Journal of Combinatorial Theory, Series B 169 (2024) 96–133

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

journal homepage: www.elsevier.com/locate/jctb

Discrepancy and sparsity ☆

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ARTICLE INFO

Article history:
Received 29 November 2021
Available online xxxx

Keywords:
Graph sparsity
First-order logic
Discrepancy
Weak coloring numbers
Definable set systems
Pointer structures
Quantifier elimination
VC-density
Epsilon-approximation
Clique coloring

ABSTRACT

We study the connections between the notions of combinatorial discrepancy and graph degeneracy. In particular, we prove that the maximum discrepancy over all subgraphs H of a graph G of the neighborhood set system of H is sandwiched between $\Omega(\log \deg(G))$ and $\mathcal{O}(\deg(G))$, where $\deg(G)$ denotes the degeneracy of G. We extend this result to inequalities relating weak coloring numbers and discrepancy of graph powers and deduce a new characterization of bounded expansion classes.

Then we switch to a model theoretical point of view, introduce pointer structures, and study their relations to graph classes with bounded expansion. We deduce that a monotone class of graphs has bounded expansion if and only if all the set systems definable in this class have bounded hereditary discrepancy.

Using known bounds on the VC-density of set systems definable in nowhere dense classes we also give a characterization of nowhere dense classes in terms of discrepancy.

☆ This paper is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 810115 – Dynasnet), from the German Research Foundation (DFG) with grant agreement No 444419611, and from the french ANR project HOSIGRA (ANR-17-CE40-0022).

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https://doi.org/10.1016/j.jctb.2024.06.001

As consequences of our results, we obtain a corollary on the discrepancy of neighborhood set systems of edge colored graphs, a polynomial-time algorithm to compute ε-approximations of size $\mathcal{O}(1/\varepsilon)$ for set systems definable in bounded expansion classes, an application to clique coloring, and even the non-existence of a quantifier elimination scheme for nowhere dense classes.

1. Introduction and some motivating examples

Discrepancy theory emerged from the study of the irregularities of statistical distributions and number sequences. It developed and became a central tool in computational geometry. Two decades ago, Matoušek [29] initiated the study of *combinatorial discrepancy*, which became a significant subject in its own right. The combinatorial discrepancy measures the inevitable irregularities of set systems and the inherent difficulty to approximate them. (We refer to Section 2 for the formal definitions of the concepts considered in this paper.)

Discrepancy theory offers powerful tools and techniques with many applications in computational geometry, probabilistic algorithms, derandomization, communication complexity, searching, machine learning, pseudorandomness, optimization, computer graphics, and more. Central notions in this theory are also the well known notions of VC-dimension, ε-nets and ε-approximations, the latter corresponding to the expected properties of a pseudorandom set. We refer the reader to the textbooks [10,14,30] for a presentation of the discrepancy method and some of its applications.

A structural theory of classes of sparse graphs emerged recently, which is based on the study of densities of shallow minors, generalized coloring numbers, and constrained orientations [35]. In this setting, two central notions are those of *classes with bounded expansion*, which generalize classes excluding a topological minor, and *nowhere dense classes*, which generalize classes locally excluding a topological minor. These classes have strong algorithmic and structural properties. In particular, in a nowhere dense class it can be checked in almost linear time whether a first-order formula is satisfied in a given graph from the class [21]. This last example is only one among others that witness a strong connection between sparsity theory and first-order logic. We refer the interested reader to [35] for a comprehensive study of these classes.

The aim of this paper is to establish a bridge between discrepancy theory and sparsity theory through a study of the combinatorial discrepancy of set systems defined from sparse classes of graphs by means of first-order formulas and further to give new characterizations of degeneracy, bounded expansion, and nowhere denseness in terms of discrepancy of definable set systems. Before proceeding, we take time for a few motivating examples.

Problem 1. Assume that a graph G has the property that for every red/blue coloring of the edges of G there exists a partition (A, B) of the vertex set of G such that the number of red (resp. blue) neighbors in A and B of any vertex differ by at most 1. Does G contain a vertex with small degree?

> It follows from our results that such a graph G is 651-degenerate (see Section 8.1).

Problem 2. Given a planar graph G, find a small subset F of edges such that, for every pair u, v of distinct vertices of G, the probability that an edge of G belongs to a uv-path of length at most 100 differs from the probability that an edge in F belongs to a uv-path of length at most 100 by at most ε.

> We prove that a set F of edges of size $\mathcal{O}(1/\varepsilon)$ with the prescribed properties can be constructed deterministically in polynomial time (see Section 8.2).

Problem 3. Does there exist a constant c such that the vertices of every map graph[1] G can be colored red or blue in such a way that the difference between the number of red and blue vertices in every maximal clique of G is at most c?

> Although there are quite a few reasons to believe that such a constant would not exist (it is not even possible in general to color the vertices of a perfect graph red and blue in such a way that no maximal clique is monochromatic [8]) we prove that such a constant c exists for map graphs (see Section 8.3).

Last, we also consider the following (seemingly completely unrelated) problem from sparse finite model theory. It is known that every class of finite graphs with bounded expansion has a quantifier elimination scheme involving unary relations and functions [17]. As it is known that the fixed-parameter tractability of first-order model-checking extends from bounded expansion classes to the more general nowhere dense classes [20,21], it is natural to ask whether the quantifier elimination scheme also extends. It has been conjectured that this is not the case, but no proof of this fact was known.

Problem 4. Give an example of a nowhere dense class $\mathscr{C}$ of graphs such that there exists no expansion σ of the signature of graphs by unary relation and function symbols with the property that every first-order formula is equivalent on a σ-expansion $\mathscr{C}^+$ of $\mathscr{C}$ to a quantifier-free first-order σ-formula.

[1] A *map graph* is the half-square of the vertex-face incidence graph of a planar map.

We prove that the class $\mathscr{C}$ of 1-subdivisions of bipartite graphs whose girth exceeds the maximum degree has the above property. Precisely, there is no expansion σ of the signature of graphs by unary relation and function symbols in which the formula $\varphi(x,y)$ expressing that x and y are at distance 2 in the graph is equivalent on a σ-expansion of $\mathscr{C}$ to a quantifier-free first-order σ-formula (see Section 8.4).

All these results are obtained as special cases of general theorems on combinatorial discrepancy of set systems definable in sparse graph classes, which we prove in this paper.

This paper is organized as follows: In Section 2 we recall some needed concepts and notation and present the results obtained in this paper. In Section 3 we relate the notions of degeneracy and discrepancy. This connection is extended in Section 4 to a relation linking generalized coloring numbers and discrepancy. From Section 5 onward, we take a (finite) model theoretic point of view, putting the results obtained in Section 3 in a wider perspective. In Section 6, we translate the quantifier elimination scheme introduced in [17] for classes with bounded expansion into a model theoretic language, which allows us to extend the result obtained in Section 4 to the general setting of set systems definable in graphs of a bounded expansion class. This section is admittedly technical and is basically a formal translation of known results. Thus, the reader unfamiliar with model theory may directly move to Theorems 6.1 and 6.2. We discuss the case of nowhere dense classes in Section 7 and applications in Section 8.

2. Preliminaries and statement of the results

2.1. Combinatorial discrepancy

Let $(U,\mathscr{S})$ be a *set system*, where $\mathscr{S}$ is a collection of subsets of the *ground set* U. When the ground set is clear from the context, we refer to the set system as $\mathscr{S}$. The *discrepancy* of a mapping $\chi\colon U\to\{-1,1\}$ on a set $S\in\mathscr{S}$ is $\mathrm{disc}_\chi(S)=\left|\sum_{v\in S}\chi(v)\right|$; the *discrepancy* of χ on $\mathscr{S}$ is the maximum of $\mathrm{disc}_\chi(S)$ over all $S\in\mathscr{S}$, that is, $\mathrm{disc}_\chi(\mathscr{S})=\max_{S\in\mathscr{S}}\mathrm{disc}_\chi(S)$. The *(combinatorial) discrepancy* of $\mathscr{S}$ is the minimum discrepancy of a mapping $\chi\colon U\to\{-1,1\}$ on $\mathscr{S}$, that is,

$$\mathrm{disc}(\mathscr{S})=\min_{\chi:U\to\{-1,1\}}\ \max_{S\in\mathscr{S}}\left|\sum_{v\in S}\chi(v)\right|.$$

Thus, the combinatorial discrepancy of a set system measures how balanced a 2-coloring of this system can be.

It is usual to consider bounds for the discrepancy of a set system $(U,\mathscr{S})$ in terms of $n=\left|\bigcup\mathscr{S}\right|$ and $m=|\mathscr{S}|$. For instance, by a celebrated result of Spencer [43], the discrepancy of a set system is in $\mathcal{O}(\sqrt{n\log(m/n)})$, and in the case where $m=n$ we have $\mathrm{disc}(\mathscr{S})\le 6\sqrt{n}$. This latter bound is tight up to the constant. Another important result is the Theorem of Beck and Fiala [5].

Theorem 2.1 *(Beck-Fiala Theorem). The discrepancy of a set system with* degree *at most d (that is, each element lies in at most d sets) is less than $2d$.*

This theorem was subsequently improved by Bednarchak and Helm [6], who gave an upper bound of $2d-3$ for $d \geq 3$. Then Bukh [7] proved that the upper bound can be decreased to $2d - \log^* d$ for sufficiently large d.

Many of the techniques to derive upper bounds for the discrepancy are non-constructive and it is difficult to efficiently find optimal discrepancy colorings. Given a set system on n elements and $m \in \mathcal{O}(n)$ sets, it is NP-hard to distinguish whether the system has discrepancy 0 or $\Omega(\sqrt{n})$ [9]. In particular, under the assumption P $\neq$ NP, one cannot compute a function χ whose discrepancy on the set system is within factor $o(\sqrt{n})$ of the discrepancy of the system. However, in polynomial time one can compute a coloring χ with discrepancy $\mathcal{O}(\sqrt{n})$ [3,26]. Also, the proof of the Beck-Fiala Theorem is constructive, and gives a polynomial time deterministic algorithm to compute a coloring χ with discrepancy smaller than twice the degree of the set system.

A standard example of set systems with high discrepancy is given by the following.

Example 1 *(Sylvester's example).* Sylvester inductively constructed Hadamard matrices H_p of order 2^p for every non-negative integer p as follows: $H_0 = (\,1\,)$, $H_1 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, and for $p \geq 1$ $H_{p+1} = \left(\begin{smallmatrix} H_p & H_p \\ H_p & -H_p \end{smallmatrix}\right) = H_1 \otimes H_p$, where $\otimes$ denotes the Kronecker product of matrices. Let

$$\mathscr{S}_p = \big\{\{j \colon (H_p)_{i,j} = 1\} \colon 1 \leq i \leq 2^p\big\}.$$

Then

$$\mathrm{disc}(\mathscr{S}_p) = \Omega(\sqrt{n}),$$

where $n = |\bigcup \mathscr{S}_p| = |\mathscr{S}_p|$.

Unfortunately, the discrepancy is known to be a *fragile* notion, as witnessed by the following standard example (see e.g. [4]): let $\mathscr{S} = \{S_1, \dots, S_m\}$ be a set system with ground set U and let $\mathscr{S}' = \{S'_1, \dots, S'_m\}$ be a copy of $\mathscr{S}$ with ground set U' disjoint from U. Then the set system $\mathscr{S}'' = \{S_1 \cup S'_1 \dots, S_m \cup S'_m\}$ has always discrepancy zero, independent of the discrepancy of $\mathscr{S}$.A more robust notion is the *hereditary discrepancy* of a set system $(U, \mathscr{S})$, defined as $\mathrm{herdisc}(\mathscr{S}) = \max_{U' \subseteq U} \mathrm{disc}(\mathscr{S}|_{U'})$, where $\mathscr{S}|_{U'}$ denotes the set system $\{S \cap U' : S \in \mathscr{S}\}$. Moreover, bounding the hereditary discrepancy allows to bound the sizes of ε-nets and ε-approximations (what is not the case for discrepancy). For this reason, we focus on hereditary discrepancy in this paper. Note that the bound in the Beck-Fiala Theorem applies to the hereditary discrepancy as well, as considering the trace over a subset of the universe does not increase the degree of a set system.

2.2. Sparse graph classes

We consider finite, simple and undirected graphs. For a graph G we write $V(G)$ for its vertex set and $E(G)$ for its edge set. We denote by $\delta(G)$ the *minimum degree* of G, by $\bar{\mathrm{d}}(G)$ the *average degree* of G, and by $\omega(G)$ the *clique number* of G.

We denote by $N_G(v)$ the *neighborhood* of v in G, that is, the set of all the vertices adjacent to v in G. Of prime interest is the *neighborhood set system* $\mathscr{S}^E(G)$ of a graph G, which is defined as $\mathscr{S}^E(G) = \{N_G(v) : v \in V(G)\}$.

We write $H \subseteq G$ (resp. $H \subseteq_i G$) if H is a subgraph (resp. an induced subgraph) of G. A graph G is *d-degenerate* if every non-empty induced subgraph of G has minimum degree at most d.The minimum integer d such that a graph G is d-degenerate is the *degeneracy* $\deg(G)$ of G.Hence, $\deg(G) = \max_{H \subseteq_i G} \delta(H)$.

A class $\mathscr{C}$ of graphs is called *monotone* if it is closed under taking subgraphs and *hereditary* if it is closed under taking induced subgraphs. A class $\mathscr{C}$ is *degenerate* if there is an integer d such that all the graphs in $\mathscr{C}$ are d-degenerate.

The *r-subdivision* of a graph G is the graph $G^{(r)}$ obtained by subdividing every edge of G exactly r times. A *$\leq r$-subdivision* of G is a graph obtained by subdividing each edge of G at most r times. A graph H is a *topological minor* of a graph G *at depth* r if a $\leq 2r$-subdivision of H is a subgraph of G. We denote by $G \,\widetilde{\nabla}\, r$ the set of all the topological minors of G at depth r, and define $\widetilde{\nabla}_r(G) = \max\{|E(H)|/|V(H)| : H \in G \,\widetilde{\nabla}\, r\}$. The two key notions in the theory of sparsity [35] are the notions of *bounded expansion* and *nowhere denseness*, defined as follows.

Definition 2.1. A class $\mathscr{C}$ of graphs has *bounded expansion* if there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ with

$$\forall G \in \mathscr{C} \quad \forall H \in G \,\widetilde{\nabla}\, r \quad \bar{\mathrm{d}}(H) \leq f(r). \tag{1}$$

Example 2. All proper minor-closed classes of graphs and all classes of graphs with bounded maximum degree have bounded expansion.

Definition 2.2. A class $\mathscr{C}$ of graphs is *nowhere dense* if there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ with

$$\forall G \in \mathscr{C} \quad \forall H \in G \,\widetilde{\nabla}\, r \quad \omega(H) \leq f(r). \tag{2}$$

Note that every class with bounded expansion is nowhere dense but the converse does not necessarily hold.

Example 3. The class of all graphs G whose girth exceeds the maximum degree is nowhere dense but does not have bounded expansion.

