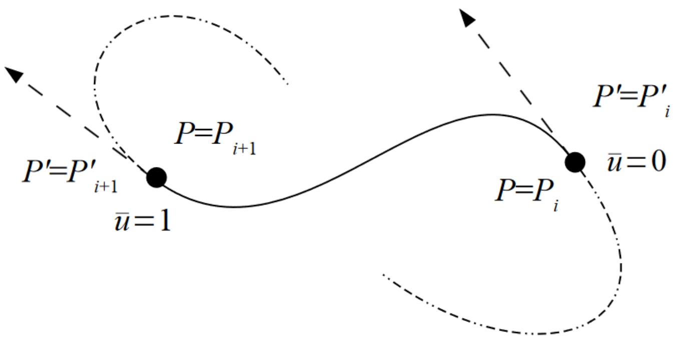
Splines



On each interval i, we thus have the following relation:

$$x_{[i]}(\bar{u})=a_{[i]0}+a_{[i]1}\bar{u}+a_{[i]2}\bar{u}^2+a_{[i]3}\bar{u}^3 \quad , \ \bar{u}\in[0,1]$$

- We pass through both control points:

$$P(\bar{u}=0)=P_i \Leftrightarrow a_{[i]0}+a_{[i]1}\bar{u}+a_{[i]2}\bar{u}^2+a_{[i]3}\bar{u}^3=x_i$$

$$P(\bar{u}=1)=P_{i+1} \Leftrightarrow a_{[i]0}+a_{[i]1}\bar{u}+a_{[i]2}\bar{u}^2+a_{[i]3}\bar{u}^3=x_{i+1}$$

- We impose both slopes :

$$P'(\bar{u}_0=0)=P'_i \Leftrightarrow a_{[i]1}+2a_{[i]2}\bar{u}+3a_{[i]3}\bar{u}^2=x_i'$$

$$P'(\bar{u}=1)=P'_{i+1} \Leftrightarrow a_{[i]1}+2a_{[i]2}\bar{u}+3a_{[i]3}\bar{u}^2=x_{i+1}'$$

- At the end :

$$\begin{cases} a_{[i]0}=x_i \\ a_{[i]1}=x_i' \\ a_{[i]2}=3(x_{i+1}-x_i)-2x_i'-x_{i+1}' \\ a_{[i]3}=2(x_i-x_{i+1})+x_i'+x_{i+1}' \end{cases}$$

Splines

Finite differences splines

$$x_i' = \frac{x_{i+1} - x_i}{2(u_{i+1} - u_i)} + \frac{x_i - x_{i-1}}{2(u_i - u_{i-1})}$$

- At the boundaries, we use finite differences (asymmetric)

$$x_0' = \frac{x_1 - x_0}{u_1 - u_0} \qquad x_{n-1}' = \frac{x_{n-1} - x_{n-2}}{u_{n-1} - u_{n-2}}$$

- The result depends on the parametrization !

Cardinal splines

$$x_i' = (1-c)\frac{x_{i+1} - x_{i-1}}{2}\ , \ 0 \le c \le 1$$

$$x_0' = (1-c)(x_1 - x_0)$$

$$x_{n-1}' = (1-c)(x_{n-1} - x_{n-2})$$

- c is a « tension » parameter. $c=0$ gives yields the so called "Catmull-Rom" spline, $c=1$ a zigzagging line.

Splines

Natural splines

- We impose the continuity of the second derivatives

$$x''_{[i-1]}(1) = x''_{[i]}(0) \Leftrightarrow 2\mathrm{a}_{[i-1]2} + 6\mathrm{a}_{[i-1]3} = 2\mathrm{a}_{[i]2}$$

- We have then a linear system with n unknowns :

$$\begin{pmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{pmatrix} \begin{pmatrix} x'_0 \\ x'_1 \\ x'_2 \\ \vdots \\ x'_{n-2} \\ x'_{n-1} \end{pmatrix} = \begin{pmatrix} 3(x_1 - x_0) \\ 3(x_2 - x_0) \\ 3(x_3 - x_1) \\ \vdots \\ 3(x_{n-1} - x_{n-3}) \\ 3(x_{n-1} - x_{n-2}) \end{pmatrix}$$

