Connectivity under vertex failure, logic and algorithm

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MAAD
Thème : Algorithmique, Graphes, Complexité

June 5, 2024

Outline

Introduction: Graphs, Algorithms, Logic

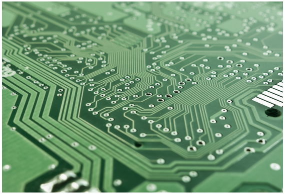
Connectivity under vertex failure

Logics between FO and MSO

Algorithmic graph theory

Given a graph G and a property P: "Does G satisfy P?"

$\rightarrow$ Is G planar?

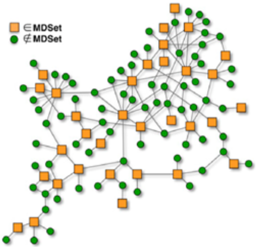


Algorithmic graph theory

Given a graph G and a property P: "Does G satisfy P?"

→ Is G planar?

→ Does G have a k-dominating set?



Algorithmic graph theory

Given a graph G and a property P: “Does G satisfy P?”

→ Is G planar?

→ Does G have a k-dominating set?

→ Is G connected?

Goal: Efficient algorithms ...

... at least for restricted graph classes and/or simple properties.

Logic

First-order (FO) logic

- → Can express k-independent set: *There are k vertices, that are not adjacent*
  $\exists x_1 \ldots \exists x_k \bigwedge_{i<j} (x_i \neq x_j \wedge \neg E(x_i, x_j))$
- → Cannot express : connectivity, planarity, 2-colorability, ...

Logic

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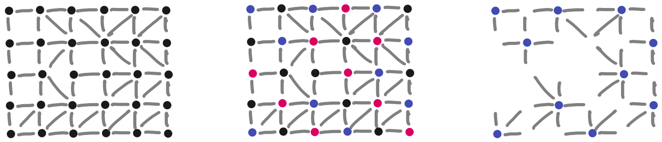
→ Cannot express : connectivity, planarity, 2-colorability, ...

Monadic second-order (MSO) logic

→ More general than FO

→ Can express : 3-colorability:
$$\exists X_1 \exists X_2 \exists X_3 \, (\forall x \bigvee_{i<3} x \in X_i) \wedge (\forall x \forall y \, E(x,y) \rightarrow \bigwedge_{i<3} (x \notin X_i \vee y \notin X_i))$$



Logic & Meta theorems

Problems can be expressed in logic. (FO, MSO,...)

The $\mathcal{L}$, $\mathcal{C}$ model-checking problem:
Given $\varphi \in \mathcal{L}$ and $G \in \mathcal{C}$, does $G \models \varphi$?

Goal: fixed parameter tractable algorithms $\mathcal{O}(f(\varphi) \cdot |G|^c)$

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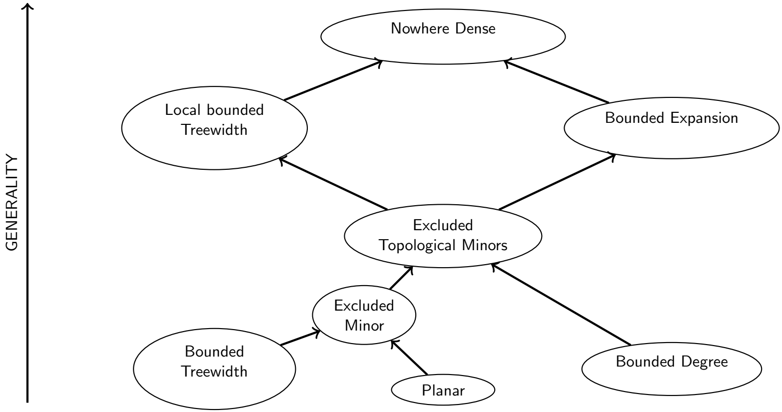
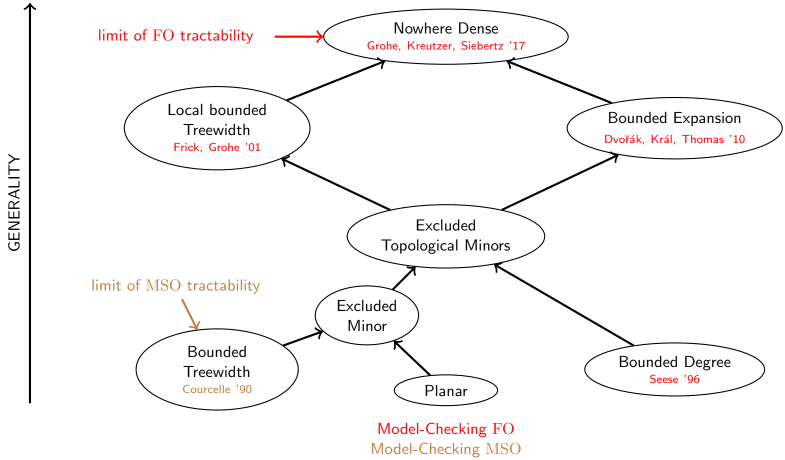
Courcelle's Theorem (1990):
for $\varphi \in \mathrm{MSO}$ and treewidth$(G) \leq k$, in time $\mathcal{O}(f(\varphi, k) \cdot |G|)$