Bounded expansion and nowhere dense classes enjoy numerous characterizations and applications (see [35]). Among others, let us mention the characterizations based on p-centered colorings [33] and on (generalized) weak coloring numbers [45].

The *depth* of a rooted forest is the maximum number of vertices in a path linking a root to a leaf. The *treedepth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest Y with vertex set $V(Y) = V(G)$ such that adjacent vertices in G belong to a path of Y linking a root to a leaf. For a positive integer p, a *p-treedepth coloring* of a graph G is a vertex coloring such that every $k \leq p$ color classes induce a subgraph with treedepth at most k. Closely related is the notion of a *p-centered coloring*, which is a coloring of the vertices such that every connected subgraph with strictly less than p colors on the vertices contains some uniquely colored vertex. It is easily checked that every $(p+1)$-centered coloring is a p-treedepth coloring (see [32,35]).

Theorem 2.2 *(Nešetřil, Ossona de Mendez [33]). A class $\mathscr{C}$ of graphs has bounded expansion if and only if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph $G \in \mathscr{C}$ admits, for each positive integer p, a p-centered coloring with at most $f(p)$ colors.*

We recall the definition of the generalized coloring numbers, which have been introduced by Kierstead and Yang [25] as a generalization of the so-called coloring number. Let G be a graph and let L be a linear ordering of $V(G)$. We say that a vertex u is *weakly d-reachable* from a vertex v if there exists in G a path P of length at most d (possibly 0) linking u and v such that u is the minimum vertex of P with respect to L, and we denote by $\mathrm{WReach}_d[G, L, v]$ the set of all vertices weakly d-reachable from v. The *weak coloring number* $\mathrm{wcol}_d(G)$ is defined as the minimum over all possible linear orderings L of $\max_{v \in V(G)} |\mathrm{WReach}_d[G, L, v]|$.

Theorem 2.3 *(Zhu, [45]). A class $\mathscr{C}$ of graphs has bounded expansion if and only if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathscr{C}$ and every positive integer d we have $\mathrm{wcol}_d(G) \leq f(d)$.*

Note that nowhere dense classes can also be characterized in terms of bounds on the weak coloring numbers.

Structural and algorithmic properties of classes with bounded expansion and nowhere dense classes have strong links with first-order logic. In particular, the fixed-parameter linear time first-order model-checking algorithm for bounded expansion classes [17] is based on a quantifier elimination scheme, which will be central to the study conducted in Section 6. This also justifies to extend the study of the discrepancy of neighborhood set systems presented in Sections 3 and 4 to first-order definable set systems.

2.3. Model theory and definable set systems

In order to establish a bridge between an approach of discrepancy based on structural graph theory and one based on (finite) model theory, we consider (from Section 5 onward) set systems defined from graphs by means of first-order formulas. As a motivating example among others, note that first-order logic is the foundation of query languages over relational databases (see e.g. [27]). In this original setting, queries are formulas and query answers form definable set systems.

For simplicity, we focus on (vertex colored) graphs, and on structures with only unary predicates and unary functions (which we call *pointer structures*). However, all definitions extend to general structures in a straightforward way. We use standard notation from graph theory [13] and finite model theory [27], and we refer to the referenced textbooks for all undefined notation and terminology.

Recall that a *signature* σ is a set of relation and function symbols, each with an attached arity. A *σ-structure* $\mathbf{M}$ is defined by its *domain* M (which is a set), and an interpretation of the symbols in σ as actual relations and functions on M.

A *monadic expansion* of a σ-structure $\mathbf{M}$ is a σ'-structure $\mathbf{M}'$ with the same domain as $\mathbf{M}$, where σ' is the union of σ and a set of unary relations, and where all the symbols in σ are interpreted the same way in $\mathbf{M}$ and $\mathbf{M}'$. By extension, a *monadic expansion* of a class $\mathscr{C}$ of σ-structures is a class $\mathscr{C}'$ of σ'-structures, such that each structure in $\mathscr{C}$ has a monadic expansion in $\mathscr{C}'$ and each structure in $\mathscr{C}'$ is a monadic expansion of some structure in $\mathscr{C}$. Monadic expansions play a key role e.g. in the quantifier-elimination scheme for bounded expansion classes, as we can encode local types of vertices by colors, and in our characterization theorems (Section 6 and Section 7). As the introduction of monadic expansions does not affect the proofs, it appeared beneficial to put our results in this general context.

The *Gaifman graph* of a structure $\mathbf{M}$ is the graph with vertex set M, where two (distinct) vertices $u, v \in M$ are adjacent if they belong together to a tuple of a relation (k-ary functions being interpreted as $(k+1)$-ary relations).

The *quantifier rank* of a first-order formula φ is the maximum depth of nested quantifiers in φ. For example, the quantifier rank of the formula $\exists x\,\big((\exists y\,(P(x,y) \vee (\forall z\,R(x,yz)))\big) \wedge (\forall t\,Z(x,t))$ is three. We say that two σ-structures $\mathbf{M}$ and $\mathbf{M}'$ are *r-equivalent* if they satisfy the same sentences with quantifier rank at most r.

A formula φ and a bi-partition of its free variables into two tuples $\bar{x}$ and $\bar{y}$ (with $\bar{x} = (x_1, \ldots, x_d)$ and $\bar{y} = (y_1, \ldots, y_\ell)$) defines a *partitioned formula* denoted by $\varphi(\bar{x}; \bar{y})$. For the sake of readability, we shall use the same symbol φ for the formula $\varphi(\bar{x}, \bar{y})$ and the partitioned formula $\varphi(\bar{x}; \bar{y})$. Let G be a graph with adjacency relation $E(x, y)$. A set $S \subseteq V(G)^d$ is *definable* in G *with parameters* if there exists a partitioned formula $\varphi(\bar{x}; \bar{y})$ and an ℓ-tuple $\bar{b} = (b_1, \ldots, b_\ell)$ of vertices of G such that a d-tuple $\bar{v} = (v_1, \ldots, v_d)$ belongs to S if and only if $G \models \varphi(\bar{v}, \bar{b})$. For $\bar{b} \in V(G)^\ell$, let $\varphi(G, \bar{b}) = \{\bar{v} \in V(G)^d : G \models \varphi(\bar{v}, \bar{b})\}$ be the subset of d-tuples of vertices of G defined by φ for the parameter $\bar{b}$. The *set system defined by* $\varphi(\bar{x}; \bar{y})$ on G is the set system $(V(G)^{|\bar{x}|}, \mathscr{S}^\varphi(G))$, where $\mathscr{S}^\varphi(G) = \{\varphi(G, \bar{b}) : \bar{b} \in V(G)^{|\bar{y}|}\}$. In particular, the neighborhood set system $\mathscr{S}^E(G)$ of a graph G is defined by the partitioned formula $E(x; y)$.

Example 4 *(Sylvester's graphs).* Let S_p be the bipartite graph with 2^{p+1} vertices, whose bipartite adjacency matrix (we have one vertex for each row and one vertex for each column and a row vertex is connected with a column vertex if the respective entry contains a 1) is obtained from the Hadamard matrix H_p (described in Example 1) by replacing each entry -1 by 0. Then we have

$$\operatorname{disc}(\mathscr{S}^E(S_p)) = \Omega\Big(\big|\bigcup \mathscr{S}^E(S_p)\big|^{1/2}\Big),$$

since $\mathscr{S}^E(S_p)$ is the set system obtained from the set system $\mathscr{S}_p$ introduced in Example 1 by duplicating every set (as H_p is symmetric).

Note that we allow formulas without parameter variables. In such a case, the set system defined by a formula $\rho(\bar{x})$ on G is the singleton $\{\rho(G)\}$, where $\rho(G) = \{\bar{v} \in V^{|\bar{x}|} : G \models \rho(\bar{v})\}$. More generally, for a finite set Φ of formulas $\varphi_i(\bar{x}, \bar{y})$ (with same $|\bar{x}|$ but possibly different $|\bar{y}|$), the set system defined by Φ on G is the set system $(V(G)^{|\bar{x}|}, \mathscr{S}^\Phi(G))$, where $\mathscr{S}^\Phi(G) = \bigcup_i \mathscr{S}^{\varphi_i}(G)$.

2.4. VC-dimension and shatter functions

The Vapnik–Chervonenkis dimension (VC-dimension) is a classical measure for the complexity of set systems [44]. We say that a subset X of the ground set U of a set system $(U, \mathscr{S})$ is *shattered* by $\mathscr{S}$ if $\mathscr{S}|_X$ contains all the subsets of X. The *VC-dimension* of $\mathscr{S}$ is the maximum size of a subset shattered by $\mathscr{S}$. Classes $\mathscr{C}$ of graphs with the property that for all first-order formulas φ, there exists a constant d such that the system $\mathscr{S}^\varphi(G)$ has VC-dimension bounded by d for all $G \in \mathscr{C}$ play a key role in model theory: they are called *dependent (or NIP) classes* [41]. Classes whose monotone closure is dependent are exactly nowhere dense classes [1].

A more precise description of the complexity of a set system $\mathscr{S}$ is given by its *(primal) shatter function* $\pi_{\mathscr{S}}$, which is defined as follows: for an integer m, we define $\pi_{\mathscr{S}}(m)$ to be the maximum over all subsets $X \subseteq U$ of size m of the number of distinct sets in the collection $\mathscr{S}|_X$. By the famous Sauer-Shelah Lemma [39,40,44] we have $\pi_{\mathscr{S}}(m) \in \mathcal{O}(m^d)$ if the VC-dimension of $\mathscr{S}$ is at most d.

A strong connection between these notions and combinatorial discrepancy is given by the next fundamental result, which is known to be optimal [2].

Theorem 2.4 *(Matoušek [28] and Matoušek, Welzl, and Wernisch [31]). Let $\mathscr{S}$ be a set system, let n be the size of the ground set of $\mathscr{S}$, and let d, C be constants, such that $\pi_{\mathscr{S}}(m) \leq Cm^d$ for all $m \leq n$. Then the discrepancy of $\mathscr{S}$ is bounded by*

$$\operatorname{disc}(\mathscr{S}) = \mathcal{O}\big(n^{1/2-1/2d}\big),\ \textit{if } d > 1,\ \textit{and } \operatorname{disc}(\mathscr{S}) = \mathcal{O}\big(\log^{5/2} n\big),\ \textit{if } d = 1.$$

Each set system $\mathscr{S}$ with ground set U can be represented by its *incidence graph* $I_{\mathscr{S}}$ with parts U and $\mathscr{S}$, where $u \in U$ is adjacent to $S \in \mathscr{S}$ if $u \in S$. The *dual* of a set system $\mathscr{S}$ with ground set U is the set system $\mathscr{S}^*$, whose incidence graph is obtained from the incidence graph of $\mathscr{S}$ by exchanging the parts. (Note that this corresponds to the notion of hypergraph duality.)

The *dual shatter function* $\pi^*_{\mathscr{S}}(m)$ of a set system $\mathscr{S}$ is defined as the primal shatter function of the dual set system $\mathscr{S}^*$. The discrepancy of a set system can also be bounded in terms of the dual shatter function (where the bound is also tight [28]). As noticed by Matoušek [28], the next theorem follows from a slight modification of the bound obtained by Matoušek, Welzl, and Wernisch [31] for the discrepancy of set systems defined by half-spaces, by using (in the abstract setting) the packing lemma due to Haussler [22], instead of the elementary lemma concerning the volume of an r-ball in an arrangement of hyperplanes due to Chazelle and Welzl [11].

Theorem 2.5. *Let $\mathscr{S}$ be a set system, let n be the size of the ground set of $\mathscr{S}$, and let d, C be constants, such that $\pi^*_{\mathscr{S}}(m) \le Cm^d$ for all $m \le n$. Then the discrepancy of $\mathscr{S}$ is bounded by*

$$\operatorname{disc}(\mathscr{S}) = \mathcal{O}\big(n^{1/2-1/2d}\sqrt{\log n}\big).$$

Bounded expansion and nowhere dense classes can be characterized by shatter functions:

Theorem 2.6 *(Pilipczuk, Siebertz, Toruńczyk [38]). Let $\mathscr{C}$ be a monotone class of graphs. Then*

- *the class $\mathscr{C}$ has bounded expansion if and only if for every formula $\varphi(\bar{x};\bar{y})$ there exists a constant C with $\pi_{\mathscr{S}^{\varphi}(G)}(m) \le Cm^{|\bar{y}|}$ and $|\mathscr{S}^{\varphi}(G)|_{A^{|\bar{x}|}}| \le C|A|^{|\bar{y}|}$ for every $G \in \mathscr{C}$ and every $A \subseteq V(G)$;*
- *the class $\mathscr{C}$ is nowhere dense if and only if for every formula $\varphi(\bar{x};\bar{y})$ and for every $\varepsilon > 0$ there exists a constant C with $\pi_{\mathscr{S}^{\varphi}(G)}(m) \le Cm^{|\bar{y}|+\varepsilon}$ and $|\mathscr{S}^{\varphi}(G)|_{A^{|\bar{x}|}}| \le C|A|^{|\bar{y}|+\varepsilon}$ for every $G \in \mathscr{C}$ and every $A \subseteq V(G)$.*

From this result, one can derive a bound on the hereditary discrepancy of sets systems definable in nowhere dense and bounded expansion classes. In the case of a nowhere dense class, the derived bound is established in Lemma 7.1; in the bounded expansion case, our result will establish a constant upper bound, clearly improving the bound derived from Theorems 2.4 and 2.6.

2.5. Our results

In Section 3, we prove that the notions of discrepancy and degeneracy are deeply linked. Precisely, we prove that the maximum discrepancy over all subgraphs H of a graph G of the neighborhood set system of H is sandwiched between $\Omega(\log \deg(G))$ and $\mathcal{O}(\deg(G))$ (see Theorem 3.1). In Section 4, we extend this result to inequalities relating weak coloring numbers and discrepancy of graph powers. Precisely, the maximum hereditary discrepancy over all subgraphs H of the power G^d of a graph G of the neighborhood set system of H is sandwiched between $\Omega(\log \mathrm{wcol}_{\lceil d/2\rceil}(G))$ and $\mathcal{O}(\mathrm{wcol}_d(G)^2)$ (Theorem 4.1), and we deduce a first characterization of bounded expansion classes in terms of discrepancy (Corollary 4.1).

In order to extend these results further, we switch to a model theoretic point of view. In Section 5, we introduce pointer structures, which are structures with only unary relations and unary functions, and prove that set systems definable by a quantifier-free formula in such structures have bounded hereditary discrepancy (Theorem 5.1). As a corollary, we deduce that every set system (induced by a finite ground set) definable in an infinite set has bounded discrepancy (Corollary 5.1) and that the neighborhood set systems of the graphs in a set-defined class have bounded hereditary discrepancy (Theorem 5.2). Then, we introduce the first-order theories $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$ for pointer structures, mimicking the characteristic properties of bounded expansion classes and prove that they have quantifier elimination (Theorem 6.1). From this, we deduce a characterization of bounded expansion classes as the monotone classes whose definable set systems have bounded hereditary discrepancy (Theorem 6.2).

