Temă din v.a continue

Ex 1: Fie X o va continuă cu densitatea de probabilitate.

I $f(x) = \begin{cases} A \cdot \sin x, & x \in [0, \pi] \\ 0, & \text{în rest} \end{cases}$ II $f(x) = \begin{cases} A\cos x, & x \in [0, \frac{\pi}{2}] \\ 0, & \text{în rest} \end{cases}$

$A \in \mathbb{R}$.

Determinați:

a) Valoarea parametrului real A.

I Dacă $f(x)$ este densitate de probabilitate, atunci:

1) $f(x) \geq 0$, $\forall x \in [0, \pi]$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

1) $\left.\begin{array}{l} A \cdot \sin x \geq 0 \\ x \in [0, \pi] \Rightarrow 0 \leq \sin x \leq 1 \end{array}\right\} \Rightarrow A \geq 0.$

2) $\int_{-\infty}^{\infty} f(x)dx = 1 \Leftrightarrow \underbrace{\int_{-\infty}^{0} f(x)dx}_{0} + \int_{0}^{\pi} f(x)dx + \underbrace{\int_{\pi}^{\infty} f(x)dx}_{0} = 1$

$\Rightarrow \int_{0}^{\pi} A \sin x\,dx = 1 \Leftrightarrow A\int_{0}^{\pi} \sin x\,dx = 1 \Leftrightarrow A \cdot (-\cos x)\Big|_{0}^{\pi} = 1$

$\Leftrightarrow A(-\cos\pi + \cos 0) = 1 \Rightarrow A(1+1) = 1 \Rightarrow A = \frac{1}{2} \geq 0$ ✓

II 1) $\left.\begin{array}{l} A\cos x \geq 0 \\ x \in [0, \pi] \Rightarrow 0 \leq \cos x \leq 1 \end{array}\right| \Rightarrow A \geq 0.$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Leftrightarrow \int_{-\infty}^{0} \overset{=0}{f(x)}dx + \int_{0}^{\frac{\pi}{2}} f(x)dx + \int_{\frac{\pi}{2}}^{\infty} \overset{=0}{f(x)}dx = 1$

$\Rightarrow A \sin x\Big|_{0}^{\frac{\pi}{2}} = 1 \Leftrightarrow A(1-0) = 1 \Rightarrow A = 1 \geq 0$ ✓

b) $\mathbb{P}(X < \frac{\pi}{3})$; $\mathbb{P}(X < \frac{\pi}{4} / X > \frac{\pi}{6})$

I O funcție se numește funcție de repartiție pt. v.a X dacă

$F(x) = \int_{-\infty}^{x} f(t)\,dt.$

obs: $F(x) = \mathbb{P}(X \leq x)$ (la fel ca în cazul v.a discrete)

$\mathbb{P}(X < \frac{\pi}{3}) = F(\frac{\pi}{3}) = \int_{-\infty}^{\frac{\pi}{3}} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{\frac{\pi}{3}} f(t)\,dt$

$= \frac{1}{2}(-\cos x)\Big|_{0}^{\frac{\pi}{3}} = \frac{1}{2}\left(-\frac{1}{2} - 1\right) = \frac{1}{4} \Rightarrow \mathbb{P}(X < \frac{\pi}{3}) = \frac{1}{4}$

II $\mathbb{P}(X < \frac{\pi}{3}) = F(\frac{\pi}{3}) = \int_{-\infty}^{\frac{\pi}{3}} f(t)\,dt \Rightarrow 1 \cdot \sin x \Big|_{0}^{\frac{\pi}{3}} = \frac{\sqrt{3}}{2}$

I $\mathbb{P}(X < \frac{\pi}{4} \mid X > \frac{\pi}{6}) = \mathbb{P}(\frac{\pi}{6} < X < \frac{\pi}{4})$

În cazu v.a continue avem:

$\mathbb{P}(a < X \leq b) = \mathbb{P}(a < X < b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a \leq X \leq b)$

Deci putem aplica: $\mathbb{P}(a < X \leq b) = \int_{a}^{b} f(x)\,dx.$

I $\mathbb{P}(\frac{\pi}{6} < X < \frac{\pi}{4}) = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2}\sin x\,dx = \frac{1}{2}(-\cos x)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} =$

$= \frac{1}{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{1}{2}\left(\frac{-\sqrt{2}+\sqrt{3}}{2}\right) = \frac{-\sqrt{2}+\sqrt{3}}{4}$

II $1 \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x\,dx = \sin x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2} - \frac{\cancel{\sqrt{3}}}{2} = \frac{\sqrt{2}-\cancel{\sqrt{3}}}{2}$

c) Media și abaterea medie pătratică a lui x.

Media $\rightsquigarrow$ $E(x) = \int_{-\infty}^{\infty} x f(x) dx$

I $E(x) = \int_{-\infty}^{0} x \underbrace{f(x)}_{=0} dx = \int_{0}^{\pi} x f(x) dx + \int_{\pi}^{\infty} x \underbrace{f(x)}_{=0} dx$

$= \int_{0}^{\pi} \frac{1}{2} x \sin(x) dx = \frac{1}{2} \int_{0}^{\pi} x \sin x dx$ $\Rightarrow$

$x = f(x) \Rightarrow f'(x) = 1$

$\sin x = g'(x) \Rightarrow g(x) = -\cos x$

$-\frac{1}{2} \cdot x \cos x \Big|_{0}^{\pi} + \frac{1}{2} \int_{0}^{\pi} \cos x dx = -\frac{1}{2}(\pi \overbrace{\cos \pi}^{-1} - 0) + \frac{1}{2}(-\sin x)\Big|_{0}^{\pi}$

$= -\frac{1}{2} \cdot \pi \cdot (-1) + \frac{1}{2}(-\overbrace{\sin \pi}^{0} + \overbrace{\sin 0}^{=0}) = \frac{\pi}{2}$

II $E(x) = \int_{0}^{\frac{\pi}{2}} 1 \cdot x \cos x dx$

$f(x) = x$, $g'(x) = \cos x$ $\Rightarrow$ $f'(x) = 1$, $g(x) = \sin x$ $\Rightarrow$

$\Rightarrow x \sin x \Big|_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} 1 \cdot \sin x dx = \frac{\pi}{2} \overbrace{\sin \frac{\pi}{2}}^{1} - \cos x \Big|_{0}^{\frac{\pi}{2}} =$

