Mathematical Programs with Complementarity Constraints and Related Problems

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Ressources

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You can find

- my slides
- and some more extensive lecture notes

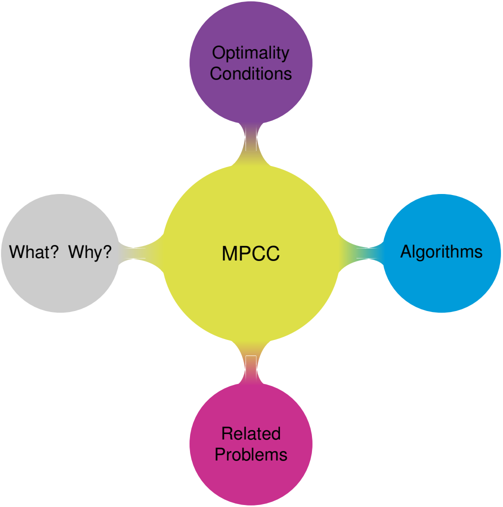
at

github.com/alexandrabschwartz/Winterschool2018

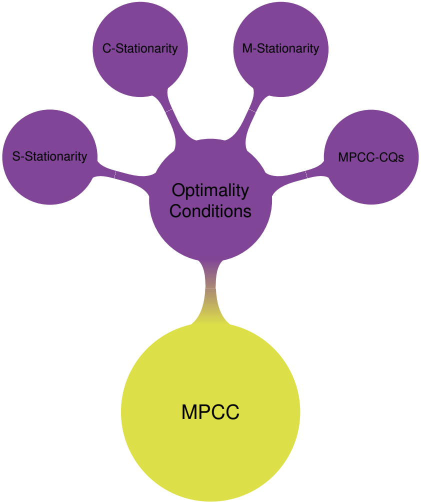
If you have any questions, please come to me during the week or contact me at

schwartz@gsc.tu-darmstadt.de

Contents of the Course



What did we do yesterday?



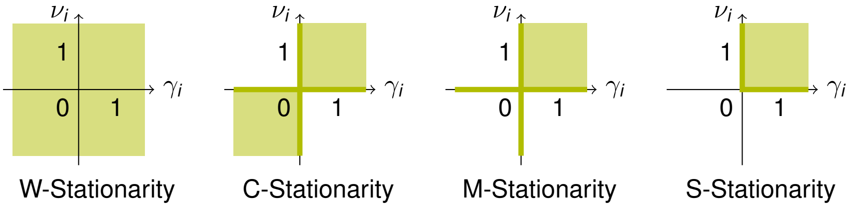
Stationarity Conditions for MPCC

A feasible point x^* of MPCC is

- W-stationary, if

$$\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu - \nabla G(x^*)\gamma - \nabla H(x^*)\nu = 0,$$
$$\lambda \geq 0 \text{ and } \lambda_i = 0 \quad \forall i \notin I_g(x^*),$$
$$\gamma_i = 0 \quad \forall i \in I_{+0}(x^*) \quad \text{and} \quad \nu_i = 0 \quad \forall i \in I_{0+}(x^*);$$

- C-stationary, if additionally $\gamma_i \cdot \nu_i \geq 0 \quad \forall i \in I_{00}(x^*)$;
- M-stationary, if additionally $\gamma_i \cdot \nu_i = 0$ or $\gamma_i \geq 0, \nu_i \geq 0 \quad \forall i \in I_{00}(x^*)$;
- S-stationary, if additionally $\gamma_i \geq 0, \nu_i \geq 0 \quad \forall i \in I_{00}(x^*)$.



Constraint Qualifications for MPCC

$$\begin{tikzcd}
 & \text{MPCC-LICQ} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \text{MPCC linear CQ} \arrow[dl, Rightarrow] \\
\text{MPCC-MFCQ} \arrow[dr, Rightarrow] & & \text{MPCC-CRCQ} \arrow[dl, Rightarrow] & \\
 & \text{MPCC-CPLD} \arrow[d, Rightarrow] & & \\
 & \text{MPCC-ACQ} \arrow[d, Rightarrow] & & \\
 & \text{MPCC-GCQ} & &
\end{tikzcd}$$

Optimality Conditions for MPCC

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S-stationarity
necessary under MPCC-LICQ

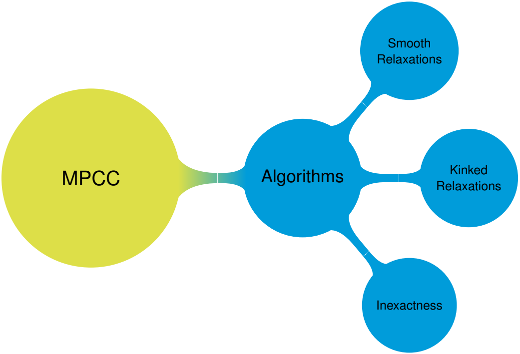
B-stationarity
necessary without any CQ

M-stationarity
necessary under MPCC-GCQ

C-stationarity
necessary under MPCC-GCQ

W-stationarity
necessary under MPCC-GCQ

What is the plan for today?



