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SUPPLEMENTARY FILE TO "VARYING COEFFICIENTS CORRELATED VELOCITY MODELS IN COMPLEX LANDSCAPES WITH BOUNDARIES APPLIED TO NARWHAL RESPONSES TO NOISE EXPOSURE"

BY ALEXANDRE DELPORTE[1,a], SUSANNE DITLEVSEN[2,c] AND ADELINE SAMSON[1,b]

[1]*Laboratoire Jean Kuntzmann, Université Grenoble-Alpes,* [a]*alexandre.delporte@univ-grenoble-alpes.fr;* [b]*adeline.leclercq-samson@univ-grenoble-alpes.fr*

[2]*Department of Mathematical Sciences, University of Copenhagen ,* [c]*susanne@math.ku.dk*

APPENDIX A: MORE DETAILS ABOUT THE DATA

This section is a complement to Section 2 of the paper. We provide more information about the narwhal movement data used for the analysis of behavioral disturbance.

The time step between consecutive GPS observations is not constant. Its median is 4.8 minutes and its mean is 9.3 minutes. We show the histogram of the time steps in Figure 1.

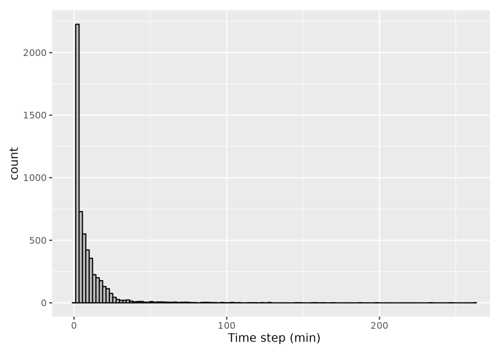

Fig 1: Histogram of time steps

The observations are divided into unexposed periods, for which the narwhals are not in line of sight with the ship; trial periods, when the narwhals are exposed to the ship and airguns are shot; and intertrial periods, when the narwhals are exposed to the ship but airguns are not shot. These periods are indicated by a categorical variable T_{ship} in the dataset. Figure 2 shows how the exposure periods are distributed among the 6 narwhals that were tracked. Our analysis in section 6 does not distinguish between intertrial and trial periods. They are both treated as exposure periods, though the nature and intensity of the behavioral response might differ for the two periods. We adopted this approach due to the lack of intertrial data as well as a potential persistence in time of the behavior shift due to airgun exposure during trial periods.

Table 1 shows how the data is distributed among the different narwhals.

The relevant covariates used for the analysis of narwhals movement are summarized in Table 2.

Keywords and phrases: behavioral response study, stochastic differential equations, mixed effect model.

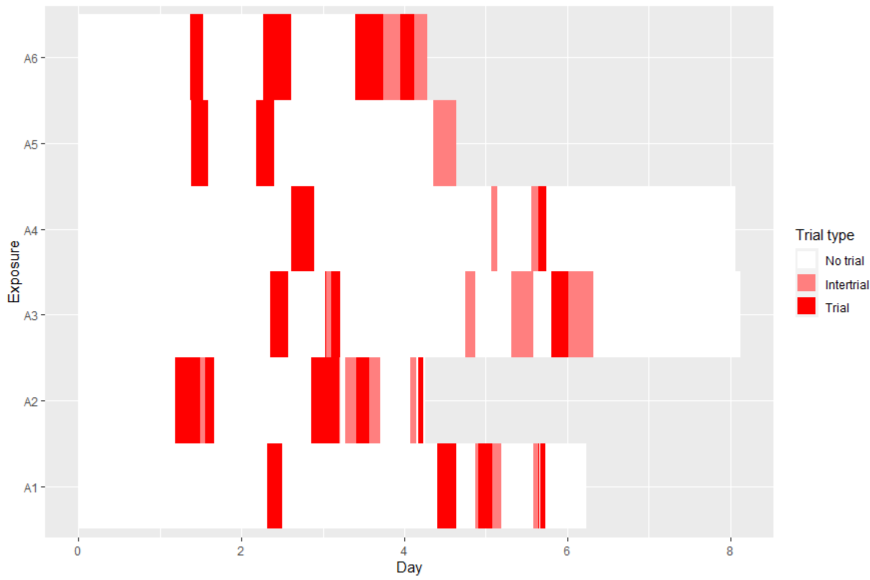

Fig 2: Trial and Intertrial periods for each narwhal

Narwhal ID	Number of measurement before exposure	Number of measurements during exposure
A1	354	576
A2	151	515
A3	397	680
A4	127	642
A5	207	419
A6	322	425
Total	**1558**	**3257**