Then, using bounds on the VC-density of set systems definable in nowhere dense classes proved in [38], we give a characterization of nowhere dense classes in terms of discrepancy (Theorem 7.1). We believe that our upper bounds on discrepancy can be improved in this case, and we propose a conjecture for the optimal bound (Conjecture 5).

In Section 8, we provide some corollaries on edge colored graphs (Corollary 8.1), ε-approximations (Corollary 8.2), clique coloring (Corollary 8.3), and quantifier elimination schemes (Corollary 8.4), which allows us to solve the motivating problems presented in the introduction.

Finally, in Section 9, we discuss some possible extensions of this work.

3. Discrepancy and degeneracy

In this section, we relate the degeneracy of a graph G to the discrepancy of the neighborhood set system $\mathscr{S}^E(G)$ of G. Our main result is the following theorem, which directly follows from Lemmas 3.1 and 3.2 proved below.

Theorem 3.1. *For every graph G we have*

$$\frac{\log_2(2\pi \deg(G))}{4} - 2 \le \max_{H\subseteq G} \mathrm{disc}(\mathscr{S}^E(H)) \le \max_{H\subseteq G} \mathrm{herdisc}(\mathscr{S}^E(H)) < 3\deg(G). \quad (3)$$

About the upper bound. The bipartite graph G with a part of size n and a part of size $\binom{n}{d}$ corresponding to all possible neighborhoods of size d in the part of size n is such that $\max_{H\subseteq G} \operatorname{disc}(\mathscr{S}^E(H)) \geq d = \deg(G)$. Hence, the upper bound of Theorem 3.1 is tight up to a constant factor.

About the lower bound. According to Example 4, there exist graphs H_n with arbitrarily large number n of vertices and $\operatorname{disc}(\mathscr{S}^E(H_n)) \in \Omega(n^{1/2})$. As H_n is a subgraph of K_n and $\deg(K_n) = n$ we deduce $\max_{H\subseteq K_n} \operatorname{disc}(\mathscr{S}^E(H)) \in \Omega(\deg(K_n)^{1/2})$. It is possible that the lower bound given in Theorem 3.1 might be improved to $\Omega(\deg(G)^c)$ for some positive constant c.

We now prove that the discrepancy of the neighborhood set system is linearly bounded by the degeneracy.

Lemma 3.1. *Let G be a graph. Then* $\operatorname{herdisc}(\mathscr{S}^E(G)) < 3\deg(G)$.

Proof. Let $d = \deg(G)$ and let $\vec{G}$ be an orientation of G with maximum out-degree d. Let $N^-(v)$ and $N^+(v)$ denote, respectively, the in-neighborhood and the out-neighborhood of a vertex v. Consider the set system $\mathscr{S}_1 = \{N^-(v) : v \in V(G)\}$. Every vertex v in $V(G)$ belongs to at most d sets in $\mathscr{S}_1$, for if $v \in N^-(u)$, then $u \in N^+(v)$. According to the Beck-Fiala Theorem, for every subset X of vertices there exists a function $\chi\colon X \to \{-1, 1\}$ such that for every vertex v we have $\left|\sum_{u\in N^-(v)\cap X} \chi(u)\right| < 2d$. As $|N^+(v) \cap X| \leq d$ we have $\left|\sum_{u\in N_G(v)\cap X} \chi(u)\right| < 3d$. Hence we have $\operatorname{disc}(\mathscr{S}^E(G)_{|X}) < 3\deg(G)$. □

Remark 3.1. Using Bukh's improvement [7], if $\deg(G)$ is sufficiently large, then the bound can be decreased to $3\deg(G) - \log^* \deg(G)$.

Lemma 3.2. *Let G be a graph and let $c = \max_{H\subseteq G} \operatorname{disc}(\mathscr{S}^E(H))$. Then G is $\frac{16^{c+2}}{2\pi}$-degenerate.*

Proof. In the following we consider G as a vertex-labeled graph and all subgraphs as vertex-labeled graphs (that is, we do not identify isomorphic subgraphs). First assume that G is bipartite. Let A and B be the two parts of the bipartition with $|A| \geq |B|$. For every subgraph $H \subseteq G$ denote by $\mathscr{S}_H$ the set system $\{N_H(v) : v \in A \cap V(H)\}$. Note that $\mathscr{S}_H \subseteq \mathscr{S}^E(H)$ and $\bigcup \mathscr{S}_H \subseteq B$. Hence $\operatorname{disc}(\mathscr{S}_H) \leq c$ with witness coloring $\gamma_H\colon B \to \{-1, 1\}$.

Let $n = |B|$, and $m = |E(G)|$. The graph G has 2^m spanning subgraphs and 2^n ways to color B (with colors in $\{-1, 1\}$). Hence, there exists a coloring γ such that the set $\mathcal{F}_\gamma$ of all spanning subgraphs H of G with $\operatorname{disc}_\gamma(\mathscr{S}_H) \leq c$ has size at least $2^m/2^n$.

For $v \in A$ and $\alpha \in \{-1, 1\}$, define $N_G^\alpha(v) = N_G(v) \cap \gamma^{-1}(\alpha)$. Then for every spanning subgraph H of G the inequality $\mathrm{disc}_\gamma(\mathscr{S}_H) \le c$ rewrites as

$$\forall v \in A \quad \big||N_H^{-1}(v)| - |N_H^1(v)|\big| \le c.$$

As G is bipartite, these conditions on the neighborhoods of the vertices in A are independent and the number of graphs H in $\mathcal{F}_\gamma$ is the product over all vertices $v \in A$ of the number of pairs $(X_v, Y_v) \subseteq N_G^{-1}(v) \times N_G^1(v)$ with $\big||X_v| - |Y_y|\big| \le c$.

Let $d^{-1}(v) = |N_G^{-1}(v)|$ and $d^1(v) = |N_G^1(v)|$. By considering supersets of $N_G^{-1}(v)$ and $N_G^1(v)$ obtained by adding c dummy elements, one easily checks that the number of pairs $(X_v, Y_v) \subseteq N_G^{-1}(v) \times N_G^1(v)$ with $\big||X_v| - |Y_v|\big| \le c$ is at most $\sum_{k=0}^{m(v)} \binom{d^{-1}(v)+c}{k}\binom{d^1(v)+c}{k}$, where $m(v) = \min(d^{-1}(v), d^1(v)) + c$. Thus,

$$|\mathcal{F}_\gamma| \le \prod_{v \in A} \sum_{k=0}^{m(v)} \binom{d^{-1}(v)+c}{k}\binom{d^1(v)+c}{k}.$$

$\triangleright$ **Claim 1.** *Let $a \le b$ be positive integers and let $s = \lceil (a+b)/2 \rceil$. Then*

$$\sum_{k=0}^{a} \binom{a}{k}\binom{b}{k} \le \binom{2s}{s}.$$

Proof of the claim. By replacing b by $b+1$ if necessary, we can assume that $a+b$ is even. Assume first that $a < b$, hence, $a + 1 \le b - 1$ (as a and b have the same parity). Then for every $0 \le k \le a$ we have $\frac{a+1}{a+1-k} \cdot \frac{b-k}{b} \ge 1$. Thus,

$$\binom{a+1}{k}\binom{b-1}{k} = \frac{a+1}{a+1-k}\binom{a}{k}\frac{b-k}{b}\binom{b}{k} \ge \binom{a}{k}\binom{b}{k}.$$

It follows that we can reduce to the case $a = b$. Then $\sum_{k=0}^{a} \binom{a}{k}\binom{b}{k} = \sum_{k=0}^{s} \binom{s}{k}^2 = \binom{2s}{s}$, which is the special case $m = n = r = s$ of the Chu-Vandermonde identity $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$. $\triangleleft$

Now we use the following upper bound.

$\triangleright$ **Claim 2** *([12, Lemma 17.5.1]). For $0 < k < n$ we have*

$$\binom{n}{k} \le \sqrt{\frac{n}{\pi k(n-k)}}\, 2^{nH(k/n)},$$

where $H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$, which is the standard entropy function. $\triangleleft$

As $H(1/2) = 1$ we have $\binom{2s}{s} \le \frac{4^s}{\sqrt{\pi s/2}}$. Let $h(v) = \lceil \frac{d(v)}{2} + c \rceil \ge \delta(G)/2$. We have

$$
\begin{aligned}
|\mathcal{F}_\gamma| &\le \prod_{v\in A} \sum_{k=0}^{m(v)} \binom{d^{-1}(v)+c}{k} \binom{d^{1}(v)+c}{k} \\
&\le \prod_{v\in A} \binom{2h(v)}{h(v)} \le \prod_{v\in A} \frac{2^{2h(v)}}{\sqrt{\pi h(v)/2}} \\
&\le \prod_{v\in A} \frac{2^{d(v)+2c+1}}{\sqrt{\pi\delta(G)/4}} \le \prod_{v\in A} \Big(\frac{4^{c+1}}{\sqrt{\pi\delta(G)}} 2^{d(v)}\Big) \\
&\le \Big(\frac{4^{c+1}}{\sqrt{\pi\delta(G)}}\Big)^{|A|} 2^m.
\end{aligned}
$$

As $|\mathcal{F}_\gamma| \ge \left(\frac{1}{2}\right)^n 2^m$ we get $\frac{4^{c+1}}{\sqrt{\pi\delta(G)}} \ge 2^{-\frac{n}{\lceil A\rceil}}$. Hence, as $n = |B| \le |A|$, we get $\delta(G) \le \frac{64}{\pi} \cdot 16^c$. As this holds for every induced subgraph, we get $\deg(G) \le \frac{64}{\pi} \cdot 16^c$.

Now, if G is not bipartite, we get that the degeneracy is at most $16^{c+2}/2\pi$, as G includes a bipartite subgraph whose degeneracy is at least one half of the degeneracy of G: first we extract a subgraph G' of G whose minimum degree is the degeneracy of G (existence of such a subgraph follows from the definition of degeneracy), then construct a bipartite subgraph G'' of G' whose minimum degree is at least half of the minimum degree of G'. $\square$

4. Discrepancy and generalized coloring numbers

In this section, we extend the result of Section 3 by considering the (generalized) weak coloring numbers. Our main result is the following.

Theorem 4.1. *Let G be a graph and let d be a positive integer. Then*

$$
\begin{aligned}
\frac{\log_2(\mathrm{wcol}_{\lceil d/2\rceil}(G))}{6(d+1)} - \frac{\log_2(d+1)}{3} - \frac{5}{4} &\le \max_{d'\le d} \max_{H\subseteq G} \mathrm{herdisc}(\mathscr{S}^E(H^{d'})) \\
&< (2d\,\mathrm{wcol}_{d-1}(G)+1)\,\mathrm{wcol}_d(G).
\end{aligned}
$$

Note that, as $\mathrm{wcol}_1(G) = \deg(G)+1$ and $\mathrm{wcol}_0(G) = 1$, Theorem 4.1 yields (for $d = 1$)

$$
\frac{\log_2(\deg(G)+1)}{12} - \frac{19}{12} \le \max_{H\subseteq G} \mathrm{herdisc}(\mathscr{S}^E(H)) < 3\deg(G)+3,
$$

to be compared with Theorem 3.1.

As a corollary of Theorem 4.1 we obtain the following characterization of classes with bounded expansion.

Corollary 4.1. *Let $\mathscr{C}$ be a monotone class of graphs. Then $\mathscr{C}$ has bounded expansion if and only if the hereditary discrepancy of $\mathscr{S}^E(G^k)$ is bounded on $\mathscr{C}$ for each positive integer k.*

The proof of Theorem 4.1 will follow from Lemmas 4.1 and 4.2, which we will prove next.

Lemma 4.1. *Let G be a graph and d a positive integer. Then*

$$\mathrm{herdisc}(\mathscr{S}^E(G^d)) < (2d\,\mathrm{wcol}_{d-1}(G) + 1)\,\mathrm{wcol}_d(G).$$

Proof. Let L be a linear ordering such that $\max_{v \in V(G)} |\mathrm{WReach}_d[G, L, v]| = \mathrm{wcol}_d(G)$. For brevity, in the following we write $\mathrm{WReach}_i[v]$ for $\mathrm{WReach}_i[G, L, v]$. For a vertex $v \in V(G)$ we define $W_0(v) = \{v\}$ and, for $1 \le i \le d$, $W_i(v) = \mathrm{WReach}_i[v] \setminus \mathrm{WReach}_{i-1}[v]$. Hence $(W_0(v), \dots, W_d(v))$ is a partition of $\mathrm{WReach}_d[v]$. In particular, $\sum_{i=0}^d |W_i(v)| = |\mathrm{WReach}_d[v]| \le \mathrm{wcol}_d(G)$. We further define, for $0 \le i \le d$, $\mathrm{WReach}^*_i[v] = \{u : v \in \mathrm{WReach}_i[u]\}$.

For every vertex v, we denote by $B_d[v]$ the set of all vertices at distance at most d from v. Note that $B_d[v] = N_{G^d}(v) \cup \{v\}$.

▷ **Claim 3.** *For every two vertices u and v, we have $u \in B_d[v]$ if and only if there exists some $0 \le i \le d$ with $W_i(v) \cap \mathrm{WReach}_{d-i}[u] \neq \emptyset$.*

Proof of the claim. Obviously, if $W_i(v) \cap \mathrm{WReach}_{d-i}[u]$ is not empty, then v is at distance at most d from u, thus $u \in B_d[v]$.

Conversely, assume $u \in B_d[v]$. Let P be a shortest path linking u and v, let m be the minimum vertex of P with respect to L, and let $i = \mathrm{dist}(m, v)$. As P is a shortest path, we have $m \in W_i(v) \cap \mathrm{WReach}_{d-i}[u] \neq \emptyset$. ◁

According to this claim we have

$$\begin{aligned} B_d[v] &= \bigcup_{i=0}^{d} \{u \mid W_i(v) \cap \mathrm{WReach}_{d-i}[u] \neq \emptyset\} \\ &= \bigcup_{i=0}^{d} \bigcup_{z \in W_i(v)} \{u \mid z \in \mathrm{WReach}_{d-i}[u]\} \\ &= \bigcup_{i=0}^{d} \bigcup_{z \in W_i(v)} \mathrm{WReach}^*_{d-i}[z]. \end{aligned}$$

Let $\mathscr{S}' = \{\mathrm{WReach}^*_i[z] : z \in V(G), 1 \le i \le d\}$. If $u \in \mathrm{WReach}^*_i[z]$, then $z \in \mathrm{WReach}_i[u]$, thus every vertex belongs to at most $d \cdot \mathrm{wcol}_d(G)$ sets $\mathrm{WReach}^*_i[z]$. According to the Beck-Fiala Theorem, for every subset X of vertices there exists a function $\chi\colon X \to \{-1, 1\}$ such that $|\mathrm{disc}_\chi(\mathrm{WReach}^*_i[z] \cap X)| \le 2d\,\mathrm{wcol}_d(G) - 1$ for all $z \in V(G)$ and $1 \le i \le d$.