Algorithms for MPCCs

- Standard NLP algorithms may fail due to lack of CQ, but nonetheless work quite often.
- Specialized MPCC algorithms include:
  - penalty and barrier methods
  - SQP methods
  - smoothing methods
  - relaxation methods
  - augmented Lagrangian methods
  - lifting methods
  - ...

Relaxation Methods

Basic Idea of a Relaxation Method

Idea: Relax the complementarity conditions and solve a sequence of nonlinear problems NLP(t_k) with $t_k \downarrow 0$.

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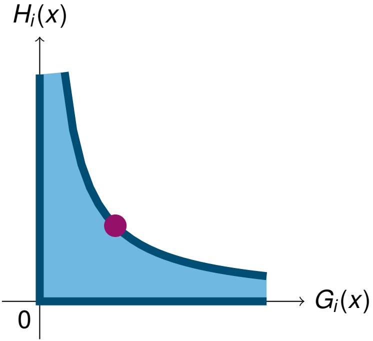

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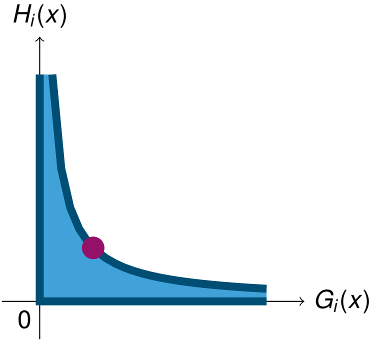

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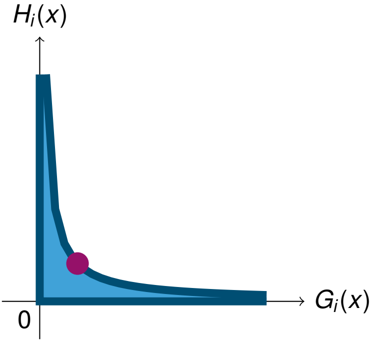

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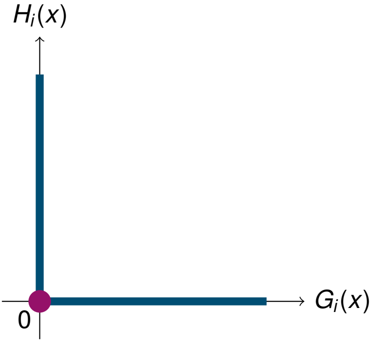

Basic Idea of a Relaxation Method

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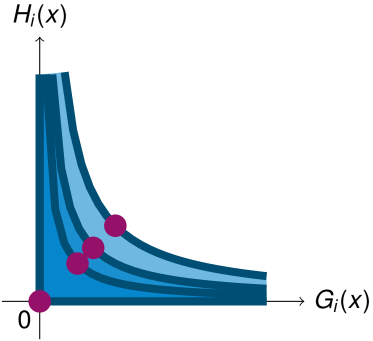
Idea: Relax the complementarity conditions and solve a sequence of nonlinear problems NLP(t_k) with $t_k \downarrow 0$.

Questions

- Are the relaxed problems easier to solve?
- What kind of points does the method converge to?



The Global Relaxation by Scholtes

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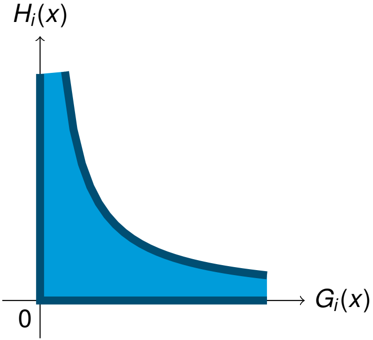
Replace the complementarity constraints by

$$G_i(x) \geq 0, H_i(x) \geq 0, G_i(x) \cdot H_i(x) \leq t_k.$$

Let $(x_k)_k$ be KKT points of the relaxed problems NLP(t_k).