$= \frac{\pi}{2} - (0 - 1) = \frac{\pi}{2} + 1$

Abaterea medie pătratică $\rightsquigarrow$ $Var(x) = E(x^2) - E(x)^2$

I $E(x^2) = \int_{0}^{\pi} \frac{1}{2} x^2 \sin x dx = \frac{1}{2} \int_{0}^{\pi} x^2 \sin x dx$ $\Rightarrow$

$f(x) = x^2 \Rightarrow f'(x) = 2x$

$g'(x) = \sin x \Rightarrow g(x) = -\cos(x)$

$-\frac{1}{2} x^2 \cos(x) \Big|_{0}^{\frac{\pi}{2}} + \frac{1}{2} \int_{0}^{\frac{\pi}{2}} 2x \cos x dx = -\frac{1}{2}\left(\frac{\pi^2}{4} \overbrace{\cos \frac{\pi}{2}}^{-1}\right) \overbrace{\int_{0}^{\frac{\pi}{2}} x \cos x dx}^{I}$

$$\int_0^{\frac{\pi}{\cancel{4}}} x\cos x\,dx = x\sin x\Big|_0^{\frac{\pi}{\cancel{4}}} - \int_0^{\frac{\pi}{\cancel{4}}} \sin x\,dx =$$

$$= \frac{\pi}{\cancel{4}}\overbrace{\sin\frac{\pi}{\cancel{4}}}^{=0} - \cos x\Big|_0^{\frac{\pi}{\cancel{4}}} = 0 - (\overbrace{\cos \cancel{\pi}}^{-1} - \overbrace{\cos 0}^{1}) = -(-1-1) = 2$$

$$\cancel{Var(x) = 2 - \frac{\pi^2}{2}} \Rightarrow E(x^2) = -\frac{\pi^2}{2} + 2$$

$$Var(x) = -\frac{\pi^2}{2} + 2 - \frac{\pi^2}{\cancel{4}} = \frac{-2\pi^2 + 8 - \pi^2}{4} = \frac{-3\pi^2+8}{4}$$

$$\sqrt{Var(x)} = \frac{\sqrt{-3\pi^2+8}}{2}$$

$$\text{II}\quad E(x^2) = \int_0^{\frac{\pi}{2}} x^2\cos x\,dx \Rightarrow$$

$$f(x) = x^2 \Rightarrow f'(x) = 2x$$

$$g'(x) = \cos x \Rightarrow g(x) = \sin x$$

$$\Rightarrow x^2\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x\sin x\,dx = \frac{\pi^2}{4}\overbrace{\sin\frac{\pi}{2}}^{1} - 0 - 2\overbrace{\int_0^{\frac{\pi}{2}} x\sin x\,dx}^{I}$$

$$I = \int_0^{\frac{\pi}{2}} x\sin x\,dx = -x\cos x\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,dx =$$

$$= -\frac{\pi}{2}\overbrace{\cos\frac{\pi}{2}}^{=0} + (-\sin x)\Big|_0^{\frac{\pi}{2}} = -\overbrace{\sin\frac{\pi}{2}}^{1} + \overbrace{\sin 0}^{0} = -1$$

$$\Rightarrow E(x^2) = \frac{\pi^2}{4}\cdot 1 + 2 = \frac{\pi^2+8}{4}$$

$$Var(x) = \frac{\pi^2+8}{4} - \left(\frac{\pi}{2}+1\right)^2 = \frac{\pi^2+8}{4} - \left(\frac{\pi^2}{4} + 2\frac{\pi}{2} + 1\right) =$$

$$= \frac{\pi^2+8}{4} - \frac{\pi^2}{4} - \pi - 1 = \frac{\cancel{\pi^2} + 8 - \cancel{\pi^2} - 4\pi - 4}{4} = \frac{-4(\pi+1-2)}{4}$$

$$= -\pi - 1 + 2 = -\pi + 1$$

$$\sqrt{Var(x)} = \sqrt{-\pi+1}$$

d) Funcția de repartiție a v.a X.

I $F(x) = \int_{-\infty}^{x} f(x)\,dx = \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{=0} + \int_{0}^{x} f(x)\,dx + \underbrace{\cancel{\int_{\pi}^{\infty} f(x)\,dx}}_{=0}$

$= \int_{0}^{x} \frac{1}{2} \sin x\,dx = \frac{1}{2}(-\cos x)\Big|_{0}^{x} = \frac{1}{2}(-\cos x + \underbrace{\cos 0}_{=1})$

$= -\frac{1}{2}\cos x + \frac{1}{2} = \frac{-\cos x + 1}{2}$

II $F(x) = \int_{0}^{x} \cos x\,dx = \sin x - \sin 0 = \sin x.$

e) Mediana și modul v.a X

~~I Mo este valoarea pt. care $f(x) \to max$~~

Def: Mediana unei variabile aleatoare $X = Me$, și verifică: $P(X \geq Me) \geq \frac{1}{2} \leq P(X \leq Me)$.

<u>Obs:</u> F este funcție de repartiție și este continuă atunc
Me se determină din ecuația $F(Me) = \frac{1}{2}$

I $F(Me) = \frac{-\cos Me + 1}{2} = \frac{1}{2} \Rightarrow -\cos Me + 1 = 1 \Rightarrow$

$\left.\begin{array}{l} \Rightarrow -\cos Me = 0 \\ Me \in [0, \pi] \end{array}\right\} \Rightarrow Me = \frac{\pi}{2}$

II $F(Me) = \sin Me = \frac{1}{2} \Rightarrow Me = \frac{\pi}{6}$

Def: Se numește modul a variabilei aleatoare X orice punct de maxim local al distribuției lui X (în cazul discret) respectiv al densității de probabilitate (în caz cont)

I $f(x) = \begin{cases} \frac{1}{2}\sin x, & x \in [0, \pi] \\ 0, & \text{altfel} \end{cases}$

$f'(x) = \begin{cases} \frac{1}{2}\cos x, & x \in [0, \pi] \\ 0, & \text{altfel} \end{cases}$

$\frac{1}{2}\cos x = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$

⇒ pct. de max

	0	$\frac{\pi}{2}$	π
f'	$+ + + + + 0$		$- - - - -$
f	$0 0 0 \nearrow$	$\frac{1}{2}$	$\searrow 0 0 0$

$f\left(\frac{\pi}{2}\right)$ val. de max

$\Rightarrow \frac{\pi}{2}$ punct de maxim al densității de prob $f \Rightarrow$

$\Rightarrow M_0(X) = \frac{\pi}{2}$ (modulul v.a. X)

II $f(x) = \begin{cases} \cos x, & x \in [0, \frac{\pi}{2}] \\ 0, & \text{altfel} \end{cases}$

$f'(x) = \begin{cases} -\sin x, & x \in [0, \frac{\pi}{2}) \\ 0, & \text{altfel} \end{cases}$

	0	$\frac{\pi}{2}$
f'	$0 - - - - -$	
f	$0 0 0 \uparrow \searrow$	$0 0$

$-\sin x = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0$

$f(0) = \cos 0 = 1$

$\Rightarrow 0$ pct. de maxim al dens. de prob $f \Rightarrow M_0(X) = 0$.

