I, \\ & G_i(x) \geq 0, H_i(x) = 0 \quad \forall i \in I_{+0}(x^*) \cup I^c,\end{aligned}$$

  where $I^c := I_{00}(x^*) \setminus I$.
- Then $x^* \in X_I$ for all $I \subseteq I_{00}(x^*)$ and there exists a radius $r > 0$ such that

$$X \cap B_r(x^*) = \bigcup_{I \subseteq I_{00}(x^*)} X_I \cap B_r(x^*).$$

Approximation of the Tangent Cone of MPCC (continued)

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- Thus we have

$$T_X(x^*)^\circ = \Big(\bigcup_{I \subseteq I_{00}(x^*)} T_{X_I}(x^*) \Big)^\circ = \bigcap_{I \subseteq I_{00}(x^*)} T_{X_I}(x^*)^\circ.$$

Approximation of the Tangent Cone of MPCC (continued)

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- We say that MPCC-LICQ at $x^* \in X$ holds, if the gradients

  $\nabla g_i(x^*)\ (i \in I_g),\ \nabla h_i(x^*)\ (i = 1, \dots, p),\ \nabla G_i(x^*)\ (i \in I_{0+} \cup I_{00}),\ \nabla H_i(x^*)\ (i \in I_{+0} \cup I_{00})$

  are linearly independent.

Approximation of the Tangent Cone of MPCC (continued)

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  are linearly independent.
- MPCC-LICQ at x^* implies LICQ for all TNLP(x^*, I) at x^* and thus

$$T_{X_I}(x^*)^\circ = L_{X_I}(x^*)^\circ.$$

Approximation of the Tangent Cone of MPCC (continued)

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L_{X_I}(x^*) = \{d \in \mathbb{R}^n \quad | \quad & \nabla g_i(x^*) \leq 0 \quad \forall i \in I_g(x^*),\\
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& \nabla H_i(x^*)^T d = 0 \quad \forall i \in I_{+0}(x^*) \cup I^c,\\
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L_{X_I}(x^*)^\circ = \{w \in \mathbb{R}^n \quad | \quad & w = \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)\gamma - \nabla H(x^*)\nu,\\
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Approximation of the Tangent Cone of MPCC (continued)

- Recall that we assume MPCC-LICQ to hold and want to show

$$T_X(x^*)^\circ = \bigcap_{I \subseteq I_{00}(x^*)} L_{X_I}(x^*)^\circ \subseteq L_X(x^*)^\circ.$$

Approximation of the Tangent Cone of MPCC (continued)

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- Consider an arbitrary $w \in T_X(x^*)^\circ$. Then

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- Consider an arbitrary $w \in T_X(x^*)^\circ$. Then

$$w \in L_{X_I}(x^*)^\circ \quad \forall I \subseteq I_{00}(x^*).$$

- Using that for all $I \subseteq I_{00}(x^*)$ we also have $I^c \subseteq I_{00}(x^*)$ and that the representation of w is unique due to MPCC-LICQ, we obtain $w \in L_X(x^*)^\circ$.

MPCC-LICQ and S-Stationarity

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Let $x^{+} \in X$ be feasible for MPCC.

- MPCC-LICQ at x^{*} implies standard Guignard CQ at x^{*}.
- If x^{*} is a local minimum of MPCC and MPCC-LICQ holds, then x^{*} is S-stationary.

Tightened and Relaxed Problem

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A closer look at the S-stationarity conditions reveals that they are the KKT conditions of the relaxed problem RNLP(x^*)

$$\begin{aligned}\min_x f(x) \quad \text{s.t.} \quad & g(x) \leq 0, h(x) = 0,\\ & G(x) = 0, H(x) \geq 0 \quad \forall i \in I_{0+}(x^*),\\ & G(x) \geq 0, H(x) = 0 \quad \forall i \in I_{+0}(x^*),\\ & G(x) \geq 0, H(x) \geq 0 \quad \forall i \in I_{00}(x^*).\end{aligned}$$

