Coloured Kac-Moody algebras, Part I

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Abstract

We introduce a parametrization of formal deformations of Verma modules of $\mathfrak{sl}_2$. A point in the moduli space is called a colouring. We prove that for each colouring ψ satisfying a regularity condition, there is a formal deformation $U_h(\psi)$ of $U(\mathfrak{sl}_2)$ acting on the deformed Verma modules. We retrieve in particular the quantum algebra $U_h(\mathfrak{sl}_2)$ from a colouring by q-numbers. More generally, we establish that regular colourings parametrize a broad family of formal deformations of the Chevalley-Serre presentation of $U(\mathfrak{sl}_2)$. The present paper is the first of a series aimed to lay the foundations of a new approach to deformations of Kac-Moody algebras and of their representations. We will employ in a forthcoming paper coloured Kac-Moody algebras to give a positive answer to E. Frenkel and D. Hernandez's conjectures on Langlands duality in quantum group theory.

1. *Introduction*

1.1. *Deformation by Tannaka duality*

The Lie algebra $\mathfrak{sl}_2$ formed by 2-by-2 matrices with zero trace is the easiest example of a semisimple Lie algebra, or more generally of a Kac-Moody algebra. The Chevalley-Serre presentation [**13**] of $\mathfrak{sl}_2$ consists of the Chevalley generators H, X^-, X^+, and of the relations

$$[H, X^{\pm}] = \pm 2X^{\pm}, \tag{1.1a}$$

$$[X^-, X^+] = H. \tag{1.1b}$$

We present in this paper a new approach, both elementary and systematic, to deformations of the universal enveloping algebra $U(\mathfrak{sl}_2)$, over a ground field K of characteristic zero. Deformations here are formal, i.e. they are considered over the power series ring $K[|h|]$. We shall give a precision. It follows from a cohomological rigidity criterion of M. Gerstenhaber [**8**] that formal deformations of the structure of associative algebra of $U(\mathfrak{sl}_2)$ are all trivial, i.e. they are conjugate to the constant formal deformation. In this paper though, we are interested in deforming a slightly richer structure, which consists of the algebra $U(\mathfrak{sl}_2)$, together with the Chevalley generators. In other words, when considering a formal deformation of $U(\mathfrak{sl}_2)$, we want to specify within it a deformation of the generators H, X^-, X^+. Equivalently, we may say that we are looking at formal deformations of the Chevalley-Serre presentation (1.1) of $U(\mathfrak{sl}_2)$.

Representations of $\mathfrak{sl}_2$ carry all the information of the algebra $U(\mathfrak{sl}_2)$, in the sense that $U(\mathfrak{sl}_2)$ can be reconstructed by Tannaka duality from the category $\mathrm{Rep}(\mathfrak{sl}_2)$ of representations of $\mathfrak{sl}_2$. More specifically, $U(\mathfrak{sl}_2)$ can be defined as the algebra of endomorphisms (namely, the natural transformations) of the forgetful functor from $\mathrm{Rep}(\mathfrak{sl}_2)$ to the category of vector spaces.

We propose to construct formal deformations of $U(\mathfrak{sl}_2)$ via Tannaka duality. In our view, the category $\mathrm{Rep}(\mathfrak{sl}_2)$ would be too large to be deformed in one go. We need to look for a more modest subcategory to start with. One first candidate that comes easily in mind is the subcategory $\mathrm{rep}(\mathfrak{sl}_2)$ of finite-dimensional representations. On the one hand, this subcategory

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is rich enough to distinguish by Tannaka duality elements in $U(\mathfrak{sl}_2)$: an element in $U(\mathfrak{sl}_2)$ is zero if and only if it acts by zero on every finite-dimensional representation of $\mathfrak{sl}_2$. On the other hand, the category rep($\mathfrak{sl}_2$) is notably elementary: all objects are completely reducible and finite-dimensional irreducible representations of $\mathfrak{sl}_2$ are classified by their dimensions. There is however a slightly larger category which appears more suited to our purpose. This category is generated by Verma modules, in a sense which we will make precise in a next paper. Let us note that we consider here only integral Verma modules, that is to say Verma modules for which the action of H has integral eigenvalues. One reason to prefer Verma modules rather than finite-dimensional representations is that the former are all equal when forgetting the action of $\mathfrak{sl}_2$ – they share the same underlying vector space. This makes deformation and Tannaka duality easier to deal with. Another reason is the definition itself of Verma modules of $\mathfrak{sl}_2$ (namely, they are the representations of $\mathfrak{sl}_2$ induced by the one-dimensional representations of a Borel subalgebra); we will show in a next paper that any deformation of Verma modules of $\mathfrak{sl}_2$ lead naturally to a deformation of the whole category Rep($\mathfrak{sl}_2$).

1.2. *Summary of the main results*

DEFINITION 1.1. A colouring is a sequence $\psi = (\psi^k(n))_{k\geq 1}$ of formal power series in $K[|h|]$, whose values depend on $n \in \mathbb{Z}$ and verify

(C1) $\psi^k(n) = k(n-k+1) \mod h$, for all k, n,

(C2) $\psi^{n+1}(n) = 0$, for all $n \geq 0$,

(C3) $\psi^{n+k+1}(n) = \psi^k(-n-2)$, for all $k \geq 1$ and all $n \geq 0$.

For $n \in \mathbb{Z}$ we denote by $M(n)$ the integral Verma module of $\mathfrak{sl}_2$ of highest weight n. When forgetting the action of X^+, the integral Verma modules of $\mathfrak{sl}_2$ become representations of the Borel subalgebra $\mathfrak{b}$ spanned by H and X^-. The action of X^+ can be retrieved from the *natural colouring* N, defined by $N^k(n) = k(n-k+1)$. In view of axiom (C1), colourings can thus be considered as formal deformations of the action of X^+ on the integral Verma modules of $\mathfrak{sl}_2$.

DEFINITION 1.2. We denote by $M_h(n,\psi)$ the $K[|h|]$-module $M(n)[|h|]$, endowed with the constant deformation of the action of $\mathfrak{b}$ on $M(n)$, together with the deformation of the action of X^+, given by the colouring ψ.

Here is where Tannaka duality comes into the picture.

DEFINITION 1.3. We denote by $U_h(\psi)$ the $K[|h|]$-algebra generated by H, X^-, X^+ and subject to the relations satisfied in every representation $M_h(n,\psi)$.

For the reader who may find unclear why this definition involves Tannaka duality, let us mention that there is a category built directly from the representations $M_h(n,\psi)$ and whose Tannaka dual algebra is canonically isomorphic to $U_h(\psi)$. Details will appear in a next paper.

We prove that $U_h(\psi)$ deforms the algebra $U(\mathfrak{sl}_2)$, provided that the colouring ψ satisfies a regularity condition; we call it then a *coloured Kac-Moody algebra*.

THEOREM 1.1. *The algebra $U_h(\psi)$ is a formal deformation of the algebra $U(\mathfrak{sl}_2)$ if and only if the colouring ψ is regular, i.e. $\psi^k(n) \in K[k,n][|h|]$.*

A coloured Kac-Moody algebra defines not only a formal deformation of the algebra $U(\mathfrak{sl}_2)$, but also a formal deformation of the Chevalley generators of $U(\mathfrak{sl}_2)$. As a result, it defines unambiguously – once we have fixed a basis of $U(\mathfrak{sl}_2)$, e.g. the PBW basis formed by the monomials $(X^-)^a(X^+)^bH^c$ $(a, b, c \geq 0)$ – a formal deformation of the Chevalley-Serre presentation of $U(\mathfrak{sl}_2)$.

THEOREM 1.2. *For ψ regular, the $K[|h|]$-algebra $U_h(\psi)$ is generated by H, X^-, X^+, and subject to the relations*

$$\begin{aligned} [H, X^{\pm}] &= \pm 2X^{\pm}, \\ X^+X^- &= \sum_{a=0}^{\infty} (X^-)^a (X^+)^a \xi^a(H), \end{aligned}$$

where the series $\xi^a(H) \in K[H][|h|]$ form the regular solution of an infinite-dimensional linear equation parametrized by ψ (see section 3).

Let $U(\mathfrak{a})$ be the K-algebra generated by H, X^-, X^+ and subject to the relations (1.1a). There is a canonical homomorphism from $U(\mathfrak{a})$ to $U(\mathfrak{sl}_2)$; we say that $U(\mathfrak{sl}_2)$ is an $\mathfrak{a}$-algebra. Relations (1.1a) hold in $U_h(\psi)$ for every colouring ψ, i.e. $U_h(\psi)$ is a $U_h(\mathfrak{a})$-algebra, where $U_h(\mathfrak{a})$ designates the $K[|h|]$-algebra $U(\mathfrak{a})[|h|]$. We may then regard a coloured Kac-Moody algebra $U_h(\psi)$ as a formal deformation of the structure of $\mathfrak{a}$-algebra of $U(\mathfrak{sl}_2)$.

To any symmetrizable Kac-Moody algebra $\mathfrak{g}$, V. Drinfel'd [**4**] and M. Jimbo [**9**] associated a formal deformation $U_h(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$.† We show in this paper that $U_h(\mathfrak{sl}_2)$ arises from the *q-colouring* N_q, which is defined by replacing natural numbers with q-numbers in the natural colouring N.