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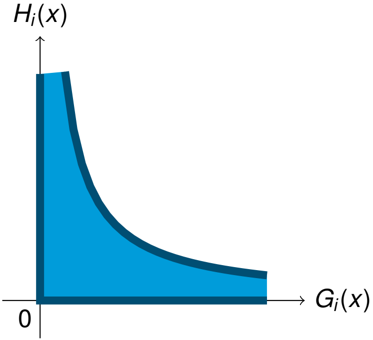

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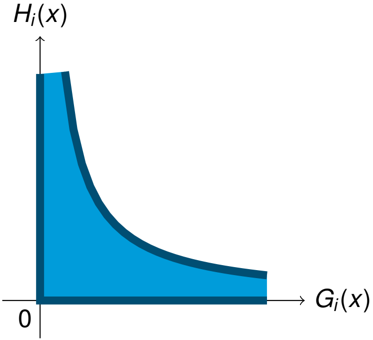
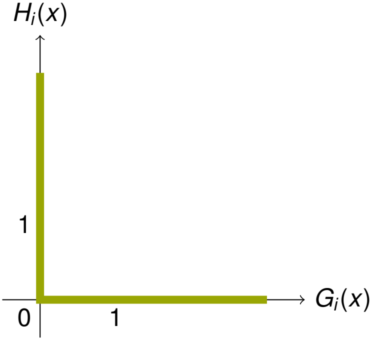


Counterexample for the Global Relaxation

Consider the MPCC

$$\min_{x\in\mathbb{R}^2} \|x-(1,1)^T\|_2^2 \quad \text{s.t.} \quad 0 \leq x_1 \perp x_2 \geq 0.$$



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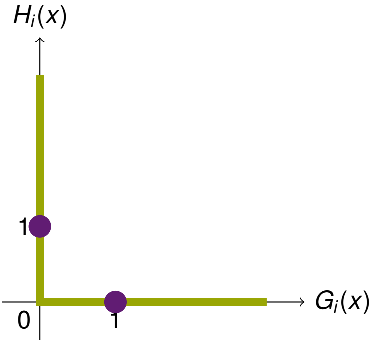

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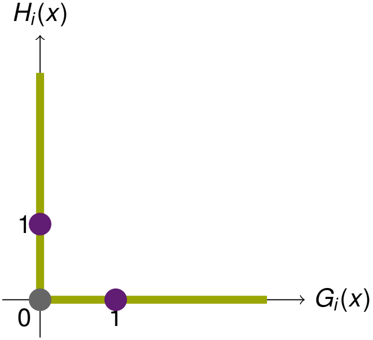
- $(1,0)^T$ is a local maximum, but C-stationary.



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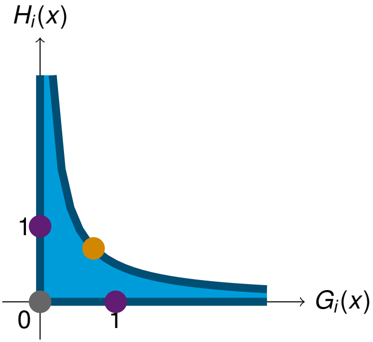
- Then $(1,0)^T$ and $(0,1)^T$ are global minima and S-stationary due to MPCC-LICQ.
- $(1,0)^T$ is a local maximum, but C-stationary.
- For $t_k > 0$ small the points $(t_k, t_k)^T$ are KKT points of NLP(t_k).



The Local Relaxation by Steffensen and Ulbrich

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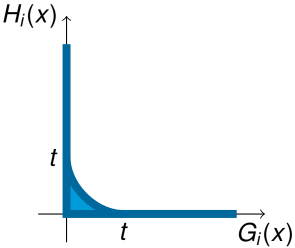
Replace the complementarity constraints by

$$G_i(x) \geq 0, H_i(x) \geq 0, \varphi(G_i(x), H_i(x); t_k) \leq 0,$$

where $\theta : [-1, 1] \to \mathbb{R}$ is a regularization function and

$$\varphi(G_i(x), H_i(x); t_k) = \begin{cases} \min\{G_i(x), H_i(x)\} & \text{if } |G_i(x) - H_i(x)| \geq t_k, \\ G_i(x) + H_i(x) - t_k\theta(\frac{G_i(x) - H_i(x)}{t_k}) & \text{if } |G_i(x) - H_i(x)| < t_k. \end{cases}$$

Let $(x_k)_k$ be KKT points of the relaxed problems NLP(t_k).