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Alternatively one can consider the *tightened program* TNLP(x^*) as

$$\begin{aligned}\min_x f(x) \quad \text{s.t.} \quad & g(x) \leq 0, h(x) = 0,\\ & G(x) = 0, H(x) \geq 0 \quad \forall i \in I_{0+}(x^*),\\ & G(x) \geq 0, H(x) = 0 \quad \forall i \in I_{+0}(x^*),\\ & G(x) = 0, H(x) = 0 \quad \forall i \in I_{00}(x^*).\end{aligned}$$

Weak Stationarity and MPCC-CQs

- The KKT conditions of TNLP(x^*) lead to W-stationarity:

$$\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)\gamma - \nabla H(x^*)\nu = 0,$$
$$\lambda \geq 0 \text{ and } \lambda_i = 0 \quad \forall i \notin I_g(x^*),$$
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- We say that MPCC-LICQ/MFCQ/CRCQ/CPLD holds at x^* if LICQ/MFCQ/CRCQ/CPLD for TNLP(x^*) holds at x^*.

Weak Stationarity and MPCC-CQs

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- We say that MPCC-LICQ/MFCQ/CRCQ/CPLD holds at x^* if LICQ/MFCQ/CRCQ/CPLD for TNLP(x^*) holds at x^*.
- A local minimum x^* of MPCC is W-stationary under any of the above MPCC-CQs.

Problem

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- S-stationarity corresponds to the KKT conditions of RNLP(x^*) and can be ensured only under MPCC-LICQ.

- Goal: We need stationarity conditions, which are stronger than W-stationarity but easier to guarantee than S-stationarity.

Problem

- S-stationarity corresponds to the KKT conditions of RNLP(x^*) and can be ensured only under MPCC-LICQ.
- W-stationarity is easier to ensure, but only corresponds to the KKT conditions of TNLP(x^*).
- Goal: We need stationarity conditions, which are stronger than W-stationarity but easier to guarantee than S-stationarity.

Clarke Stationarity

NCP Functions

- A function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is called an *NCP function (nonlinear complementarity problem)*, if

$$\varphi(a, b) = 0 \quad \Longleftrightarrow \quad 0 \leq a \perp b \geq 0.$$

NCP Functions

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$$\varphi(a, b) = 0 \quad \Longleftrightarrow \quad 0 \le a \perp b \ge 0.$$
- The two most prominent examples for NCP functions are:
  - *minimum function*: $\varphi(a, b) = \min\{a, b\}$
  - *Fischer-Burmeister function*: $\varphi(a, b) = \sqrt{a^2 + b^2} - (a + b)$

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  - *Fischer-Burmeister function*: $\varphi(a, b) = \sqrt{a^2 + b^2} - (a + b)$
- We can use an NPC function to reformulate the complementarity constraints as
$$0 \leq G(x) \perp H(x) \geq 0 \quad \Longleftrightarrow \quad \varphi(G_i(x), H_i(x)) = 0 \quad \forall i = 1, \dots, q.$$

NCP Functions

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- We can use an NPC function to reformulate the complementarity constraints as

$$0 \leq G(x) \perp H(x) \geq 0 \quad \Longleftrightarrow \quad \varphi(G_i(x), H_i(x)) = 0 \quad \forall i = 1, \dots, q.$$

- Problem: NPC functions usually are nondifferentiable at $(a, b) = (0, 0)$.

Fritz-John Conditions of Clarke

- We use $\varphi(a,b) = \min\{a,b\}$ and consider the problem

$$\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \quad h(x) = 0,$$
$$\varphi(G_i(x), H_i(x)) = 0 \quad \forall i = 1, \ldots, q.$$

- Applying the Fritz-John conditions of Clarke yields

$$0 \in \alpha \nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu + \sum_{i=1}^{q} \delta_i \partial^C \varphi(G_i(x^*), H_i(x^*)),$$
$$\alpha \geq 0, \quad \lambda \geq 0, \quad \lambda_i = 0 \quad \forall i \notin I_g(x^*),$$
$$(\alpha, \lambda, \mu, \delta) \neq (0, 0, 0, 0).$$