TABLE 1
Distribution of the data among the 6 individuals

Covariate	Unit	Description	Domain
$D^{ship}(t)$	km	distance in kilometers between the narwhal and the ship at time t	$\mathbb{R}^+$
$E^{ship}(t) = \frac{1}{D^{ship}(t)}$	km^{-1}	global exposure level of the narwhal to the ship disturbance at time t	$\mathbb{R}^+$
$D^{shore}(t)$	km	distance between the narwhal and the nearest point on the shore at time t	$\mathbb{R}^+$
$\Theta(t)$	rad	angle between the vector that goes from the nearest shore point to the narwhal's position and the empirical velocity vector at time t	$[-\pi, \pi]$

TABLE 2
Summary of the covariates

APPENDIX B: PROOF OF PROPOSITION 3.1

Here, we prove Proposition 3.1 The proof is inspired by the results in (Gurarie et al., 2017; Johnson et al., 2008) and the proof of the transition density of the velocity process in (Albertsen, 2018).

PROOF. The velocity process is an Ornstein-Uhlenbeck process. For $t \geq 0$ and $\Delta > 0$,

(1)
$$V(t+\Delta) = \exp(-A\Delta)V(t) + (I_2 - \exp(-A\Delta))\mu + \frac{2\nu}{\sqrt{\pi\tau}} \int_t^{t+\Delta} \exp(A(s-(t+\Delta)))dW(s)$$

It has Gaussian transition density with mean

(2)
$$\mathbb{E}(V(t+\Delta|V(t))) = \exp(-A\Delta)V(t) + (I_2 - \exp(-A\Delta))\mu$$

and covariance matrix

$$\begin{aligned} Var(V(t+\Delta)|V(t)) &= \frac{4\nu^2}{\pi\tau} \int_t^{t+\Delta} \exp(A(s-(t+\Delta))) \exp(A(s-(t+\Delta)))^\top ds \\ &= \frac{4\nu^2}{\pi\tau} \int_0^{\Delta} \exp(-Au) \exp(-Au)^\top du \end{aligned}$$

An integration by part gives

$$\int_0^{\Delta} \exp(-Au) \exp(-Au)^\top du = A^{-1}(I_2 - \exp(-A\Delta))(I_2 - \exp(-A\Delta))^\top A^{-\top}$$

. In the sequel, we denote $S(\Delta) = A^{-1}(I_2 - \exp(-A\Delta))$. Since $\exp(-Au) = \exp(-\frac{u}{\tau})R_{-\omega u}$ where $R_{-\omega u}$ is the rotation matrix with angle $-\omega u$, we can deduce that the two components of the velocity are independent and have the same variance, denoted $q_2(\Delta)$. This variance is

(3)
$$q_2(\Delta) = \frac{2\nu^2}{\pi} \left(1 - \exp\left(-\frac{2\Delta}{\tau}\right)\right).$$

These results are found in Gurarie et al. (2017). In the sequel, we use the notation

$$\zeta(t,s) = \frac{2\nu}{\sqrt{\pi\tau}} \int_t^s \exp(A(u-s))dW(u).$$

Using that $V(s) = \mu + \exp(-A(s-t))(V(t)-\mu) + \zeta(t,s)$, we have

$$\begin{aligned} X(t+\Delta) &= X(t) + \int_t^{t+\Delta} V(s)\,ds \\ &= X(t) + \mu\Delta + \int_t^{t+\Delta} \exp(-A(s-t))(V(t)-\mu)\,ds \\ &\quad + \int_t^{t+\Delta} \zeta(t,s)\,ds \\ &= X(t) + \mu\Delta + \left(A^{-1}(V(t)-\mu) - A^{-1}\exp(-A\Delta)(X(t)-\mu)\right) \\ &\quad + \int_t^{t+\Delta} \zeta(t,s)\,ds \end{aligned}$$