As $\bigcup_{z\in W_d(v)} \mathrm{WReach}^*_0[z] = W_d(v)$, we have

$$\begin{aligned}
\mathrm{disc}_\chi(B_d[v] \cap X) &\le \Big(\sum_{i=0}^{d-1} \sum_{z\in W_i(v)} \mathrm{disc}_\chi(\mathrm{WReach}^*_{d-i}[z] \cap X)\Big) + \mathrm{disc}_\chi(W_d(v) \cap X) \\
&\le \Big(\sum_{i=0}^{d-1} |W_i(v)|\Big)\,(2d\,\mathrm{wcol}_d(G) - 1) + |W_d(v)| \\
&\le \mathrm{wcol}_{d-1}(G)\,(2d\,\mathrm{wcol}_d(G) - 1) + (\mathrm{wcol}_d(G) - 1) \\
&\le (2d\,\mathrm{wcol}_{d-1}(G) + 1)\,\mathrm{wcol}_d(G) - 2
\end{aligned}$$

As $N_{G^d}(v) = B_d[v] \setminus \{v\}$ we have $\mathrm{disc}_\chi(N_{G^d}(v) \cap X) \le \mathrm{disc}_\chi(B_d[v] \cap X) + 1$. □

Lemma 4.2. *Let G be a graph and let ℓ be a positive integer. Then there exists a subgraph H of G and an integer $d' < 2\ell$ with*

$$\mathrm{herdisc}(\mathscr{S}^E(H^{d'})) \ge \frac{1}{12\ell}\log_2(\mathrm{wcol}_\ell(G)) - \frac{1}{3}\log_2 \ell - \frac{19}{12}.$$

Proof. We make use of the following inequalities relating the weak coloring numbers and shallow topological minor average degrees to ℓ-admissibility[2] [42, Theorem 4.1.3] and [19].

$$\begin{aligned}
\mathrm{wcol}_\ell(G) &\le \mathrm{adm}_\ell(G)^\ell \\
\mathrm{adm}_\ell(G) &\le 6\ell\,\lceil \widetilde{\nabla}_{\ell-1}(G)\rceil^3.
\end{aligned}$$

Let $\alpha = \left(\frac{1}{6\ell}\right)^{1/3} \mathrm{wcol}_\ell(G)^{1/3\ell}$. Let T_1 be a shallow topological minor of G at depth $(\ell - 1)$ witnessing $\widetilde{\nabla}_{\ell-1}(G) \ge \alpha$.

▷ **Claim 4.** *There is an integer $1 \le d' \le 2\ell - 1$, such that the graph T_1 has a subgraph T_2 with degeneracy at least α/ℓ, all the edges of which appear in G as paths of length d'.*

Proof of the claim. Each edge of T_1 appears in G as a path of length $i + 1$, for some $0 \le i \le 2(\ell - 1)$. By the pigeonhole principle, for some $1 \le d' \le 2\ell - 1$, at least

[2] We refer the interested reader to [19] for a definition of ℓ-admissibility. Note that the definition of $\mathrm{adm}_\ell(G)$ and $\mathrm{wcol}_\ell(G)$ used in our paper and the two quoted papers differs by 1 from the original definition of [16].

$|E(T_1)|/(2\ell-1)$ edges of T_1 appears in G as a path of length d'. Let T_2 be the subgraph of T_1 induced by these edges. Then, we have $\deg(T_2) \geq 2|E(T_2)|/|V(T_2)| \geq 2|E(T_2)|/|V(T_1)| \geq \alpha/\ell$. $\lhd$

By Theorem 3.1, T_2 has a subgraph T_3 with $\mathrm{disc}(\mathscr{S}^E(T_3)) \geq \frac{1}{4}\log_2(2\pi\alpha/\ell) - 2$. The $(d'-1)$-subdivision H of T_3 is a subgraph of G, and $\mathrm{disc}(\mathscr{S}^E(T_3)) \leq \mathrm{herdisc}(\mathscr{S}^E(H^{d'}))$ (as T_3 is isomorphic to $H^{d'}[X]$, where X is the set of the vertices of H corresponding to vertices of T_3). Hence, the result follows, as

$$\begin{aligned}\frac{1}{4}\log_2(2\pi\alpha/\ell) - 2 &= \frac{1}{4}\Big(\log_2(2\pi/\ell) - \frac{1}{3}\log_2(6\ell) + \frac{1}{3\ell}\log_2(\mathrm{wcol}_\ell(G))\Big) - 2\\ &= \frac{1}{12\ell}\log_2(\mathrm{wcol}_\ell(G)) - \frac{1}{3}\log_2\ell + \Big(\frac{\log_2(2\pi)}{4} - \frac{\log_2 6}{12} - 2\Big)\\ &\geq \frac{1}{12\ell}\log_2(\mathrm{wcol}_\ell(G)) - \frac{1}{3}\log_2\ell - \frac{19}{12}. \quad \square\end{aligned}$$

Proof of Theorem 4.1. If $H \subseteq G$ and $d' \leq d$, then $\mathrm{wcol}_{d'}(H) \leq \mathrm{wcol}_d(G)$. Thus, we deduce the upper bound of Theorem 4.1 from Lemma 4.1.

According to Lemma 4.2 applied to G and $\ell = \lceil d/2 \rceil$, there exists a subgraph H of G and an integer $d' \leq d$ with

$$\begin{aligned}\mathrm{herdisc}(\mathscr{S}^E(H^{d'})) &\geq \frac{1}{12\lceil d/2\rceil}\log_2(\mathrm{wcol}_{\lceil d/2\rceil}(G)) - \frac{1}{3}\log_2\lceil d/2\rceil - \frac{19}{12}\\ &\geq \frac{1}{6(d+1)}\log_2(\mathrm{wcol}_{\lceil d/2\rceil}(G)) - \frac{1}{3}\log_2(d+1) - \frac{5}{4}. \quad \square\end{aligned}$$

5. A model theoretic approach to degeneracy: pointer structures

A *pointer structure* is a unary structure, that is, a structure with a (possibly infinite) signature σ that consists of unary relation and unary function symbols. It will be convenient to assume that the function symbols in σ are the elements of a monoid $\mathcal{F}_\sigma$. We fix the signature σ and we will consider only σ-structures satisfying the theory T_σ that consists of the sentences $\forall x\, f(g(x)) = h(x)$, where f and g range over $\mathcal{F}_\sigma$ and $h = f \circ g$. Note that this is not really a restriction, but rather a way to simplify the notations.

Observation 5.1. For every d-degenerate graph G there is a pointer structure $\mathbf{M}$ with d unary functions such that $\mathscr{S}^E(G) = \mathscr{S}^\eta(\mathbf{M})$, where $\eta(x,y)$ is the quantifier-free partitioned formula

$$\eta(x;y) := \neg(x = y) \wedge \bigvee_{i=1}^{d}\big((f_i(x) = y) \vee (f_i(y) = x)\big).$$

Proof. If G is d-degenerate, then G has an orientation $\vec{G}$ with maximum out-degree d. Consider such an orientation $\vec{G}$ of G and an arbitrary labeling of the arcs of $\vec{G}$ by integers between 1 and d such that the arcs with the same origin have pairwise different labels. We define $\mathbf{M}$ as the pointer structure with domain $V(G)$ such that for every vertex v and every integer $i \in [d]$ we have $f_i(v) = w$ if (v, w) is an arc of $\vec{G}$ with label i, and $f_i(v) = v$ if no arc with origin v has label i. That $G \models E(u, v)$ is equivalent to $\mathbf{M} \models \eta(u; v)$ is straightforward. □

For a pointer structure $\mathbf{M}$ (with signature σ) and a subset A of the domain of $\mathbf{M}$, we define the pointer structure $\mathbf{M}[A]$ *weakly induced* by A on $\mathbf{M}$ as the pointer structure with domain A (and signature σ), such that for every relation symbol $P \in \sigma$, every function symbol $f \in \sigma$, and all distinct u and v in A, we have

$$\mathbf{M}[A] \models P(u) \iff \mathbf{M} \models P(u)$$
$$\mathbf{M}[A] \models f(u) = v \iff \mathbf{M} \models f(u) = v$$

(It follows that if $u \in A$ and the image by f of u in $\mathbf{M}$ is not in A, then u is a fixed point of f in $\mathbf{M}[A]$.) Note that the Gaifman graph of $\mathbf{M}[A]$ is the subgraph of the Gaifman graph of $\mathbf{M}$ induced by A.

5.1. Hereditary discrepancy of QF-definable set systems

The goal of this section is to prove the following theorem.

Theorem 5.1. *For every finite set $\Phi = \{\varphi_1(\bar{x}; \bar{y}), \ldots, \varphi_p(\bar{x}; \bar{y})\}$ of quantifier free partitioned σ-formulas (with same $|\bar{x}|$ but possibly different $|\bar{y}|$) there exists a constant C_Φ such that for every σ-structure $\mathbf{M}$ we have*

$$\mathrm{herdisc}(\mathscr{S}^\Phi(\mathbf{M})) \leq C_\Phi.$$

The proof goes by the following key steps. One of the building blocks is the Beck-Fiala Theorem. However, we cannot directly apply this result as it requires set systems to have bounded degree. We generalize the theorem to set systems that are Boolean combinations of set systems with bounded degree (Lemmas 5.1 and 5.2). Finally, we prove that we can decompose set systems defined by quantifier free formulas into Boolean combinations of set systems that have bounded degree (Lemma 5.3), which concludes the proof of Theorem 5.1.

The two following lemmas extend the Beck-Fiala Theorem to handle set systems built from sets in a set system of bounded degree.

Lemma 5.1. *Let $(U, \mathscr{S}_0)$ be a set system of degree at most d. Then the set system $(U, \mathscr{S}_1)$, where $\mathscr{S}_1$ is the collection of all intersections of sets in $\mathscr{S}_0$ (with the convention that U is the empty intersection) has degree at most 2^d.*

Proof. As each $v \in U$ belongs to at most d sets $S_1, \dots, S_d$ in $\mathscr{S}_0$, it belongs to at most 2^d sets in $\mathscr{S}_1$, namely the sets of the form $\bigcap_{i \in I} S_i$, for $I \subseteq [d]$. Thus, the set system $(U, \mathscr{S}_1)$ has degree at most 2^d. $\square$

Lemma 5.2. *Let $(U, \mathscr{S}_1)$ and $(U, \mathscr{S})$ be two set systems over the same ground set U such that $\mathscr{S}_1$ is closed under taking intersections (including U, the empty intersection) and such that every set in $\mathscr{S}$ is a Boolean combination of at most k sets from $\mathscr{S}_1$. Then* $\mathrm{herdisc}(\mathscr{S}) \le 4^k \,\mathrm{herdisc}(\mathscr{S}_1)$.

Proof. Let $S \in \mathscr{S}$. By assumption, S is a Boolean combination of at most k sets $S_1, \dots, S_k$ in $\mathscr{S}_1$. This Boolean combination can be written as a disjoint union of at most 2^k terms of the form $T_I = \bigcap_{i \in I} S_i \cap \bigcap_{j \in [k] \setminus I} (U \setminus S_j)$, where $I \subseteq [k]$. Let $\mathbb{I}_Y : U \to \{0, 1\}$ denote the indicator function of a subset Y of U. For a subset J of $[k]$, let $S_J = \bigcap_{j \in J} S_j$. Note that $S_J \in \mathscr{S}_1$. Then we have

$$\mathbb{I}_{T_I} = \mathbb{I}_{S_I} \cdot \prod_{j \in [k] \setminus I} (\mathbb{I}_U - \mathbb{I}_{S_j}) = \sum_{J \subseteq [k] \setminus I} (-1)^{|J|} \, \mathbb{I}_{S_{I \cup J}}.$$

To see this, consider some $u \in U$. Assume first that u does not appear in any of the sets S_j for $j \in [k] \setminus I$. Then the only term in the sum with $\mathbb{I}_{S_{I \cup J}}(u) \neq 0$ is that with $J = \emptyset$, so we obtain exactly I_{S_I}. Otherwise, the set $K = \{j \in [k] \setminus I : u \in S_j\}$ is non-empty and $u \in S_J$ if and only if $J \subseteq K$. Thus we count u exactly $\sum_{J \subseteq K} (-1)^{|J|} = \sum_{i=0}^{|K|} \binom{|K|}{i} (-1)^i = (1-1)^{|K|} = 0$ times by the binomial theorem.

Hence, for any $X \subseteq U$ we have $|\chi(T_I \cap X)| \le \sum_{J \subseteq [k] \setminus I} |\chi(S_{I \cup J} \cap X)| \le 2^k \cdot \mathrm{herdisc}(\mathscr{S}_1)$. As $\chi(S \cap X)$ is the sum of at most 2^k terms of the form $\chi(T_I \cap X)$ we deduce $\mathrm{herdisc}(\mathscr{S}) \le 4^k \,\mathrm{herdisc}(\mathscr{S}_1)$. $\square$

We now turn to Lemma 5.3, which is the main ingredient from the logical point of view.

Lemma 5.3. *Let $\varphi(\bar{x}; \bar{y})$ be a quantifier-free partitioned σ-formula. There exists a finite set*

$$\Psi = \{\rho_i(\bar{x}) : 1 \le i \le k\} \cup \{\psi_j(\bar{x}; \bar{z}) : 1 \le j \le \ell\}$$

of quantifier-free partitioned σ-formulas (with same $|\bar{x}|$, but possibly different $|\bar{z}|$ for the partitioned formulas ψ_j) where each $\psi_j(\bar{x}; \bar{z})$ has the form

$$\psi_j(\bar{x}; \bar{z}) := \bigwedge_{r=1}^{|\bar{z}|} \left(f_{\alpha_{j,r}}(x_{i_{j,r}}) = z_r \right)$$

and there is an integer p such that for every σ-structure $\mathbf{M}$, every set in $\mathscr{S}^{\varphi}(\mathbf{M})$ is a Boolean combination of at most p sets in $\mathscr{S}^{\Psi}(\mathbf{M})$, and $\mathscr{S}^{\Psi}(\mathbf{M})$ has degree at most $|\Psi| = k + \ell$.

Note that we do not translate φ into an equivalent formula. An example for the construction can be found below the proof and it may be helpful to consider it in parallel while reading the proof.