THEOREM 1.3. *The quantum algebra $U_h(\mathfrak{sl}_2)$ is isomorphic as a $U_h(\mathfrak{a})$-algebra to the coloured Kac-Moody algebra $U_h(N_q)$.*

It has been proved by V. Drinfel'd [**5**] that for $\mathfrak{g}$ semisimple, $U_h(\mathfrak{g})$ is a $\mathfrak{h}$-trivial formal deformation of $U(\mathfrak{g})$, i.e. there exists an equivalence of formal deformation between $U(\mathfrak{g})[|h|]$ and $U_h(\mathfrak{g})$ fixing the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We establish in the present paper that regular colourings classify all $\mathfrak{h}$-trivial formal deformations of the structure of $\mathfrak{a}$-algebra of $U(\mathfrak{sl}_2)$.

THEOREM 1.4. *The map $\psi \mapsto U_h(\psi)$ is a bijection between colourings and isomorphism classes of $\mathfrak{h}$-trivial deformations of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$.*

As a corollary, we obtain a new‡ rigidity result for $U(\mathfrak{sl}_2)$.

COROLLARY 1.5. *A $\mathfrak{h}$-trivial formal deformation A of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ is also $\mathfrak{b}$-trivial, i.e. there exists an equivalence of formal deformation between $U(\mathfrak{sl}_2)[|h|]$ and A fixing both H and X^-, and A moreover admits only one such equivalence.*

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†The quantum algebra $U_h(\mathfrak{sl}_2)$ first appeared in works of P. Kulish and E. Sklyanin; see [**10**].

‡To the best of the author's knowledge.

1.3. *Coloured Kac-Moody algebras*

The present paper is the first of a series aimed to lay the foundations of a new approach to deformations of Kac-Moody algebras, and of their representations. We present here results in the rank one case, focusing on one-parameter formal deformations of the $\mathfrak{a}$-algebra structure of $U(\mathfrak{sl}_2)$. We will investigate in a next paper formal deformations of the structure of Hopf algebra of $U(\mathfrak{sl}_2)$. It will be proved that regular (di)colourings provide a classification of formal deformations of the Hopf algebra $U(\mathfrak{sl}_2)$ (together with formal deformations of the Chevalley generators). More generally, we will show that there is a natural group action on the set of colourings and that the resulting action groupoid is equivalent to the groupoid formed by formal deformations of the Hopf algebra $U(\mathfrak{sl}_2)$. These results will be generalized in subsequent papers from $\mathfrak{sl}_2$ to any symmetrizable Kac-Moody algebra $\mathfrak{g}$. Let us precise that we won't be concerned with all the deformations of the Hopf algebra $U(\mathfrak{g})$, as we will restrict ourselves to those deformations which preserve the grading of $U(\mathfrak{g})$ by the weight lattice of $\mathfrak{g}$.

Coloured Kac-Moody algebras are defined by Tannaka duality. In a next paper, we will explain how a colouring ψ induces in an elementary way a closed monoidal category $\mathrm{Rep}(\mathfrak{g},\psi)$, and we will show that $\mathrm{Rep}(\mathfrak{g},\psi)$ is a deformation of the category of all representations of $\mathfrak{g}$. The coloured Kac-Moody algebra $U(\mathfrak{g},\psi)$ will be defined as the Hopf algebra corresponding by Tannaka duality to the category $\mathrm{Rep}(\mathfrak{g},\psi)$. Let us note that whereas the construction of the category $\mathrm{Rep}(\mathfrak{g},\psi)$ is aimed to be as elementary as possible, the coloured Kac-Moody algebra $U(\mathfrak{g},\psi)$ itself may be in general difficult to describe explicitly (consider for example the Chevalley-Serre presentation of $U_h(\psi)$; see theorem 4.2).

Coloured Kac-Moody algebras should be understood as multi-parameters deformations of usual Kac-Moody algebras, with as many deformation parameters as there are degrees of freedom in the choice of a colouring. We will show in a next paper that all constructions and results obtained over the power series ring $K[|h|]$ hold over more general rings. There is in fact a generic coloured Kac-Moody algebra, which is universal in the sense that every other coloured Kac-Moody algebra can be obtained from it by specializing generic deformation parameters. Specializations will be a key feature of coloured Kac-Moody algebras, with several applications, as for example crystallographic Kac-Moody algebras to name one (we will show that there is a natural correspondence between representations of crystallographic Kac-Moody algebras and crystals of representations of quantum Kac-Moody algebras).

1.4. *Langlands interpolation*

P. Littelmann [**11**] and K. McGerty [**12**] have revealed the existence of relations between representations of a symmetrizable Kac-Moody algebra $\mathfrak{g}$ and representations of its Langlands dual ${}^L\mathfrak{g}$ (the Kac-Moody algebra defined by transposing the Cartan matrix of $\mathfrak{g}$). They have proved that the action of the quantum algebra $U_q(\mathfrak{g})$ on certain representations interpolates between an action of $\mathfrak{g}$ and an action of ${}^L\mathfrak{g}$. Namely, they have shown that the actions of $\mathfrak{g}$ and ${}^L\mathfrak{g}$ can be retrieved from the action of $U_q(\mathfrak{g})$ by specializing the parameter q to 1, and to some root of unity ϵ, respectively. In the case where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, E. Frenkel and D. Hernandez introduced in [**6**] an algebra depending on an additional parameter t, and they conjectured the existence of representations for this algebra interpolating between representations of the quantum algebras $U_q(\mathfrak{g})$ and $U_t({}^L\mathfrak{g})$. They besides conjectured that the constructions could be extended to any symmetrizable Kac-Moody algebra $\mathfrak{g}$. They lastly suggested a Langlands duality for crystals.

We will give in a forthcoming paper a positive answer to these conjectures. More precisely, we will show that for any symmetrizable Kac-Moody algebra $\mathfrak{g}$ there exists a coloured Kac-Moody algebra $U(\mathfrak{g}, N_{q,t})$ whose representations possess the predicted interpolation property. Examples of such an algebra have been explicitly constructed by the author of this paper in the case of $\mathfrak{sl}_2$ (within the more general context of isogenies of root data); see [**1**]. Using crystallographic Kac-Moody algebras, we will moreover confirm manifestations of Langlands duality at the level of crystals. Let us precise that the coloured Kac-Moody algebra $U(\mathfrak{g}, N_{q,t})$ have relations with E. Frenkel and D. Hernandez's algebra only when $\mathfrak{g} = \mathfrak{sl}_2$. For $\mathfrak{g}$ of higher ranks, the two algebras differ significantly.

We lastly mention that Langlands duality for quantum groups might have promising connections with the geometric Langlands correspondence; see [**6**] and [**7**].

1.5. *Organization of the paper*

In section 2, we introduce colourings and we construct deformed Verma modules $M_h(n, \psi)$ associated to a colouring ψ. We briefly recall the notion of formal deformation of associative algebras, and we give several definitions suited to the context of this paper. We define the algebra $U_h(\psi)$, and we prove that $U_h(\psi)$ is a formal deformation of an extension of $U(\mathfrak{sl}_2)$. In section 3, we express the action of $U_h(\psi)$ on $M_h(n, \psi)$ in terms of infinite-dimensional linear equations (proposition 3.3). We prove that these equations always admit regular solutions if and only if ψ is regular (proposition 3.4); this is the key technical result of this paper. In section 4, we prove that $U_h(\psi)$ is a formal deformation of $U(\mathfrak{sl}_2)$ if and only if the colouring ψ is regular (theorem 4.1). We give a Chevalley-Serre presentation of the coloured Kac-Moody algebra $U_h(\psi)$ (theorem 4.2). We show that the constant formal deformation $U(\mathfrak{sl}_2)[|h|]$ and the quantum algebra $U_h(\mathfrak{sl}_2)$ can be realized as coloured Kac-Moody algebras (theorem 4.3). We prove that coloured Kac-Moody algebras are $\mathfrak{b}$-trivial deformations of $U(\mathfrak{sl}_2)$, and admit unique $\mathfrak{b}$-trivializations (theorem 4.4). We prove that regular colourings classify $\mathfrak{h}$-trivial formal deformations of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ (theorem 4.5). As a corollary, we obtain a rigidity result for $U(\mathfrak{sl}_2)$; namely, every $\mathfrak{h}$-trivial formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ is also $\mathfrak{b}$-trivial, and admits a unique $\mathfrak{b}$-trivialization (corollary 4.6).

2. *Preliminaries*

In this section, we introduce colourings, and we construct deformed Verma modules $M_h(n, \psi)$ associated to a colouring ψ. We briefly recall the notion of formal deformations of associative algebras, and we give a few definitions suited to the context of this paper. We define the algebra $U_h(\psi)$, and we prove that $U_h(\psi)$ is a formal deformation of an extension of the algebra $U(\mathfrak{sl}_2)$.

Notations and conventions

(i) The integers are elements of $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. The non-negative integers are elements of $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. An empty sum is equal to zero, and an empty product is equal to one. We recall that K designates a field of characteristic zero. We denote by $K^{\mathbb{Z}}$ the K-vector space formed by functions from $\mathbb{Z}$ to K.

