Soft Actor Critic: Off-Policy Maximum Entropy Deep Reinforcement Learning with a Stochastic Actor

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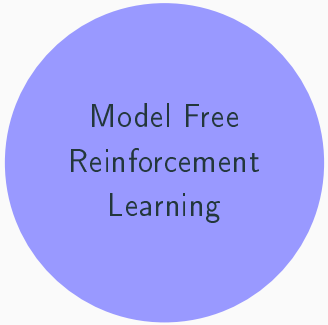
Based on the article: (Haarnoja et al. 2018)

Introduction and context

Accurate Recap of the context



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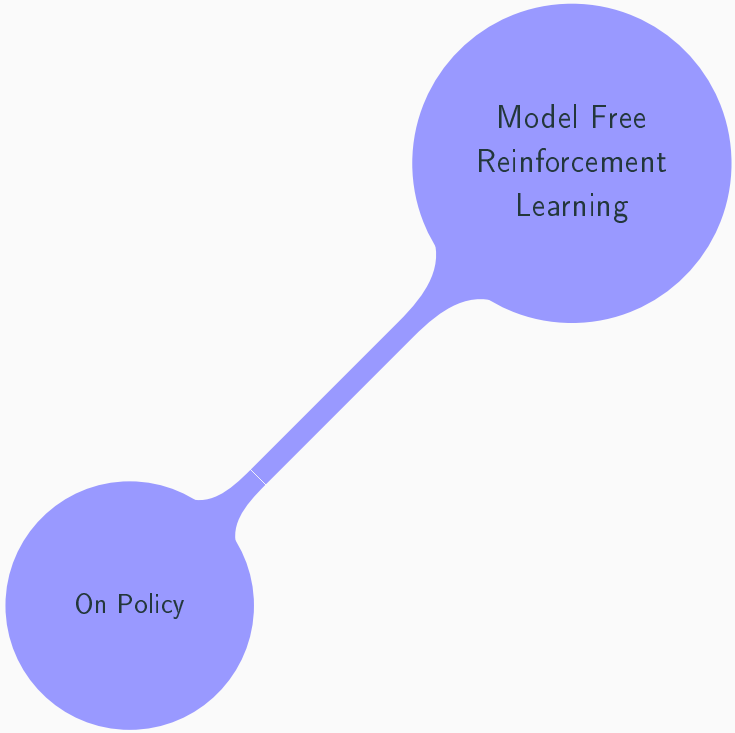