Proof. The partitioned formula $\varphi(\bar{x};\bar{y})$ is equivalent (modulo T_σ) to a Boolean combination of formulas of the form

$$\zeta(\bar{x};\bar{y}) := \rho(\bar{x}) \wedge \eta(\bar{y}) \wedge \bigwedge_{(f,g,i,j)\in\mathcal{F}} f(x_i) = g(y_j),$$

where $\mathcal{F}$ is a finite subset of $\mathcal{F}_\sigma \times \mathcal{F}_\sigma \times [|\bar{x}|] \times [|\bar{y}|]$. This is because we may assume that φ uses no composition of functions, as $\mathcal{F}_\sigma$ is a monoid, and because every atomic formula can either use a unary relation symbol (and then is in ρ or η), or express equality of two terms. If both terms are of the form $f(x_i)$ (resp. $g(y_j)$), then this atomic formula is in ρ (resp. in η), otherwise it must be of the form $f(x_i) = g(y_j)$. Moreover, we can assume that each pair (g, j) belongs to at most one tuple $(f, g, i, j) \in \mathcal{F}$ as $(f(x_i) = g(y_j)) \wedge (h(x_{i'}) = g(y_j))$ is equivalent to $(f(x_i) = h(x_{i'})) \wedge (f(x_i) = g(y_j))$. Then for every structure $\mathbf{M}$ and every tuple $\bar{b}$ we have that $\varphi(\mathbf{M};\bar{b})$ is a Boolean combination of sets of the form $\zeta(\mathbf{M};\bar{b})$.

Now for every $\zeta(\bar{x};\bar{y}) = \rho(\bar{x}) \wedge \eta(\bar{y}) \wedge \bigwedge_{(f,g,i,j)\in\mathcal{F}} f(x_i) = g(y_j)$ of the Boolean combination, we add ρ and ψ to Ψ, where we define $\bar{z}$ and $\psi(\bar{x};\bar{z})$ from $\bigwedge_{(f,g,i,j)\in\mathcal{F}} f(x_i) = g(y_j)$ by replacing each appearing term $g(y_j)$ by a new variable $z_{g,j}$. Note that $\bar{z}$ may be longer than $\bar{y}$, however its length is at most the number of conjuncts in the big conjunction in ζ. Then ψ has the required form. Moreover, for every tuple $\bar{b}$, we can define an assignment $\bar{c}$ of $\bar{z}$ (let $c_{g,j} := g(b_j)$) such that

$$\zeta(\mathbf{M};\bar{b}) = \begin{cases} \rho(\mathbf{M}) \cap \psi(\mathbf{M};\bar{c}) & \text{if } \mathbf{M} \models \eta(\bar{b}) \\ \emptyset & \text{otherwise.} \end{cases}$$

Finally, observe that for every tuple $\bar{a}$, there do not exist two distinct tuples $\bar{c}, \bar{c}'$ with $\bar{a} \in \psi(\mathbf{M};\bar{c}) \cap \psi(\mathbf{M};\bar{c}')$. If $\bar{a} \in \psi(\mathbf{M};\bar{c}) \cap \psi(\mathbf{M};\bar{c}')$, then for all g, j of the conjunction, $c_{g,j} = f(a_i) = c_{g,j'}$, hence $\bar{c} = \bar{c}'$. Hence the degree of each $\mathscr{S}^\psi$ is 1. Obviously, the degree of each $\mathscr{S}^\rho$ is also 1 (as this set system contains only one set). As we have k formulas ρ and ℓ partitioned formulas ψ, we have that the degree of $\mathscr{S}^\Psi(\mathbf{M})$ is at most $k + \ell$. $\square$

Example 5. Let $\varphi(x_1, x_2, x_3; y_1, y_2) = R(x_3) \wedge B(y_1) \wedge (h(x_1) = f(y_2)) \wedge (h(x_2) = g(y_2))$. We replace $f(y_2)$ by z_1, $g(y_2)$ by z_2, and obtain $\rho(\bar{x}) = R(x_3)$ and $\psi(\bar{x};\bar{z}) = (h(x_1) = z_1)$ $\wedge(h(x_2) = z_2)$.

Let $\mathbf{M}$ be a σ-structure, and let b_1, b_2 be two elements of the universe of $\mathbf{M}$, from which we derive $c_1 = f(b_2)$ and $c_2 = g(b_2)$. Note that for any tuple $\bar{a}$, if $\mathbf{M} \models \varphi(\bar{a};\bar{b})$, then $\mathbf{M} \models \rho(\bar{a}) \wedge \psi(\bar{a};\bar{c})$. Note that the inverse is not true, for example if $B(b_1)$ is not satisfied. Nonetheless, for any fixed b_1, b_2, the set $\varphi(\mathbf{M};\bar{b})$ is either empty or equal to $\rho(\mathbf{M}) \cap \psi(\mathbf{M};\bar{c})$.

For the degree bound, take two tuples $\bar{c}$ and $\bar{c}'$ such that $\psi(\mathbf{M};\bar{c}) \cap \psi(\mathbf{M};\bar{c}') \neq \emptyset$. Take a tuple $\bar{a}$ in the intersection. We have that $h(a_1) = c_1 = c'_1$ and $h(a_2) = c_2 = c'_2$, hence $\bar{c} = \bar{c}'$. So all sets in $\psi(\mathbf{M};\bar{z})$ are pairwise disjoint.

Example 6. Let $\varphi(x;y) = A(x) \wedge \Big(\big(B(y) \wedge (f(x) = y)\big) \vee \big(\neg B(y) \wedge (f(x) = f(y))\big)\Big)$. Then we can choose $\Psi = \{\rho(x), \psi(x;z)\}$, where $\rho(x) = A(x)$ and $\psi(x;z) = (f(x) = z)$, as for every b we have

$$\varphi(G;b) = \begin{cases} \rho(G) \cap \psi(G;b) & \text{if } b \in B(G) \\ \rho(G) \cap \psi(G;f(b)) & \text{otherwise.} \end{cases}$$

We finally deduce a uniform bound for set systems defined on σ-structures by quantifier-free partitioned formulas, the main result of this section.

Proof of Theorem 5.1. According to Lemma 5.3 (applied to each partitioned formula in Φ) there exists a finite set Ψ of quantifier free partitioned formulas and integers k, t such that for every σ-structure $\mathbf{M}$ the set system $\mathscr{S}^\Psi(\mathbf{M})$ has degree at most t, and every set in $\mathscr{S}^\Phi$ is a Boolean combination of at most k sets in $\mathscr{S}^\Psi$. Let $\mathbf{M}$ be any σ-structure, and let $\mathscr{S}_0$ be the collection of all intersections of sets in $\mathscr{S}^\Psi(\mathbf{M})$. According to Lemma 5.1, the set system $\mathscr{S}_0$ has degree at most 2^t. Thus, according to Theorem 2.1 we have $\operatorname{herdisc}(\mathscr{S}_0) < 2^{t+1}$. Then it follows from Lemma 5.2 that $\operatorname{herdisc}(\mathscr{S}^\Phi) < 4^k \operatorname{herdisc}(\mathscr{S}_0) < 2^{2k+t+1}$, so we can choose $C_\Phi = 2^{2k+t+1}$. $\square$

Note that this theorem is tight in the sense that for every signature σ with a relation or a function of arity at least two there exists a quantifier-free formula with unbounded discrepancy on the class of all σ-structures: for instance, consider the structure with domain $X \cup 2^X$ and the relation $E(x,y) := x \in y$ or the structure with domain $X \cup 2^X \cup \{\top, \bot\}$ with $f(x,y) = \top$ if $x \in y$, and $f(x,y) = \bot$, otherwise. In both cases, the discrepancy of the set system defined by E or f is at least $|X|/2$.

5.2. Pointer structures and set-defined graphs

Pointer structures are intrinsically related to set-defined classes. Recall that the *age* of an infinite graph is the class of all its finite induced subgraphs. A class $\mathscr{C}$ (of finite graphs) is *set-defined* if it is included in the age of an (infinite) graph definable in $\mathbb{N}$ (seen as a model of an infinite set in the empty language) [24]. In other words, $\mathscr{C}$ is set defined if there exists a partitioned formula $\varphi(\bar{x};\bar{y})$ with $|\bar{x}| = |\bar{y}| = d$ such that every graph in $\mathscr{C}$ is an induced subgraph of the graph with vertex set $\mathbb{N}^d$ in which $\bar{u}$ and $\bar{v}$ are adjacent if and only if $\mathbb{N} \models \varphi(\bar{u};\bar{v})$. Set-defined classes have bounded VC-dimension (they are actually both semi-algebraic and stable). Set-defined classes of graphs generalize classes of degenerate graphs. Indeed, it is proved in [24] that a class is degenerate if and only if it is *biclique-free* (i.e. excludes some biclique as a subgraph) and set-defined.

Lemma 5.4. *Let $\mathscr{C}$ be a class of pointer structures, and let $q(\bar{x};\bar{y})$ be a quantifier-free partitioned formula in the language of $\mathscr{C}$. Then, there is a set-defined class $\mathscr{D}$ of graphs and a mapping $f:\mathscr{C}\to\mathscr{D}$, such that $\mathscr{S}^q(\mathbf{M})\subseteq\mathscr{S}^E(f(\mathbf{M}))$, for all $\mathbf{M}\in\mathscr{C}$.*

Conversely, if $\mathscr{D}$ is a set-defined class of graphs, then there exists a class $\mathscr{C}$ of pointer structures, a quantifier-free partitioned formula $q(\bar{x};\bar{y})$ in the language of $\mathscr{C}$, and a mapping $g:\mathscr{D}\to\mathscr{C}$, such that $\mathscr{S}^E(G)\subseteq\mathscr{S}^q(g(G))$, for all $G\in\mathscr{D}$.

Proof. Let $\mathscr{C}$ be a class of pointer structures, and let $q(\bar{x};\bar{y})$ be a quantifier-free partitioned formula in the language of $\mathscr{C}$. We can assume that the formula q contains no composition of functions (as we assumed that the functions in the signature form a monoid). Let $f_1,\dots,f_a$ (resp. $P_1,\dots,P_b$) be the functions (resp. the predicates) appearing in q. Moreover, we can assume that the domains of the structures in $\mathscr{C}$ are all subsets of $\mathbb{N}\setminus\{1,2\}$. Let Λ_1 and Λ_2 be the functions that map each pair $(\mathbf{M},u)$ with $\mathbf{M}\in\mathscr{C}$ and $u\in M$ to

$$\Lambda_1(\mathbf{M},u)=(1,u,f_1^{\mathbf{M}}(u),\dots,f_a^{\mathbf{M}}(u),1,p_1^{\mathbf{M}}(u),\dots,p_b^{\mathbf{M}}(u))\text{ and}$$
$$\Lambda_2(\mathbf{M},u)=(2,u,f_1^{\mathbf{M}}(u),\dots,f_a^{\mathbf{M}}(u),1,p_1^{\mathbf{M}}(u),\dots,p_b^{\mathbf{M}}(u)),$$

where $f_i^{\mathbf{M}}(u)=v$ if $\mathbf{M}\models f_i(u)=v$ and $p_i^{\mathbf{M}}(u)=1$ if $\mathbf{M}\models P_i(u)$, and $p_i^{\mathbf{M}}(u)=2$, otherwise.

It is easily checked that there exists a quantifier-free partitioned formula $Q(\bar{s};\bar{t})$ in the language of $\mathbb{N}$, such that, for every $\mathbf{M}\in\mathscr{C}$, every $\bar{u}\in M^{|\bar{x}|}$, and every $\bar{v}\in M^{|\bar{y}|}$, we have

$$\mathbf{M}\models q(\bar{u};\bar{v})\quad\iff\quad\mathbb{N}\models Q(\Lambda_1(\mathbf{M},u_1),\dots,\Lambda_1(\mathbf{M},u_{|\bar{x}|});\Lambda_2(\mathbf{M},v_1),\dots,\Lambda_2(\mathbf{M},v_{|\bar{y}|})).$$

For example, to check the atomic formula $f_i(x_1)=f_j(y_2)$ we need to check that the $(i+2)$nd coordinate of the part of $\bar{s}$ encoding x_1 is equal to the $(j+2)$nd coordinate of $\bar{t}$ encoding y_2 (the $(i+2)$nd coordinate of the tuple $\Lambda_1(\mathbf{M},x_1)$ contains the value $f_i(x_1)$ and analogously, the $(j+2)$nd coordinate of the tuple $\Lambda_2(\mathbf{M},y_2)$ contains the value $f_j(y_2)$). Similarly, $\mathbf{M}\models P_i(x_1)$ if and only if the coordinate $(a+3+i)$ of the part of $\bar{s}$ encoding x_1 contains 1. We use the 1 in coordinate $(a+3)$ to test this, by checking if the $(a+3)$rd coordinate of the tuple $\Lambda_1(\mathbf{M},x_1)$ is equal to the $(a+3+i)$th coordinate of the tuple $\Lambda_1(\mathbf{M},x_1)$.

Conversely, let $\mathscr{D}$ be a set-defined class of graphs. Without loss of generality, we may assume that the graphs in $\mathscr{D}$ are induced subgraphs of an infinite graph U defined on $\mathbb{N}^d$ by a symmetric partitioned formula $\varphi(\bar{x};\bar{y})$ (in the language of $\mathbb{N}$). We may assume $|\bar{x}|=|\bar{y}|=d>1$. We may also assume that φ is quantifier free (as the theory of infinite sets has quantifier elimination, see [23]).

Let G be a finite induced subgraph of U. Let A be the set of vertices of G, which is a finite subset of $\mathbb{N}^d$. We define $g(G)$ as the pointer structure with domain $A\cup X$, where X is the set of all elements of $\mathbb{N}$ appearing in a tuple of A, and where $f_i\colon A\cup X\to X$

is defined by: $f_i(\bar{v}) = v_i$ if $\bar{v} \in A$ and $f_i(u) = u$ if $u \in X$. Note that an element of $A \cup X$ is a fixed-point of f_i if and only if it belongs to X. It is easily checked that there is a quantifier-free partitioned first-order formula $q(\bar{x}; \bar{y})$, such that $\mathscr{S}^E(G)$ is included in $\mathscr{S}^q(g(G))$, for every induced subgraph G of U. Indeed, the domain of G is defined as the subset of the domain of $g(G)$ defined by $f_i(x) \neq x$, and each equality $s_i = t_j$ for $\bar{s}, \bar{t} \in \mathbb{N}^d$ translates in $g(G)$ as $f_i(x) = f_j(y)$. $\square$

By Theorem 5.1 and Lemma 5.4, by considering the age of an infinite graph definable on $\mathbb{N}$, we deduce the following result.

Corollary 5.1. *Let $(U, \mathscr{S})$ be a set system definable in an infinite set X, i.e. $\mathscr{S} = \mathscr{S}^\varphi(X)$, for some partitioned formula $\varphi(\bar{x}; \bar{y})$ in the language of infinite sets (with empty signature). Then there exists a constant C such that for every finite subset A of U we have* $\mathrm{disc}(\mathscr{S}_{|A}) \leq C$.

Theorem 5.2. *For every set-defined class $\mathscr{C}$ there exists a constant C such that for every $G \in \mathscr{C}$*

$$\mathrm{herdisc}(\mathscr{S}^E(G)) \leq C.$$

Proof. The statement is a direct consequence of Lemma 5.4 and Corollary 5.1. $\square$

We now use Example 4 to give a lower bound on the discrepancy of neighborhood set systems of set-definable graphs in terms of the length of the parameter tuples used in the formula defining the set system.