(ii) We denote by $K[|h|]$ the power series ring in the variable h over the field K. An element λ in $K[|h|]$ is of the form $\lambda = \sum_{m \geq 0} \lambda_m h^m$ with $\lambda_m \in K$. We regard K as a subring of $K[|h|]$.

(ii) Associative algebras and their homomorphisms are unital. Representations are left. Let $\mathfrak{g}$ be a Lie algebra over K, its universal enveloping algebra is denoted by $U(\mathfrak{g})$. We identify as usual representations of $\mathfrak{g}$ with representations of $U(\mathfrak{g})$.

(iv) Let B be a R-algebra ($R = K, K[|h|]$). A B-algebra is a R-algebra A, together with a structural R-algebra homomorphism $f : B \to A$. Let A' be another B-algebra, a B-algebra homomorphism from A to A' is a R-algebra homomorphism $g : A \to A'$ such that $g \circ f = f'$, where f' designates the structural homomorphism from B to A'. By $\mathfrak{g}$-algebra (for $\mathfrak{g}$ a Lie algebra over K) we understand $U(\mathfrak{g})$-algebra.

(v) For V_0 a K-vector space, we denote by $V_0[|h|]$ the $K[|h|]$-module formed by series of the form $v = \sum_{m\geq 0} v_m h^m$ with $v_m \in V_0$. We regard V_0 as a K-vector subspace of $V_0[|h|]$. A structure of K-algebra on V_0 induces a structure of $K[|h|]$-algebra on $V_0[|h|]$. Similarly, a representation V_0 of a K-algebra B_0 yields a representation $V_0[|h|]$ of the $K[|h|]$-algebra $B_0[|h|]$. More generally, a B_0-algebra A_0 yields a $B_0[|h|]$-algebra $A_0[|h|]$.

(vi) For V a $K[|h|]$-module, we denote by $V_{h=0}$ the K-vector space V/hV. For $f : V \to W$ a $K[|h|]$-linear map, we denote by $f_{h=0}$ the induced K-linear map from $V_{h=0}$ to $W_{h=0}$. A structure of $K[|h|]$-algebra on V induces a structure of K-algebra on $V_{h=0}$. Similarly, a representation V of a $K[|h|]$-algebra B yields a representation $V_{h=0}$ of the K-algebra $B_{h=0}$, and a B-algebra A yields a $(B_{h=0})$-algebra $A_{h=0}$.

(vii) The h-adic topology of a $K[|h|]$-module V is the linear topology whose local base at zero is formed by the $K[|h|]$-submodules $h^m V$ $(m \in \mathbb{Z}_{\geq 0})$. A $K[|h|]$-module V is said topologically free if V is isomorphic to $V_0[|h|]$ for some K-vector space V_0, or equivalently, if V is Hausdorff, complete and torsion-free; see for example [**2**]. A $K[|h|]$-algebra A is said topologically generated by a subset A', if A is equal to the closure of the $K[|h|]$-subalgebra generated by A'.

(viii) We will often make use of the following two facts for a $K[|h|]$-linear map $f : V \to W$:
- $f_{h=0}$ surjective implies f surjective, if V is complete, and if W is Hausdorff,
- $f_{h=0}$ injective implies f injective, if V is Hausdorff, and if W is torsion-free.

(ix) The Chevalley generators X^-, X^+, H form a basis of $\mathfrak{sl}_2$. It then follows from the Poincaré-Birkhoff-Witt theorem that the monomials $(X^-)^a (X^+)^b H^c$ $(a, b, c \geq 0)$ form a basis of $U(\mathfrak{sl}_2)$; we call it the PBW basis.

2.1. *Colourings*

We call the relations

$$[H, X^{\pm}] = \pm 2X^{\pm} \tag{1.1a}$$

the *non-deformed relations* of $\mathfrak{sl}_2$. We denote by $\mathfrak{a}$ the Lie algebra over K generated by H, X^-, X^+, and subject to the non-deformed relations of $\mathfrak{sl}_2$. We are interested in this paper in deformations of the algebra $U(\mathfrak{sl}_2)$, where the non-deformed relations of $\mathfrak{sl}_2$ still hold. Representations of such deformations of $U(\mathfrak{sl}_2)$ are in particular representations of the constant formal deformation $U(\mathfrak{a})[|h|]$, which we will denote by $U_h(\mathfrak{a})$.

We denote by $\mathfrak{b}^+$ the Borel subalgebra of $\mathfrak{sl}_2$ spanned by H and X^+. We recall that a Verma module of $\mathfrak{sl}_2$ is a representation of $\mathfrak{sl}_2$ induced from a one-dimensional representation of $\mathfrak{b}^+$. For $n \in \mathbb{Z}$ the (integral) Verma module $M(n)$ of highest weight n is equal to $\bigoplus_{k\geq 0} K b_k$ as a vector space, and the action of $\mathfrak{sl}_2$ is given by

$$\begin{aligned} H.b_k &= (n-2k)b_k, \\ X^-.b_k &= b_{k+1}, \\ X^+.b_k &= \begin{cases} 0 & \text{if } k = 0, \\ k(n-k+1)b_{k-1} & \text{if } k \geq 1. \end{cases} \end{aligned} \tag{2.1}$$

We remark that H, X^-, X^+ act on $M(n)$ as scalar multiplications between $\mathbb{Z}_{\geq 0}$ copies of K:

$$\begin{tikzcd} K \arrow[loop above, "n"] \arrow[r, bend left, "1"] & K \arrow[l, bend left, "n"] \arrow[loop above, "n-2"] \arrow[r, bend left, "1"] & \cdots \arrow[l, bend left, "2(n-1)"] \arrow[r, bend left, "1"] & K \arrow[l, bend left, "k(n-k+1)"] \arrow[loop above, "n-2k"] \arrow[r, bend left, "1"] & K \arrow[l, bend left, "(k+1)(n-k)"] \arrow[loop above, "n-2k-2"] \arrow[r, bend left] & \cdots \arrow[l, bend left] \end{tikzcd}$$

Let $\mathcal{B}^+$ designate the quiver formed by the bottom arrows of the previous graph. We can think of the action of X^+ on the integral Verma modules of $\mathfrak{sl}_2$ as a $\mathbb{Z}$-graded representation of the quiver $\mathcal{B}^+$. This representation, which we denote by N, assigns to each vertex of $\mathcal{B}^+$ the K-vector space $\bigoplus_{n\in\mathbb{Z}} K$, and assigns to the k-th arrow ($k \geq 1$) the K-linear map represented by the diagonal matrix $\mathrm{diag}(N^k(n); n \in \mathbb{Z})$ where $N^k(n) = k(n-k+1)$.

Fixing the actions of H and X^-, a formal deformation of the action of the Lie algebra $\mathfrak{a}$ on the integral Verma modules of $\mathfrak{sl}_2$ corresponds to a formal deformation of the $\mathbb{Z}$-graded representation N of the quiver $\mathcal{B}^+$. Such a deformation is specified by a deformation $\psi^k(n)$ in $K[|h|]$ of the scalar $N^k(n)$, for each $k \geq 1$ and for each $n \in \mathbb{Z}$:

$$\bullet \xleftarrow{\psi^1(n)} \bullet \xleftarrow{\psi^2(n)} \cdots \xleftarrow{\psi^k(n)} \bullet \xleftarrow{\psi^{k+1}(n)} \bullet \longleftarrow \cdots$$

We recall that for each $n \geq 0$ there is a non-zero morphism from $M(-n-2)$ to $M(n)$. In terms of the representation N, the property becomes $N^{n+1}(n) = 0$ and $N^{n+k+1}(n) = N^k(-n-2)$ for all $k \geq 1$ and all $n \geq 0$. A formal deformation of the representation N preserving these conditions is called a colouring.

DEFINITION 2.1. A colouring is a sequence $\psi = (\psi^k)_{k\geq 1}$ with values in $K^{\mathbb{Z}}[|h|]$, verifying

(C1) $\psi^k(n) = k(n-k+1) \mod h$, for all k, n,

(C2) $\psi^{n+1}(n) = 0$, for all $n \geq 0$,

(C3) $\psi^{n+k+1}(n) = \psi^k(-n-2)$, for all $k \geq 1$ and all $n \geq 0$.

As explained before, a colouring is meant to encode a formal deformation of the action of X^+ on the integral Verma modules of $\mathfrak{sl}_2$, in such a way that the non-deformed relations of $\mathfrak{sl}_2$ remain satisfied.

DEFINITION 2.2. Let ψ be a colouring, and let $n \in \mathbb{Z}$. We denote by $M_h(n, \psi)$ the representation of $U_h(\mathfrak{a})$, whose underlying $K[|h|]$-module is $(\bigoplus_{k\geq 0} Kb_k)[|h|]$, and where the action of $U_h(\mathfrak{a})$ is given by

$$\begin{aligned} H.b_k &= (n-2k)b_k, \\ X^-.b_k &= b_{k+1}, \\ X^+.b_k &= \begin{cases} 0 & \text{if } k = 0, \\ \psi^k(n)b_{k-1} & \text{if } k \geq 1. \end{cases} \end{aligned}$$

EXAMPLE 2.3. We call *natural colouring*, and we denote by N, the unique colouring with values in $K^{\mathbb{Z}}$; it is defined by $N^k(n) = k(n-k+1)$. The natural colouring encodes the action of $\mathfrak{a}$ on the integral Verma modules of $\mathfrak{sl}_2$: $M_h(n, N) = M(n)[|h|]$ as representations of $U_h(\mathfrak{a})$ for all $n \in \mathbb{Z}$.