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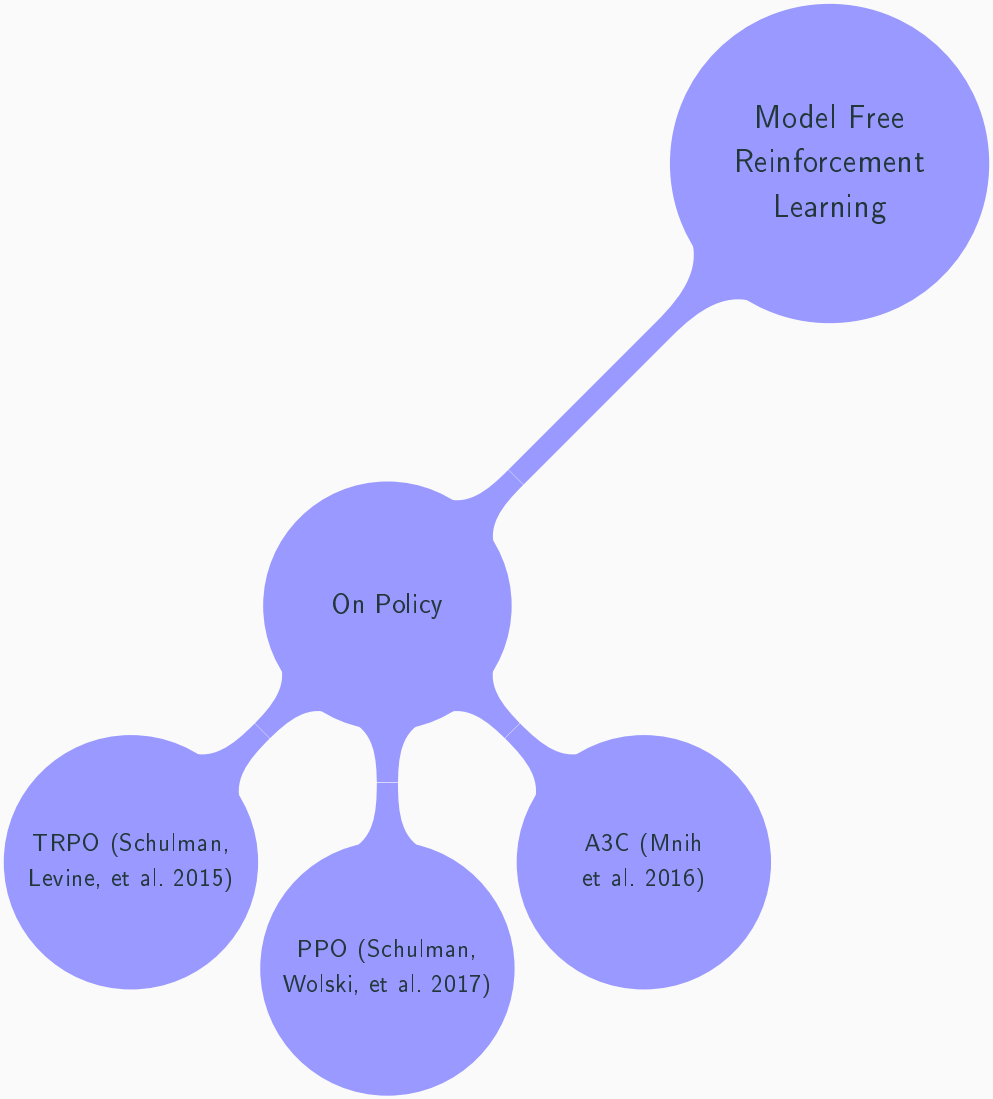
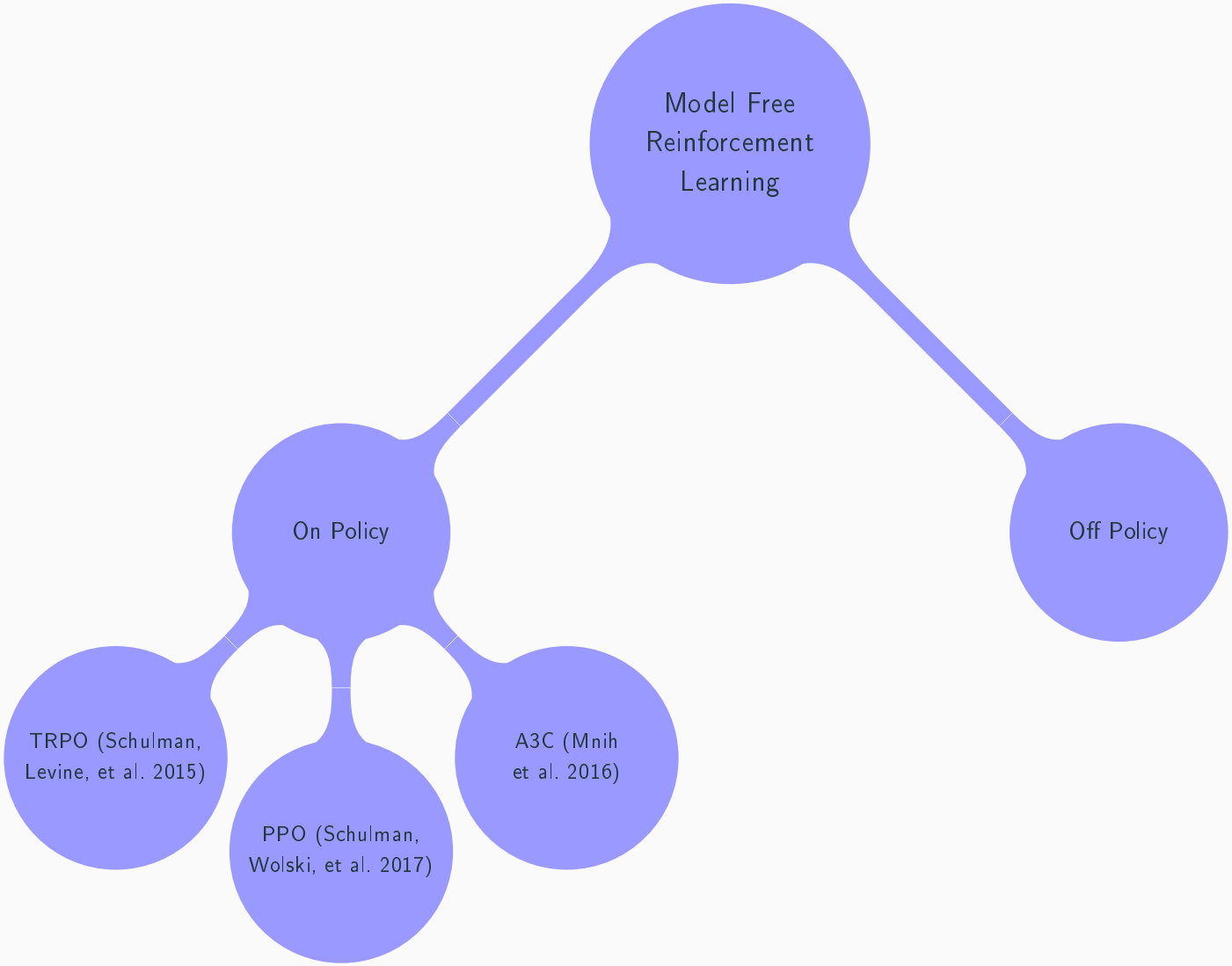
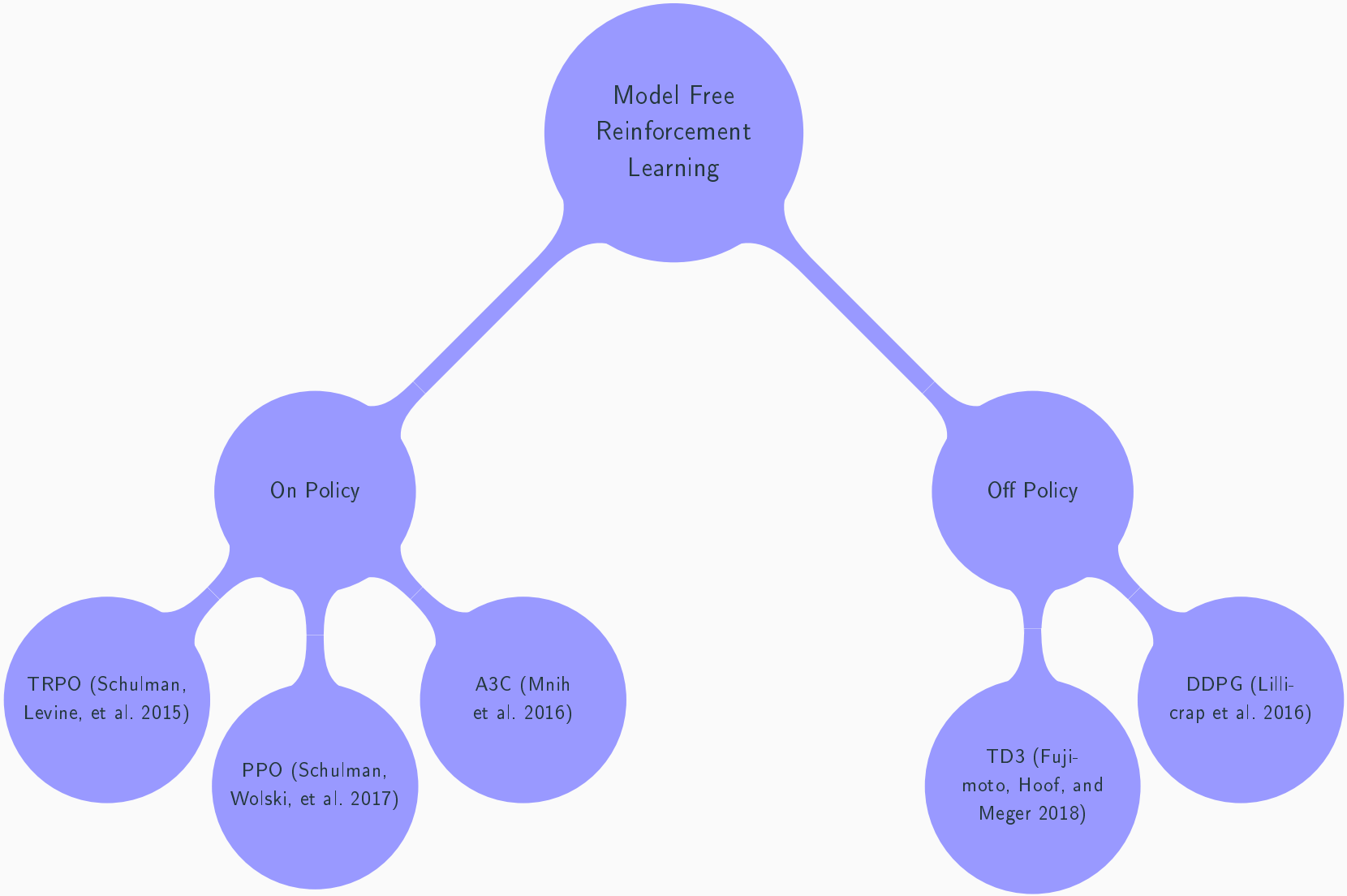