Fritz-John Conditions of Clarke (continued)

- Here, we have

$$\partial^C \varphi(G_i(x^*), H_i(x^*)) = \begin{cases} \nabla G_i(x^*) & \text{if } i \in I_{0+}(x^*), \\ \nabla H_i(x^*) & \text{if } i \in I_{+0}(x^*), \\ \operatorname{conv}\{\nabla G_i(x^*), \nabla H_i(x^*)\} & \text{if } i \in I_{00}(x^*), \end{cases}$$

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- A feasible point $x^* \in X$ of MPCC is called C-stationary, if there exist $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$, $\gamma, \nu \in \mathbb{R}^p$ with

$$\begin{aligned} &\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)\gamma - \nabla H(x^*)\nu = 0, \\ &\lambda \geq 0 \text{ and } \lambda_i = 0 \quad \forall i \notin I_g(x^*), \\ &\gamma_i = 0 \quad \forall i \in I_{+0}(x^*), \\ &\nu_i = 0 \quad \forall i \in I_{0+}(x^*), \\ &\gamma_i \cdot \nu_i \geq 0 \quad \forall i \in I_{00}(x^*). \end{aligned}$$

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$$\partial^C \varphi(G_i(x^*), H_i(x^*)) = \begin{cases} \nabla G_i(x^*) & \text{if } i \in I_{0+}(x^*), \\ \nabla H_i(x^*) & \text{if } i \in I_{+0}(x^*), \\ \operatorname{conv}\{\nabla G_i(x^*), \nabla H_i(x^*)\} & \text{if } i \in I_{00}(x^*), \end{cases}$$

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$$\begin{aligned} &\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)\gamma - \nabla H(x^*)\nu = 0, \\ &\lambda \geq 0 \text{ and } \lambda_i = 0 \quad \forall i \notin I_g(x^*), \\ &\gamma_i = 0 \quad \forall i \in I_{+0}(x^*), \\ &\nu_i = 0 \quad \forall i \in I_{0+}(x^*), \\ &\gamma_i \cdot \nu_i \geq 0 \quad \forall i \in I_{00}(x^*). \end{aligned}$$

- A local minimum x^* of MPCC is C-stationary under MPCC-LICQ/MFCQ.

Mordukhovich or Limiting Stationarity

MPCC Analogues of Abadie and Guignard CQ

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- Problem with Abadie CQ: $L_X(x^*)$ is convex, $T_X(x^*)$ can be nonconvex.

MPCC Analogues of Abadie and Guignard CQ

- Problem with Abadie CQ: $L_X(x^*)$ is convex, $T_X(x^*)$ can be nonconvex.
- Idea: Reuse the tightened problems TNLP(x^*, I) for $I \subseteq I_{00}(x^*)$ and define the MPCC linearized tangent cone as

$$
\begin{aligned}
L_X^{\text{MPCC}}(x^*) \quad &:= \quad \bigcup_{I \subseteq I_{00}(x^*)} L_{X_I}(x^*) \\
&= \quad \{d \in \mathbb{R}^n \mid \nabla g_i(x^*)^T d \leq 0 \quad \forall i \in I_g(x^*), \\
&\qquad\qquad \nabla h_i(x^*)^T d = 0 \quad \forall i = 1, \dots, p, \\
&\qquad\qquad \nabla G_i(x^*)^T d = 0 \quad \forall i \in I_{0+}(x^*), \\
&\qquad\qquad \nabla H_i(x^*)^T d = 0 \quad \forall i \in I_{+0}(x^*), \\
&\qquad\qquad 0 \leq \nabla G_i(x^*)^T d \perp \nabla H_i(x^*)^T d \geq 0 \quad \forall i \in I_{00}(x^*)\}, \\
&= \quad \{d \in L_X(x^*) \mid (\nabla G_i(x^*)^T d)(\nabla H_i(x^*)^T d) = 0 \quad \forall i \in I_{00}(x^*)\}.
\end{aligned}
$$