EXAMPLE 2.4. The quantum algebra $U_h(\mathfrak{sl}_2)$ is the $U_h(\mathfrak{a})$-algebra topologically generated by H, X^-, X^+, and subject to the relation

$$[X^+, X^-] = \frac{q^H - q^{-H}}{q - q^{-1}} \quad \text{with } q = \exp(h) \text{ and } q^H = \exp(hH),$$

i.e. $U_h(\mathfrak{sl}_2)$ is the quotient of $U_h(\mathfrak{a})$ by the smallest closed (for the h-adic topology) two-sided ideal containing the previous relation. We denote by N_q the colouring defined by

$$N_q^k(n) = [k]_q[n-k+1]_q \quad \text{where } [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}};$$

we call it the *q-colouring*. The q-colouring encodes the action of $U_h(\mathfrak{a})$ on the integral Verma modules of $U_h(\mathfrak{sl}_2)$: for all $n \in \mathbb{Z}$, the representation $M_h(n, N_q)$ is the Verma module of $U_h(\mathfrak{sl}_2)$ of highest weight n, when viewed as a representation of $U_h(\mathfrak{a})$.

2.2. *Formal deformation of associative algebras*

By formal deformation of a K-algebra A_0, one usually designates a topologically free $K[|h|]$-algebra A, together with a K-algebra isomorphism f_0 from A_0 to $A_{h=0}$. Two formal deformations (A, f_0) and (A', f'_0) are said equivalent if there exists a $K[|h|]$-algebra isomorphism g from A to A' such that $g_{h=0} \circ f_0 = f'_0$.

As already mentioned, we are interested in this paper in formal deformations of $U(\mathfrak{sl}_2)$ where the non-deformed relations (1.1a) of $\mathfrak{sl}_2$ still hold. In other words, the deformations of interest will be the formal deformations of $U(\mathfrak{sl}_2)$ within the category of $\mathfrak{a}$-algebras ($U(\mathfrak{sl}_2)$ has a canonical structure of $\mathfrak{a}$-algebra, induced by the projection map from $\mathfrak{a}$ to $\mathfrak{sl}_2$). Let us remark that specifying a B-algebra structure on an algebra A not only forces every relation in the algebra B to be satisfied in A, but also fixes pointwise in A the image of B. A formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ should therefore be understood as a formal deformation of the K-algebra $U(\mathfrak{sl}_2)$, together with a formal deformation of the Chevalley generators H, X^-, X^+ within the deformed algebra, in such a way that the non-deformed relations of $\mathfrak{sl}_2$ are preserved.

Definition 2.5. Let B_0 be a K-algebra, and let A_0 be a B_0-algebra. We suppose that the structural homomorphism from B_0 to A_0 is surjective. A formal deformation of the B_0-algebra A_0 is a $B_0[|h|]$-algebra A such that

(D1) the $K[|h|]$-module A is topologically free,

(D2) the B_0-algebras $A_{h=0}$ and A_0 are isomorphic.

Axiom (D2) may need a precision: the structure of $B_0[|h|]$-algebra on A induces a structure of $(B_0[|h|]_{h=0})$-algebra on $A_{h=0}$, and thus a structure of B_0-algebra (the K-algebras $B_0[|h|]_{h=0}$ and B_0 are canonically isomorphic).

As the structural homomorphism from B_0 to A_0 is surjective, there is a unique way to identify the B_0-algebras $A_{h=0}$ and A_0. This shows that definition 2.5 extends the usual definition of a formal deformation of a K-algebra. Let us also remark that in view of axiom (D1) the structural homomorphism from $B_0[|h|]$ to A is necessarily surjective.

Example 2.6. The quantum algebra $U_h(\mathfrak{sl}_2)$, as defined in example 2.4, is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$.

Definition 2.7. Let B_0 be a K-algebra and let V_0 be a representation of B_0. A formal deformation of V_0 along $B_0[|h|]$ is a representation V of $B_0[|h|]$ such that

(D1') the $K[|h|]$-module V is topologically free,

(D2') the representations $V_{h=0}$ and V_0 are isomorphic.

Let A be a formal deformation of a B_0-algebra A_0. We suppose that the action of B_0 on V_0 factorises through A_0. We say that V is a formal deformation of V_0 along A if the action of $B_0[|h|]$ on V factorises through A.

Example 2.8. For every $n \in \mathbb{Z}$, the representation $M_h(n, N_q)$, where N_q designates the q-colouring (see example 2.4), is a formal deformation along $U_h(\mathfrak{sl}_2)$ of $M(n)$.

The following lemma gives further examples of formal deformations of representations along the algebra $U_h(\mathfrak{a})$. It follows immediately from the first colouring axiom.

LEMMA 2.9. *Let ψ be a colouring. For every $n \in \mathbb{Z}$, the representation $M_h(n, \psi)$ is a formal deformation along $U_h(\mathfrak{a})$ of the Verma module $M(n)$, when viewed as a representation of $U(\mathfrak{a})$.*

2.3. *The algebra $U_h(\psi)$*

Integral Verma modules of $\mathfrak{sl}_2$ distinguish elements in the algebra $U(\mathfrak{sl}_2)$, i.e. an element in $U(\mathfrak{sl}_2)$ is zero if and only if it acts by zero on $M(n)$ for all $n \in \mathbb{Z}$. This remains true if we replace integral Verma modules of $\mathfrak{sl}_2$ with finite-dimensional irreducible representations of $\mathfrak{sl}_2$. These are known facts. We give a proof of them, for the reader's convenience.

PROPOSITION 2.10. *Let $x \in U(\mathfrak{sl}_2)$. The following assertions are equivalent.*
(i) *The element x is zero.*
(ii) *The element x acts by zero on all the integral Verma modules of $\mathfrak{sl}_2$.*
(iii) *The element x acts by zero on all the finite-dimensional irreducible representations of $\mathfrak{sl}_2$.*

Proof. The implication (i) $\Rightarrow$ (ii) is immediate. For $n \geq 0$ we denote by $L(n)$ the unique (up to isomorphism) irreducible representation of $\mathfrak{sl}_2$ of dimension $n+1$. The representation $L(n)$ is a quotient of the integral Verma module $M(n)$. As a consequence, assertion (ii) implies assertion (iii). Let us prove that assertion (iii) implies assertion (i). Let x be a non-zero element in $U(\mathfrak{sl}_2)$, and let us suppose for the sake of contradiction that x acts by zero on $L(n)$ for all $n \in \mathbb{Z}$. The representation $L(n)$ is equal to $\bigoplus_{k=0}^{n} K b_k$ as a vector space, and the action of $\mathfrak{sl}_2$ is given by

$$\begin{aligned} H.b_k &= (n-2k)b_k, \\ X^-.b_k &= \begin{cases} b_{k+1} & \text{if } k \leq n-1, \\ 0 & \text{if } k = n, \end{cases} \\ X^+.b_k &= \begin{cases} 0 & \text{if } k = 0, \\ k(n-k+1)b_{k-1} & \text{if } k \geq 1. \end{cases} \end{aligned} \tag{2.2}$$

The K-algebra $U(\mathfrak{sl}_2)$ has a $\mathbb{Z}$-gradation, defined by $\deg(H) = 0$ and $\deg(X^{\pm}) = \pm 1$. According to the way $U(\mathfrak{sl}_2)$ acts on $L(n)$, we can assume without loss of generality that x is a homogeneous element of $U(\mathfrak{sl}_2)$. Let d designate the degree of x. In view of the PBW basis of $U(\mathfrak{sl}_2)$, $x = \sum_{a=a_1}^{a_2} (X^-)^{a-d}(X^+)^a \xi^a(H)$ for some $\xi^{a_1}(H), \dots, \xi^{a_2}(H) \in K[H]$, with $a_2 \geq a_1 \geq \max(0, d)$. As x is non-zero, we can assume that $\xi^{a_1}(H) \neq 0$. Let $n \geq a_1, a_1 - d$. According to (2.2), the action of x on $L(n)$ satisfies

$$x.b_{a_1} = (X^-)^{a_1-d}(X^+)^{a_1}\xi^{a_1}(H).b_{a_1} = \frac{a_1!\, n!}{(n-a_1)!}\xi^{a_1}(n-2a_1)b_{a_1-d}.$$

It follows that $\xi^{a_1}(n-2a_1)$ is zero for all $n \geq a_1, a_1 - d$. The polynomial $\xi^{a_1}(H)$ is therefore zero, which is a contradiction. $\square$

In other words, one can define the algebra $U(\mathfrak{sl}_2)$ as the quotient of $U(\mathfrak{a})$ by I, where I designates the two-sided ideal of $U(\mathfrak{a})$ formed by the elements acting by zero on all the integral Verma modules of $\mathfrak{sl}_2$, when viewed as representations of $U(\mathfrak{a})$. This construction of $U(\mathfrak{sl}_2)$ may be viewed as an expression of a Tannaka duality between the algebra $U(\mathfrak{sl}_2)$ on the one side, and the Verma modules $M(n)$ on the other. We propose to consider the same construction, where integral Verma modules of $\mathfrak{sl}_2$ now carry a "coloured" action.