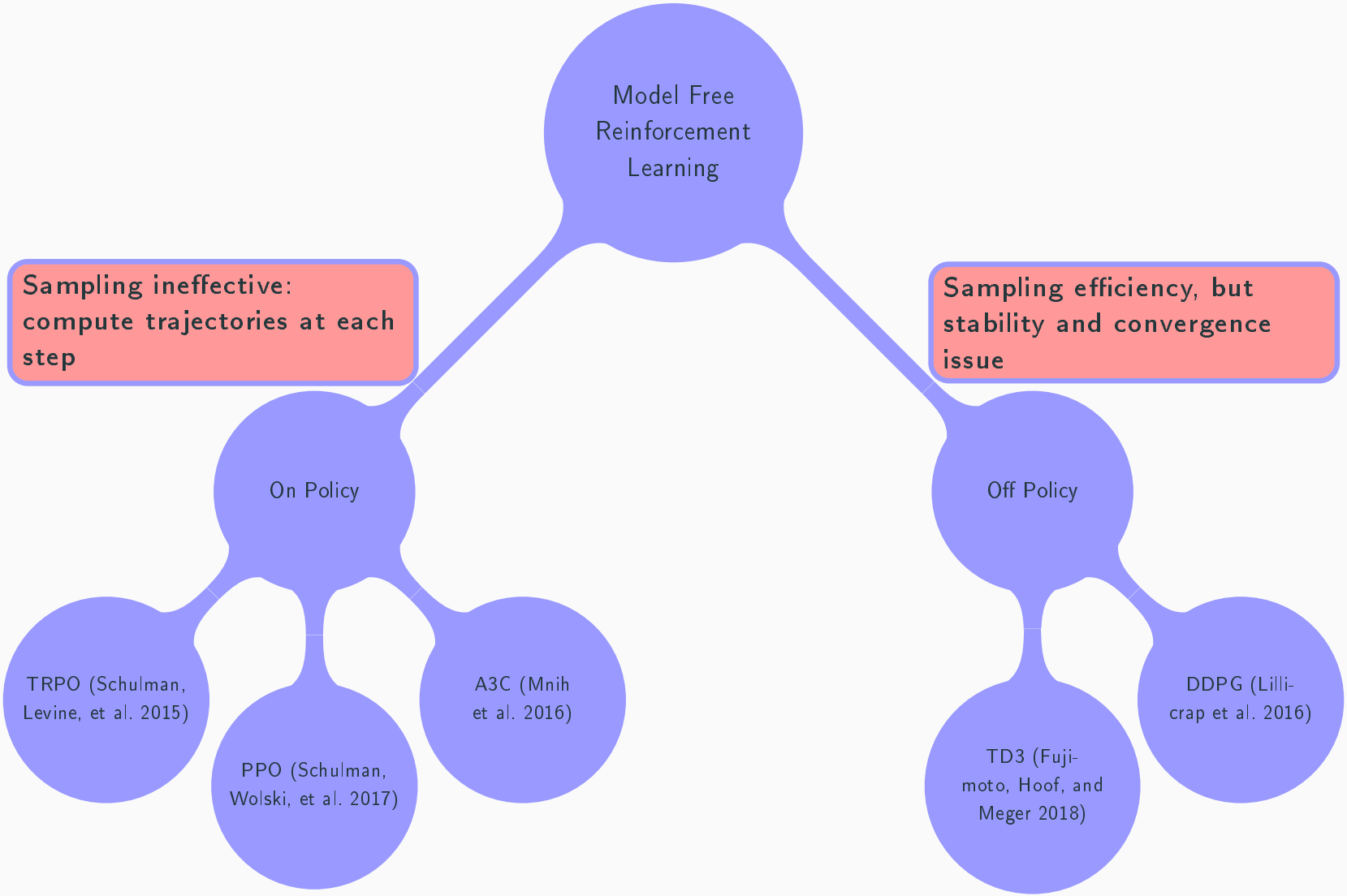
Accurate Recap of the context



Accurate Recap of the context



Accurate Recap of the context



Soft Policy Iteration

Augmented reward

The objective function is defined as:

$$J(\pi) = \sum_{t=0}^{T} \mathbb{E}_{s_t \sim \mu, a_t \sim \pi}(r(s_t, a_t) + \alpha \mathcal{H}(\pi(\cdot|s_t))) \tag{1}$$

Where the entropy is the following:

$$\mathcal{H}(\pi(\cdot|s_t)) = -\mathbb{E}_{a_t \sim \pi}\big[\log \pi(a_t \mid s_t)\big] \tag{2}$$

Introducing a reward *augmented* reward $r_\pi(s, a) = r(s, a) + \mathbb{E}_{s' \sim \mu}(\mathcal{H}(\pi(\cdot|s')))$ for all $(s, a) \in \mathcal{S} \times \mathcal{A}$, one can rewrite the objective function as the well-known finite horizon objective function for MDPs: $J(\pi) = \mathbb{E}_{s \sim \mu}(v_\pi(s))$.