$\rightarrow$ Generalize many known results, ex:
Arnborg, Proskurowski 1989:
independent sets, dominating sets, graph coloring, Hamiltonian, ...
are linear on partial k-tree.

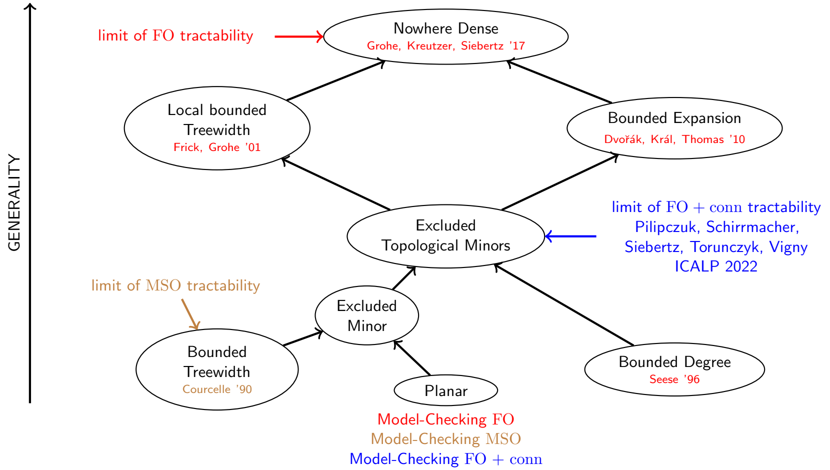
Monotone graph classes



Monotone graph classes



Monotone graph classes



Connectivity under vertex failure

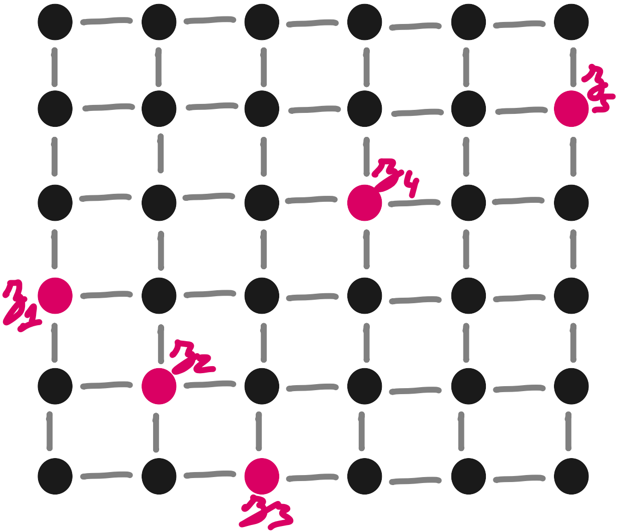
Input: A graph G, and k vertices $v_1, \ldots, v_k$

$\rightarrow$ 1st precomputation.

$\rightarrow$ 2nd Given x, y: are they connected in $G \setminus \{z_1, \ldots, z_k\}$?