Thus,

(4)
$$X(t+\Delta) = X(t) + \mu\Delta + A^{-1}\left(I_2 - \exp(-A\Delta)\right)(V(t)-\mu) + \xi(t,t+\Delta)$$

where $\xi(t,t+\Delta) = \int_t^{t+\Delta} \zeta(t,s)ds$. The location process is also Gaussian with mean

(5)
$$\mathbb{E}(X(t+\Delta)|V(t),X(t)) = X(t) + \mu\Delta + A^{-1}\left(I_2 - \exp(-A\Delta)\right)(V(t)-\mu).$$

To get an expression of the covariance matrix, first rewrite

$$\begin{aligned}\xi(t,t+\Delta) &= \int_t^{t+\Delta} \frac{2\nu}{\sqrt{\pi\tau}} \left(\int_t^s \exp(-A((u-s))dW(u)\right) ds \\ &= \frac{2\nu}{\sqrt{\pi\tau}} \int_t^{t+\Delta} (A^{-1} - A^{-1}\exp(A(u-t-\Delta)))dW(u) \\ &= \frac{2\nu}{\sqrt{\pi\tau}} \int_t^{t+\Delta} A^{-1}(I_2 - \exp(A(u-t-\Delta)))dW(u)\end{aligned}$$

Then use Ito's isometry

$$\begin{aligned}\mathrm{Var}(X(t+\Delta)|X(t),V(t)) &= \frac{4\nu^2}{\pi\tau} \int_t^{t+\Delta} A^{-1}(I_2 - \exp(A(u-t-\Delta))) \\ &\quad \times (I_2 - \exp(A(u-t-\Delta)))^\top A^{-\top}\, du \\ &= \frac{4\nu^2}{\pi\tau} \int_0^{\Delta} A^{-1}(I_2 - \exp(-Ar)) \\ &\quad \times (I_2 - \exp(-Ar))^\top A^{-\top}\, dr.\end{aligned}$$

This integral can be computed explicitly since

$$A^{-1}(I_2 - \exp(-Ar))(I_2 - \exp(-Ar))^\top A^{-\top} = \frac{1}{C} f(r) I_2$$

where $f(r) = 1 - 2\exp\left(-\frac{r}{\tau}\right)\cos(\omega r) + \exp\left(\frac{-2r}{\tau}\right)$ and $C = \frac{1}{\tau^2} + \omega^2$. We obtain that $X_1(t+\Delta)$ and $X_2(t+\Delta)$ are independent and have the same variance, denoted $q_1(\Delta)$. Writing $\sigma = \frac{2\nu}{\sqrt{\pi\tau}}$, the variance is

$$\begin{aligned}q_1(\Delta) = \frac{\sigma^2}{C}\Bigg(\Delta - 2\frac{\omega\sin(\omega\Delta) - \frac{1}{\tau}\cos(\omega\Delta)}{\frac{1}{\tau^2}+\omega^2}\exp\left(-\frac{\Delta}{\tau}\right) \\ +\frac{\tau}{2}\left(\frac{\omega^2 - \frac{3}{\tau^2}}{\frac{1}{\tau^2}+\omega^2} - \exp\left(-\frac{2\Delta}{\tau}\right)\right)\Bigg)\end{aligned}$$

Now we compute the covariance between X and V to get the full covariance matrix of U:

$$\begin{aligned}\Gamma(\Delta) &= \frac{4\nu^2}{\pi\tau}\mathbb{E}\Bigg(\left(\int_t^{t+\Delta} A^{-1}(I_2 - \exp(A(u-t-\Delta)))\, dW(u)\right) \\ &\quad \times \left(\int_t^{t+\Delta} \exp(A(s-(t+\Delta)))\, dW(s)\right)^\top\Bigg) \\ &= \int_t^{t+\Delta} A^{-1}(I_2 - \exp(A(u-(t+\Delta))))\exp(A(u-(t+\Delta)))^\top\, du \\ &= \frac{4\nu^2}{\pi\tau}\int_0^{\Delta} A^{-1}(I_2 - \exp(-Ar))\exp(-Ar)^\top\, dr.\end{aligned}$$