DEFINITION 2.11. Let ψ be a colouring. We denote by $I_h(\psi)$ the two-sided ideal of $U_h(\mathfrak{a})$ formed by the elements acting by zero on all the representations $M_h(n,\psi)$ ($n \in \mathbb{Z}$). We denote by $U_h(\psi)$ the quotient of $U_h(\mathfrak{a})$ by $I_h(\psi)$.

The algebra $U_h(\psi)$ has a natural structure of $U_h(\mathfrak{a})$-algebra, given by the projection map from $U_h(\mathfrak{a})$ to $U_h(\psi)$, and it follows from the definition that the action of $U_h(\mathfrak{a})$ on $M_h(n,\psi)$ factorises through $U_h(\psi)$. The algebra $U_h(\psi)$ is universal for this property.

PROPOSITION 2.12. *Let ψ a colouring. If A is a $U_h(\mathfrak{a})$-algebra such that*
(i) *the structural homomorphism from $U_h(\mathfrak{a})$ to A is surjective,*
(ii) *the action of $U_h(\mathfrak{a})$ on $M_h(n,\psi)$ factorises through A for all $n \in \mathbb{Z}$,*
then there is a unique surjective $U_h(\mathfrak{a})$-algebra homomorphism from A to $U_h(\psi)$.

Proof. Let f and g be the structural homomorphisms from $U_h(\mathfrak{a})$ to A and $U_h(\psi)$, respectively. As f and g are surjective, a $U_h(\mathfrak{a})$-algebra homomorphism from A to $U_h(\psi)$ is necessarily unique and surjective. Let x be an element in $U_h(\mathfrak{a})$ such that $f(x) = 0$. Since the action of $U_h(\mathfrak{a})$ on $M_h(n,\psi)$ factorises through A for all $n \in \mathbb{Z}$, it follows that x acts by zero on $M_h(n,\psi)$ for all $n \in \mathbb{Z}$. It implies, by definition of the algebra $U_h(\psi)$, that the image of x in $U_h(\psi)$ is zero. Put in other words, the map g factorises through f. □

The algebra $U_h(\psi)$ is not in general a formal deformation of $U(\mathfrak{sl}_2)$. We will give in section 4 a sufficient and necessary condition on the colouring ψ for $U_h(\psi)$ to be a formal deformation of $U(\mathfrak{sl}_2)$. However, the algebra $U_h(\psi)$ always satisfies the first axiom of a formal deformation; namely, $U_h(\psi)$ is topologically free.

LEMMA 2.13. *For any colouring ψ, the $K[|h|]$-module $U_h(\psi)$ is topologically free.*

Proof. The $K[|h|]$-module $U_h(\mathfrak{a})$ is by definition topologically free. It is in particular complete (for the h-adic topology). As the algebra $U_h(\psi)$ is a quotient of $U_h(\mathfrak{a})$, it is also complete. For $n \in \mathbb{Z}$ we denote by $E(n)$ the $K[|h|]$-algebra $\mathrm{End}_{K[|h|]}(M_h(n,\psi))$ and we denote by f_n the $K[|h|]$-algebra homomorphism from $U_h(\mathfrak{a})$ to $E(n)$ given by the representation $M_h(n,\psi)$. We denote by f the product of the f_n's ($n \in \mathbb{Z}$). The ideal $I_h(\psi)$ is equal to $\ker(f)$. The $K[|h|]$-modules $M_h(n,\psi)$ are by definition topologically free. They are in particular Hausdorff (for the h-adic topology) and torsion-free. Hence so is $E = \prod_{n\in\mathbb{Z}} E(n)$. As $I_h(\psi) = \ker(f)$, the algebra $U_h(\psi)$ is isomorphic to a $K[|h|]$-subalgebra of E. Therefore, $U_h(\psi)$ is Hausdorff and torsion-free. In conclusion, $U_h(\psi)$ is Hausdorff, complete and torsion-free. It is thus topologically free. □

Proving that $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ consists from now in proving that $U_h(\psi)_{h=0}$ is isomorphic as an $\mathfrak{a}$-algebra to $U(\mathfrak{sl}_2)$. As mentioned earlier, this is not true for a general colouring ψ. Nevertheless, we can show that the algebra $U_h(\psi)_{h=0}$ is always an extension of $U(\mathfrak{sl}_2)$.

LEMMA 2.14. *For any colouring ψ, there is a unique surjective $\mathfrak{a}$-algebra homomorphism from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)$.*

Proof. The structural homomorphisms from $U(\mathfrak{a})$ to the $\mathfrak{a}$-algebras $U_h(\psi)_{h=0}$ and $U(\mathfrak{sl}_2)$ are surjective. It implies that an $\mathfrak{a}$-algebra homomorphism from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)$ is necessarily unique and surjective. Let us consider the functor $(\bullet)_{h=0}$ from the category of $K[|h|]$-modules to the category of K-vector spaces. It is is a right-exact functor. Hence, there is a natural isomorphism between $U_h(\psi)_{h=0}$ and the quotient of $U_h(\mathfrak{a})_{h=0}$ by $I_0(\psi)$, where $I_0(\psi)$ designates the image of $I_h(\psi)_{h=0}$ in $U_h(\mathfrak{a})_{h=0}$. Using the canonical identification between $U_h(\mathfrak{a})_{h=0}$ and $U(\mathfrak{a})$, proving the lemma then reduces to proving that every element of $I_0(\psi) \subset U(\mathfrak{a})$ is zero in $U(\mathfrak{sl}_2)$. For all $n \in \mathbb{Z}$, the representation $M_h(n, \psi)$ is a formal deformation along $U_h(\mathfrak{a})$ of the Verma module $M(n)$, when viewed as representation of $U(\mathfrak{a})$ (lemma 2.9). This implies that every element $x \in I_0(\psi)$ acts by zero on all the integral Verma modules of $\mathfrak{sl}_2$, and in consequence that x is zero in $U(\mathfrak{sl}_2)$ (proposition 2.10). $\square$

3. *The equation $\psi \ltimes \xi = \theta$*

We have established in section 2 that for every colouring ψ the algebra $U_h(\psi)$ is a formal deformation of an extension of the $\mathfrak{a}$-algebra $U_h(\psi)$. Namely, we proved that $U_h(\psi)$ is a topologically free $K[|h|]$-module, and we proved that there is a surjective $\mathfrak{a}$-algebra homomorphism from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)$. It follows that $U_h(\psi)$ is a formal deformation of $U(\mathfrak{sl}_2)$ if and only if the aforementioned homomorphism is also injective. It is equivalent to prove that there is a relation in $U_h(\psi)$ which deforms the relation $[X^+, X^-] = H$ of $U(\mathfrak{sl}_2)$, or, that the element X^+X^- can be expressed in $U_h(\psi)$ as a linear combination (more precisely, as a limit of linear combinations) of the monomials $(X^-)^a(X^+)^bH^c$ $(a, b, c \geq 0)$; see section 4. In order to address this problem, we introduce in this section infinite-dimensional linear equations which encode the action of $U_h(\psi)$ on $M_h(n, \psi)$ (proposition 3.3). We prove that these equations always admit regular solutions if and only if the colouring ψ is regular (proposition 3.4); this is the key technical result of this paper.

3.1. *Definitions and notations*

We designate by $\mathbb{X}^d$ $(d \in \mathbb{Z}_{\geq 0})$ the $K[|h|]$-module formed by sequences with values in $K^{\mathbb{Z}}[|h|]$, of the form $f = (f^k)_{k \geq d}$ with $f^k = \sum_{m \geq 0} f^k_m h^m$ and $f^k_m \in K^{\mathbb{Z}}$. We say that $f \in \mathbb{X}^d$ is *summable* if f^k tends to zero (for the h-adic topology) as k goes to infinity. We say that f is of *Verma type* if it verifies

$$f^k(n) = 0 \text{ for all } k \geq d \text{ and all } n \geq 0, \text{ such that } n + 1 \leq k \leq n + d, \tag{3.1}$$

$$f^{n+k+1}(n) = f^k(-n-2) \text{ for all } k \geq d \text{ and all } n \geq 0. \tag{3.2}$$

For each $d \geq 0$ we denote by $\mathbb{X}^d_\infty$ and $\mathbb{X}^d_V$ the $K[|h|]$-submodules of $\mathbb{X}^d$ formed by the summable sequences, and by the sequences of Verma type, respectively. Let us remark that colourings form a subset of $\mathbb{X}^1_V$.

A sequence f in $\mathbb{X}^d$ is said *quasi-regular* if $f^k(n) \in K[n][|h|]$ for all k (i.e. the value $f^k_m(n) \in K$ depends polynomially on n for all k, m). We say that f is *regular* if there exists a sequence g in $\mathbb{X}^0_\infty$ such that $f^k(n) = \sum_{a=0}^{\infty} g^a(k) n^a$ for all $k \geq d$ and all n (the series is convergent as g is summable). Let us remark that f is regular if and only if it is quasi-regular, and if for each m the degree of the polynomial $f^k_m(n)$ is a function of k bounded above.