Ex2:

Fie X o v.a continuă cu densitatea de probab.:

$$f(x) = \begin{cases} k\cdot(e^{-x}+e^{x}), & x\in[0,1], k\in\mathbb{R}. \\ 0 & , \text{in rest} \end{cases}$$

Determinați:

a) Valoarea param. k

b) $F(X)$ și $\mathbb{P}(X<\frac{1}{2} \mid X>\frac{1}{4})$

c) Media și dispersia v.a X.

Sol: a) $f(x)$ - densitatea de prob. $\Rightarrow$

$\Rightarrow$ 1) $f(x)\geq 0$, ~~$x\in[0,1]$~~

2) $\int_{-\infty}^{\infty} f(x) = 1.$

1) $k\cdot(\underbrace{e^{-x}}_{>0}+\underbrace{e^{x}}_{>0})\geq 0$, $x\in[0,1] \Rightarrow k\geq 0$

2) $\int_{-\infty}^{\infty} k(e^{-x}+e^{x}) = 1 \Leftrightarrow \int_0^1 k(e^{-x}+e^{x})dx = 1$

$\Leftrightarrow k\int_0^1 e^{-x}+e^{x}dx = 1 \Leftrightarrow k(-e^{-x}\Big|_0^1 + e^{x}\Big|_0^1) = 1$

$\Leftrightarrow k(-e^{-1}+\cancel{e^{-0}}^{=1} + e^{1}-\cancel{e^{0}}^{=1}) = 1 \Leftrightarrow k(-e^{-1}+e^{1}) = 1$

$\Leftrightarrow k(-\frac{1}{e}+e) = 1 \Rightarrow k\cdot\frac{-1+e^2}{e} = 1 \Rightarrow$ ~~$-1+e^2$~~ $\Rightarrow$

$k = \frac{e}{-1+e^2} \geq 0$

b) $F(X)$, $\mathbb{P}(X<\frac{1}{2} \mid X>\frac{1}{4})$

$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x \frac{e}{-1+e^2}(e^{-t}+e^{t})\,dt$

$= \int_0 \frac{e}{-1+e^2}(-e^{-t}\Big|_0^x + e^{t}\Big|_0^x) = \frac{e}{-1+e^2}(-e^{-x}-\cancel{1}+e^{x}+\cancel{1})$

$$\mathbb{P}(X<\tfrac{1}{2} \mid X>\tfrac{1}{4}) = \mathbb{P}(\tfrac{1}{4}<X<\tfrac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x)\,dx$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{e}{-1+e^2}(e^{-x}+e^{x})\,dx = \frac{e}{-1+e^2}\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-x}+e^{x}\,dx$$

$$= \frac{e}{-1+e^2}\left(-e^{-x}\Big|_{\frac{1}{4}}^{\frac{1}{2}} + e^{x}\Big|_{\frac{1}{4}}^{\frac{1}{2}}\right) = \frac{e}{-1+e^2}\left(-e^{-\frac{1}{2}}+e^{-\frac{1}{4}}+e^{\frac{1}{2}}-e^{\frac{1}{4}}\right)$$

$$= \frac{e}{-1+e^2}\left(-\frac{1}{e^{\frac{1}{2}}}+\frac{1}{e^{\frac{1}{4}}}+e^{\frac{1}{2}}-e^{\frac{1}{4}}\right)$$

$= \dots\dots$

c) Media și dispersia v.c. X.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\cdot\frac{e}{-1+e^2}(e^{-x}+e^{x})\,dx$$

$$= \frac{e}{-1+e^2}\int_0^1 x\cdot e^{-x}+xe^{x}\,dx = \frac{e}{-1+e^2}\left(\int_0^1 x\cdot e^{-x}\,dx+\int_0^1 xe^{x}\,dx\right)$$

$f(x)=x \Rightarrow f'(x)=1$ $\qquad h'(x)=e^{-x} \Rightarrow h(x)=-e^{-x}$

$g'(x)=e^{x} \Rightarrow g(x)=e^{x}$

$$= \frac{e}{e^2-1}\left[\left(xe^{x}\Big|_0^1-\int_0^1 e^{x}\,dx\right)+\left(-xe^{-x}-\int_0^1 -e^{-x}\,dx\right)\right]$$

$$= \frac{e}{e^2-1}\left(1e^{1}-0-(e^{1}-e^{0})+(-1e^{-1}+0)+\left(-e^{-x}\Big|_0^1\right)\right)$$

$$= \frac{e}{e^2-1}\left(\cancel{e}-\cancel{e}+1-\frac{1}{e}-\frac{1}{e}+1\right) = \frac{e}{e^2-1}\left(-\frac{2}{e}+2\right) = \frac{-2\cancel{e}}{\cancel{e}(e^2-1)}+\frac{2e(e^2-1)}{e^2-1}$$

$$= \frac{-2+2e(e^2-1)}{e^2-1} = \frac{\cancel{e}}{e^2-1}\left(\frac{-2+2e}{\cancel{e}}\right) = \frac{2(\cancel{e-1})}{(\cancel{e-1})(e+1)} = \frac{2}{e+1}$$

$$\operatorname{Var}(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_0^1 x^2 f(x)\,dx = \frac{e}{-1+e^2}\int_0^1 x^2(e^{-x} + e^{x})\,dx$$

$$= \frac{e}{-1+e^2}\left(\int_0^1 x^2 e^{-x}\,dx + \int_0^1 e^{x}\cdot x^2\,dx\right)$$

$$= \frac{e}{-1+e^2}\left[\left(-x^2 e^{-x}\Big|_0^1 + \int_0^1 2x e^{-x}\,dx\right) + \left(x^2 e^{x}\Big|_0^1 - \int_0^1 2x e^{x}\,dx\right.\right.$$