Inside the code GNURBS

1. Instantiate Curve
2. Set CP and u (fill `val` and `pos`)
3. Compute slopes (fill `der`)
4. Draw with interpolation

Splines (general)

- We pass through both points :

$$P(\bar{u}_0=0)=P_i \Leftrightarrow a_{[i]0}+a_{[i]1}\bar{u}_0+a_{[i]2}\bar{u}_0^2+a_{[i]3}\bar{u}_0^3=x_i$$

$$P(\bar{u}_1=1)=P_{i+1} \Leftrightarrow a_{[i]0}+a_{[i]1}\bar{u}_1+a_{[i]2}\bar{u}_1^2+a_{[i]3}\bar{u}_1^3=x_{i+1}$$

- We impose both slopes :

$$P'(\bar{u}_0=0)=\frac{dP}{du}(0)=\frac{dP}{d\bar{u}}(0)\frac{1}{h_i}=P'_i \Leftrightarrow a_{[i]1}+2\mathrm{a}_{[i]2}\bar{u}_0+3\mathrm{a}_{[i]3}\bar{u}_0^2=x'_i h_i$$

$$P'(\bar{u}_1=1)=\frac{dP}{du}(1)=\frac{dP}{d\bar{u}}(1)\frac{1}{h_i}=P'_{i+1} \Leftrightarrow a_{[i]1}+2\mathrm{a}_{[i]2}\bar{u}_1+3\mathrm{a}_{[i]3}\bar{u}_1^2=x'_{i+1} h_i$$

- Finally :

$$\begin{cases} a_{[i]0}=x_i \\ a_{[i]1}=x'_i h_i \\ a_{[i]2}=3(x_{i+1}-x_i)-2x'_i h_i-x'_{i+1}h_i \\ a_{[i]3}=2(x_i-x_{i+1})+x'_i h_i+x'_{i+1}h_i \end{cases}$$

with $h_i = u_{i+1} - u_i$

Natural splines (general)

$$x_{[i]}(\bar{u}) = a_{[i]0} + a_{[i]1}\bar{u} + a_{[i]2}\bar{u}^2 + a_{[i]3}\bar{u}^3 \quad , \ \bar{u} \in [0,1]$$

Using an arbitrary parametrization: $\bar{u} = \dfrac{u - u_i}{u_{i+1} - u_i}$ $\qquad \dfrac{d\bar{u}}{du} = \dfrac{1}{u_{i+1} - u_i} = \dfrac{1}{h_i}$

We impose continuity of second derivatives: $x''_{[i-1]}(1) = x''_{[i]}(0)$

And, at the boundaries:

$$x''_{[0]}(0) = 0$$
$$x''_{[n-2]}(1) = 0$$

$$\Rightarrow \begin{cases} a_{[i]0} = x_i \\ a_{[i]1} = x'_i h_i \\ a_{[i]2} = 3\left(x_{i+1} - x_i\right) - 2x'_i h_i - x'_{i+1} h_i \\ a_{[i]3} = 2\left(x_i - x_{i+1}\right) + x'_i h_i + x'_{i+1} h_i \end{cases}$$

A system with *n* unknonws has to be solved to get the first derivatives:

$$\begin{pmatrix} 2h_0 & h_0 & & & & \\ h_1 & 2h_1 + 2h_0 & h_0 & & & \\ & & \ddots & & & \\ & & & h_{n-2} & 2h_{n-2} + 2h_{n-3} & h_{n-3} \\ & & & & h_{n-2} & 2h_{n-2} \end{pmatrix} \begin{pmatrix} x'_0 \\ x'_1 \\ \vdots \\ x'_{n-2} \\ x'_{n-1} \end{pmatrix} = \begin{pmatrix} 3\left(x_1 - x_0\right) \\ 3\dfrac{h_1}{h_0}\left(x_1 - x_0\right) + 3\dfrac{h_0}{h_1}\left(x_2 - x_1\right) \\ \vdots \\ 3\dfrac{h_{n-2}}{h_{n-3}}\left(x_{n-2} - x_{n-3}\right) + 3\dfrac{h_{n-3}}{h_{n-2}}\left(x_{n-1} - x_{n-2}\right) \\ 3\left(x_{n-1} - x_{n-2}\right) \end{pmatrix}$$