Example 7. For every non-negative integer d there exists a set system $(U, \mathscr{S})$ definable in an infinite set X by a partitioned formula $\varphi(\bar{x}; \bar{y})$ with $|\bar{x}| = |\bar{y}| = d + 1$, such that $\mathrm{disc}(\mathscr{S}_{|A}) = \Omega(\sqrt{d})$ for some subset A of size 2^{d+1}.

Proof. For a non-negative integer p, let S_p be Sylvester's graph, introduced in Example 4. The above statement will follow from the lower bound on $\mathrm{disc}(\mathscr{S}^E(S_p))$ given there and the property that S_p is an induced subgraph of the graph defined on $\mathbb{N}$ by some partitioned formula $\varphi_p(\bar{x}, \bar{y})$, where $|\bar{x}| = |\bar{y}| = p + 1$, which we prove now.

We define inductively the parts U_p and V_p of S_p, as well as the partitioned formula φ_p. We let $U_0 = \{1\}$, $V_0 = \{2\}$, and $\varphi_0(x; y) = \neg(x = y)$. Assume S_k is the subgraph of the graph defined by φ_k induced by U_k and V_k. We define

$$\begin{aligned}
&\varphi_{k+1}(x_0, \dots, x_{k+1}; y_0, \dots, y_{k+1}) \\
&\quad = \varphi_k(x_1, \dots, x_k; y_1, \dots, y_k) \leftrightarrow ((x_{k+1} = y_0) \vee (y_{k+1} = x_0)), \\
&U_{k+1} = \{(x_0, \dots, x_{k+1}) \in \{1, 2\}^{k+2} : x_0 = 1\}, \\
&\text{and } V_{k+1} = \{(y_0, \dots, y_{k+1}) \in \{1, 2\}^{k+2} : y_0 = 2\}. \quad \square
\end{aligned}$$

6. A model theoretic approach to classes with bounded expansion

In this section we translate the quantifier elimination scheme introduced in [17] for classes with bounded expansion into a model theoretic language, which allows us to extend the result obtained in Section 4 to the general setting of set systems definable in graphs of a bounded expansion class. Important model theoretic properties of classes of graphs with bounded expansion (like the existence of a quantifier elimination scheme) are based on the existence of so-called "transitive fraternal augmentations" with bounded in-degree. Here, we work with transitive fraternal augmentations with *bounded out-degree* instead, as arcs (u, v) will represent functions and bounded out-degree corresponds to a bounded number of functions in a structure. This can be formalized as follows (see [35, Section 7.4] and also [33]): A class $\mathscr{C}$ of finite graphs has bounded expansion if and only if there exists a function $F\colon \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathscr{C}$ there exists a directed supergraph $\overrightarrow{G}^+$ of an orientation of G and a *fraternity function* $r\colon E(\overrightarrow{G}^+) \to \mathbb{N}$ satisfying:

(P1) **founding:** an arc $e = (u, v)$ of $\overrightarrow{G}^+$ is such that $r(e) = 1$ if and only if it links two vertices that are adjacent in G;
(P2) **fraternity:** for every two arcs $e_1 = (u, v)$ and $e_2 = (u, w)$ of $\overrightarrow{G}^+$ there is an arc $e_3 = (v, w)$ or (w, v) in $\overrightarrow{G}^+$ with $r(e_3) \leq r(e_1) + r(e_2)$;
(P3) **boundedness:** every vertex of $\overrightarrow{G}^+$ has at most $F(i)$ outgoing arcs e with $r(e) \leq i$.

Note that the fraternal augmentation algorithm presented in [35] (see also [34]) can be iteratively used to compute a supergraph $\overrightarrow{G}^+$ and a fraternity function r of a finite graph G in time $O(n^4)$ satisfying the properties 1 and 2. Moreover, for every class $\mathscr{C}$ of finite graphs with bounded expansion there exists a function $F_{\mathscr{C}}$ such that for graphs in $\mathscr{C}$ the property 3 will be satisfied by the computed $\overrightarrow{G}^+$ and r (with $F = F_{\mathscr{C}}$). Note that we do not use the finiteness of G in the definition of fraternity functions and we will use all defined notation also for infinite graphs.

We now introduce theories $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$, which are intrinsically related to the notion of bounded expansion. A *ranked signature* is a pair (σ, ρ), where σ is a signature and $\rho\colon \sigma \to \mathbb{N}$ is a *(signature) ranking*. A *ranked* pointer structure is a pointer structure with a ranked signature. For a positive integer i, we denote by σ_i the subset of symbols in σ with rank at most i, and we define the *rank-i shadow* of a σ-structure $\mathbf{M}$ as the σ_i-reduct of $\mathbf{M}$, that is the σ_i-structure obtained from $\mathbf{M}$ by "forgetting" the functions and relations whose symbols are not in σ_i. Note that the structures $\mathbf{M}$ considered in this section may be infinite.

We consider signatures $\varsigma \subset \sigma$, where ς is a signature consisting in a monoid $\mathcal{F}_\sigma$ of function symbols, and sets $\{M_\alpha\colon \alpha \in \mathbb{N}\}$ of unary relations, and σ is obtained from ς by adding a unary predicate P_φ for each (equivalence class[3] of) formula $\varphi \in \mathrm{FO}_1(\varsigma)$. The notation $\mathcal{F}_\sigma$ is motivated by the fact that $\mathcal{F}_\sigma$ is the set of all function symbols in σ, and that (conversely) the full signature σ can be defined from it.

Moreover, we require σ to be ranked, and that the rank function ρ on σ is such that every σ_i is finite. Note that this implies that σ is countable.

We define the theory $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$ of *fraternal* σ-structures by the following axioms, which mirror the properties of fraternity functions.

(A1) **fraternity:** For all functions $f, g \in \mathcal{F}_\sigma$, and for every x there exists a function h (possibly depending on x) with $h(f(x)) = g(x)$ or $h(g(x)) = f(x)$ and $\rho(h) \le \rho(f) + \rho(g)$.

As each σ_i is finite, this axiom can be expressed by the sentences $\theta^{\mathrm{frat}}_{i,j}$ (for $i, j \in \mathbb{N}$), where

$$\theta^{\mathrm{frat}}_{i,j} \coloneqq \bigwedge_{f\in\sigma_i}\bigwedge_{g\in\sigma_j}\Big(\forall x \bigvee_{h\in\sigma_{i+j}} (g(x) = h(f(x)) \vee (f(x) = h(g(x))\Big).$$

(A2) **transitive coloring:** The unary predicates M_α with $\rho(M_\alpha) = i$ define a partition of the domain, and for every sequence of $k \le i$ functions $f_1, \ldots, f_k$ where $\rho(f_j) \le i$ for all $j \le k$ we have $M_\alpha(x) \wedge (f_1 \circ \cdots \circ f_k(x) \ne x) \to \neg M_\alpha(f_1 \circ \cdots \circ f_k(x))$.

(A3) **typing:** For every formula $\varphi \in \mathrm{FO}_1(\varsigma)$ the predicate P_φ satisfies $P_\varphi(x) \leftrightarrow \varphi(x)$.

Recall that a *p-centered coloring* of a graph G is a coloring of the vertices such that every connected subgraph with at most p colors on the vertices contains some vertex with a unique color.

Lemma 6.1. *Let* $\mathbf{M}$ *be a fraternal* σ*-structure and let* p, q *be positive integers. Then the unary predicates* M_α *with* $\rho(M_\alpha) = q \cdot (2^{p-1} + 2)$ *define a* $(p+1)$*-centered coloring of the Gaifman graph of the rank-*q *shadow of* $\mathbf{M}$*.*

Proof. Denote by $\vec{G}$ the (possibly infinite) directed graph whose vertex set is the domain of $\mathbf{M}$, in which there is an arc from a vertex u to a (distinct) vertex v if there exists $f \in \mathcal{F}_\sigma$ with $f(u) = v$ and $\rho(f) \le q$, and let G be its underlying undirected graph (which is the Gaifman graph of the rank-q shadow of $\mathbf{M}$). Note that for every pair of distinct vertices u and v, there is at most one arc from u to v and at most one arc from v to u since we consider simple graphs only. Let $\vec{G}^+$ be the directed graph obtained in a similar way but without the constraint $\rho(f) \le q$. For each arc $e = (u, v)$ of $\vec{G}^+$ define $r(e) = \lceil \rho(f)/q \rceil$, where $f \in \mathcal{F}_\sigma$ is such that $f(u) = v$ and $\rho(f)$ is minimal for this property. This implies that the axiom (P1) is satisfied by $\vec{G}^+$ and r. Then, for every two arcs $e_1 = (u, v)$ and $e_2 = (u, w)$ of $\vec{G}^+$, we deduce from axiom (A1) that at least

[3] meaning that we keep only one representative of each equivalence class of the relation "is logically equivalent to".

one of (v, w) and (w, v) is an arc of $\vec{G}^+$ with r-value at most $r(e_1) + r(e_2)$. It follows that the axiom (P2) is satisfied by $\vec{G}^+$ and r. The axiom (P3) is also satisfied by $\vec{G}^+$ and r, with $F(i) = |\{f \in \mathcal{F}_\sigma : \rho(f) \leq i\}|$. Then, it follows from [35, Lemma 7.8] that every vertex coloring c with $c(u) \neq c(v)$ whenever there is a directed path of length at most p using only arcs e with $r(e) \leq 2^{p-1} + 2$ linking distinct vertices u and v is a $(p+1)$-centered coloring of G. Note that the proof of [35, Lemma 7.8] does not involve any argument based on the finiteness of the considered graphs, and thus remains valid in an infinite graph setting. The considered arcs correspond to functions with rank ρ at most $q \cdot (2^{p-1} + 2)$ and we conclude that the predicates M_α with $\rho(M_\alpha) \leq q \cdot (2^{p-1} + 2)$ define a coloring, which is as desired. □

The connection between ranked pointer structures models of $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$ and classes with bounded expansion follows.

Lemma 6.2. *A class $\mathscr{C}$ (of finite graphs) has bounded expansion if and only if there exists a ranked signature (σ, ρ) such that for every graph $G \in \mathscr{C}$ there exists a model $\mathbf{M}$ of $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$ with the property that G is the Gaifman graph of the rank-1 shadow of $\mathbf{M}$.*

Proof. Assume that there exists a ranked signature (σ, ρ) such that for every graph $G \in \mathscr{C}$ there exists a model $\mathbf{M}$ of $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$ with the property that G is the Gaifman graph of the rank-1 shadow of $\mathbf{M}$. Then $\mathscr{C}$ has bounded expansion according to Lemma 6.1 (for $q = 1$) and the property that classes with bounded p-centered colorings have bounded expansion.

Conversely, assume $\mathscr{C}$ has bounded expansion, $G \in \mathscr{C}$ and $\vec{G}^+$ is a directed supergraph of G with properties (P1) to (P3). At each vertex u we number arbitrarily the outgoing arcs $e = (u, v)$ with $r(e) = i$ by distinct values $\lambda(e) \in [F(i)]$. We define the mappings $\zeta_{i,j} : V(G) \to V(G)$ by $\zeta_{i,j}(u) = v$ if $e = (u, v) \in E(\overrightarrow{G^+})$, $r(e) = i$, and $\lambda(e) = j$; $\zeta_{i,j}(u) = u$, otherwise. As the maximum out-degree of arcs with r-value i is $F(i)$, there is a vertex coloring γ_i with at most $2iF(i)^i + 1$ colors such that no directed path of length at most i with arcs e with $r(e) \leq i$ links two vertices with the same γ_i-value. To see this, observe that the ith power of a directed graph with out-degree $F(i)$ has out-degree at most $\sum_{j=1}^{i} F(i)^j \leq iF(i)^i$, such that the underlying undirected graph is $2iF^i$ degenerate and the greedy coloring uses at most $2iF(i)^i + 1$ colors. Let $\mathcal{F}_\sigma$ be the free monoid generated by the symbols $\{f_{i,j} : i \in \mathbb{N}, j \in [F(i)]\}$, let $\sigma_0 = \mathcal{F}_\sigma \cup \{M_{i,j} : i \in \mathbb{N}, j \in [(2F(i)^i + 1]\}$ and $\sigma = \sigma_0 \cup \{P_\psi : \psi \in \mathrm{FO}_1[\sigma_0]\}$. We define the rank ρ on σ by $\rho(f_{i,j}) = \rho(M_{i,j}) = i$ and $\rho(P_\psi)$ is the quantifier rank of ψ. Let $\mathbf{M}$ be the σ-structure with domain $V(G)$ where we interpret $f_{i,j}(x)$ by $\zeta_{i,j}(x)$, $M_{i,j}(x)$ by $(\gamma_i(x) = j)$, and $P_\psi(x)$ by $\psi(x)$. By construction, $\mathbf{M}$ is a model of $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$, and G is the Gaifman graph of the rank-1 shadow of $\mathbf{M}$. □

Lemma 6.3. *Let p be a positive integer, let σ be the signature of a pointer structure, and let $\psi(\bar{x})$ be a formula of the form*

$$\psi(\bar{x}) \coloneqq \exists z_1 \dots \exists z_p \psi'(\bar{x}, \bar{z}),$$

where ψ' is a conjunction of atomic formulas of the form $z_i = f(z_j)$ or $z_i = x_j$ and a Boolean combination of formulas of the form $z_i = z_j$ or $\alpha(z_i)$.

Then there exists a formula $\zeta(\bar{x})$, which is a Boolean combination of quantifier-free formulas $q_i(\bar{x})$ and single variable formulas $\eta_i(x_j)$, such that ψ and ζ are equivalent on σ-structures whose Gaifman graphs have tree-depth at most p.

Proof. By removing all the symbols that are not used in ψ, we can restrict to the case where the signature is finite. It follows from [35, Section 6.8] that for every two integers r and p there is an integer $C(r, p)$ such that every colored graph[4] with treedepth at most p is r-equivalent to one of its induced subgraphs of order at most $C(r, p)$. (Recall that two structures are r-equivalent if they satisfy the same sentences with quantifier rank at most r, see Section 2.3.) The lemma thus follows from the finite case, for which we refer the reader (for instance) to [17]. $\square$

We continue to prove that $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$ has quantifier elimination. Following the standard quantifier elimination proof scheme (see e.g. [23, Lemma 2.3.1]), we can reduce to the case of eliminating a single existential quantifier.

Lemma 6.4. *Let $\varphi(\bar{x}) := \exists y\, \varphi'(\bar{x}, y)$ be a formula, where φ' is a quantifier-free σ-formula. Then $\varphi(\bar{x})$ is equivalent to a quantifier-free formula $\psi(\bar{x})$ on all models of $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$.*

Proof. Let $\sigma_1 \subseteq \sigma$ be the (finite) set of symbols used in φ. We consider an arbitrary (possibly infinite) model $\mathbf{M}$ of $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$ and construct a formula ψ such that $\mathbf{M} \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$. Our construction of ψ will be independent of the choice of the model $\mathbf{M}$. Let p be the minimum integer such that φ' is equivalent to a formula of the form $\exists z_1 \dots \exists z_p\, \psi'(\bar{x}, y, \bar{z})$, where ψ' is a conjunction of atomic formulas of the form $z_i = f(z_j)$ or $z_i = x_j$ or $z_i = y$ and a Boolean combination of formulas of the form $z_i = z_j$ or $\alpha(z_i)$.