Soft Policy evaluation: classic policy evaluation but modified reward

One can derive the policy evaluation presented in the paper using the Bellman equation (Sutton and Barto 1998).

$$
\begin{aligned}
q(s,a) &= r_\pi(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) v(s') \text{ (Bellman equation)} \\
&= r(s,a) - \gamma \sum_{s' \in \mathcal{S}} \sum_{a' \in \mathcal{A}} p(s'|s,a) \pi(a'|s') \log(\pi(a'|s')) \\
&+ \gamma \sum_{s' \in \mathcal{S}} \sum_{a' \in \mathcal{A}} p(s'|s,a) \pi(a'|s') q(s',a')
\end{aligned}
$$

One deduces:

$$
q(s,a) = r(s,a) + \gamma \mathbb{E}_{s \sim \mu}(\bar{v}(s)) \tag{3}
$$

with the *augmented* value function defined as:

$$
\bar{v}(s) = \mathbb{E}_{a \sim \pi}(q(s,a) - \log(\pi(a|s))) \tag{4}
$$

Soft Policy improvement

The policy improvement step is the following for each state $s \in \mathcal{S}$

$$\pi_{\text{new}}(a|s) = \arg\min_{\pi' \in \Pi} D_{\text{KL}} \left(\pi'(\cdot|s) \middle\| \frac{\exp(q^{\pi^{\text{old}}}(s, \cdot))}{Z^{\pi^{\text{old}}}(s)} \right) \tag{5}$$

Where the Kullback-Leibler divergence ensures a projection of the improve policy over a desired set of policies Π. One can easily show that:

Proposition (Soft Policy improvement)
If a policy $\pi^{new}(\cdot|s)$ fulfilled 5, and Π is the set of distribution over $\mathcal{A}$, then:

$$\pi_{new}(a|s) = \frac{\exp(q^{\pi^{old}}(s, a))}{\sum_{a \in \mathcal{A}} \exp(q^{\pi^{old}}(s, a))} \tag{6}$$

Proof.
See in appendix A (page 23).

□

Convergence of the algorithm

The policy evaluation algorithm converges due to two facts:

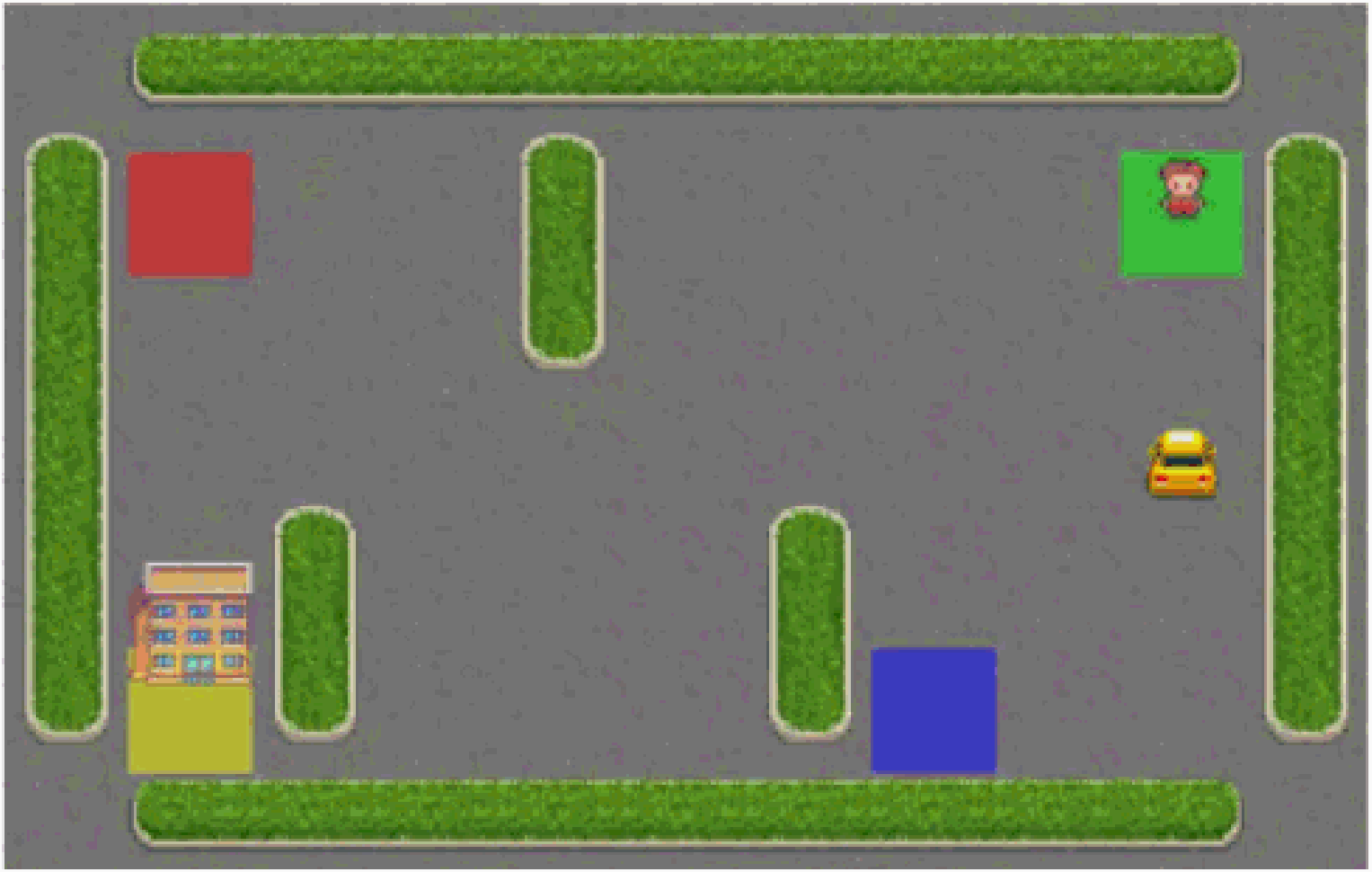
1. Contraction of the *soft Bellman operator* which ensures the convergences of the q-function.
2. The soft policy improvement which ensures that we improve the q-function at each step.