$$= \frac{e}{-1+e^2}\left[-1\cdot\frac{1}{e} + 2\underbrace{\int_0^1 x e^{-x}\,dx}_{1-\frac{2}{e}} + e - 2\underbrace{\int_0^1 x e^{x}\,dx}_{1}\right\}$$

$$= \frac{e}{-1+e^2}\left(-\frac{1}{e} + \cancel{2} - \frac{4}{e} + e - \cancel{2}\right) = \frac{e}{e^2-1}\left(-\frac{5}{e} + e\right)$$

$$= \frac{\cancel{e}}{e^2-1}\cdot\frac{-5+e^2}{e^{\cancel{2}}} = \frac{-5+e^2}{\cancel{e}(e^2-1)}$$

$$\operatorname{Var}(X) = \frac{e^2-5}{e^2-1} - \frac{4}{(e+1)^2} = \frac{\overset{(e+1)^2}{e^2-5}}{e^2-1} - \frac{\overset{e^2-1}{4}}{e^2-2e+1}$$

$$= \frac{(e^2-5)(e^2+2e+1) - 4e^2 + 4}{(e^2-1)(e^2-2e+1)}$$

$$= \frac{e^4+2e^3+e^2-5e^2-10e-5-4e^2+4}{e^4-2e^3+\cancel{e^2}-\cancel{e^2}+2e-1}$$

$$= \frac{e^4+2e^3-8e^2-10e-1}{e^4-2e^3+2e-1}$$

Ex3: Fie $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} k \cdot x^{a-1}(1-x)^{b-1}, & x \in (0,1) \\ 0, & \text{în rest} \end{cases}$

Determinați:

a) Valoarea param. real k a.î. f să fie densitate de prob. a X

b) Media, dispersia, modul și momentul inițial de ordin r a v.a. cont X, $r \in \mathbb{N}^*$.

c) Funcția de repartiție a v.a X și $P(X<\frac{1}{2})$; $P(X>\frac{1}{3})$; $P(X<\frac{1}{2} \mid X>\frac{1}{4})$ pt. $a=2$ și $b=3$.

Sol: a) $f(x)$ dens. de prob $\to$ 1) $f(x) \geq 0$

2) $\int_{-\infty}^{\infty} f(x) = 1$

1) $f(x) \geq 0 \Leftrightarrow k \cdot \underset{\substack{\geq 0 \\ x \in (0,1)}}{x^{a-1}} \underset{\geq 0}{(1-x)^{b-1}} \Rightarrow k \geq 0$

2) $\int_{-\infty}^{\infty} f(x) = 1 \Leftrightarrow \int_0^1 k \cdot x^{a-1}(1-x)^{b-1} dx = k \underbrace{\int_0^1 x^{a-1}(1-x)^{b-1} dx}_{\text{Integrală Beta}}$

$a, b > 0$

$\Rightarrow k \int_0^{\infty} \cancel{\frac{x^{a-1}}{(1+x)^{a+b}}} dx = 1 \Leftrightarrow k \cdot \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)} = 1 \Rightarrow$

$\Leftrightarrow k \cdot \frac{(a-1)!(b-1)!}{(a+b-1)!} = 1 \Rightarrow k = \frac{(a+b-1)!}{(a-1)!(b-1)!}$

b) $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 \frac{(a+b-1)!}{(a-1)!(b-1)!} \cdot x \cdot x^{a-1} (1-x)^{b-1} dx$

$= \underbrace{\frac{(a+b-1)!}{(a-1)!(b-1)!}}_{M} \int_0^1 x \cdot x^{\overset{=}{a-1}} (1-x)^{b-1} dx = M \int_0^1 x^{\overset{=c-1>0}{a}} (1-x)^{b-1} dx$

$= \frac{(a+b-1)!}{(a-1)!(b-1)!} \cdot \frac{\Gamma(c) \cdot \Gamma(b)}{\Gamma(c+b)} = \frac{(a+b-1)!}{(a-1)!(b-1)!} \cdot \frac{(c-1)!(b-1)!}{(c+b-1)!}$

$$= \cancel{\frac{(a+b-1)!}{(a-1)!(b-1)!}} \cdot \frac{\overset{(a-1)\,a}{\cancel{a!}}\cancel{(b-1)!}}{\underset{(a+b)}{(a+b)!}} = \frac{a}{a+b}.$$

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 \cdot \cancel{x}^{a-1}(1-x)^{b-1}\,dx \qquad d = a+2$$

$$= \int_0^1 k\,x^{\overset{=d-1}{a+1}}(1-x)^{b-1}\,dx = k\,\frac{\Gamma(d)\cdot\Gamma(b)}{\Gamma(d+b)} =$$

$$= \frac{(a+b-1)!}{(a-1)!\cancel{(b-1)!}} \cdot \cancel{\frac{(d-1)!(b-1)!}{\Gamma(d+b-1)!}} = \cancel{\frac{(a+b-1)!}{(a-1)!(b-1)!}} \cdot \frac{\overset{a}{a!}}{(a+b)!} = \frac{a}{a+b}.$$

~~Var(x)~~

$$= \cancel{\frac{(a+b-1)!}{(a-1)!(b-1)!}} \cdot \frac{\overset{a(a+1)}{\cancel{(a+1)!}}\cancel{(b-1)!}}{\underset{(a+b+1)(a+b)}{(a+b+1)!}}$$

$$= \frac{a(a+1)}{(a+b)(a+b+1)}$$

$$Var(x) = \frac{\overset{(a+b)}{a(a+1)}}{(a+b)(a+b+1)} - \frac{\overset{(a+b+1)}{a^2}}{(a+b)^2}$$

$$= \frac{a(a+1)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)} = \frac{a(a^2+ab+a+b) - a^3 - a^2b - a^2}{(a+b)^2(a+b+1)}$$

$$= \frac{\cancel{a^3} + \cancel{a^2b} + \cancel{a^2} + ab - \cancel{a^3} - \cancel{a^2b} - \cancel{a^2}}{(a+b)^2(a+b+1)} = \frac{\cancel{a^2} + ab - \cancel{a}}{(a+b)^2(a+b+1)}$$

$$f(x) = \begin{cases} k \cdot x^{a-1}(1-x)^{b-1}, & x \in (0,1),\ a,b>0 \\ 0, \text{ in rest} \end{cases}$$

$$f'(x) = \begin{cases} k\left((a-1)x^{a-2}(1-x)^{b-1} + x^{a-1}(b-1)(1-x)^{b-2}\cdot(-1)\right) \\ 0, \text{ in rest} \end{cases}$$

$$k(a-1)x^{a-2}(1-x)^{b-1} - x^{a-1}(b-1)(1-x)^{b-2}$$

$\cancel{k(a-1)}$

$k \cdot x^{a-2}(1-x)^{b-2}\{(a-1)(1-x) - (\cancel{1-x})(b-1)\} = 0$

$\cancel{\Leftrightarrow k \cdot x^{a-2}(1-x)^{b-2}(1-x)\{(a-1)-(b-1)\}=0}$

$\underbrace{k \cdot x^{a-2}(1-x)^{b-2}}_{\neq 0}\{(a-1)(1-x) - x(b-1)\} = 0 \Rightarrow$