For $f \in \mathbb{X}^d$ we denote by $f[-1]$ the sequence in $\mathbb{X}^{d+1}$ defined by $(f[-1])^k = f^{k-1}$. We denote by $[+1]$ the inverse $K[|h|]$-linear map, from $\mathbb{X}^{d+1}$ to $\mathbb{X}^d$. Let us remark that the maps $[-1]$, $[+1]$

preserve summability and regularity. Let us furthermore remark that $[+1]$ sends sequences of Verma type to sequences of Verma type (this is not true for $[-1]$ in general).

Let $\psi \in \mathbb{X}_V^1$ and let $\xi \in \mathbb{X}^d$. We denote by $\psi \ltimes \xi$ the sequence in $\mathbb{X}_V^d$ defined for $k \geq d$ and for $n \in \mathbb{Z}$ by

$$(\psi \ltimes \xi)^k(n) = \sum_{a=d}^{k} \left(\prod_{b=k-a+1}^{k} \psi^b(n) \right) \xi^a(n-2k).$$

Let us note that the product $\prod_{b=k-a+1}^{k} \psi^b(n)$ is empty when a is zero, and thus equal to one by convention.

Remark 3.1. A sequence f in $\mathbb{X}_V^d$ ($d \in \mathbb{Z}_{\geq 0}$) is regular if and only if $f^k(n) \in K[k,n][|h|]$ (i.e. the value $f_m^k(n) \in K$ depends polynomially on k and n, for each m). Whereas we do not really use this fact in the present paper, the author believes that it is interesting in its own.

Proof of remark 3.1. Using the maps $[+1]$ and $[-1]$, we can assume without loss of generality that $d = 0$. Let us fix $m \geq 0$. On the one hand, it follows from the definition of regularity that there are functions $c^0, c^1, \ldots, c^p$ ($p \geq 0$) from $\mathbb{Z}_{\geq 0}$ to K such that $f_m^k(n) = \sum_{a=0}^{p} c^a(k) n^a$ for all k, n. On the other hand, f being of Verma type, it follows from condition (3.2) that $f_m^{n+k+1}(n) \in K[n]$ for all $k \geq 0$. The sequence $(f_m^k)_{k \geq 0}$ therefore verifies the assumptions of lemma 3.2 below. This proves that if f is regular, then $f^k(n) \in K[k,n][|h|]$. The converse implication is immediate. □

Lemma 3.2. *Let $(f^k)_{k\geq 0}$ be a sequence with values in $K^{\mathbb{Z}}$. We suppose that there are functions $c^0, c^1 \ldots, c^p$ ($p \geq 0$) from $\mathbb{Z}_{\geq 0}$ to K such that $f^k(n) = \sum_{a=0}^{p} c^a(k) n^a$, and we suppose that $f^{n+k+1}(n) \in K[n]$ for all $k \geq 0$. Then $f^k(n) \in K[k,n]$.*

Proof. Let us assume by induction that the claim holds for all $p' < p$. Induction starts at $p = 0$. We denote by δ the K-linear endomorphism of $K^{\mathbb{Z}}$ defined by $(\delta g)(n) = g(n) - g(n-1)$ for $g \in K^{\mathbb{Z}}$ and $n \in \mathbb{Z}$. We denote by δ^p the p-th power of δ. By hypothesis, $f^{n+k+1}(n) \in K[n]$ for $k \geq 0$. Shifting n by k, we obtain that $f^{n+1}(n-k) \in K[n]$ for all $k \geq 0$. We obtain in particular that the value

$$\sum_{k=0}^{p} \frac{(-1)^k}{k!(p-k)!} f^{n+1}(n-k) = \frac{(\delta^p f^{n+1})(n)}{p!} = c^p(n+1)$$

depends polynomially on n. It follows that $f_*^{n+k+1}(n) \in K[n]$ for all $k \geq 0$, where $f_* \in K^{\mathbb{Z}}$ is defined for $n \in \mathbb{Z}$ by $f_*^k(n) = \sum_{a=0}^{p-1} c^a(k) n^k$. One concludes by using the induction hypothesis. □

3.2. *Interpretation*

Let ψ be a colouring, and let $d \in \mathbb{Z}$. We designate by d^+ the non-negative integer $\max(d, 0)$. We say that an element x in $U_h(\psi)$ is of degree d if for every $k \geq d^+$ and for every $n \in \mathbb{Z}$, $x.b_k$ is equal to $\psi(x)^k(n) b_{k-d}$ in $M_h(n, \psi)$ for some $\psi(x)^k(n) \in K[|h|]$, and if $x.b_k$ is zero for all $0 \leq k < d^+$. Let us remark that the degree d of x is unique (by definition of $U_h(\psi)$ the element x is zero if and only if it acts by zero on $M_h(n, \psi)$ for all $n \in \mathbb{Z}$). Let us also remark that H and $X^{\pm}$ are of degrees 0 and ± 1 in $U_h(\psi)$, respectively. As $U_h(\psi)$ is topologically generated by H, X^-, X^+, it then follows from colouring axioms (C2), (C3), and from the definition of $M_h(n, \psi)$, that the values $\psi(x)^k(n)$ ($k \geq d^+$, $n \in \mathbb{Z}$) define a sequence $\psi(x)$ in $\mathbb{X}_V^{d^+}$.

Proposition 3.3. *Let ψ be a colouring, let $d \in \mathbb{Z}$, and let $\xi \in \mathbb{X}_\infty^{d^+}$ be a regular sequence. The series $\sum_{a \geq d^+} (X^-)^{a-d}(X^+)^a \xi^a(H)$ converges to a unique element x in $U_h(\psi)$ – for the h-adic topology, and the sequence ξ is a solution of the equation $\psi \ltimes \xi = \psi(x)$.*

Proof. It follows from regularity and summability that $\xi^a(n)$ $(a \geq d^+)$ define a sequence in $K[n][|h|]$, converging to zero (for the h-adic topology). Therefore, $\xi^a(H)$ defines an element in $U_h(\psi)$ which tends to zero (for the h-adic topology), as a goes to infinity. As $U_h(\psi)$ is topologically free (lemma 2.13), the series $\sum_{a \geq d^+} (X^-)^{a-d}(X^+)^a \xi^a(H)$ then converges to a unique element x in $U_h(\psi)$, of degree d. It follows from the definition of $\psi(x)$ that

$$\sum_{a=d^+}^{\infty} (X^-)^{a-d}(X^+)^a \xi^a(H).b_k \;=\; \psi(x)^k(n) b_{k-d}$$

holds in the representation $M_h(n, \psi)$ for all $k \geq d^+$ and all $n \in \mathbb{Z}$. In other words,

$$\sum_{a=d^+}^{k} \left(\prod_{b=k-a+1}^{k} \psi^b(n) \right) \xi^a(n-2k) \;=\; \psi(x)^k(n)$$

holds for all $k \geq d^+$ and all $n \in \mathbb{Z}$. □

3.3. *Regular solutions*

Let ψ be a colouring. As already mentioned, proving that $U_h(\psi)$ is a formal deformation of $U(\mathfrak{sl}_2)$ amounts to proving that there is a relation in $U_h(\psi)$ which deforms the relation $[X^+, X^-] = H$ of $U(\mathfrak{sl}_2)$. In particular (equivalently, in fact), we want to be able to express the element X^+X^- in $U_h(\psi)$ as a linear combination (more precisely, as a limit of linear combinations) of the monomials $(X^-)^a(X^+)^b H^c$ $(a, b, c \geq 0)$. In view of proposition 3.3, and since $\psi(X^+X^-) = \psi[+1]$, this leads us to look for regular solutions ξ of $\psi \ltimes \xi = \psi[+1]$.

We prove here that the equation $\psi \ltimes \xi = \psi[+1]$ admits a regular summable solution ξ if and only if the colouring ψ is regular. We moreover prove that the equation $\psi \ltimes \xi = \theta$ has a unique regular summable solution ξ, for every regular sequence θ of Verma type, provided that ψ is regular. This is the key technical result of this paper.

Proposition 3.4. *Let ψ be colouring.*
1. *The equation $\psi \ltimes \xi = \psi[+1]$ admits a regular solution ξ in $\mathbb{X}_\infty^0$ if and only if ψ is regular.*
2. *Let us suppose that ψ is regular. For each $d \geq 0$ the map $\xi \mapsto \psi \ltimes \xi$ induces a $K[|h|]$-linear isomorphism from regular sequences in $\mathbb{X}_\infty^d$ to regular sequences in $\mathbb{X}_V^d$.*

Proof of proposition 3.4. The proof of the proposition is in six steps.

Step 1. Let $\xi \in \mathbb{X}_\infty^0$. If ψ and ξ are regular, then $\psi \ltimes \xi$ is.