Algorithm 1: Soft Policy Iteration

Input: Initialise q and π

foreach i *in* $[\![1, N_{iter}]\!]$ **do**
- **Policy evaluation**
- **foreach** j *in* $[\![1, N_q]\!]$ **do**
  - **foreach** s *in* $\mathcal{S}$ **do**
    - **foreach** a *in* $\mathcal{A}$ **do**
      - $\text{acc} \leftarrow 0$
      - **foreach** s' *in* $\mathcal{S}$ **do**
        - **foreach** a' *in* $\mathcal{A}$ **do**
          - $\text{acc} \leftarrow p(s'|s,a)\pi(a'|s')q(s',a') + p(s'|s,a)\pi(a'|s')\log(\pi(a'|s'))$
      - $q(s,a) \leftarrow r(s,a) + \gamma \times \text{acc}$
- **Policy improvement** $\pi(a|s) \leftarrow \dfrac{\exp(q^{\pi}(s,a))}{\sum_{a \in \mathcal{A}} \exp(q^{\pi}(s,a))}$

Numerical results with `env = gym.make("Taxi-v3")`



Soft Policy iteration only works with finite states and actions space

=> need to developp more robust methods for continuous spaces

Soft Actor Critic

Soft Actor Critic

For continous spaces, one need to introduce specific function approxiators:

- Value function:

$$J_{\bar{v}}(\phi) = \mathbb{E}_{s_t \sim \mathcal{D}} \left[\frac{1}{2} \left(\bar{v}_\phi(s_t) - \underbrace{\mathbb{E}_{a_t \sim \pi_\phi}[q_\theta(s_t, a_t) - \log(\pi_\phi(a_t|s_t))]}_{\text{Approximation of the augmented value function given by 4}} \right)^2 \right] \tag{7}$$

- q function:

$$J_q(\theta) = \mathbb{E}_{(s_t, a_t) \sim \mathcal{D}} \left[\frac{1}{2} \left(Q_\theta(s_t, a_t) - \underbrace{(r(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim \mu}[\bar{v}_{\bar{\psi}}(s_{t+1})])}_{\text{Approximation of the real } q \text{ value given by 3}} \right)^2 \right] \tag{8}$$

- π:

$$J_\pi(\phi) = \mathbb{E}_{s_t \sim \mathcal{D}} \left[D_{\mathrm{KL}} \left(\pi_\phi(\cdot|s_t) \middle\| \frac{q_\theta(s_t, \cdot)}{Z_\theta(s_t)} \right) \right] \tag{9}$$

One can now perform a stochastic gradient descent.

Algorithm

Algorithm 2: Soft Actor-Critic (SAC)

Initialize:

Parameters $\psi, \theta_1, \theta_2, \phi$

Replay buffer $\mathcal{D} \leftarrow \emptyset$ `// used for Off-Policy learning`

for *each iteration* **do**

for *each environment step* **do**

$a_t \sim \pi_\phi(a_t|s_t)$ `// Sample action`

$s_{t+1} \sim p(s_{t+1}|s_t, a_t)$ `// Step environment`

$\mathcal{D} \leftarrow \mathcal{D} \cup \{(s_t, a_t, r_t, s_{t+1})\}$

for *each gradient step* **do**

`// Neural Networks over a batch` B `selected over` $\mathcal{D}$ `and compute gradient` *via* `backpropagation`

$\psi \leftarrow \psi - \lambda_V \hat{\nabla}_\psi J_V(\psi)$

$\phi \leftarrow \phi - \lambda_\pi \hat{\nabla}_\phi J_\pi(\phi)$ `// Update Policy`

$\theta_i \leftarrow \theta_i - \lambda_q \hat{\nabla}_{\theta_i} J_q(\theta_i)$ for $i \in \{1, 2\}$ `// Update q-networks`

$\bar{\psi} = \tau\psi + (1 - \tau)\bar{\psi}$

Pratical implementation of the code

The code involves several ideas:

- Use function approximation to deal with continuous functions.
- Two differents Neural Networks for the policy (actor) and the q function (Critic).
- Perform Stochastic gradient descent.
- Replay buffer to perform off policy learning.

as well as keeping this idea of entropy to help for stabilization.

Practical Implementation of the Code

- `next_values = next_q_values - entropy_scale * next_log_pis`
  and `q_targets = rewards + discounts * next_values`
  corresponds respectively to the estimation of the value and of the estimated $\hat{q}$: $v(s) = \mathbb{E}_{a\sim\pi}(q(s,a) - \log(\pi(a|s)))$ (4),
  $\hat{q}(s_t, a_t) = r(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1}\sim p}[v_{\bar{\phi}}(s_{t+1})]$
- **Update of θ (Critic):**
  `q_losses = 0.5*(tf.losses.MSE(y_true=q_targets, y_pred=q_values))`
  corresponds to: $J_q(\theta) = \mathbb{E}_{(s_t,a_t)\sim\mathcal{D}}\left[\frac{1}{2}\left(q_\theta(s_t,a_t) - \hat{q}(s_t,a_t)\right)^2\right]$
- **Update of ϕ (Actor):**
  `policy_losses = entropy_scale * log_pis - q_log_targets`
  corresponds to:
  $J_\pi(q) = \mathbb{E}_{s_t\sim\mathcal{D},\epsilon_t\sim\mathcal{N}}[\log \pi_\phi(f_\phi(\epsilon_t; s_t)|s_t) - q_\theta(s_t, f_\phi(\epsilon_t; s_t))]$.

In fact, a gradient descent on ϕ is unuseful.

