Solutions Chapter 5 - Prior

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October 2019

5.1 Dodgy coins

Suppose there are three coins in a bag. The first coin is biased towards heads, with a 75% probability of a heads occurring if the coin is flipped. The second is fair, so a 50% chance of heads occurring. The third coin is biased towards tails, and has a 25% probability of coming up heads. Assume that it is impossible to identify which coin is which from looking at or touching them.

Problem 5.1.1 - 5.1.8 were rather straightforward so I did them quickly on paper instead of saving my solutions in latex. But now here is an interesting question!

Problem 5.1.9

. For the case when we flip the coin once and obtain X = H, using the uniform prior on C, determine the posterior predictive distribution for a new coin flip with result $\widetilde{X}$, using the below expression,

$$Pr(\widetilde{X}|X=H) = \sum_{c=1}^{3} Pr(\widetilde{X}|C) \cdot Pr(C|X=H) \tag{1}$$

Answer: So, here we have an informative prior based on the posterior of previously collected data. First, the prior is defined as:

$$\begin{cases} Pr(C=1|X=H) = \frac{1}{2} \\ Pr(C=2|X=H) = \frac{1}{3} \\ Pr(C=3|X=H) = \frac{1}{6} \end{cases} \tag{2}$$

The likelihood becomes for heads:

$$\begin{cases} Pr(\widetilde{X}=H|C=1) = \frac{}{4} \\ \quad \text{Pr}(\widetilde{X}=H|C=2) = \frac{1}{2} Pr(\widetilde{X}=H|C=3) = \}4 \end{cases} \tag{3}$$

The likelihood becomes for tails:

$$\begin{cases} Pr(\widetilde{X}=T|C=1) = \frac{1}{4} \\ Pr(\widetilde{X}=T|C=2) = \frac{1}{2} \\ Pr(\widetilde{X}=T|C=3) = \frac{3}{4} \end{cases} \tag{4}$$

There are two possible outcomes for $Pr(\widetilde{X}|X=H)$, thus the posterior predictive distribution is over the possible outcomes heads or tails.

$$Pr(\widetilde{X}=H|X=H) = \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{6} = \frac{7}{12} \tag{5}$$

$$Pr(\widetilde{X}=T|X=H) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{6} = \frac{5}{12} \tag{6}$$

Problem 5.1.10. (Optional)

Justify the use of the expression in the previous question.

Answer: We do this by deriving (1)

$$Pr(\widetilde{X}|X = H) = \sum_{c=1}^{3} Pr(\widetilde{X}, C = c|X = H) \tag{7}$$

Now we use that:

$$P(A, B) = P(A|B) \cdot P(B) \tag{8}$$

or

$$P(A, B|C) = P(A|B, C) \cdot P(B|C) \tag{9}$$

This gives us:

$$Pr(\widetilde{X}|X = H) = \sum_{c=1}^{3} Pr(\widetilde{X}|X = H, C = c)Pr(C = c|X = H) \tag{10}$$

Now, we use the fact that given C, $\widetilde{X}$ is conditionally independent of $X = H$, since it is completely determined by C.

$$Pr(\widetilde{X}|X = H) = \sum_{c=1}^{3} Pr(\widetilde{X}|C = c)Pr(C = c|X = H) \tag{11}$$

the first part of this expression is the likelihood and the second is the posterior (our prior) derived from previous research.

5.2 Left-handedness

Suppose that we are interested in the prevalence of left-handedness in a particular population.

Problem 5.2.1.

We begin with a sample of one individual whose dexterity we record as X = 1 for left-handed, X = 0 otherwise. Explain why the following probability distribution makes sense here:

$$Pr(X|\theta) = \theta^X (1 - \theta)^{1-X} \tag{12}$$

where θ is the probability that a randomly chosen individual is left-handed.

Answer: Well is is quite intuitive, but if the first individual is left handed we get:

$$Pr(X|\theta) = \theta^1 (1 - \theta)^{1-1} = \theta \tag{13}$$

If the next is also left handed we get:

$$Pr(X|\theta) = \theta^2 \tag{14}$$

and so on.

Problem 5.2.2.

Suppose we hold θ constant. Demonstrate that under these circumstances the above distribution is a valid probability distribution. What sort of distribution is this?

Answer: So, if we have a constant $\theta = \theta_0$ we have two possible outcomes for the resulting distribution: $X = 0$ or $X = 1$

$$\begin{cases} Pr(X = 1|\theta = \theta_0) = \theta \\ Pr(X = 0|\theta = \theta_0) = 1 - \theta \end{cases} \tag{15}$$

Now a valid distribution has to fulfill two criteria. First, it has to sum to 1:

$$\sum_{i=1}^{2} Pr(X = x|\theta = \theta_0) = Pr(X = 1|\theta = \theta_0) + Pr(X = 0|\theta = \theta_0) = \theta + 1 - \theta = 1$$

Next, all the values of the probability distribution should be non-negative. Which they are since θ can be either 1 or 0. This is a Bernoulli probability distribution.

Problem 5.2.3.

Now suppose we randomly sample a person who happens to be left-handed. Using the above function calculate the probability of this occurring.

Answer: $Pr(X = 1|\theta = \theta_0) = \theta$.

Problem 5.2.4.

Show that when we vary the above distribution does not behave as a valid probability distribution. Also, what sort of distribution is this?

Answer: We show this through a counter-example. We know that θ is the probability that a randomly chosen individual is left-handed and thus it will have a value in the range [0,1]. Thus, our counter-example is to integrate this distribution.

$$\int_0^1 \theta d\theta = [\frac{\theta^2}{2}]_0^1 = \frac{1}{2} \tag{16}$$

This is a likelihood.

References