Proof. Let θ designate the sequence $\psi \ltimes \xi$:

$$\theta^k(n) \;=\; \sum_{a=0}^{k} \left(\prod_{b=k-a+1}^{k} \psi^b(n) \right) \xi^a(n-2k), \text{ for } k \geq 0 \text{ and } n \in \mathbb{Z}.$$

We suppose that ψ and ξ are regular. They are in particular quasi-regular, and so then is the sequence θ. Let $m \in \mathbb{Z}_{\geq 0}$. The sequence ξ being by hypothesis summable, there is $a(m) \geq 0$

such that $\xi^a(n) \in h^{m+1}K[n][|h|]$ for all $a > a(m)$. Therefore, the equality

$$\theta^k(n) \;=\; \sum_{a=0}^{a(m)} \left(\prod_{b=k-a+1}^{k} \psi^b(n) \right) \xi^a(n-2k)$$

holds in $K[n][|h|]/h^{m+1}K[n][|h|]$ for all $k \geq a(m)$. The sequence ψ being by assumption regular, it follows that the degree of the polynomial $\theta^k_m(n)$ is a function of k bounded above. □

STEP 2. Let $\theta \in \mathbb{X}^0_V$. If ψ and θ are quasi-regular, then the equation $\psi \ltimes \xi = \theta$ admits a quasi-regular solution ξ in $\mathbb{X}^0$.

Proof. We suppose that ψ and θ are quasi-regular. Let $k \geq 0$ and let us assume by induction that there exist $\xi^0(n), \xi^1(n), \ldots, \xi^{k-1}(n) \in K[n][|h|]$ verifying

$$\sum_{a=0}^{l} \left(\prod_{b=l-a+1}^{l} \psi^b(n) \right) \xi^a(n-2l) \;=\; \theta^l(n) \tag{3.3}$$

for all $0 \leq l \leq k-1$. It follows from colouring axioms (C1) and (C2) that $\psi^b(n)$ is equal to $(n-b+1)f^b(n)$ for some invertible element $f^b(n)$ in $K[n][|h|]$. Therefore, there exists $\xi^k(n)$ in $K[n][|h|]$ such that

$$\sum_{a=0}^{k-1} \left(\prod_{b=k-a+1}^{k} \psi^b(n) \right) \xi^a(n-2k) \;+\; \left(\prod_{b=1}^{k} \psi^b(n) \right) \xi^k(n-2k) \;=\; \theta^k(n)$$

if only if the equality

$$\sum_{a=0}^{k-1} \left(\prod_{b=k-a+1}^{k} \psi^b(n') \right) \xi^a(n'-2k) \;=\; \theta^k(n') \tag{3.4}$$

holds in $K[|h|]$ for all $n' \in \{0, 1, \ldots, k-1\}$. For such n', the left-hand side of (3.4) is equal to

$$\begin{aligned}
&\sum_{a=0}^{k-n'-1} \left(\prod_{b=k-a+1}^{k} \psi^b(n') \right) \xi^a(n'-2k) && \text{by (C2)}\\
&= \sum_{a=0}^{k-n'-1} \left(\prod_{b=k-a+1}^{k} \psi^{b-n'-1}(-n'-2) \right) \xi^a(n'-2k) && \text{by (C3)}\\
&= \sum_{a=0}^{k-n'-1} \left(\prod_{b=k-n'-a}^{k-n'-1} \psi^{b}(-n'-2) \right) \xi^a((-n'-2)-2(k-n'-1)).
\end{aligned}$$

The sequence θ being of Verma type, it follows from (3.2) that for all $n' \in \{0, 1, \ldots, k-1\}$, equality (3.4) holds if and only if the following one does:

$$\sum_{a=0}^{k-n'-1} \left(\prod_{b=k-n'-a}^{k-n'-1} \psi^{b}(-n'-2) \right) \xi^a((-n'-2)-2(k-n'-1)) \;=\; \theta^{k-n'-1}(-n'-2).$$

The latter is equality (3.3) for $l = k-n'-1$ and $n = -n'-2$. One then concludes using the induction hypothesis. □

STEP 3. Let $\theta \in \mathbb{X}^0_V$. If ψ and θ are regular, then the equation $\psi \ltimes \xi = \theta$ admits a regular solution ξ in $\mathbb{X}^0_\infty$.

Proof. We suppose that ψ and θ are regular. It follows from step 2 that there exists a quasi-regular sequence ξ in $\mathbb{X}^0$ such that $\psi \ltimes \xi = \theta$. It suffices to prove that ξ is summable (a quasi-regular summable sequence is in particular regular). Let $m \geq 0$ and let us assume by induction that there is $a(m) \geq 0$ such that the polynomial $\xi^a_{m'}(n)$ is zero for all $a \geq a(m)$ and for all $m' < m$. We designate by $\tilde{\theta}$ the quasi-regular sequence in $\mathbb{X}^{a(m)}$ defined for $k \geq a(m)$ by

$$\tilde{\theta}^k(n) \;=\; \theta^k(n) \;-\; \sum_{a=0}^{a(m)-1} \left(\prod_{b=k-a+1}^{k} \psi^b(n) \right) \xi^a(n-2k) \tag{3.5}$$

Using the induction hypothesis, it then follows from $\psi \ltimes \xi = \theta$ that

$$\sum_{a=a(m)}^{k} \left(\prod_{b=k-a+1}^{k} \psi^b_0(n) \right) \xi^a_m(n-2k) \;=\; \tilde{\theta}^k_m(n) \tag{3.6}$$

for all $k \geq a(m)$. As ψ and θ are by assumption both regular, it besides follows from (3.5) that there is $p \geq 0$ such that the degree of the polynomial $\tilde{\theta}^k_m(n)$ is at most p for all k. On the other hand, according to the first colouring axiom, the polynomial $\psi^b_0(n)$ is of degree 1 for all $b \geq 1$. Therefore, equality (3.6) implies, by induction on k, that $p(k) + k \leq p$ for all $k \geq a(m)$, where $p(k)$ designates the degree of the polynomial $\xi^k_m(n)$. It follows that $\xi^k_m(n)$ is zero for sufficiently large k. □

Step 4. Let $\theta \in \mathbb{X}^0_V$. The equation $\psi \ltimes \xi = \theta$ admits at most one quasi-regular solution ξ.

Proof. Let ξ be a solution in $\mathbb{X}^0$ of the equation $\psi \ltimes \xi = 0$:

$$\sum_{a=0}^{k} \left(\prod_{b=k-a+1}^{k} \psi^b(n) \right) \xi^a(n-2k) \;=\; 0, \text{ for all } k \geq 0 \text{ and all } n \in \mathbb{Z}. \tag{3.7}$$

We suppose that ξ is quasi-regular. Let $k \geq 0$ and let us assume by induction that $\xi^{k'}$ is zero for all $k' < k$. It then follows from (3.7) that $(\prod_{b=1}^{k} \psi^b(n))\xi^k(n-2k)$ is zero for all $n \in \mathbb{Z}$. On the other hand, according to the first colouring axiom, $\psi^b(n)$ is zero only if $n = b - 1$. Therefore, $\xi^k(n-2k)$ is zero for infinitely many values of n. This implies that ξ^k is zero, as $\xi^k(n) \in K[n][|h|]$ according to the quasi-regularity assumption. □

Step 5. If the equation $\psi \ltimes \xi = \psi[+1]$ admits a regular solution ξ in $\mathbb{X}^0_\infty$, then ψ is regular.

Proof. Let ξ be a solution in $\mathbb{X}^0$ of the equation $\psi \times \xi = \psi[+1]$:

$$\sum_{a=0}^{k} \left(\prod_{b=k-a+1}^{k} \psi^b(n) \right) \xi^a(n-2k) \;=\; \psi^{k+1}(n), \text{ for all } k \geq 0 \text{ and all } n \in \mathbb{Z}. \tag{3.8}$$

We suppose that ξ is regular and summable. It is in particular quasi-regular, and it follows from (3.8), by induction on k, that ψ is quasi-regular.

Let us recall that $N = (N^k)_{k \geq 1}$ designates the colouring with values in $K^{\mathbb{Z}}$, defined by $N^k(n) = k(n-k+1)$ for $n \in \mathbb{Z}$. There is an evident quasi-regular solution f in $\mathbb{X}^0$ of the equation $N \ltimes f = N[+1]$, which is given by $f^0(n) = n$, $f^1(n) = 1$ and $f^k(n) = 0$ for $k \geq 2$. This is the unique quasi-regular solution (step 4). On the other hand, it follows from the first colouring axiom that $(\xi^k_0)_{k \geq 0}$ is another solution, also quasi-regular, since ξ is by assumption. This proves that $\xi^0_0(n) = n$, $\xi^1_0(n) = 1$ and $\xi^k_0(n) = 0$ for all $k \geq 2$.

Let $m \geq 0$ and let us assume by induction that there is $p \geq 0$ such that the degree of the polynomial $\psi^k_{m'}(n)$ is at most p for all $k \geq 1$ and for all $m' < m$. As ξ is by assumption summable, there is $a(m) \geq 0$ such that $\xi^a(n) \in h^{m+1}K[n][|h|]$ for all $a > a(m)$. Let $p' \geq 0$ such that the degree of the polynomial $\xi^a_{m'}(n)$ is at most p' for all $a \leq a(m)$ and all $m' \leq m$. The following equation holds in $K[|h|]/h^{m+1}K[|h|]$ for all $k \geq a(m)$:

$$\sum_{a=0}^{a(m)} \left(\prod_{b=k-a+1}^{k} \psi^b(n) \right) \xi^a(n-2k) \;=\; \psi^{k+1}(n).$$

Using that $\xi_0^1(n) = 1$, that $\xi_0^k(n) = 0$ for $k \geq 2$, and using the induction hypothesis, it follows that for all $k \geq a(m)$, $\psi_m^{k+1}(n) = \xi_m^0(n-2k) + \psi_m^k(n) + g^k(n)$ for some $g^k(n) \in K[n]$ of degree at most $a(m)p + p'$. This proves, by induction on k, that the degree of the polynomial $\psi_m^k(n)$ is a function of k bounded above $\square$

STEP 6. For $f \in \mathbb{X}^d$ $(d \in \mathbb{Z}_{\geq 0})$ let $f\{-1\}$ designate the sequence in $\mathbb{X}^{d+1}$ defined by $(f\{-1\})^k(n) = \psi^k(n) f^{k-1}(n)$ for $k \geq d+1$ and $n \in \mathbb{Z}$.