$\rightarrow$ 3rd Update vertices $z_1, \ldots, z_k$
Be ready for 2nd as soon as possible.

Examples



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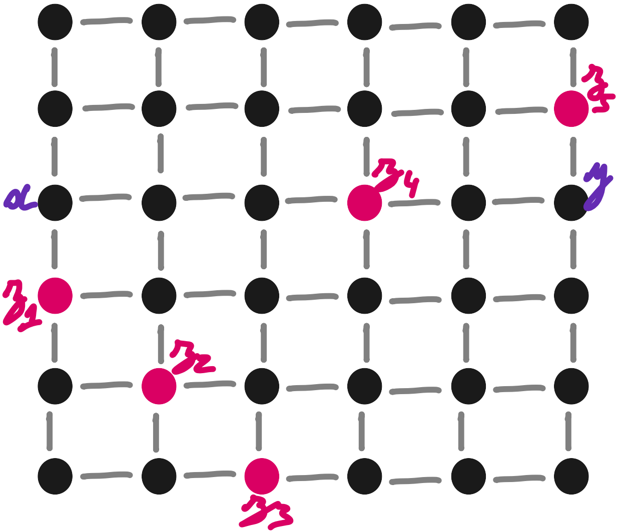

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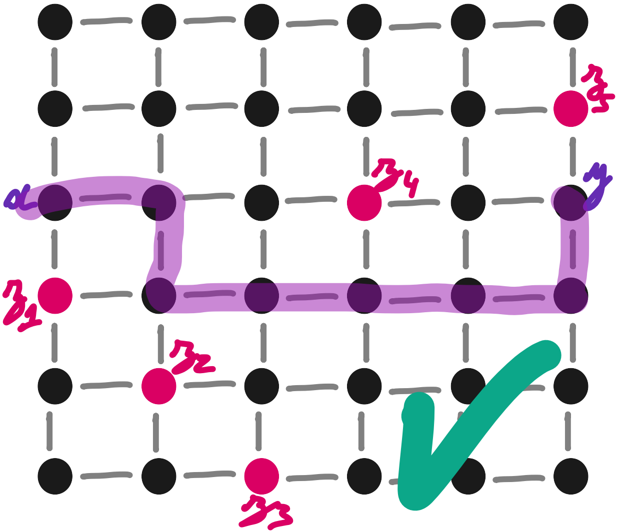

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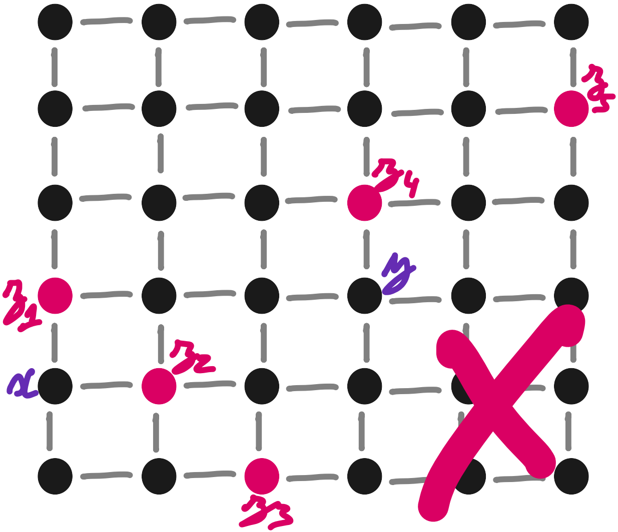

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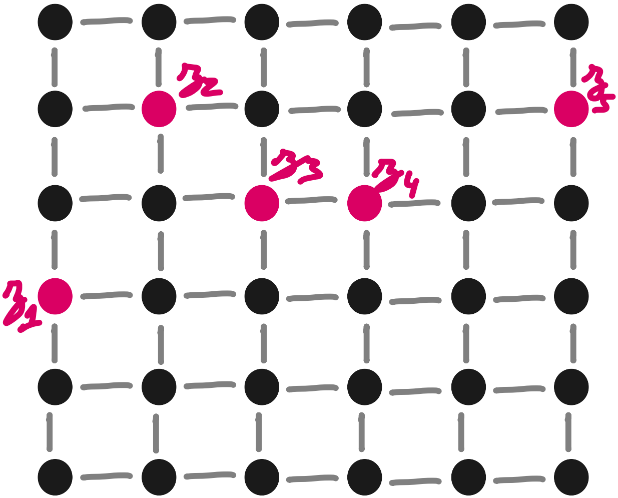