Then,

$$A^{-1}(I_2 - \exp(-Ar))\exp(-Ar)^\top = \frac{1}{C}\exp\left(-\frac{r}{\tau}\right)\begin{pmatrix} g(r) & h(r) \\ -h(r) & g(r)\end{pmatrix}$$

where

$$g(r) = \frac{1}{\tau}\left(\cos(\omega r) - \exp\left(-\frac{r}{\tau}\right)\right) + \omega \sin(\omega r),$$
$$h(r) = -\frac{1}{\tau}\sin(\omega r) + \omega\left(\cos(\omega r) - \exp\left(-\frac{r}{\tau}\right)\right).$$

Finally we get

$$\gamma_1(\Delta) = \frac{\sigma^2}{2C}\left(1 + \exp\left(-\frac{2\Delta}{\tau}\right) - 2\exp\left(-\frac{\Delta}{\tau}\right)\cos(\omega\Delta)\right),$$

$$\gamma_2(\Delta) = \frac{\sigma^2}{C}\left(\exp\left(-\frac{\Delta}{\tau}\right)\sin(\omega\Delta) - \frac{\omega\tau}{2}\left(1 - \exp\left(-2\frac{\Delta}{\tau}\right)\right)\right).$$

In the specific case $\omega = 0$, we obtain $C = \frac{1}{\tau^2}$ and the variance of X becomes

$$q_1(\Delta) = \sigma^2\tau^2\left(\Delta + 2\tau\exp\left(-\frac{\Delta}{\tau}\right) + \frac{\tau}{2}\left(-3 - \exp\left(-\frac{2\Delta}{\tau}\right)\right)\right).$$

Writing $\beta = \frac{1}{\tau}$ and reorganizing the terms, we obtain

$$q_1(\Delta) = \frac{\sigma^2}{\beta^2}\left(\Delta - 2\frac{1-\exp(-\beta\Delta)}{\beta} + \frac{1-\exp(-2\beta\Delta)}{2\beta}\right). \tag{6}$$

This result match equation (6) in (Johnson et al., 2008). Similarly, in the case $\omega = 0$, we get $\gamma_2 = 0$ and the expression for γ_1 match equation (7) in (Johnson et al., 2008).

□

APPENDIX C: APPROXIMATE TRANSITION DENSITY

We now suppose that the angular velocity ω varies with time. Denote $\mathcal{A}(0,t) = \int_0^t A(s)ds$ the time integral of the drift matrix. A similar calculation as before gives

$$V(t+\Delta) = \exp(-\mathcal{A}(t,t+\Delta))V(t) + \sigma\int_t^{t+\Delta}\exp(-\mathcal{A}(s,t+\Delta))dW(s)$$

Therefore, $V(t+\Delta)|V(t)$ is normally distributed with mean

$$\mathbb{E}(V(t+\Delta)|V(t)) = \exp(-\mathcal{A}(t,t+\Delta))V(t)$$

and covariance matrix

$$\mathrm{Var}(V(t+\Delta)|V(t)) = \sigma^2\int_t^{t+\Delta}\exp(-\mathcal{A}(s,t+\Delta))\exp(-\mathcal{A}(s,t+\Delta))^\top ds$$

The exact formula for the process X is

$$X(t+\Delta) = X(t) + \int_t^{t+\Delta}\exp(-\mathcal{A}(t,s))V(t)ds + \sigma\int_t^{t+\Delta}\int_t^s\exp(-\mathcal{A}(u,s))dW(u)ds$$

Consequently, the process is gaussian with mean given by

$$\mathbb{E}(X(t+\Delta)|X(t)) = X(t) + \int_t^{t+\Delta}\exp(-\mathcal{A}(t,s))V(t)ds$$

and covariance matrix

$$\mathrm{Var}(X(t+\Delta)|X(t)) = \sigma^2\int_t^{t+\Delta}\{\int_u^{t+\Delta}\exp(-\mathcal{A}(u,s))ds\}\{\int_u^{t+\Delta}\exp(-\mathcal{A}(u,s))ds\}^\top du$$