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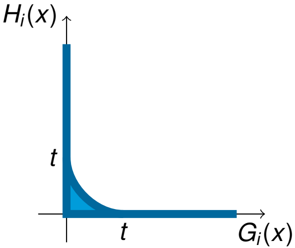

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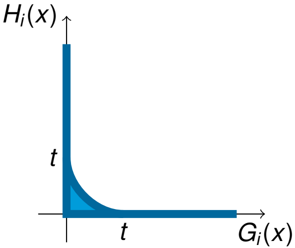

The Nonsmooth Relaxation by Kardani et al.

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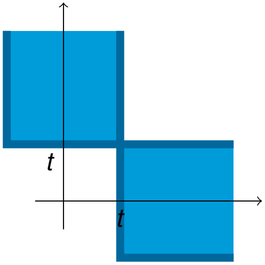

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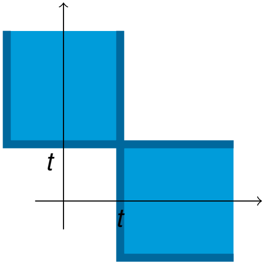

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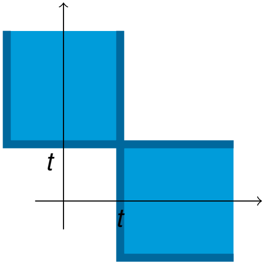

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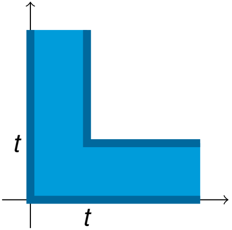

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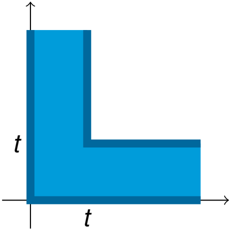

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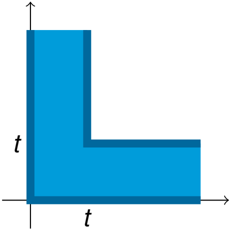
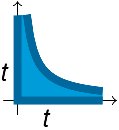
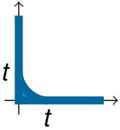
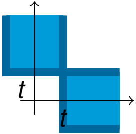
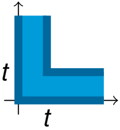


Comparison of the Relaxation Methods

relaxation	global	local	nonsmooth	kinked
authors	Scholtes	Steff./Ulbr.	Kadrani et al.	Kanzow, S.
geometry				
stationarity under CQ	C-stat. MPEC-MFCQ	C-stat. MPEC-CPLD	M-stat. MPEC-CPLD	M-stat. MPEC-CPLD

Numerical Comparison

Implementation

- We implemented all methods in MATLAB using `snopt` to solve the relaxed problems NLP(t_k).
- The relaxation algorithm was terminated if either $t_k < 10^{-15}$ or

$$\max\left\{\left|\min\{G(x^k), H(x^k)\}\right|\right\} \leq 10^{-6}.$$

- A termination was considered successful if

$$\max\left\{\max\{g(x^k), 0\}, |h(x^k)|, \left|\min\{G(x^k), H(x^k)\}\right|\right\} \leq 10^{-6}.$$

- We used the parameters

relaxation	global	local	nonsmooth	kinked
t_0	0.25	$\frac{\pi}{\pi-2}$	1	2
σ	0.01	0.01	0.01	0.01

- We applied these methods to 126 test problems from the MacMPEC library.
- For comparison, we also applied `snopt` directly to the test problems.

Optimal Function Value

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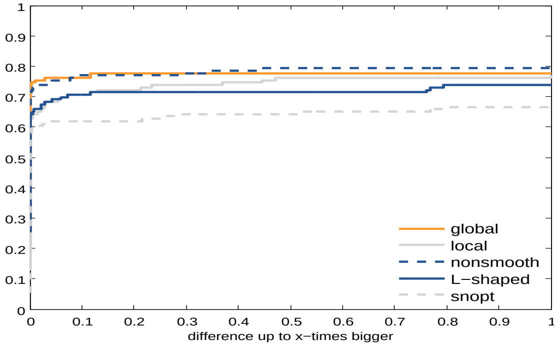
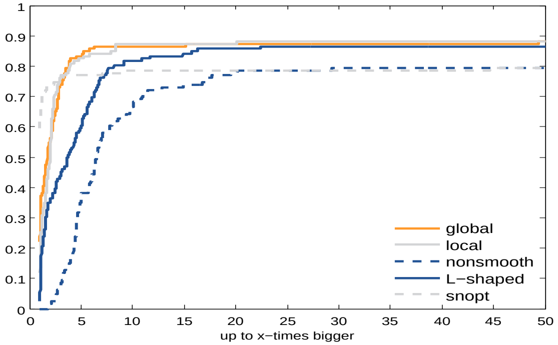


We considered the relative difference to the best solution found by any of the five methods.