Let $\mathbf{M}_1$ be the σ_1-reduct of $\mathbf{M}$ and let $q = \max\{\rho(f) \colon f \in \sigma_1\}$, and let $t = q(2^{p-1}+2)$. According to Lemma 6.1, the predicates M_α with $\rho(M_\alpha) = t$ define a partition of the domain of $\mathbf{M}_1$, such that any p classes weakly induce a pointer structure whose Gaifman graph has tree-depth at most p. Let $T = \{\alpha \colon \rho(M_\alpha) = t\}$. We rewrite the formula $\exists y \left(\exists z_1 \dots \exists z_p\, \psi'(\bar{x}, y, \bar{z})\right)$ as

$$\bigvee_{t_0 \in T} \bigvee_{t_1 \in T} \cdots \bigvee_{t_p \in T} \exists y \exists z_1 \dots \exists z_p \left(M_{t_0}(y) \wedge \Big(\bigwedge_{i=1}^{p} M_{t_i}(z_i) \Big) \wedge \psi'(\bar{x}, y, \bar{z}) \right).$$

[4] By a colored graph, we mean a graph whose vertices and edges are colored using a finite set of colors.

According to Lemma 6.3 there exist formulas $\zeta_{t_0,\dots,t_p}(\bar{x})$, which are Boolean combinations of quantifier-free formulas and single variable formulas, such that the above formula is equivalent to

$$\bigvee_{t_0\in T}\cdots\bigvee_{t_p\in T}\zeta_{t_0,\dots,t_p}(\bar{x}).$$

By replacing in each $\zeta_{t_0,\dots,t_p}$ each single free variable formula $\eta(x)$ by the corresponding predicate P_η, we get a quantifier-free formula $\widehat{\zeta}_{t_0,\dots,t_p}(\bar{x})$ that is equivalent to $\zeta_{t_0,\dots,t_p}(\bar{x})$ on every model of $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$. Thus, we have $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho} \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$, where $\psi(\bar{x})$ is the quantifier-free formula $\bigvee_{t_0\in T}\cdots\bigvee_{t_p\in T}\widehat{\zeta}_{t_0,\dots,t_p}(\bar{x})$. $\square$

The main result of this section now follows by an easy induction from Lemma 6.4.

Theorem 6.1. *For every ranked signature (σ,ρ) and every formula $\varphi(\bar{x})$ there exists a quantifier free formula $q(\bar{x})$ such that φ and q are equivalent on every model of $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$. In other words, the theory $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$ has quantifier elimination.*

We now can derive a model theoretical characterization of bounded expansion classes.

Theorem 6.2. *Let $\mathscr{C}$ be a monotone class of (finite) graphs. Then the following are equivalent:*

(i) The class $\mathscr{C}$ has bounded expansion;
(ii) for every monadic expansion $\mathscr{C}^+$ of $\mathscr{C}$ and every formula $\varphi(\bar{x})$ in the language of $\mathscr{C}^+$ there exists a constant C_φ such that the hereditary discrepancy of every set system $\mathscr{S}^\varphi(G)$ (for $G\in\mathscr{C}^+$) is bounded by C_φ;
(iii) for each positive integer k, there exists an integer C_k such that the hereditary discrepancy of every set system $\mathscr{S}^E(G^k)$ (for $G\in\mathscr{C}$) is bounded by C_k.

Proof. Assume (i). According to Lemma 6.2 there exists a ranked signature (σ,ρ) such that for every graph $G\in\mathscr{C}$ there exists a model $\mathbf{M}(G)$ of $\mathrm{Th}^{\mathrm{TF}}_{\sigma,\rho}$ with the property that G is the Gaifman graph of the rank-1 shadow of $\mathbf{M}(G)$. Let $\mathscr{D}=\{\mathbf{M}(G) : G\in\mathscr{C}\}$. Let $\varphi(\bar{x})$ be a first-order formula in the language of $\mathscr{C}$. As the adjacency in $G\in\mathscr{C}$ can be checked by using functions in $\mathbf{M}(G)$ with rank at most 1, there is a formula $\varphi'(\bar{x})$ such that, for all $\bar{v}\in V(G)^{|\bar{x}|}$ we have $G\models\varphi(\bar{v}) \iff \mathbf{M}\models\varphi'(\bar{v})$. According to Theorem 6.1, $\varphi'(\bar{x})$ is equivalent (on structures in $\mathscr{D}$) to a quantifier-free formula $q(\bar{x})$. Now, according to Lemma 5.4 and Theorem 5.2 there exists a constant C such that the hereditary discrepancy of $\mathscr{S}^\varphi(G)=\mathscr{S}^q(\mathbf{M}(G))$ is bounded by C. Consequently, (i)⇒(ii).

Moreover, (ii)⇒(iii) as $\mathscr{S}^E(G^k)$ is definable on $\mathscr{C}$, and (iii)⇒(i) as a consequence of Corollary 4.1. $\square$

7. Discrepancy in nowhere dense classes

Let $\varphi(\bar{x};\bar{y})$ be a partitioned first-order formula and let $\mathscr{C}$ be a nowhere dense class. We now give two bounds on the discrepancy of $\mathscr{S}^{\varphi}(G)$ (for $G \in \mathscr{C}$), in terms of the lengths of the tuples $\bar{x}$ and $\bar{y}$, respectively.

Lemma 7.1. *Let $\mathscr{C}$ be a nowhere dense class and let $\varphi(\bar{x};\bar{y})$ be a partitioned first-order formula. Then for every $\varepsilon > 0$ there exists a constant C with*

$$\mathrm{disc}(\mathscr{S}^{\varphi}(G)_{|A}) \leq C\,|A|^{\frac{1}{2}-\frac{1}{2|\bar{y}|}+\varepsilon}$$

for every $G \in \mathscr{C}$ and $A \subseteq V(G)^{|\bar{x}|}$.

Proof. Let A be a subset of $V(G)^{|\bar{x}|}$ and let $m = |A|$. Then $A \subseteq U^{|\bar{x}|}$, where $U \subseteq V(G)$ is the set of all the elements appearing in some tuple in A. Obviously $|U| \leq |\bar{x}| \cdot |A|$. According to Theorem 2.6, for every $\varepsilon > 0$ there exists a constant c such that for every $G \in \mathscr{C}$ and every $U \subseteq V(G)$ we have $\left|\mathscr{S}^{\varphi}(G)_{|U^{|\bar{x}|}}\right| \leq c\,|U|^{|\bar{y}|+\varepsilon}$. Thus, we have $\left|\mathscr{S}^{\varphi}(G)_{|A}\right| \leq c\,(m\,|\bar{x}|)^{|\bar{y}|+\varepsilon}$. It follows that the primal shatter function $\pi(m)$ of $\mathscr{S}^{\varphi}(G)$ (and, more generally, of $\mathscr{S}^{\varphi}(G)_{|A}$ for $A \subseteq V(G)^{|\bar{x}|}$) is bounded by $c' m^{|\bar{y}|+\varepsilon}$, where $c' = c\,|\bar{x}|^{|\bar{y}|}$, and the result follows from Theorem 2.4 (as A is the ground set of $\mathscr{S}^{\varphi}(G)_{|A}$). $\square$

Lemma 7.2. *Let $\mathscr{C}$ be a nowhere dense class and let $\varphi(\bar{x};\bar{y})$ be a partitioned first-order formula. Then for every $\varepsilon > 0$ there exists a constant C with*

$$\mathrm{disc}(\mathscr{S}^{\varphi}(G)_{|A}) \leq C\,|A|^{\frac{1}{2}-\frac{1}{2|\bar{x}|}+\varepsilon}$$

for every $G \in \mathscr{C}$ and $A \subseteq V(G)^{|\bar{x}|}$.

Proof. Let $\varphi^*(\bar{y};\bar{x}) = \varphi(\bar{x};\bar{y})$. As in the proof of Lemma 7.1, the shatter function of $\mathscr{S}^{\varphi^*}(G)$ is bounded by $c\,m^{|\bar{x}|+\varepsilon}$ for some constant c. As $\mathscr{S}^{\varphi^*}(G)$ is the dual of the set system of $\mathscr{S}^{\varphi}(G)$, the dual shatter function $\pi^*(m)$ of $\mathscr{S}^{\varphi}(G)$ is bounded by $c\,m^{|\bar{x}|+\varepsilon}$ as well, and the result follows from Theorem 2.5. $\square$

Theorem 7.1. *For a monotone class of graphs $\mathscr{C}$ the following are equivalent:*

(1) $\mathscr{C}$ is nowhere dense;
(2) for every monadic expansion $\mathscr{C}^+$ of $\mathscr{C}$, for every partitioned formula $\varphi(\bar{x};\bar{y})$ in the language of $\mathscr{C}^+$, for every $\alpha > \frac{1}{2} - \frac{1}{2\min(|\bar{x}|,|\bar{y}|)}$, and for $G \in \mathscr{C}$, we have

$$\mathrm{herdisc}(\mathscr{S}^{\varphi}(G^+)) \in \mathcal{O}\Big(\left|\bigcup \mathscr{S}^{\varphi}(G^+)\right|^{\alpha}\Big);$$

(3) for every positive integer k, every $\varepsilon > 0$, and for $G \in \mathscr{C}$, we have

$$\mathrm{herdisc}(\mathscr{S}^E(G^k)) \in \mathcal{O}(|G|^\varepsilon);$$

(4) there exists a positive integer r such that for every partitioned formula $\varphi(\bar{x};\bar{y})$ with $|\bar{x}| = |\bar{y}| = r$ and for $G \in \mathscr{C}$ we have $\mathrm{herdisc}(\mathscr{S}^\varphi(G)) \in o\Big(\big|\bigcup \mathscr{S}^\varphi(G)\big|^{1/2}\Big)$.

Proof. The proof follows from the following implications:

(1)⇒(2) This immediately follows from Lemmas 7.1 and 7.2, with $A = \bigcup \mathscr{S}^\varphi(G^+)$ (that is, the ground set of $\mathscr{S}^\varphi(G^+)$).

(2)⇒(3) This immediately follows from the special case of the set system defined by a formula $\varphi(x, y)$ expressing that the distance between x and y is at most k.

(3)⇒(1) (by contradiction): Assume the class $\mathscr{C}$ is not nowhere dense. As $\mathscr{C}$ is monotone there exists a positive integer k such that the class $\mathscr{C}$ contains the $\le (k-1)$-subdivision of all graphs. By a standard Ramsey argument, that the class $\mathscr{C}$ contains the $(k-1)$-subdivision of all graphs (for a possibly smaller k). In particular, $\mathscr{C}$ contains the $(k-1)$-subdivision of all the graphs S_p (defined in Example 4). Let G_p be the $(k-1)$-subdivision of S_p and let $A_p \subseteq V(G_p)$ be the set of the principal vertices of G_p (i.e. the vertices of S_p). Then $\mathrm{herdisc}(\mathscr{S}^E(G^k)) = \Omega(|A_p|^{1/2}) = \Omega(|G|^{1/4})$. Hence $\neg(1) \Rightarrow \neg(3)$.

(2)⇒(4) This is trivial as we can always choose $\alpha < 1/2$ in (2).

(4)⇒(1) (by contradiction): Assume the class $\mathscr{C}$ is not nowhere dense. As above, $\mathscr{C}$ contains the $(k-1)$-subdivision of all the graphs S_p (defined in Example 4), for some positive integer k.

For $p > 1$, consider the graph G_p obtained from S_p by adding a new vertex s adjacent to all the vertices in one of the parts of S_p, and a vertex r adjacent only to s. Let $\mu_A(x)$ be the formula expressing that x has degree greater than one and is adjacent to a vertex adjacent to a vertex with degree one, and let $\mu_B(x)$ be the formula expressing that no neighbor of x is adjacent to a vertex with degree one.

For $d \in \mathbb{N}$ we define the partitioned formula $\zeta_d(\bar{x};\bar{y})$ with $|\bar{x}| = |\bar{y}| = d$ inductively, as follows (where $\bar{x}' = (x_1, \dots, x_{d-1})$ and $\bar{y}' = (y_1, \dots, y_{d-1})$):

$$\zeta_1(x, y) := \mu_A(x) \wedge \mu_B(y) \wedge E(x, y)$$

and, for $d > 1$ we define

$$\zeta_d(\bar{x}, \bar{y}) := \Big(\zeta_{d-1}(\bar{x}', \bar{y}') \leftrightarrow E(x_d, y_d)\Big) \;\wedge\; \bigwedge_{i=1}^{d} (\mu_A(x_i) \wedge \mu_B(y_i)).$$

It is easily checked that for $p > 1$ the graph defined on $V(G_p)^d$ where $\bar{a}$ is adjacent to $\bar{b}$ if $G_p \models \zeta_d(\bar{a}, \bar{b})$ is isomorphic to the union of G_{dp} and isolated vertices. In particular, we have $\operatorname{disc}(\mathscr{S}^{\zeta_d}(G_p)) \in \Omega\Big(\sqrt{|V(G_p)|^d}\Big)$.

For $p > 2$, let H_p be the $(k-1)$-subdivision of G_p, and let $\hat{\zeta}_d(\bar{x}; \bar{y})$ be the partitioned formula obtained from ζ_d by replacing $E(x, y)$ by a formula expressing that $\operatorname{dist}(x, y) = k$, and by conditioning all the vertices to have degree at least 3. Then $\mathscr{S}^{\hat{\zeta}_d}(H_p) = \mathscr{S}^{\zeta_d}(G_p)$. Thus, the ground set $\bigcup \mathscr{S}^{\hat{\zeta}_d}(H_p)$ of $\mathscr{S}^{\hat{\zeta}_d}(H_p)$ is nothing but $V(G_p)^d$, and $\operatorname{disc}(\mathscr{S}^{\hat{\zeta}_d}(H_p)) \in \Omega\Big(\big|\bigcup \mathscr{S}^{\hat{\zeta}_d}(H_p)\big|^{1/2}\Big)$. As $\mathscr{C}$ contains all the graphs H_p, we deduce $\neg(1) \Rightarrow \neg(4)$. $\square$

Conjecture 5. *A monotone class $\mathscr{C}$ is nowhere dense if and only if for every partitioned formula $\varphi(\bar{x}; \bar{y})$, for every $\varepsilon > 0$, and for $G \in \mathscr{C}$ we have* $\operatorname{herdisc}(\mathscr{S}^{\varphi}(G)) \in \mathcal{O}\Big(\big|\bigcup \mathscr{S}^{\varphi}(G)\big|^{\varepsilon}\Big)$.

8. Applications

We now consider the problems introduced in Section 1 in the light of the results obtained in the previous sections.