Denote $U(t) = (X(t), V(t))$. It is a Markov process with transition density

$$U(t+\Delta)|U(t) = u \sim \mathcal{N}(T(\Delta)u, Q(\Delta))$$

where

$$T(\Delta) = \begin{pmatrix} I_2 & \int_t^{t+\Delta} \exp(-\mathcal{A}(t,s))ds \\ 0_2 & \exp(-\mathcal{A}(t,t+\Delta)) \end{pmatrix} \tag{7}$$

$$Q(\Delta) = \begin{pmatrix} Q_{11}(\Delta) & Q_{12}(\Delta) \\ Q_{21}(\Delta) & Q_{22}(\Delta) \end{pmatrix} \tag{8}$$

where

$$Q_{11}(\Delta) = \sigma^2 \int_t^{t+\Delta} \left(\int_u^{t+\Delta} \exp(-\mathcal{A}(u,s))ds \right) \left(\int_u^{t+\Delta} \exp(-\mathcal{A}(u,s))ds \right)^\top du$$

$$Q_12(\Delta) = \sigma^2 \int_t^{t+\Delta} \left(\int_u^{t+\Delta} \exp(-\mathcal{A}(u,s))ds \right) \exp(-\mathcal{A}(u,t+\Delta))^\top du$$

$$Q_{21}(\Delta) = Q_{12}(\Delta)^\top$$

$$Q_{22}(\Delta) = \sigma^2 \int_t^{t+\Delta} \exp(-\mathcal{A}(s,t+\Delta)) \exp(-\mathcal{A}(s,t+\Delta))^\top ds$$

Suppose we have approximations of the time varying matrix A at times $t_0 = t < t_1 < \cdots < t_n = t+\Delta$ and denote $\Delta_i = t_{i+1} - t_i$ for $i \in \{0, \cdots, n-1\}$. Then we can approximate

$$\exp(-\mathcal{A}(t,t+\Delta)) = \prod_{i=0}^{n-1} \exp(-A_i \Delta_i)$$

where $A_i = A(t_i)$. Each matrix $\exp(-A_i\Delta_i)$ is a weighted rotation matrix with angle $-\omega_i\Delta_i$. For conciseness, in the sequel we denote $E_i = \exp(-A_i\Delta_i)$, $P_{jk} = \prod_{i=j}^{k-1} E_i$ and $S_j = S(\Delta_j)$.

The covariance matrix of V is approximated by

$$Q_{22}(\Delta) \simeq \sum_{j=0}^{n-1} \sigma^2 P_{j+1,n} P_{j+1,n}^\top S_j S_j^\top$$

The mean for the process X can be estimated with the following formula

$$\mathbb{E}(X(t+\Delta)|X(t)) \simeq X(t) + \sum_{k=0}^{n-1} P_{0,k} S_k V(t)$$

The covariance can also be approximated. The first step is to approximate $\mathcal{A}(u,s)$ for $u \in [t, t+\Delta]$ and $s \in [u, t+\Delta]$. For a fixed $u \in [t_{j-1}, t_j]$,

$$\exp(-\mathcal{A}(u,s)) \simeq \exp(-A_{j-1}(t_j - u)) \mathbb{1}_{[u,t_j]}(s) +$$
$$\sum_{k=j}^{n-1} \exp(-A_{j-1}(t_j - u)) \times \prod_{i=j}^{k-1} \exp(-A_i\Delta_i) \times \exp(-A_k(s - t_k)) \mathbb{1}_{[t_k,t_{k+1}]}(s)$$

This gives the following approximation for any $u \in [t_{j-1}, t_j]$

$$\int_u^{t+\Delta} \exp(-\mathcal{A}(u,s))ds \simeq A_{j-1}^{-1}(I_2 - \exp(-A_{j-1}(t_j - u))) +$$
$$\sum_{k=j}^{n-1} \exp(-A_{j-1}(t_j - u)) \prod_{i=j}^{k-1} \exp(-A_i\Delta_i) A_k^{-1}(I_2 - \exp(A_k\Delta_k))$$

We then deduce an approximation of the covariance matrix.