Computation Time



The number of objective function evaluations and gradient evaluations lead to the same results.

Comparison of the Methods

- The relaxation methods lead to better results than the standard NLP solver `snopt` alone.
- The local relaxation is often very successful but somewhat instable.
- The global relaxation is fast, stable and leads to good objective function values.
- The kinked relaxation behaves similar to the global relaxation but is not quite as good.
- The nonsmooth relaxation is slower than the others and leads to less feasible solutions which, however, have the smallest objective function values.

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- The global relaxation is fast, stable and leads to good objective function values.
- The kinked relaxation behaves similar to the global relaxation but is not quite as good.
- The nonsmooth relaxation is slower than the others and leads to less feasible solutions which, however, have the smallest objective function values.

Inexact KKT Points

ε-KKT Points

Consider a nonlinear program (NLP)

$$\min f(x) \quad \text{s.t.} \quad g(x) \leq 0, h(x) = 0.$$

Definition: A feasible point x^* is said to be a KKT point if there are multipliers $\lambda \geq 0$ and μ such that $\lambda_i = 0$ for all $i \notin I_g$ and

$$\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu = 0.$$

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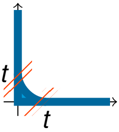
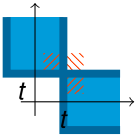
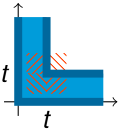
Definition: Let $\varepsilon > 0$. A point x^* is said to be an ε-KKT point if there are multipliers λ and μ such that

$$\begin{aligned}
&\left\|\nabla f(x^*) + \nabla g(x^*)\lambda + \nabla h(x^*)\mu\right\|_\infty \leq \varepsilon,\\
&g_i(x^*) \leq \varepsilon,\ \lambda_i \geq -\varepsilon,\ |\lambda_i g_i(x^*)| \leq \varepsilon, \quad \forall i = 1, \dots, m\\
&|h_i(x^*)| \leq \varepsilon \quad \forall i = 1, \dots, p.
\end{aligned}$$

Inexact Relaxation Methods

Let $t_k \downarrow 0$ and $\varepsilon_k = o(t_k)$. Let x^k be ε_k-KKT points of NLP(t_k) and assume $x^k \to x^*$. Then the limit x^* satisfies:

relaxation	global	local	nonsmooth	kinked
authors	Scholtes	Steff./Ulbr.	Kadrani et al.	Kanzow, S.
geometry				
stationarity		W-stat.	W-stat.	W-stat.
under CQ		MPEC-MFCQ	MPEC-MFCQ	MPEC-MFCQ

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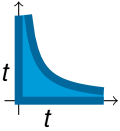
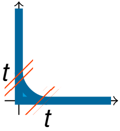
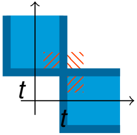
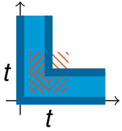
geometry				
stationarity	C-stat.	W-stat.	W-stat.	W-stat.
under CQ	MPEC-MFCQ	MPEC-MFCQ	MPEC-MFCQ	MPEC-MFCQ

An Exemplary Proof for the Scholtes Relaxation

TECHNISCHE UNIVERSITÄT DARMSTADT

Theorem

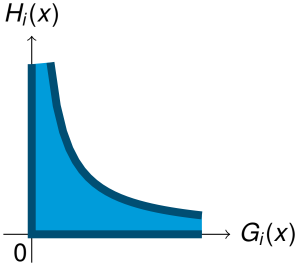
Let $(t_k)_k \downarrow 0$ and x^k be KKT points of the relaxed problems LNP(t_k) according to Scholtes. Then every accumulation point x^*, where MPCC-MFCQ holds, is C-stationary.

Recall that we replace the complementarity constraints by

$$G_i(x) \geq 0, H_i(x) \geq 0, G_i(x) \cdot H_i(x) \leq t_k.$$

Define the index sets

$$\begin{aligned} I_G(x^k) &:= \{i \mid G_i(x^k) = 0\}, \\ I_H(x^k) &:= \{i \mid H_i(x^k) = 0\}, \\ I_{GH}(x^k) &:= \{i \mid G_i(x^k)H_i(x^k) = t_k\} \end{aligned}$$



What should you remember?

- There is a vast amount of algorithms tailored to MPCCs.
- Relaxation methods improve the results of standard NLP solvers.
- Different relaxations converge to different kinds of stationary point sunder different assumptions.
- Inexactness changes the properties of some methods drastically.

What is the plan for tomorrow?