Difficulties Cloning the GitHub Repository

Creating the SAC environment using `conda env create -f environment.yml` was impossible for the SAC repository (even for the most recent version), due to outdated dependencies. That results in the following errors:

```
Could not solve for environment specs.
The following packages are incompatible:
- joblib ==0.10.3
  - joblib 0.10.3 requires
    - python =3.5 *, which can be installed;
  - joblib 0.10.3 requires
    - python =3.4 *, which does not exist
      (possibly due to a missing channel);
  - joblib 0.10.3 requires
    - python =2.7 *, but no viable options are available:
      - python [2.7.13|2.7.14|...|2.7.18] requires
        - vc =9 *, which can be installed;
      - python 2.7.12 conflicts with all previously
        installable versions...
```

Results and conclusion

Results presented in the article: Comparaison between different algorithms

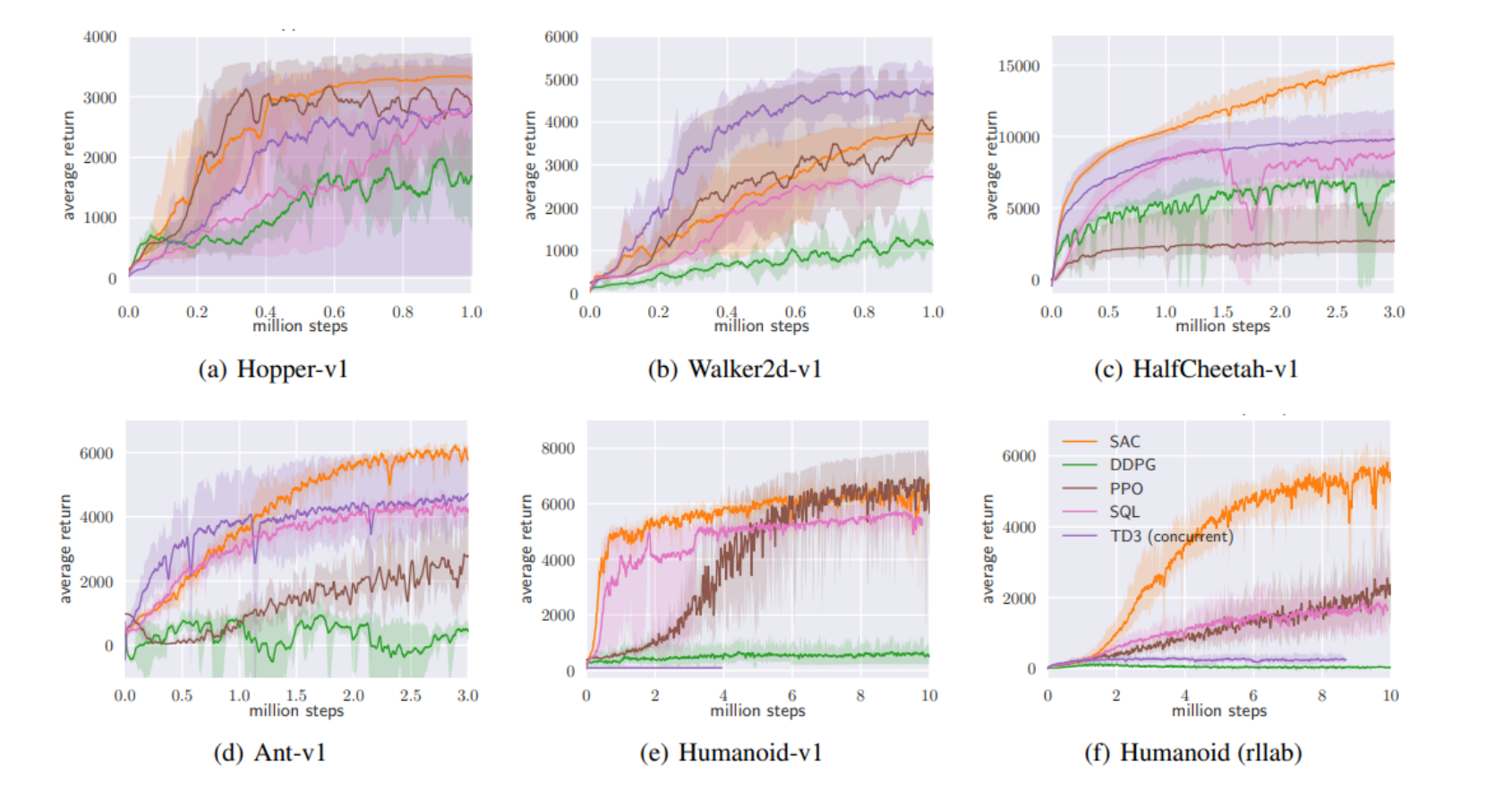


Figure 1: Results: Comparaison between different algorithms on the Mujoco environment