$$\begin{aligned}\mathrm{Var}(X(t+\Delta)|X(t)) \simeq & \sum_{j=1}^{n} \sigma^2 A_{j-1}^{-1} A_{j-1}^{-\top}(\Delta_{j-1} I_2 - S_{j-1} - S_{j-1}^\top + S_{j-1} S_{j-1}^\top) \\ & + \sigma^2 M_{jn}^\top A_{j-1}^{-1}(S_{j-1}^\top - S_{j-1}S_{j-1}^\top) M_j M_j^\top + \sigma^2 M_{jn} A_{j-1}^{-\top}(S_{j-1} - S_{j-1}S_{j-1}^\top) \\ & + M_{jn} M_{jn}^\top S_{j-1} S_{j-1}^\top\end{aligned}$$

where $M_{jn} = \sum_{k=j}^{n-1} P_{jk} S_k$.

Finally, we have the following formula to approximate the covariance matrix between X and V.

$$\mathrm{Var}(X(t+\Delta), V(t+\Delta)|X(t), V(t)) = \sum_{j=1}^{n} \sigma^2 P_{j,n}^\top A_{j-1}^{-1}(S_{j-1}^\top - S_{j-1}S_{j-1}^\top) + \sigma^2 M_{jn} P_{j,n}^\top S_{j-1} S_{j-1}^\top$$

We denote

$$\tilde{T}_n = \begin{pmatrix} I_2 & \sum_{k=0}^{n-1} P_{0,k} S_k \\ 0_2 & \prod_{i=0}^{n-1} E_i \end{pmatrix}$$

and

$$\tilde{Q}_n = \begin{pmatrix} \tilde{Q}_n^{11} & \tilde{Q}_n^{12} \\ \tilde{Q}_n^{21} & \tilde{Q}_n^{22} \end{pmatrix}$$

where

$$\begin{aligned}\tilde{Q}_n^{11} = & \sum_{j=1}^{n} \sigma^2 A_{j-1}^{-1} A_{j-1}^{-\top}(\Delta_{j-1} I_2 - S_{j-1} - S_{j-1}^\top + S_{j-1} S_{j-1}^\top) \\ & + \sigma^2 M_{jn}^\top A_{j-1}^{-1}(S_{j-1}^\top - S_{j-1}S_{j-1}^\top) + \sigma^2 M_{jn} A_{j-1}^{-\top}(S_{j-1} - S_{j-1}S_{j-1}^\top) \\ & + \sigma^2 M_{jn} M_{jn}^\top S_{j-1} S_{j-1}^\top \\ \tilde{Q}_n^{12} = & \sum_{j=1}^{n} \sigma^2 P_{j,n}^\top A_{j-1}^{-1}(S_{j-1}^\top - S_{j-1}S_{j-1}^\top) + \sigma^2 M_{jn} P_{j,n}^\top S_{j-1} S_{j-1}^\top \\ \tilde{Q}_n^{21} = & (\tilde{Q}_n^{12})^\top \\ \tilde{Q}_n^{22} = & \sum_{j=0}^{n-1} \sigma^2 P_{j+1,n} P_{j+1,n}^\top S_j S_j^\top\end{aligned}$$

We can express $\tilde{T}_n$ and $\tilde{Q}_n$ in terms of the elementary matrices

$$T_i = \begin{pmatrix} I_2 & S_i \\ 0_2 & E_i \end{pmatrix} \text{ and } Q_i = \begin{pmatrix} \sigma^2 A_i^{-1} A_i^{-\top}(\Delta_i I_2 - S_i - S_i^\top + S_i S_i^\top) & \sigma^2 A_i^{-1}(S_i^\top - S_i S_i^\top) \\ \sigma^2 A_i^{-\top}(S_i - S_i S_i^\top) & \sigma^2 S_i S_i^\top \end{pmatrix}$$

$$\tilde{T}_n = T_{n-1} \times \cdots \times T_1 \times T_0 \tag{9}$$

$$\tilde{Q}_n = \sum_{k=1}^{n-1} T_{n-1}^\top \cdots T_{k+1}^\top Q_k T_{k+1} \cdots T_{n-1} \tag{10}$$