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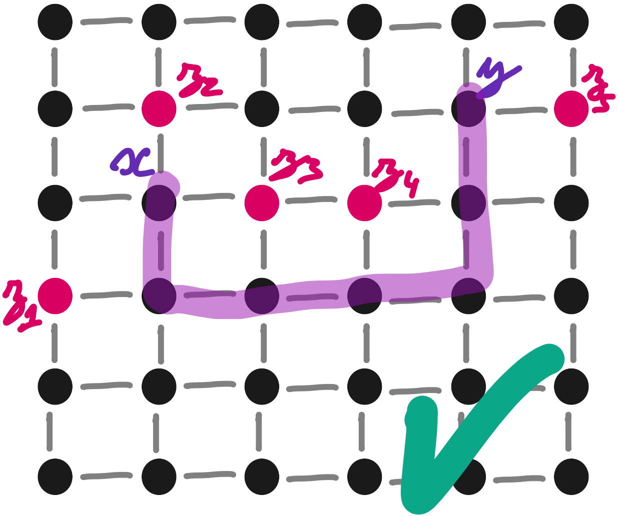

Computational complexity

	precomputation	query	update
Naive	$\mathcal{O}(1)$	$\mathcal{O}(\|G\|)$	$\mathcal{O}(1)$
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Computational complexity

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Naive	$\mathcal{O}(1)$	$\mathcal{O}(\|G\|)$	$\mathcal{O}(1)$
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Duan, Pettie '20	$\mathcal{O}(\lvert G\rvert \|G\| \cdot \log \lvert G\rvert)$	$\mathcal{O}(k)$	$\mathcal{O}(k^3 \log^3 \lvert G\rvert)$
Brand Saranurak '19 (randomized)	$\mathcal{O}(\lvert G\rvert^\omega)$	$\mathcal{O}(k^2)$	$\mathcal{O}(k^\omega)$
contribution 1	$2^{2^{\mathcal{O}(k)}} \cdot \lvert G\rvert^2 \cdot \|G\|$	$2^{2^{\mathcal{O}(k)}}$	$2^{2^{\mathcal{O}(k)}}$
contribution 2	$2^{\mathcal{O}(k \log k)} \cdot \lvert G\rvert^{\mathcal{O}(1)}$	$k^{\mathcal{O}(1)}$	$k^{\mathcal{O}(1)}$

Adding Connectivity to FO

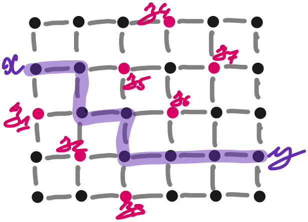
Schirrmacher, Siebertz, Vigny '21 and Bojanczyk '21

Syntax

→ Uses : FO and $\mathbf{conn}_k(x, y, z_1, \ldots, z_k)$

Meaning

→ x and y are connected after the deletion of $z_1, \ldots, z_k$.



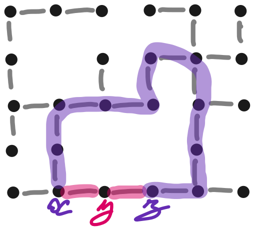
Expressive power of $\mathrm{FO} + \mathrm{conn}$

→ connectivity

$$\forall x \forall y \; \mathrm{conn}_0(x, y)$$

→ cycle

$$\varphi_{cycle} := \exists x \exists y \exists z \big(E(x, y) \wedge E(y, z) \wedge z \neq x \wedge \mathrm{conn}_1(z, x, y) \big)$$



→ Not expressible

planarity, bipartiteness, Hamiltonicity, . . .