8.1. Simultaneous neighborhood discrepancy for red and blue edges

Let G be a graph. To a 2-coloring $\gamma\colon E(G) \to \{1, 2\}$ we associate the set system $\mathscr{S}^{\gamma}(G)$ whose members are the 1-neighborhoods and the 2-neighborhoods of the vertices in G, where by *i-neighborhood* of a vertex v (with $i \in \{1, 2\}$) we mean the set of all the vertices u adjacent to v by an edge with γ-color i. For every subgraph $H \subseteq G$ we can consider the coloring γ with $\gamma(e) = 1$ if $e \in E(H)$. Then $\mathscr{S}^{E}(H) \subseteq \mathscr{S}^{\gamma}(G)$. It follows that

$$\max_{H \subseteq G} \operatorname{disc}(\mathscr{S}^{E}(H)) \leq \max_{\gamma} \operatorname{disc}(\mathscr{S}^{\gamma}(G)).$$

On the other hand, let G^{γ} be the bipartite graph with parts V and $W = V \times \{1, 2\}$, where u is adjacent to (v, i) if $uv \in E(G)$ and uv is colored i. It is easily checked that $\deg(G^{\gamma}) \leq \deg(G)$. Moreover, it is easily checked that $\mathscr{S}^{\gamma}(G)$ is included in $\mathscr{S}^{E}(G^{\gamma})$. Thus, any upper bound on $\operatorname{disc}(\mathscr{S}^{E}(G^{\gamma}))$ based on its degeneracy provides an upper bound on $\operatorname{disc}(\mathscr{S}^{\gamma}(G))$ in terms of $\deg(G)$.

From this and Theorem 3.1, we deduce the following (partial) answer to Problem 1.

Corollary 8.1. *For every graph G we have*

$$\frac{\log_2(\pi \deg(G))}{4} - 2 \le \max_{\gamma} \operatorname{disc}(\mathscr{S}^{\gamma}(G)) < 3\deg(G), \tag{4}$$

where the maximum runs over 2-colorings $\gamma\colon E(G) \to \{1,2\}$ and $\mathscr{S}^{\gamma}(G)$ is the set system, whose members are the 1-neighborhoods and the 2-neighborhoods of the vertices in G.

Problem 1. Assume that a graph G has the property that for every red/blue coloring of the edges of G there exists a partition (A, B) of the vertex set of G such that the number of red (resp. blue) neighbors in A and B of any vertex differ by at most 1. Does G contain a vertex with small degree?

According to Corollary 8.1, we deduce from $\max_\gamma \operatorname{disc}(\mathscr{S}^{\gamma}(G)) \le 1$ that the degeneracy of G is at most $\lfloor 2^{12}/2\pi \rfloor = 651$.

8.2. Epsilon-nets and epsilon-approximations

An important application of the notion of discrepancy is that of an ε-net. A subset N of the ground set U of a set system $(U, \mathscr{S})$ is an *ε-net* for $\mathscr{S}$ if N intersects all the sets $S \in \mathscr{S}$ that have at least $\varepsilon\,|U|$ elements. A related concept is that of an ε-approximation. A subset $A \subseteq U$ is an *ε-approximation* for $\mathscr{S}$ if $\left|\frac{|A\cap S|}{|A|} - \frac{|S|}{|U|}\right| \le \varepsilon$ for every $S \in \mathscr{S}$. The notions of ε-nets and ε-approximations play a key role in the approximation of sets from the systems by smaller subsets. Upper bounds for the sizes of ε-nets and ε-approximations of a set system can be derived from bounds on its hereditary discrepancy. Matoušek, Welzl, and Wernisch proved [31, Lemma 2.2] that if $(U, \mathscr{S})$ is a set system with $U \in \mathscr{S}$ and f is a function such that $\operatorname{disc}(\mathscr{S}_{|X}) \le f(|X|)$ for all $X \subseteq U$, then for every integer $t \ge 0$ there exists an ε-approximation A for $\mathscr{S}$ with $|A| = \lceil \frac{|U|}{2^t} \rceil$ and $\varepsilon \le \frac{2}{|U|} \cdot \sum_{i=0}^{t-1} 2^i f\big(\lceil \frac{|U|}{2^i} \rceil\big) < \frac{4}{|U|} \operatorname{herdisc}(\mathscr{S})$. In particular, every set system $(U, \mathscr{S})$ has an ε-approximation of size at most $\lceil \frac{4\,\operatorname{herdisc}(\mathscr{S})}{\varepsilon} \rceil$. Furthermore, this construction is efficient and an ε-approximation with the stated size can be computed in polynomial time. Thus, as definable set systems can be constructed in polynomial time in bounded expansion classes (by using the model checking algorithm of [17]), we get the following.

Corollary 8.2. *For every class $\mathscr{C}$ with bounded expansion, there exists an integer c and an algorithm that computes, for a partitioned formula $\varphi(\bar{x}; \bar{y})$, a graph $G \in \mathscr{C}$, and a real $\varepsilon > 0$, an ε-approximation of $\mathscr{S}^{\varphi}(G)$ of size at most C/ε in time $\mathcal{O}(|V(G)|^c)$, where $C = C(\mathscr{C}, \varphi)$ depends on $\mathscr{C}$ and φ, and where the constant hidden in the $\mathcal{O}$-notation depends on $\mathscr{C}$, φ, and ε.*

As the class of planar graphs has bounded expansion, we deduce the following answer to Problem 2.

Problem 2. Given a planar graph G, find a small subset F of edges such that, for every pair u, v of distinct vertices of G, the probability that an edge of G belongs to a uv-path of length at most 100 differs from the probability that an edge in F belongs to a uv-path of length at most 100 by at most ε.

> Consider the partitioned formula $\varphi(\bar{x};\bar{y})$ with $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$, which asserts that x_1 and x_2 are adjacent, and that there exists a path of length at most 100 linking y_1 and y_2 including the edge x_1x_2. For any fixed bounded expansion class $\mathscr{C}$, like the class of planar graphs, there exists a constant c such that for every $G \in \mathscr{C}$ we have $\operatorname{herdisc}(\mathscr{S}^\varphi(G)) \le c$. It follows that the set system $(E(G), \mathscr{S}^\varphi(G))$ has an ε-approximation with size $\lceil \frac{4c}{\varepsilon} \rceil$, which can be computed in polynomial time.

8.3. Clique coloring

A *clique coloring* of a graph G is a coloring of the vertices of G such that no maximal clique is monochromatic [15]. If a hereditary class of graphs has a bounded clique coloring number, it is obviously (exponentially) χ-bounded. However, while it was conjectured [15] that every perfect graph is 3-colorable, Charbit et al. [8] showed they do not have a bounded clique coloring number.

As a variant of clique coloring, one can ask whether a graph admits a vertex coloring by k colors in such a way that no maximal clique K contains more than $\lceil (1-\varepsilon)\,|K| \rceil$ vertices[5] of same color (for some fixed integer k and positive real ε). For instance, an easy induction shows that cographs do not have such a coloring in general, while classes with bounded shrubdepth (and, more generally, classes with structurally bounded expansion) do.

In this context, discrepancy colorings appear as a very strong property, by requiring that the vertices can be 2-colored in such a way that in every maximal clique the difference between the numbers of vertices of each color is bounded by some constant c. Even classes with bounded shrubdepth do not have this property. However, we shall prove the following.

> **Corollary 8.3.** *For every class $\mathscr{C}$ with bounded expansion and every integer k there is a constant c such that for every graph $G \in \mathscr{C}$, the set system formed by all maximal cliques of G^k has hereditary discrepancy at most c.*

Proof. According to [36], there exists an integer p (depending on $\mathscr{C}$ and k) such that for every $G \in \mathscr{C}$ and every maximal clique K of G^k there exist $v_1, \dots, v_p \in K$ with $K = \bigcap_{i=1}^p N_{G^k}[v_i]$. On the other hand, given p vertices $u_1, \dots, u_p$ it is easily checked that the property that $\bigcap_{i=1}^p N_{G^k}[u_i]$ induces a maximal clique of G^k can be checked by a first-order sentence $\kappa(x_1, \dots, x_p)$. Hence, the set system of all maximal cliques of G^k

[5] Note that the rounding allows to introduce a threshold, meaning that small cliques (of size smaller than $\lfloor 1/\varepsilon \rfloor$) are allowed to be monochromatic.

(plus the empty set) is definable in G as $\mathscr{S}^{\varphi}(G)$, where $\varphi(\bar{x};\bar{y}) := (\bar{x}=\bar{y}) \vee \kappa(\bar{y})$. Thus, the result follows from Theorem 6.2. □

Problem 3. Does there exist a constant c such that the vertices of every map graph G can be colored red or blue in such a way that the difference between the number of red and blue vertices in every maximal clique of G is at most c?

Let G be a map graph, that is, the half-square of the vertex-face incidence graph of a planar map. By definition, G is an induced subgraph of the square H^2 of a planar graph H. According to Corollary 8.3, the hereditary discrepancy of the set system of the maximal cliques of H^2 is bounded by some constant c. As every maximal clique of G is the trace on $V(G)$ of some maximal clique of H^2 we deduce that the set system of all maximal cliques of G has bounded discrepancy.

8.4. Quantifier elimination

Our results on hereditary discrepancy unexpectedly allow us to answer Problem 4.

Problem 4. Give an example of a nowhere dense class $\mathscr{C}$ of graphs such that there exists no expansion σ of the signature of graphs by unary relation and function symbols with the property that every first-order formula is equivalent on a σ-expansion $\mathscr{C}^+$ of $\mathscr{C}$ to a quantifier-free first-order σ-formula.

Corollary 8.4. *Let $\mathscr{C}$ be the class of the 1-subdivisions of all bipartite graphs, whose girth is greater than their maximum degree. The class $\mathscr{C}$ is nowhere dense and such that there does not exist an expansion σ of the signature of graphs by unary relations and functions, an expansion F of graphs into σ-structures and a quantifier-free partitioned formula $q(x,y)$ in the language of σ-structures with*

$$\forall G \in \mathscr{C}\ \forall a,b \in V(G) \quad (G \models \varphi(a;b)) \iff (F(G) \models q(a;b)), \tag{5}$$

where $\varphi(a;b)$ is a partitioned formula expressing that a and b are at distance 2 in G.

Proof. Let $\mathscr{C}_0$ be the class of all bipartite graphs whose girth is greater than the maximum degree. It is well known that this class is monotone, nowhere dense, but has unbounded degeneracy. Let $\mathscr{C}$ be the class of the 1-subdivisions of the graphs in $\mathscr{C}_0$. This class is obviously also nowhere dense.

Let σ_0 be the signature consisting of two unary functions f_1 and f_2. According to Observation 5.1, we can associate to each $G \in \mathscr{C}$ (which is the 1-subdivision of a bipartite graph H with parts V_1 and V_2) the σ_0-structure $\mathbf{M}$ defined as follows: if v is a principal vertex (i.e. a vertex of H) then $f_1(v) = f_2(v) = v$, while if v is the subdivision vertex of the edge $\{u_1,u_2\} \in V_1 \times V_2$ then $f_1(v) = u_1$ and $f_2(v) = u_2$. Let $\mathscr{D}$ be the class of all the σ_0-structures $\mathbf{M}$ obtained from the graphs $G \in \mathscr{C}$.

Assume towards a contradiction that there is an expansion σ of the signature of graphs by unary relations and functions, an expansion F of graphs into σ-structures and a quantifier-free partitioned formula $q(x;y)$ in the language of σ-structures such that equation (5) holds. Then this defines an expansion σ' of σ_0, an expansion F' of σ_0-structures into σ'-structures, and a quantifier free partitioned formula $q'(x;y)$, such that

$$\forall \mathbf{M} \in \mathscr{D}\ \forall a,b \in M \quad (\mathbf{M} \models \varphi'(a;b)) \iff (F'(\mathbf{M}) \models q'(a;b)),$$

where the partitioned formula $\varphi'(u;v)$ expresses that u and v are at distance 2 in the Gaifman graph of $\mathbf{M}$. According to Theorem 5.1 there is a constant C such that, for all $\mathbf{M} \in \mathscr{D}$, we have $\mathrm{disc}(\mathscr{S}^{q'}(F'(\mathbf{M})) \leq \mathrm{herdisc}(\mathscr{S}^{q'}(F'(\mathbf{M})) \leq C$. However, it is easily checked that if $\mathbf{M}$ is the σ_0-structure corresponding to the 1-subdivision G of a graph H we have $\mathscr{S}^E(H) \subseteq \mathscr{S}^{\varphi}(G) = \mathscr{S}^{\varphi'}(\mathbf{M})$. As the class $\mathscr{C}_0$ is monotone and contains graphs with arbitrarily large degeneracy, it follows from Theorem 3.1 that the set systems $\mathscr{S}^E(H)$ with $H \in \mathscr{C}_0$ have unbounded discrepancy, but that contradicts $\mathrm{disc}(\mathscr{S}^{\varphi'}(\mathbf{M})) = \mathrm{disc}(\mathscr{S}^{q'}(F'(\mathbf{M})) \leq C$. $\square$

9. Concluding remarks

As mentioned in Section 5.2, the set systems definable by a quantifier-free formula in pointer structures are basically the same as the neighborhood set systems of set-defined graphs.

A *k-copy* of a graph G is the graph obtained from k copies of G by adding an edge between any two clone vertices. A *simple (first-order) interpretation* I is a pair of formulas $(\eta(x,y),\nu(x))$; it maps a (possibly colored) graph G into the graph $\mathsf{I}(G)$ with vertex set $\nu(G) = \{v \in G\colon G \models \nu(v)\}$ and edge set $\eta(G) \cap \nu(G)^2 = \{\{u,v\} \in \nu(G)^2\colon G \models \eta(u,v) \vee \eta(v,u)\}$ (where the formulas η and ν can use the colors). A *(first-order) transduction* T is a pair (k,I), where k is an integer and I is a simple interpretation; it maps a graph G to the class of all graphs that can be obtained from G by computing a k-copy, then coloring this k-copy in an arbitrary way, then applying the interpretation I. We refer the interested reader to [18,37] for a discussion on first-order transductions.

For a monotone class $\mathscr{C}$, it follows from [24] and Theorem 6.2 that the four following properties are equivalent:

1. $\mathscr{C}$ has bounded expansion;
2. every transduction of $\mathscr{C}$ is set-defined;
3. every transduction of $\mathscr{C}$ is linearly χ-bounded;
4. every set system definable in $\mathscr{C}$ has bounded hereditary discrepancy.

The equivalence between the first three items is known to hold for biclique-free classes of graphs. We conjecture that this extends to the equivalence with the last item.

Conjecture 6. *A biclique-free class of graphs has bounded expansion if and only if every set system definable in $\mathscr{C}$ has bounded hereditary discrepancy.*

An intriguing related problem is the following, for which a positive answer would imply Conjecture 6.

Problem 7. Is it true that for every non-degenerate biclique-free class $\mathscr{C}$ of graphs there exists a transduction T such that $\mathsf{T}(\mathscr{C})$ includes a monotone non-degenerate class of graphs?

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors wish to express their gratitude to the referees for their careful review and many constructive suggestions.

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