PROOF. $\tilde{T}_n = T_0$ and $\tilde{Q}_1 = Q_0$. Then, for $n \geq 2$,

$$\begin{aligned}\tilde{T}_n &= \begin{pmatrix} I_2 & P_{0,n-1}S_{n-1} + \sum_{j=0}^{n-2} P_{0,j}S_j \\ 0_2 & P_{0,n}E_{n-1} \end{pmatrix} = \begin{pmatrix} I_2 & S_{n-1} \\ 0_2 & E_{n-1} \end{pmatrix} \begin{pmatrix} I_2 & \sum_{j=0}^{n-2} P_{0,j}S_j \\ 0_2 & P_{0,n-1} \end{pmatrix} \\ &= T_{n-1}\tilde{T}_{n-1}\end{aligned}$$

Then, using that $M_{n,n} = 0_2$ and $P_{n,n} = I_2$ we get

$$\tilde{Q}_n = Z_{n-1} + Q_{n-1}$$

where

$$Z_{n-1} = \begin{pmatrix} \sum_{j=1}^{n-1} Q_{j-1}^{11} + M_{jn}^\top Q_{j-1}^{12} + M_{jn}Q_{j-1}^{21} + M_{j,n}M_{j,n}^\top Q_{j-1}^{22} & \sum_{j=1}^{n-1} P_{j,n}^\top Q_{j-1}^{12} + M_{j,n}P_{j,n}^\top Q_{j-1}^{22} \\ \sum_{j=1}^{n-1} P_{j,n}Q_{j-1}^{21} + M_{jn}^\top P_{jn}Q_{j-1}^{22} & \sum_{j=1}^{n-1} P_{jn}^\top P_{jn}Q_{j-1}^{22} \end{pmatrix}$$

Then, since $M_{j,n} = M_{j,n-1} + P_{j,n-1}S_{n-1}$ we get

$$Z_{n-1} = \begin{pmatrix} \tilde{Q}_{n-1}^{11} + S_{n-1}^\top Q_{n-1}^{12} + S_{n-1}\tilde{Q}_{n-1}^{21} + S_{n-1}S_{n-1}^\top \tilde{Q}_{n-1}^{22} & E_{n-1}^\top \tilde{Q}_{n-1}^{12} + E_{n-1}^\top S_{n-1}\tilde{Q}_{n-1}^{12} \\ E_{n-1}\tilde{Q}_{n-1}^{21} + E_{n-1}S_{n-1}^\top \tilde{Q}_{n-1}^{22} & E_{n-1}E_{n-1}^\top \end{pmatrix}$$

And this is equal to $T_{n-1}^\top \tilde{Q}_{n-1} T_{n-1}$.

□

APPENDIX D: MEASUREMENT ERROR

In the application in Section 6, different values of the measurement error were estimated for the data before and after exposure: 35 m before and 48 m after. Both these values are consistent with the results in (Wensveen, Thomas and Miller, 2015). However, the post exposure estimation gave non-positive definite Hessian matrix for the negative log-likelihood, which prevents from using the information matrix equality to get confidence intervals of the estimates. Fixing a 35 m measurement error value when fitting the response model led to the same issue. We therefore tried different values of the measurement error, and kept the one that gave a positive definite hessian matrix and had the highest log-likelihood value. It turned out to be 50 m, very close to the initially estimated 48 m. Table 3 shows these results. In comparison, the final values of the log-likelihood when σ_{obs} is estimated from the data are respectively 4273 and 8043 before and after exposure, while the estimate of τ_0 is 1.10, and the estimates of α_τ and α_ν are respectively -4.19 and 0.66, which is in the confidence interval of the final estimations we kept (those obtained for $\sigma_{obs} = 50$ m).