$\Rightarrow (a-1)(1-x) - x(b-1) = 0 \Leftrightarrow$

$\Leftrightarrow \cancel{a} - ax - \cancel{a} + x - xb + x = 0 \Rightarrow -ax - xb + 2x \cancel{=a} \;\; +a-1$

$\Rightarrow x(2-a-b) = 1-a$

I $a+b=2 \Rightarrow 1-a=0 \Rightarrow a=1 \Rightarrow b=1 \Rightarrow$

$f(x) = \begin{cases} k \cdot x^0 \cdot x^0 = k, & x \in [0,1] \\ 0, & \text{in rest} \end{cases} \Rightarrow \begin{cases} 1, & x \in [0,1] \\ 0, & \text{in rest} \end{cases}$

Deci $M_0(X) = (0,1)$

II $a+b \neq 2 \Rightarrow x = \frac{1-a}{2-a-b}$

Momentul inițial de ordin r

$$m_r = E(X^r) = \int_{-\infty}^{\infty} k x^r x^{a-1}(1-x)^{b-1}\,dx, \quad r \in \mathbb{N}^*$$

$$= k\int_0^1 x^{a+r-1}(1-x)^{b-1}\,dx, \quad c = a+r > 0$$

$$= k\,\frac{(c-1)!\,(b-1)!}{(b+c-1)!} = \frac{(a+b-1)!}{(a-1)!\,\cancel{(b-1)!}} \cdot \frac{(a+r-1)!\,\cancel{(b-1)!}}{(a+r+b-1)!}$$

c) $F(x)$, $P(X<\frac{1}{2})$, $P(X>\frac{1}{3})$, $P(X\le\frac{1}{2}\mid X>\frac{1}{4})$ $\quad a=2,\ b=3$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x k t^1 (1-t)^2\,dt$$

$$= 12\cdot t\cdot\frac{-(1-t)^3}{3}\Big|_0^x + \frac{12}{3}\int_0^x (1-t)^3\,dt$$

$$= -\frac{12}{3}x(1-x)^3 - \frac{12}{3}\,\frac{(1-t)^4}{4}\Big|_0^x$$

$$= -4\left(x(1-x)^3 + \frac{(1-x)^4}{4} - \frac{1}{4}\right)$$

$$= -4\cdot\frac{4x(1-x)^3 + (1-x)^3(1-x) - 1}{4}$$

$$= -\left(4x(1-x)^3 + (1-x)^3 - x(1-x)^3 - 1\right)$$

$$= -(3x+1)(1-x)^3 + 1$$

$$P(X<\tfrac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} f(x)\,dx = \int_0^{\frac{1}{2}} f(x)\,dx = -\left(3\cdot\tfrac{1}{2}+1\right)\left(1-\tfrac{1}{2}\right)^3 + 1$$

$$= -\frac{5}{2}\cdot\frac{1}{8} + \frac{16}{1} = \frac{16-5}{16} = \frac{11}{6}$$

$$P\left(X > \frac{1}{3}\right) = \int_{\frac{1}{3}}^{\infty} f(x)\,dx = \int_{\frac{1}{3}}^{1} f(x)\,dx = \int_{\frac{1}{3}}^{1} 12\,x\,(1-x)^2\,dx$$

$$f(x) = x \Rightarrow f'(x) = 1$$

$$g'(x) = (1-x)^2 \Rightarrow g(x) = -\frac{(1-x)^3}{3}$$

$$12\left(-x\,\frac{(1-x)^3}{3}\Big|_{\frac{1}{3}}^{1}\right) - \frac{12}{3}\int_{\frac{1}{3}}^{1} -(1-x)^3\,dx$$

$$= \frac{12}{3}\left(\underbrace{-1(1-1)^3}_{=0} + \frac{1}{3}\left(\frac{\left(1-\frac{1}{3}\right)^3}{\cancel{3}}\right) - \frac{(1-x)^4}{4}\Big|_{\frac{1}{3}}^{1}\right)$$

$$= \frac{12}{3}\left(\frac{1}{3}\cdot\frac{8}{27} + \frac{\left(1-\frac{1}{3}\right)^4}{4}\right) = 4\left(\frac{8}{81} + \frac{16}{81}\cdot\frac{1}{4}\right)$$

$$\cancel{= 4\left(\frac{128}{1296} + \frac{1}{1296}\right) = 4\cdot\frac{129}{1296} = \frac{129}{324}}$$

$$\cancel{= 4\left(\frac{8}{81} + \frac{4}{81}\right) = 4\cdot\frac{36}{324} = \frac{36}{81} = \frac{4}{9}}$$

$$= \cancel{4}\ \frac{32}{81} + \frac{16}{81} = \frac{48}{81}$$

$$P\left(X \le \frac{1}{2} \,\middle|\, X > \frac{1}{4}\right) = P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x)\,dx =$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} 12x(1-x)^2\,dx = 12\int_{\frac{1}{4}}^{\frac{1}{2}} x(1-x)^2$$

$$= 12\left[\left(-x\,\frac{(1-x)^3}{3}\Big|_{\frac{1}{4}}^{\frac{1}{2}}\right] - \frac{12}{3}\int_{\frac{1}{4}}^{\frac{1}{2}} -(1-x)^3\,dx$$

$$= \frac{12}{3}\left[\left(-\frac{1}{2}\left(1-\frac{1}{2}\right)^3 + \frac{1}{4}\left(1-\frac{1}{4}\right)^3\right) - \left(\frac{\left(1-\frac{1}{2}\right)^4}{4} - \frac{\left(1-\frac{1}{4}\right)^4}{4}\right)\right.$$