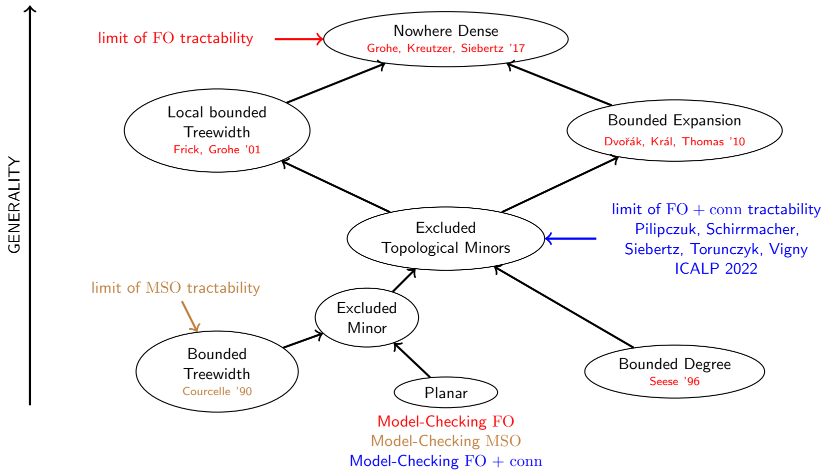
Main result

Theorem: Pilipczuk, Schirrmacher, Siebertz, Torunczyk, Vigny

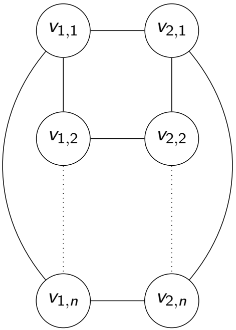
$\rightarrow$ Model-checking for properties in $\mathrm{FO} + \mathrm{conn}$ over graph classes excluding a topological minor is solvable in time FPT.

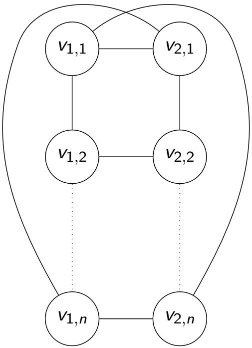
$\rightarrow$ Model-checking is not FPT for more general graph classes.
Under complexity assumptions

Monotone graph classes



Planarity is not expressible in $\mathrm{FO} + \mathrm{conn}_k$





Non-expressibility results

→ Ehrenfeucht-Fraïssé games

→ Construct ad-hoc counter examples

More logics: disjoint-paths logic $(\mathrm{FO} + \mathrm{DP})$

Syntax

→ Uses : FO and $\boldsymbol{\text{disjoint-paths}_k(x_1, y_1, \ldots, x_k, y_k)}$

Meaning

→ x_i and y_i are connected by internally vertex-disjoint paths.

Expressive power of disjoint-paths logic $(\mathrm{FO} + \mathrm{DP})$

→ connectivity

$$\forall x \forall y \text{ disjoint-paths}_1(x, y)$$

→ cycle

$$\exists x \exists y \exists z \text{ disjoint-paths}_3((x, y), (y, z), (z, x))$$

→ connectivity operators

$$\mathrm{conn}_k(x, y, z_1, \ldots, z_k) := \text{disjoint-paths}_{k+1}[(x, y), (z_1, z_1), \ldots, (z_k, z_k)]$$

Disjoint-paths logic ($\mathrm{FO} + \mathrm{DP}$)

Expressible

- $\rightarrow$ connectivity
- $\rightarrow$ (topological) minors,
- $\rightarrow$ planarity, ...

Not expressible

- $\rightarrow$ bipartiteness

strict hierarchy

- $\rightarrow$ $\mathrm{FO} + \mathrm{DP}_1 \subsetneq \mathrm{FO} + \mathrm{DP}_2 \subsetneq \dots$

Model checking?

- $\rightarrow$ For planar graphs?
- $\rightarrow$ For excluded minor?

New results

Theorem: Schirrmacher, Siebertz, Stamoulis, Thilikos, Vigny

$\rightarrow$ Model-checking for properties in $\mathrm{FO} + \mathrm{DP}$ over graph classes excluding a topological minor is solvable in time FPT.

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Thank you very much for your attention!