APPENDIX E: CODE EXAMPLE

We illustrate briefly how to fit our baseline SDE model and obtain the results with `smoothSDE` R package Michelot et al. (2021). The version of the package including our new model is available here . We suppose the package has been loaded. We consider a dataframe `dataBE` containing the preprocessed observations before exposure to the ship in the columns `x` and `y`, an animal identifier in a column `ID` and columns `Ishore` and `AngleNormal` for

σ_{obs} (m)	$\hat{\tau_0}$	Baseline llk	Response llk	P.d hessian	α_τ	α_ν
30	0.96 ± 0.15	4261	8038	No	0.29	2.17
40	1.18 ± 0.16	4266	8014	No	-2.06	0.60
45	1.29 ± 0.17	4243	7965	No	-3.64	0.49
50	$\mathbf{1.35 \pm 0.16}$	**4208**	**7861**	**Yes**	$\mathbf{-3.43 \pm 0.70}$	$\mathbf{0.74 \pm 0.27}$
75	1.63 ± 0.19	3944	7225	Yes	-3.88 ± 0.73	0.76 ± 0.30
100	1.85 ± 0.22	3640	6590	Yes	-4.27 ± 0.74	0.63 ± 0.29

TABLE 3

Estimate for the baseline and response models for several fixed measurement errors.

the covariates I^{shore} and Θ. The first step consists in choosing initial SDE parameters and model formulas. For the model we consider, there are five parameters μ_1, μ_2, τ, ν, and ω, and each of them needs a formula. Specification of the formulas is identical to the `R` package `mgcv`. Among the parameters, μ_1, μ_2 will be set to 0, while τ, ν and ω are expressed as in section 4.1.

```
#number of observation
n_pre<-nrow(dataBE)

#initial parameters
par0 <- c(0,0,1,4,0)

#model formulas
formulas <- list(mu1 = ~1 ,mu2 =~1,tau =~s(ID,bs="re"),nu=~s(ID,bs="re"),
omega=~ti(AngleNormal,k=5,bs="cs")+ti(Ishore,k=5,bs="cs")+
ti(AngleNormal,Ishore,k=c(5,5),bs="cs"))
```

We then specify the measurement error for each observation in an array of covariance matrices. We suppose they are all diagonal with the same standard deviation `sigma_obs`. We will fix this measurement error.

```
# 50m measurement error
sigma_obs=0.05
H=array(rep(sigma_obs^2*diag(2),n_pre),dim=c(2,2,n_pre))
```

We can then create the SDE object as in Michelot et al. (2021). We choose the type of SDE in the argument `type`. Here, it is `RACVM` (see Section 3.1) since we want to include a non zero rotation parameter ω. The name of the columns where the observations are found is specified in the `response` argument. We specify the measurement error matrix `H` in the argument `other_data`. Fixed parameters are indicated in the argument `fixpar`.

```
#create SDE object
baseline_50m<- SDE$new(formulas = formulas,data = dataBE,type = "RACVM",
response = c("x","y"),par0 = par0,other_data=list("H"=H),
fixpar=c("mu1","mu2"))
```

To fix specific parameters in the statistical model, we need to use the `map` attribute. Here we use it to specify that the smoothing parameters should be fixed. Then we update the smoothing parameters to 1, and fit the SDE model.

```
#update map to fix smoothig parameters
baseline_50m$update_map(list("log_lambda"=factor(c(1,2,rep(NA,4)))))

#update smoothing parameters values
init_lambda=rep(1,6)
baseline_50m$update_lambda(init_lambda)

#fit the model
baseline_50m$fit()
```

The results of the optimization are stored in the attribute `tmb_rep`. We can extract the estimated parameters along with the standard errors.

```
#estimates
estimates_bas_50m=as.list(baseline_50m$tmb_rep(),what="Est")
#standard error
std_bas_50m=as.list(baseline_50m$tmb_rep(),what="Std")
```

Finally, we would like to plot all the smooth parameters as a function of the covariates. We can do it with the `get_all_plots` method. We only need to specify the range of each covariate value we want to plot, a link function if we don't want to have directly the covariate on the x-axis but rather a function of the covariate, and the x-axis label of the plots. We put the option `show_CI="pointwise"` to show the pointwise confidence intervals on the plots.

```
#range of the covariates
D_low=0.073
D_up=3
xmin=list("Ishore"=1/D_up)
xmax=list("Ishore"=1/D_low)
#link function
link=list("Ishore"=(\(x) 1/x))
#label
xlabel=list("Ishore"="Distance to shore")

#draw plots
plots_bas_50m=baseline_50m$get_all_plots(model_name="baseline_50m",
xmin=xmin,xmax=xmax,link=link,xlabel=xlabel,show_CI="pointwise",save=TRUE)
```

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