$=$

$$= \frac{12}{3}\left(-\frac{1}{2}\cdot\frac{1}{8}+\frac{1}{4}\cdot\frac{27}{64}\right)-\frac{12}{3}\left(\frac{1}{4}\cdot\frac{1}{16}-\frac{81}{256\cdot 4}\right)$$

$$= -4\cdot\frac{1}{16}+\frac{27}{64}-\frac{1}{16}+\frac{81}{256}$$

$$= {}^{64)}-\frac{1}{4}+{}^{4)}\frac{27}{64}-{}^{16)}\frac{1}{16}+\frac{81}{256} = \frac{-64+108-16+81}{256} = \frac{109}{256}$$

Ex4: Fie $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} k\cdot x\cdot e^{-\frac{x^2}{2a^2}}, & x\geq 0,\ k\in\mathbb{R},\ a>0 \\ 0, & x<0\end{cases}$

Determinați:

a) Valoare param. k a.î. f să fie dens. de prob.

f dens. de prob $\Leftrightarrow$ 1) $f(x)\geq 0$

2) $\int_{-\infty}^{\infty} f(x) = 1$

$$\int_{-\infty}^{\infty} f(x) = 1 \Leftrightarrow \int_0^{\infty} k\cdot x\cdot e^{\frac{-x^2}{2a^2}}dx = k\cdot\int_0^{\infty} x\cdot e^{\frac{-x^2}{2a^2}}dx$$

$$\cancel{\left(e^{-\frac{x^2}{2a^2}}\right)' = -\frac{x^2}{2a^2}\cdot e^{-\frac{x^2-1}{2a^2}}}$$

$$\left(\frac{-x^2}{2a^2}\right)' = -\frac{1}{\cancel{2}a^2}\cdot\cancel{2}x = \frac{-x}{a^2}$$

$\Rightarrow$

$$\Rightarrow k\cdot\int_0^{\infty} -\frac{x}{a^2}\cdot e^{\frac{-x^2}{2a^2}}\cdot(-a^2)dx \Leftrightarrow -a^2k\int_0^{\infty}\left(\frac{-x^2}{2a^2}\right)'\cdot e^{\frac{-x^2}{2a^2}}dx$$

$$= -a^2k\cdot e^{\frac{-x^2}{2a^2}}\Big|_0^{\infty} = -a^2k\cdot\left(\underbrace{e^{-\infty}}_{=0}-\underbrace{e^{0}}_{=1}\right) = +a^2k = 1 \Rightarrow k = \frac{1}{a^2}$$

$$\underbrace{k\cdot x}_{\geq 0}\cdot\underbrace{e^{\frac{-x^2}{2a^2}}}_{\geq 0}\geq 0 \Rightarrow k\geq 0$$

$k = \frac{1}{a^2} > 0 \Rightarrow k\geq 0$ ✓

b) Funcția de repartiție $F(x)$

$$F(x)=\int_{-\infty}^{x} f(t)\,dt=\int_{0}^{x} k\cdot t\cdot e^{\frac{-t^2}{2a^2}}\,dt$$

$$=\frac{-1}{a^2}\cdot a^2\cdot e^{\frac{-x^2}{2a^2}}\Big|_{0}^{x}=-\Big(e^{-\frac{x^2}{2a^2}}-\underbrace{e^{-0}}_{=1}\Big)=-e^{\frac{-x^2}{2a^2}}+1$$

c) $E(X)$, $Var(X)$ și momentul inițial de ordin r, $r\in\mathbb{N}^*$

$$E(X)=\int_{-\infty}^{\infty} x f(x)\,dx=\int_0 kx\cdot x\cdot e^{-\frac{x^2}{2a^2}}\,dx=\int_0 k\cdot x^2\cdot e^{\frac{-x^2}{2a^2}}\,dx$$

$$=k\int_0^{\infty} x^2\cdot e^{\frac{-x^2}{2a^2}}\,dx=\int_0^{\infty} x\cdot\frac{x}{a^2}\cdot e^{\frac{-x^2}{2a^2}}\,dx=\int_0^{\infty} x\cdot\left(e^{\frac{-x^2}{2a^2}}\right)'dx$$

$f(x)=x \Rightarrow f'(x)=1$

$g'(x)=\cancel{x}\,e^{\frac{-x^2}{2a^2}} \Rightarrow g(x)=-\cancel{x}\,e^{\frac{-x^2}{2a^2}}$

$$=\cancel{\frac{1}{a^2}\Big(\underbrace{x\cdot\big(-e^{\frac{-x^2}{2a^2}}\big)\Big|_0^{\infty}}_{0}+\int_0^{\infty} e^{\frac{-x^2}{2a^2}}\,dx\Big)}=$$

$x^2=2a^2t \Rightarrow x=\sqrt{2}at$

$$=\quad \cancel{-t=\frac{x}{\sqrt{2}a}}\qquad \frac{x^2}{2a^2}=t \Rightarrow \cancel{\frac{x\,dx}{a^2}=dt} \Rightarrow dx=\sqrt{2}a\,dt$$

$$=\cancel{\frac{1}{a^2}}:\underbrace{-x\cdot e^{\frac{-x^2}{2a^2}}\Big|_0^{\infty}}_{0}+\int_0^{\infty} e^{-\frac{x^2}{2a^2}}\,dx$$

$dx=\sqrt{2}a\,dt$

$$=\int_0^{\infty}\sqrt{2}a\,e^{-t^2}\,dt=\sqrt{2}a\cdot\frac{\sqrt{\pi}}{2}=a\sqrt{\frac{\pi}{2}}$$

$$E(x^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx=\int_0^{\infty}\frac{x}{a^2}\cdot x^2\cdot e^{\frac{-x^2}{2a^2}}dx$$

$$=2a^2\int_0^{\infty}\frac{x}{a^2}\cdot\frac{x^2}{2a^2}\cdot e^{\frac{-x^2}{2a^2}}dx$$

$$\frac{x^2}{2a^2}=t\Rightarrow\frac{x}{a^2}dx=dt$$

$$=2a^2\int_0^{\infty}t\cdot e^{-t}dt=2a^2[\underbrace{(-te^{-t})\Big|_0^{\infty}}_{0}+\int_0^{\infty}e^{-t}dt]$$

$$=2a^2(-e^{-t})\Big|_0^{\infty}=2a^2$$

- $Var(X)=2a^2-\frac{\pi a^2}{2}=\frac{a^2(4-\pi)}{2}$

- $m_r=E(x^r)=\int_{-\infty}^{\infty}x^r f(x)\,dx=\int_0\frac{x}{a^2}\cdot x^r\cdot e^{-\frac{x^2}{2a^2}}dx.$

