# B-splines

B-splines basis

$$P(u)=\sum_{i=0}^{n} P_i N_i^d(u)$$

- The basis functions $N_i^d$ are piecewise polynomials
- Have a *compact support* + satisfy partition of the unity
- The continuity is defined at the basis function's level.

- The number of control points is fixed by the relation

$n+1=m-d$ degree

Last knot number

# B-splines

- The functions are such as : (recurrence formula of Cox – de Boor)

$$N_i^0(u)=\begin{cases} 1 \quad \text{if} & u_i \le u < u_{i+1} \\ 0 \quad \text{otherwise} & \end{cases}$$

$$N_i^d(u)=\frac{u-u_i}{u_{i+d}-u_i}N_i^{d-1}(u)+\frac{u_{i+d+1}-u}{u_{i+d+1}-u_{i+1}}N_{i+1}^{d-1}(u)$$

- Where $u_{i+d}$ - $u_i$=0, necessarily $N_i^{d-1}(u)\equiv 0$

  By convention, we set in this case $\frac{0}{0}=0$ when the limit is undefined.

# B-splines

- (simplified) Cox-de Boor's Algorithm :

Determine the interval of $u$ : $u \in [u_i, u_{i+1}[$

Initialization of $P_j^0$ $\quad i \in \{d, d+1, \cdots, m-d-1\}$

For $k$ from 1 to $d$

For $j$ from i to i-$d$+$k$

$$P_j^k = \left(\frac{u - u_j}{u_{j+d+1-k} - u_j}\right) P_j^{k-1} + \left(\frac{u_{j+d+1-k} - u}{u_{j+d+1-k} - u_j}\right) P_{j-1}^{k-1}$$

Endfor

Endfor

$P_i^d$ is the point that is sought.

- What is its complexity ?
  - quadratic in function of the degree $d$.

# B-spline and the Cox-de Boor algorithm ?

For u in $[u_i, u_{i+1}[$

For $k$ from 1 to $d$

  For $j$ from i to i-$d$+$k$

$$P_j^k = \left(\frac{u - u_j}{u_{j+d+1-k} - u_j}\right) P_j^{k-1} + \left(\frac{u_{j+d+1-k} - u}{u_{j+d+1-k} - u_j}\right) P_{j-1}^{k-1}$$

  Endfor

Endfor

$$
\begin{array}{lcccccc}
 & Q_0^0 & & & & & \\
 & \vdots & & & & & \\
 & Q_{i-d}^0 & & & & & \\
 & Q_{i-d+1}^0 & & Q_{i-d+1}^1 & & & \\
Q_j^k & Q_{i-d+2}^0 & & Q_{i-d+2}^1 & & Q_{i-d+2}^2 & \dots \quad Q_i^d \\
 & \vdots & & \vdots & & \vdots & \\
 & Q_{i-1}^0 & & & & Q_i^2 & \\
 & Q_i^0 & & Q_i^1 & & & \\
 & \vdots & & & & & \\
 & Q_n^0 & & & & &
\end{array}
$$

# B-splines

- Types of nodal sequence...
  - Uniform – The gap between two successive knots is constant

$$U=\{u_0, u_1, \cdots, u_{m-d-1}\} \quad , \quad u_{i+1}-u_i=k$$

  - Periodic - The gap between the knots at the start of a nodal sequence is identical to the one at the end of the nodal sequence

$$U=\{\underbrace{u_{0,} \cdots, u_d}_{d+1}, u_{d+1}, \cdots, u_{m-d-1}, \underbrace{u'_{0,} \cdots, u'_d}_{d+1}\} \quad , \quad u'_i-u_i=k$$

  - Non uniform, interpolating – first and last control point are interpolated

$$U=\{\underbrace{a, \cdots, a}_{d+1}, u_{d+1}, \cdots, u_{m-d-1}, \underbrace{b, \cdots, b}_{d+1}\}$$