Results presented in the article: Stochastic VS deterministic

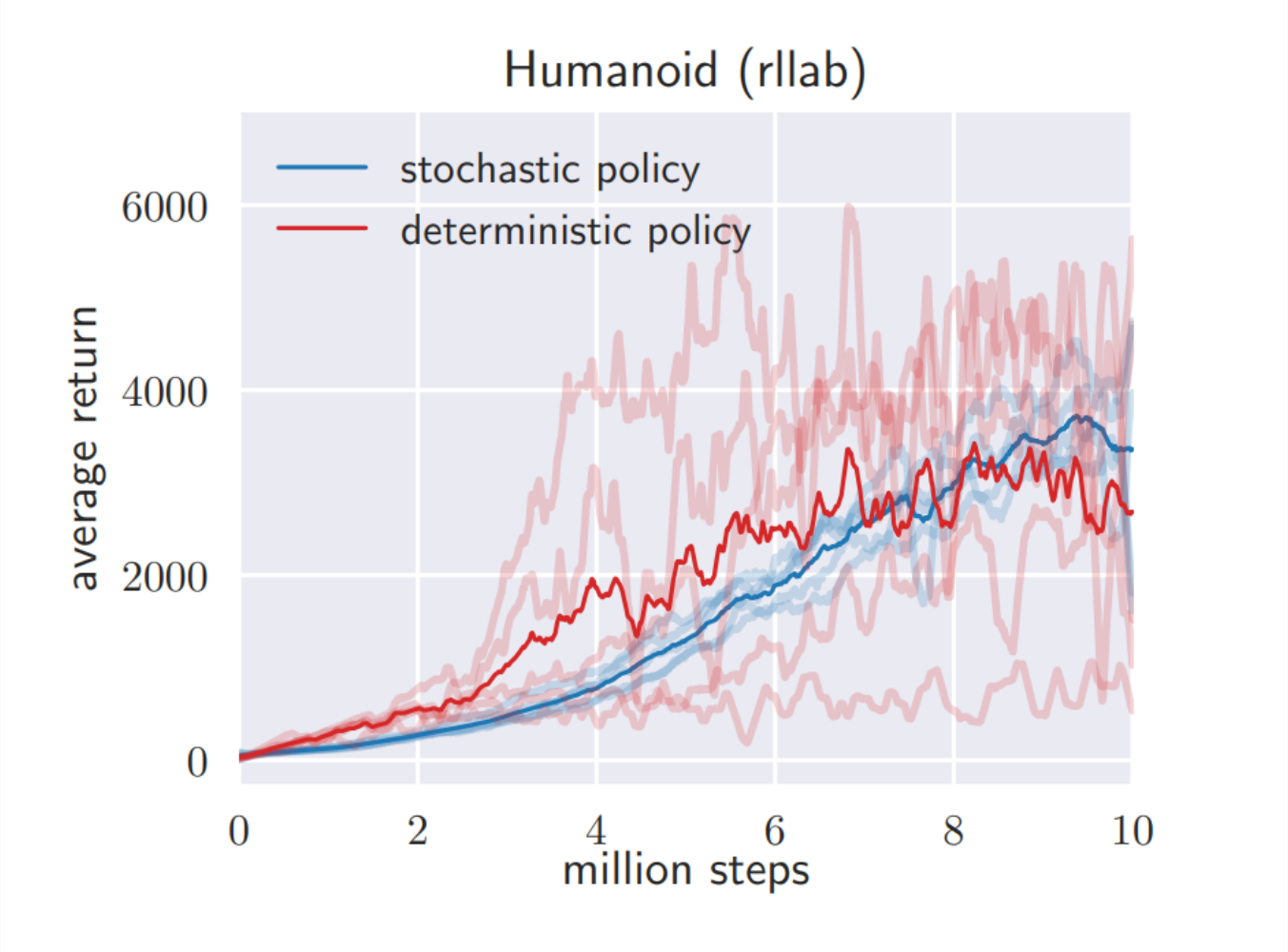


Figure 2: Results: Comparaison between different stochastic policy and deterministic policy for the SAC algorithm