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&\qquad 0 \leq \nabla G_i(x^*)^T d \perp \nabla H_i(x^*)^T d \geq 0 \quad \forall i \in I_{00}(x^*) \}, \\
&= \{ d \in L_X(x^*) \mid (\nabla G_i(x^*)^T d)(\nabla H_i(x^*)^T d) = 0 \quad \forall i \in I_{00}(x^*) \}.
\end{aligned}
$$

- MPCC-Abadie CQ: $T_X(x^*) = L_X^{\mathrm{MPCC}}(x^*)$
- MPCC-Guignard CQ: $T_X(x^*)^\circ = L_X^{\mathrm{MPCC}}(x^*)^\circ$

Relations between the MPCC Constraint Qualifications

$$
\begin{tikzcd}
 & \text{MPCC-LICQ} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \text{MPCC linear CQ} \arrow[dl, Rightarrow] \\
\text{MPCC-MFCQ} \arrow[dr, Rightarrow] & & \text{MPCC-CRCQ} \arrow[dl, Rightarrow] & \\
 & \text{MPCC-CPLD} \arrow[d, Rightarrow] & & \\
 & \text{MPCC-ACQ} \arrow[d, Rightarrow] & & \\
 & \text{MPCC-GCQ} & &
\end{tikzcd}
$$

Optimality Condition under MPCC-GCQ

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Let x^* be a local minimum of MPCC

$\Longrightarrow$ x^* is B-stationary, i.e.

$$\nabla f(x^*)^T d \geq 0 \quad \forall d \in T_X(x^*) \qquad \iff \qquad -\nabla f(x^*) \in T_X(x^*)^\circ$$

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$\Longrightarrow$ Under MPCC-GCQ we know $T_X(x^*)^\circ = L_X^{\text{MPCC}}(x^*)^\circ$ and thus

$$-\nabla f(x^*) \in L_X^{\text{MPCC}}(x^*)^\circ \qquad \Longleftrightarrow \qquad \nabla f(x^*)^T d \geq 0 \quad \forall d \in L_X^{\text{MPCC}}(x^*)$$

Optimality Condition under MPCC-GCQ

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$$-\nabla f(x^*) \in L_X^{\text{MPCC}}(x^*)^\circ \qquad \Longleftrightarrow \qquad \nabla f(x^*)^T d \geq 0 \quad \forall d \in L_X^{\text{MPCC}}(x^*)$$

$\Longrightarrow$ $d^* = 0$ is a minimum of the linear MPCC

$$\min_d \nabla f(x^*)^T d \quad \text{s.t.} \quad d \in L_X^{\text{MPCC}}(x^*)$$

Optimality Condition under MPCC-GCQ (continued)

Rewrite $L_X^{\text{MPCC}}(x^*)$ as $D = D_1 \cap D_2$ with

$$
\begin{aligned}
D_1 = \big\{(d,u,v) \in R^{n+2|I_{00}|} \quad \big| \quad & \nabla g_i(x^*)^T d \leq 0 \quad \forall i \in I_g(x^*), \\
& \nabla h_i(x^*)^T d = 0 \quad \forall i = 1, \dots, p, \\
& \nabla G_i(x^*)^T d = 0 \quad \forall i \in I_{0+}(x^*), \\
& \nabla H_i(x^*)^T d = 0 \quad \forall i \in I_{+0}(x^*), \\
& \nabla G_i(x^*)^T d - u_i = 0 \quad \forall i \in I_{00}(x^*), \\
& \nabla H_i(x^*)^T d - v_i = 0 \quad \forall i \in I_{00}(x^*)\big\}, \\
D_2 = \big\{(d,u,v) \in R^{n+2|I_{00}|} \quad \big| \quad & 0 \leq u \perp v \geq 0\big\}
\end{aligned}
$$

$\Longrightarrow$ $(d^*, u^*, v^*) = (0,0,0)$ is a minimum of the linear MPCCs

$$
\min_{d,u,v} \nabla f(x^*)^T d \quad \text{s.t.} \quad (d,u,v) \in D = D_1 \cap D_2.
$$