$$\frac{x^2}{2a^2}=t\Rightarrow dt=\frac{x}{a^2}dx$$

$$=2a^2\int_0^{\infty}\frac{x}{a^2}\cdot\frac{x^r}{2a^2}\cdot e^{-\frac{x^2}{2a^2}}dx$$

$$=(2a^2)^{\frac{r}{2}}\int_0^{\infty}\frac{x}{a^2}\left(\frac{x^2}{2a^2}\right)^{\frac{r}{2}}\cdot e^{\frac{-x^2}{2a^2}}dx$$

$$=(2a^2)^{\frac{r}{2}}\int_0^{\infty}t^{\frac{r}{2}}\cdot e^{-t}dx=(2a^2)^{\frac{r}{2}}\cancel{\Gamma}(\tfrac{r}{2})\,\Gamma(\tfrac{r}{2})$$

$$\frac{r}{2}=c-1\Rightarrow c=\frac{r}{2}+1$$

d) $\mathbb{P}(X<2a)$, $\mathbb{P}(X>a)$, $\mathbb{P}(X\le 4a\mid X>2a)$

$$\mathbb{P}(X<2a)=F(2a)=\int_{-\infty}^{2a}f(x)\,dx=\int_0^{2a}\frac{1}{a^2}\cdot x\cdot e^{\frac{-x^2}{2a^2}}dx$$

$$=-e^{\frac{-x^2}{2a^2}}\Big|_0^{2a}=-e^{\frac{-4a^2}{2a^2}}+e^0=-e^{-2}+1=1-\frac{1}{e^2}$$

$$\mathbb{P}(X>a)=\int_a^{\infty}f(x)\,dx=\int_a^{\infty}\frac{x}{a^2}\cdot e^{\frac{-x^2}{2a^2}}dx=-e^{-\frac{x^2}{2a^2}}\Big|_a^{\infty}$$

$$=-e^{-\infty}+e^{\frac{-a^2}{2a^2}}=e^{-\frac{1}{2}}=\frac{1}{\sqrt{e}}$$

$$P(X\le 4a \mid X>2a) = P(2a<X<4a) = \int_{2a}^{4a} \frac{x}{2a}\, e^{-\frac{x^2}{2a^2}}\,dx$$

$$= -e^{-\frac{x^2}{2a^2}}\Big|_{2a}^{4a} = -e^{-\frac{\overset{8}{\cancel{16a^2}}}{\cancel{2a^2}}} + e^{-\frac{\overset{2}{\cancel{4a^2}}}{\cancel{2a^2}}} = -e^{-8}+e^{-2} = \frac{1}{e^2}-\frac{1}{e^8}$$

e) Momentul centrat de ordin 3, mediana și modulul.

$$\mu_3 = \int_{-\infty}^{\infty}\left(x-a\sqrt{\tfrac{\pi}{2}}\right)^3 \cdot f(x)\,dx = \int_0^{\infty}\left(x-a\sqrt{\tfrac{\pi}{2}}\right)^3\cdot\frac{x}{a^2}\cdot e^{-\frac{x^2}{2a^2}}\,dx$$

$$= \int_0^{\infty}\left(x^3 - 3x^2 a\sqrt{\tfrac{\pi}{2}} + 3x a^2\tfrac{\pi}{2} - a^3\left(\sqrt{\tfrac{\pi}{2}}\right)^3\right)\cdot\frac{x}{a^2}\cdot e^{-\frac{x^2}{2a^2}}\,dx$$

$$= \underbrace{\int_0^{\infty} x^3\cdot\frac{x}{a^2}e^{-\frac{x^2}{2a^2}}\,dx}_{E(X^3)} - \overbrace{\int_0^{\infty} 3x^2 a\sqrt{\tfrac{\pi}{2}}\cdot\frac{x}{a^2}\cdot e^{-\frac{x^2}{2a^2}}\,dx}^{3\cdot a\sqrt{\frac{\pi}{2}}\cdot E(X^2)} + \overset{3a^2\frac{\pi}{2}\cdot E(X)}{\overset{=}{\int 3xa^2\tfrac{\pi}{2}\cdot\frac{x}{a^2}\cdot e^{-\frac{x^2}{2a^2}}\,dx}}$$

$$- \underbrace{\int_0^{\infty} a^3\left(\sqrt{\tfrac{\pi}{2}}\right)^3\cdot\frac{x}{a^2}\cdot e^{-\frac{x^2}{2a^2}}\,dx}_{a^3\left(\sqrt{\frac{\pi}{2}}\right)^3\cdot \cancel{F(x)}\ \underbrace{\int_0^{\infty} f(x)\,dx}_{1}}$$

$$= E(x^3) - 3a\sqrt{\tfrac{\pi}{2}}E(X^2) + 3a^2\tfrac{\pi}{2}E(X) - a^3\left(\sqrt{\tfrac{\pi}{2}}\right)^3\cdot F(x)$$

$$E(x^3) = (2a^2)^{\frac{3}{2}}\,\Gamma\left(\tfrac{3}{2}\right)\left(\tfrac{3}{2}\right) = 2^{\frac{3}{2}}\cdot a^3\left(\tfrac{3}{2}\right)\Gamma(3,2)$$

$$= \sqrt{2^3}\cdot a^3\cdot\tfrac{3}{2}\cdot\left(\tfrac{3}{2}-1\right)\Gamma\left(\tfrac{3}{2}-1\right)$$

$$= \sqrt{8}\cdot a^3\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\underbrace{\Gamma\left(\tfrac{1}{2}\right)}_{=\sqrt{\pi}} = 2\sqrt{2}\,a^3\cdot\tfrac{3}{4}\sqrt{\pi} = \frac{\overset{3}{\cancel{6}}\sqrt{2}\,a^3\sqrt{\pi}}{\underset{2}{\cancel{4}}}$$

$$= \frac{3\sqrt{2}a^3\sqrt{\pi}}{3} - 3a\sqrt{\tfrac{\pi}{2}}\cdot 2a^2 + 3a^2\tfrac{\pi}{2}\,a\sqrt{\tfrac{\pi}{2}} - a^3\left(\sqrt{\tfrac{\pi}{2}}\right)^3$$

$$= 3\frac{a^3\sqrt{2\pi}}{2} - 6a^3\sqrt{\tfrac{\pi}{2}} + 3a^3\tfrac{\pi}{2}\sqrt{\tfrac{\pi}{2}} - a^3\left(\sqrt{\tfrac{\pi}{2}}\right)^3$$