Results presented in the article: Parameters influence

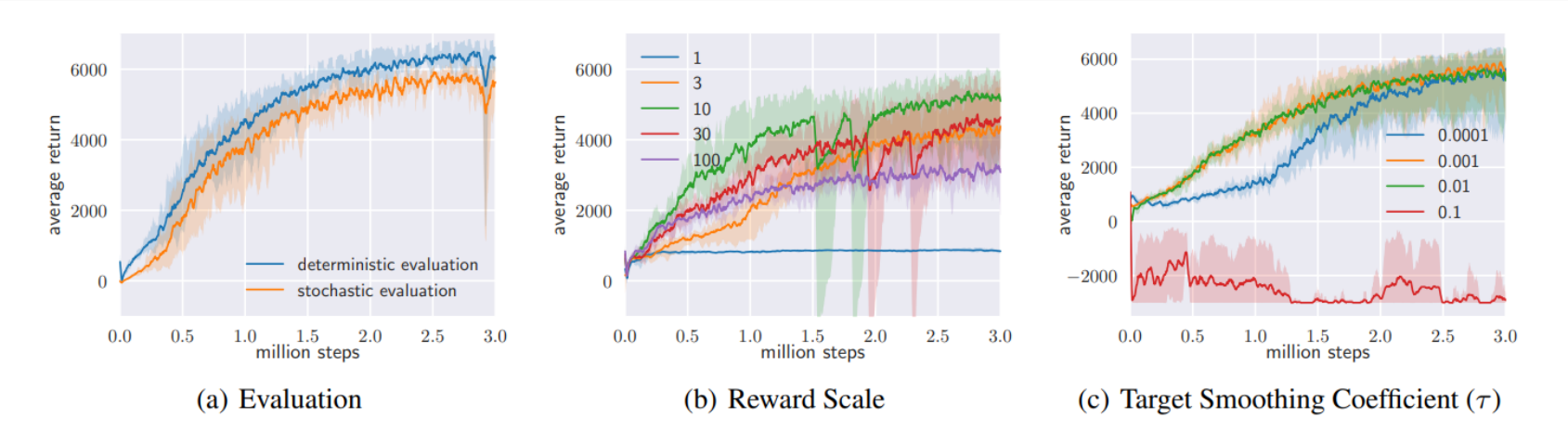


Figure 3: Results: Parameters influence of SAC

Our attempt to reproduce the results

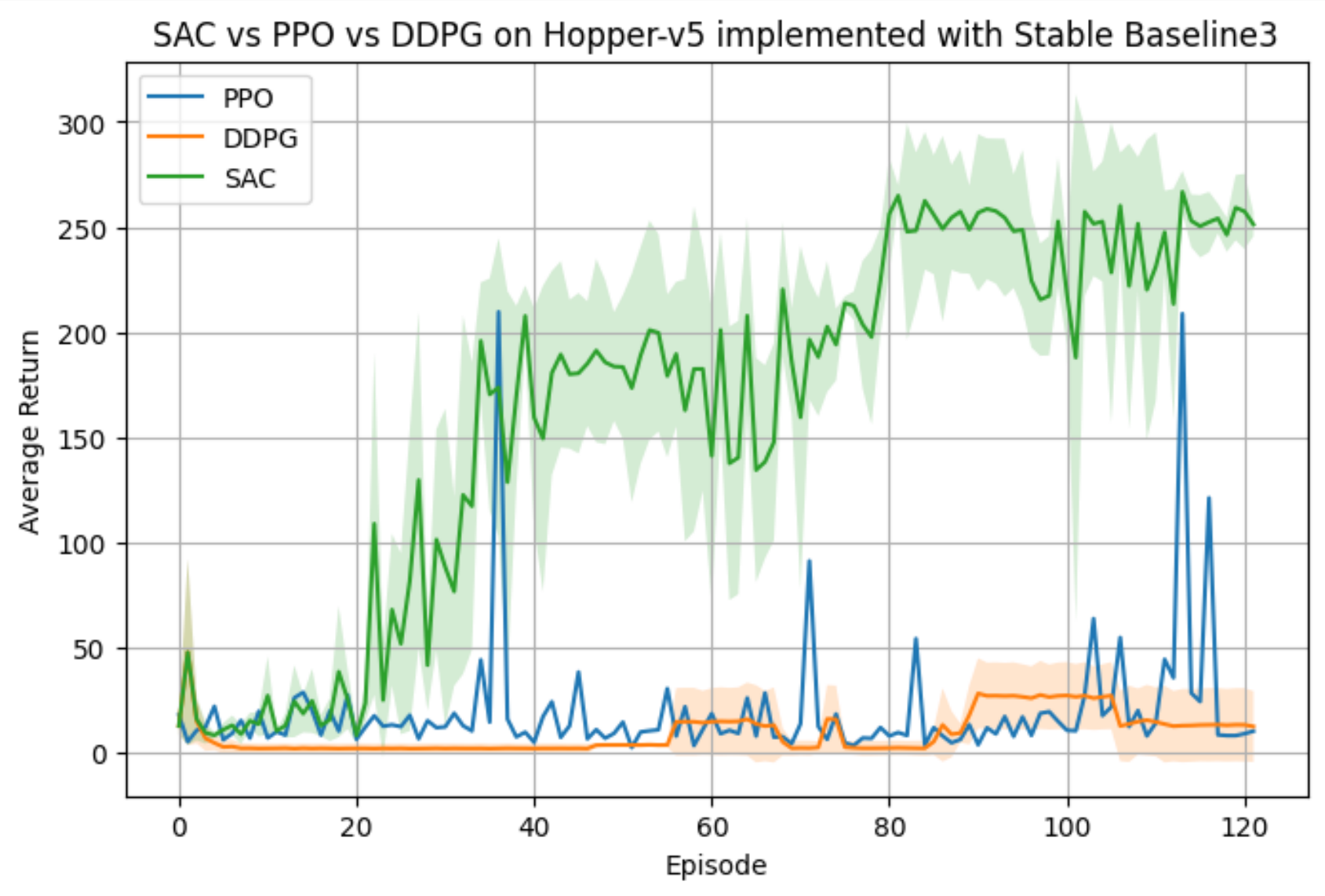


Figure 4: Personnal result implemented with the Stable Baseline3 library

Appendix A: Proof for proposition 1

We want to compute 5 $\arg\min_{\pi' \in \Pi} D_{\mathrm{KL}}\left(\pi'(\cdot|s) \middle\| \frac{\exp(q^{\pi^{\mathrm{old}}}(s,\cdot))}{Z^{\pi^{\mathrm{old}}}(s)}\right)$ One has:

$$
\begin{aligned}
D_{\mathrm{KL}}\left(\pi'(\cdot|s) \middle\| \frac{\exp(q^{\pi^{\mathrm{old}}}(s,\cdot))}{Z^{\pi^{\mathrm{old}}}(s)}\right) &= \sum_{a\in\mathcal{A}} \pi'(a|s) \log\left(\frac{\pi'(a|s) Z^{\pi^{\mathrm{old}}}(s)}{\exp(q^{\pi^{\mathrm{old}}}(s,a))}\right) \\
&= \sum_{a\in\mathcal{A}} \pi'(a|s) \log \pi'(a|s) \\
&\quad - \sum_{a\in\mathcal{A}} \pi'(a|s) q^{\pi^{\mathrm{old}}}(s,a) + \log Z^{\pi^{\mathrm{old}}}(s) \\
&:= J(\pi'(\cdot|s))
\end{aligned}
$$

Appendix A: Proof for proposition 1

J is a strictly convex and continuous function over the compact set which is the simplex, so it has a unique minimum. We consider the Lagrangian function $\mathcal{L}(\pi', \lambda) = J(\pi'(\cdot|s)) + \lambda \left(\sum_{a \in \mathcal{A}} \pi'(a|s) - 1 \right)$

The minimum is reached when the gradient w.r.t. π' vanishes:
$\frac{\partial \mathcal{L}}{\partial \pi'(a|s)} = \log \pi'(a|s) + 1 - q^{\pi^{\text{old}}}(s,a) + \lambda = 0, \quad \forall a \in \mathcal{A}.$

Taking into account the condition $\sum_{a \in \mathcal{A}} \pi'(a|s) = 1$ helps to find λ

which leads to: $\pi^*(a|s) = \frac{\exp(q^{\pi^{\text{old}}}(s,a))}{\sum_{b \in \mathcal{A}} \exp(q^{\pi^{\text{old}}}(s,b))}$.

Appendix B: TRPO and PPO

Using like a Taylor-expansion, one get that:

$$V(\pi') = V(\pi) + (1-\gamma)\,\mathbb{E}_{S\sim d_\pi,\, A\sim\pi}\left[\frac{\pi'(A|S)}{\pi(A|S)}\, a_\pi(S,A)\right] + O\left(\mathbb{E}_{S\sim d_\pi}\left[D_{\mathrm{KL}}\left(\pi'(\cdot|S)\,\|\,\pi(\cdot|S)\right)\right]\right)$$

One aims to achieve directly a policy update:

$$\theta_{k+1} \in \arg\max_\theta L_{\pi_{\theta_k}}(\pi_\theta) \;\triangleq\; \mathbb{E}_{\pi_{\theta_k}}\left[\sum_{t=0}^{\infty}\gamma^t \frac{\pi_\theta(A_t|S_t)}{\pi_{\theta_k}(A_t|S_t)} a_{\pi_{\theta_k}}(S_t,A_t)\right]$$
$$\text{s.t.}\quad \mathbb{E}_{S\sim d_{\pi_{\theta_k}}}\left[D_{\mathrm{KL}}(\pi_\theta(\cdot|S)\,\|\,\pi_{\theta_k}(\cdot|S))\right] \le \delta \tag{10}$$

This is the main idea in PPO (Schulman, Levine, et al. 2015), while in TPO achieves gradient clipping over (Schulman, Wolski, et al. 2017).

Appendix C : DDPG and TD3

Let's recap the optimal Bellman equation for the q function.

$$q^*(s,a) = \mathbb{E}_{s' \sim \mu}\left[r(s,a) + \gamma \max_{a'}(q^*(s',a'))\right] \tag{11}$$

From this idea, one can derive a squared Bellman error (MSBE):

$$L(\phi, \mathcal{D}) = \underset{(s,a,r,s',d)\sim\mathcal{D}}{\mathrm{E}}\left[\left(q_\phi(s,a) - \left(r + \gamma(1-d)\max_{a'} q_\phi(s',a')\right)\right)^2\right] \tag{12}$$

The $(d-1)$ is set for computation reasons. $\mathcal{D}$ is a replay-buffer which helps to perform Off-policy learning.
TD3 is based on this idea but introduces Double-q Learning to reduce positive bias.

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