Optimality Condition under MPCC-GCQ (continued)

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$\Longrightarrow$ $(d^*, u^*, v^*) = (0, 0, 0)$ is B-stationary, i.e.

$$-\begin{pmatrix} \nabla f(x^*) \\ 0 \\ 0 \end{pmatrix} \in T_D(0,0,0)^\circ = T_{D_1 \cap D_2}(0,0,0)^\circ = N^F_{D_1 \cap D_2}(0,0,0).$$

- The set-valued map

$$M(y) := \left\{(d, u, v) \in D_1 \;\middle|\; (d, u, v) + y \in D_2\right\}$$

  is polyhedral and thus calm.
- Consequently

$$N^F_{D_1 \cap D_2}(0,0,0) \subseteq N^M_{D_1 \cap D_2}(0,0,0) \subseteq N^M_{D_1}(0,0,0) + N^M_{D_2}(0,0,0).$$

M-Stationarity

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- A feasible point $x^* \in X$ of MPCC is called M-stationary, if there exist $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$, $\gamma, \nu \in \mathbb{R}^p$ with

$$\begin{aligned}
&\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)\gamma - \nabla H(x^*)\nu = 0,\\
&\lambda \geq 0 \text{ and } \lambda_i = 0 \quad \forall i \notin I_g(x^*),\\
&\gamma_i = 0 \quad \forall i \in I_{+0}(x^*),\\
&\nu_i = 0 \quad \forall i \in I_{0+}(x^*),\\
&\gamma_i \cdot \nu_i = 0 \text{ or } \gamma_i \geq 0, \nu_i \geq 0 \quad \forall i \in I_{00}(x^*).
\end{aligned}$$

M-Stationarity

TECHNISCHE UNIVERSITÄT DARMSTADT

- A feasible point $x^* \in X$ of MPCC is called M-stationary, if there exist $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$, $\gamma, \nu \in \mathbb{R}^p$ with

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&\gamma_i \cdot \nu_i = 0 \text{ or } \gamma_i \geq 0, \nu_i \geq 0 \quad \forall i \in I_{00}(x^*).
\end{aligned}
$$

- A local minimum x^* of MPCC is M-stationary under MPCC-Guignard CQ.

Comparison of Stationarity Conditions for MPCC

A feasible point x^* of MPCC is

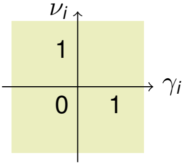
- W-stationary, if

$$\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)\gamma - \nabla H(x^*)\nu = 0,$$

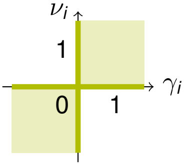
$$\lambda \geq 0 \text{ and } \lambda_i = 0 \quad \forall i \notin I_g(x^*),$$

$$\gamma_i = 0 \quad \forall i \in I_{+0}(x^*) \quad \text{and} \quad \nu_i = 0 \quad \forall i \in I_{0+}(x^*);$$

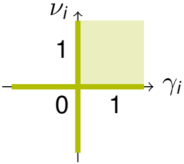
- C-stationary, if additionally $\gamma_i \cdot \nu_i \geq 0 \quad \forall i \in I_{00}(x^*)$;
- M-stationary, if additionally $\gamma_i \cdot \nu_i = 0$ or $\gamma_i \geq 0, \nu_i \geq 0 \quad \forall i \in I_{00}(x^*)$;
- S-stationary, if additionally $\gamma_i \geq 0, \nu_i \geq 0 \quad \forall i \in I_{00}(x^*)$.



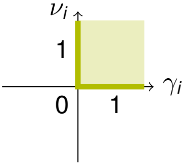
W-Stationarity



C-Stationarity



M-Stationarity



S-Stationarity

What should you remember?

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- The different stationarity conditions for MPCCs, their origins and their differences.
- The MPCC constraint qualifications and their relations.
- The different necessary optimality conditions for MPCC.

What is the plan for tomorrow?