$$= a^3\left(\frac{3\sqrt{2\pi}}{2} - 6\sqrt{\tfrac{\pi}{2}} + 3\tfrac{\pi}{2}\sqrt{\tfrac{\pi}{2}} - \tfrac{\pi}{2}\sqrt{\tfrac{\pi}{2}}\right)$$

$$= a^3\left(\frac{3\sqrt{\cancel{2}\pi}}{\cancel{2}\sqrt{2}} - 6\sqrt{\tfrac{\pi}{2}} + 2\tfrac{\pi}{2}\sqrt{\tfrac{\pi}{2}}\right) = a^3\left(-3\sqrt{\tfrac{\pi}{2}} + \cancel{2}\tfrac{\pi}{\cancel{2}}\sqrt{\tfrac{\pi}{2}}\right) = a^3\sqrt{\tfrac{\pi}{2}}\left(-3+\pi\right)$$

Mediana.

$$F(Me(X)) = \frac{1}{2} \Leftrightarrow -e^{\frac{-x^2}{2a^2}}\Big|_0^{Me(x)}$$

$$= -e^{-\frac{Me^2(x)}{2a^2}} + 1 = \frac{1}{2} \Rightarrow e^{-\frac{Me^2(x)}{2a^2}} = 1-\frac{1}{2} = \frac{1}{2} \Rightarrow$$

$$\Rightarrow \frac{-Me^2(x)}{2a^2} = \ln\frac{1}{2} \Rightarrow +Me^2(x) = -\cancel{2a}^2 \ln\frac{1}{2}^{-2a^2} \Rightarrow$$

$$\Rightarrow Me(x) = \sqrt{\ln\frac{1}{2}^{-2a^2}} \to \text{med.}$$

Modulul:

$$f(x) = \begin{cases} k\cdot x\cdot e^{\frac{-x^2}{2a^2}}, & x \geq 0 \\ 0, & x<0 \end{cases} \Rightarrow \frac{1}{a^2}x\cdot e^{\frac{-x^2}{2a^2}}$$

$$f'(x) = \begin{cases} \frac{1}{a^2}\cdot e^{\frac{-x^2}{2a^2}} + \frac{1}{a^2}\cdot x^2 \ln \frac{x^2}{2a^2}, & x\geq 0 \\ 0, & x<0 \end{cases}$$

$$f'(x) = 0 \Rightarrow \frac{1}{a^2}\left(e^{\frac{-x^2}{2a^2}} + x^2 e^{\frac{-x^2}{2a^2}}\right) = 0$$

$$\Leftrightarrow \frac{1}{a^2} e^{\frac{-x^2}{2a^2}}(1+x^2) = 0 \Leftrightarrow x = 1$$

x		0	1	
f'	0 0	0 0	0	
f	0 0	0 0 $\nearrow$		$\searrow$

$Mo(X) = 1$
pct. de maxim.

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d) O margine inferioară a $P(0 < X < a\sqrt{2\pi})$

$$P(0 < X < a\sqrt{2\pi}) = \int_0^{a\sqrt{2\pi}} \frac{x}{a^2} \cdot e^{\frac{-x^2}{2a^2}} dx$$

$$= -e^{\frac{-x^2}{2a^2}} \Big|_0^{a\sqrt{2\pi}} = -e^{-\frac{a^2 2\pi}{2a^2}} + e^0 = -e^{-\pi} + 1 = 1 - \frac{1}{e^{\pi}}$$

Ex5: Fie v.a X_m dată prin densitatea $f_m : \mathbb{R} \to \mathbb{R}$,

$f_m(x) = p \cdot x^{\frac{m}{2}-1} \cdot e^{-\frac{x}{3}}$, $x \geq 0$, $p \in \mathbb{R}$, $m \in \mathbb{R}^*$

a) Determinați param. real p.

b) Dacă Y este o v.a. cont. cu dens. de prob. f_m, $m \in \mathbb{N}^*$

Det. dens. de prob. a v.a $X+Y$, știind că X și Y indep.

Indicație: Folosiți formula convoluției!

Sol.: a) f_m dens de prob: 1) $f_m(x) \geq 0$

2) $\int_{-\infty}^{\infty} f_m(x) = 1$

1) $p \cdot \underbrace{x^{\frac{m}{2}-1}}_{\geq 0} \underbrace{e^{-\frac{x}{3}}}_{\geq 0} \Rightarrow p \geq 0$, $\forall x \geq 0 \Rightarrow \frac{m}{2} - 1 > 0 \Rightarrow \frac{m}{2} > 1$

2) $\int_{-\infty}^{\infty} f_m(x) = 1 \Leftrightarrow \int_0^{\infty} p\, x^{\frac{m}{2}-1} e^{-\frac{x}{3}} dx = p \int_0^{\infty} x^{\frac{m}{2}-1} e^{-\frac{x}{3}} dx$

$\frac{x}{3} = t \Rightarrow x = 3t \Rightarrow dx = 3dt$

$$= p \int_0^{\infty} (3t)^{\frac{m}{2}-1} e^{-t} 3dt = p\, 3^{\frac{m}{2}-1} 3 \int_0^{\infty} t^{\overbrace{\frac{m}{2}-1}^{a>1}} e^{-t} dt$$

$$= p\, 3^{a-1} \cdot 3 \int_0^{\infty} t^{a-1} e^{-t} dt = p\, 3^{\frac{m}{2}} \cdot \left(\frac{m}{2}-1\right) \Gamma\left(\frac{m}{2}-1\right) = 1 \Rightarrow$$

$$\Rightarrow p = \frac{1}{3^{\frac{m}{2}}\left(\frac{m}{2}-1\right)\Gamma\left(\frac{m}{2}-1\right)}$$

b)

$$f_{X+Y} = \frac{1}{3^{\frac{n}{2}}\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)} \cdot \frac{1}{3^{\frac{m}{2}}\left(\frac{m}{2}-1\right)\Gamma\left(\frac{m}{2}-1\right)} \cdot x^{\frac{n+m}{2}-2} \cdot e^{-\frac{2x}{3}}$$

$$= \frac{1}{3^{\frac{n+m}{2}}\left(\frac{n}{2}-1\right)\left(\frac{m}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(\frac{m}{2}-1\right)} \cdot x^{\frac{n+m}{2}-2} \cdot e^{-\frac{x}{3}}$$

$=$