I. The equality $\psi \ltimes (\xi[-1]) = (\psi \ltimes \xi)\{-1\}$ holds for all $\xi \in \mathbb{X}^d$.

II. Let us suppose that ψ is regular. The map $f \mapsto f\{-1\}$ induces a $K[|h|]$-linear isomorphism from regular sequences in $\mathbb{X}_V^d$ to regular sequences in $\mathbb{X}_V^{d+1}$.

Proof. Point I is straightforward calculations. Let us prove point II. We suppose that ψ is regular. It follows from colouring axioms (C2) and (C3) that if f is of Verma type, then $f\{-1\}$ is. Also, as the colouring ψ is by assumption regular, f regular implies $f\{-1\}$ regular. This proves that $f \mapsto f\{-1\}$ defines a $K[|h|]$-linear map, from regular sequences in $\mathbb{X}_V^d$ to regular sequences in $\mathbb{X}_V^{d+1}$. Let θ be a regular sequence in $\mathbb{X}_V^{d+1}$. Then $n-k$ divides $\theta^{k+1}(n)$ in $K[n][|h|]$ for all $k \geq d$. On the other hand, it follows from colouring axioms (C1) and (C2), and as ψ is by assumption regular, that for all $k \geq 0$, $\psi^{k+1}(n) = (k+1)(n-k)g^k(n)$ for some unique invertible element $g^k(n)$ in $K[n][|h|]$. Therefore, for all $k \geq d$, there is a unique element $(\theta\{+1\})^k(n)$ in $K[n][|h|]$ such that $\theta^{k+1}(n) = \psi^{k+1}(n)(\theta\{+1\})^k(n)$. In other words, there is a unique quasi-regular sequence $\theta\{+1\}$ in $\mathbb{X}^d$ such that $\theta = (\theta\{+1\})\{-1\}$. Let $\tilde{g}^k(n)$ $(k \geq 0)$ designates the inverse of $g^k(n)$ in $K[n][|h|]$. It follows from the definition of $\theta\{+1\}$ that

$$(k+1)(n-k)(\theta\{+1\})^k(n) = \tilde{g}^k(n)\theta^{k+1}(n), \text{ for all } k \geq d. \tag{3.9}$$

As the colouring ψ is regular, so then is the sequence $\tilde{g} = (\tilde{g}^k)_{k \geq 0}$. As θ is also regular, it follows from (3.9) that $\theta\{+1\}$ is. Let us fix $k \geq d$ and $n \geq 0$. As θ is of Verma type, it follows from (3.9) that $(k+1)(n-k)(\theta\{+1\})^k(n)$ is zero if $n+1 \leq k+1 \leq n+d+1$. In particular, $(\theta\{+1\})^k(n)$ is zero if $n+1 \leq k \leq n+d$. Colouring axiom (C3) implies that $\tilde{g}^{n+k+1}(n) = \tilde{g}^k(-n-2)$. Using that θ is of Verma type, it then follows from (3.9) that $(\theta\{+1\})^{n+k+1}(n) = (\theta\{+1\})^k(-n-2)$. We thus have proved that $\theta \mapsto \theta\{+1\}$ defines a map from regular sequences in $\mathbb{X}_V^{d+1}$ to regular sequences in $\mathbb{X}_V^d$, and that $\{+1\}$ is a right inverse of the map $\{-1\}$. It follows by definition that $\{+1\}$ is also a left inverse. $\square$

Conclusion. Step 3 for $\theta = \psi[+1]$, together with step 5, prove point 1 of the proposition. Let us suppose that ψ is regular. Step 1 then implies that $\xi \mapsto \psi \ltimes \xi$ defines a $K[|h|]$-linear map from regular sequences in $\mathbb{X}_\infty^0$ to regular sequences in $\mathbb{X}_V^0$. Steps 3 and 4 prove that the map is surjective and injective, respectively. This establishes point 2 of the proposition for $d = 0$. The general case follows, by induction on d, from step 6. $\square$

REMARK 3.5. Let ψ a colouring and let $\theta \in \mathbb{X}_V^d$ $(d \in \mathbb{Z}_{\geq 0})$. We do not assume here that ψ and θ are regular. We see from the proof of proposition 3.4 (steps 2 and 6) that the equation $\psi \ltimes \xi = \theta$ admits as many solutions ξ in $\mathbb{X}^d$ as they are choices for the values $\xi^{k+d}(-2k), \ldots, \xi^{k+d}(-k-1)$ $(k \geq 0)$. We see in particular that if not regular, a solution is not unique. It may be interesting to find an explicit condition on those values of ξ characterising regularity for the solution ξ, when ψ and θ are regular.

4. *Coloured Kac-Moody algebras of rank one*

We present here the main results of this paper. We prove that $U_h(\psi)$ is a formal deformation of $U(\mathfrak{sl}_2)$ if and only if the colouring ψ is regular (theorem 4.1). We give a Chevalley-Serre presentation of $U_h(\psi)$ for ψ regular (theorem 4.2). We show that the constant formal deformation $U(\mathfrak{sl}_2)[|h|]$ and the quantum algebra $U_h(\mathfrak{sl}_2)$ can both be realized as coloured Kac-Moody algebras (theorem 4.3). We prove that coloured Kac-Moody algebras are $\mathfrak{b}$-trivial deformations of $U(\mathfrak{sl}_2)$, and admit unique $\mathfrak{b}$-trivializations (theorem 4.4). We prove that regular colourings classify $\mathfrak{h}$-trivial formal deformations of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ (theorem 4.5). As a corollary, we obtain a rigidity result for $U(\mathfrak{sl}_2)$; namely, every $\mathfrak{h}$-trivial formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ is also $\mathfrak{b}$-trivial, and admits a unique $\mathfrak{b}$-trivialization (corollary 4.6).

4.1. *Formal deformations of $U(\mathfrak{sl}_2)$*

We recall that a colouring ψ is said regular if $\psi^k(n) \in K[k,n][|h|]$ (see remark 3.1). For ψ regular, we call the algebra $U_h(\psi)$ a *coloured Kac-Moody algebra.*

THEOREM 4.1. *Let ψ be a colouring. The following three assertions are equivalent.*
(i) *The colouring ψ is regular.*
(ii) *The algebra $U_h(\psi)$ is a formal deformation of the K-algebra $U(\mathfrak{sl}_2)$.*
(iii) *The algebra $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$.*

Proof of theorem 4.1.
Let f be the surjective $\mathfrak{a}$-algebra homomorphism from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)$ (lemma 2.14). We already proved that $U_h(\psi)$ is topologically free (lemma 2.13), it is therefore sufficient to prove that f is injective in order to prove that $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$. We propose to prove the following implications: (i) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i). The implication (iii) $\Rightarrow$ (ii) is immediate.

1. Assertion (i) implies assertion (iii).

Proof. Let us suppose that ψ is regular. Then, the equation $\psi \ltimes \xi = \psi[+1]$ admits a regular solution ξ in $\mathbb{X}^0_\infty$ (proposition 3.4). The series $\sum_{a\geq 0}(X^-)^a(X^+)^a\xi^a(H)$ converges to a unique element x in $U_h(\psi)$ (for the h-adic topology), such that $\psi(x) = \psi[+1]$ (proposition 3.3). As $\psi[+1] = \psi(X^+X^-)$, this means that for every $n \in \mathbb{Z}$, the elements X^+X^- and x act identically on $M_h(n,\psi)$, when viewed as a representation of $U_h(\psi)$. Hence, by definition of $U_h(\psi)$, the relation $X^+X^- = x$ holds in $U_h(\psi)$. It follows that $U_h(\psi)_{h=0}$ is spanned by the monomials $(X^-)^a(X^+)^bH^c$ $(a,b,c \geq 0)$. On the other hand, these monomials form the PBW basis of $U(\mathfrak{sl}_2)$. In other words, f sends a spanning subset to a basis. This implies that f is injective. □

2. Assertion (ii) implies assertion (iii).

Proof. We suppose that $U_h(\psi)$ is a formal deformation of the K-algebra $U(\mathfrak{sl}_2)$. Then there is a K-algebra isomorphism f' from $U(\mathfrak{sl}_2)$ to $U_h(\psi)_{h=0}$. Let g designate the map $f \circ f'$, it a surjective K-algebra endomorphism of $U(\mathfrak{sl}_2)$. We denote by $L(n)^g$ the pullback by g of the $(n+1)$-dimensional irreducible representation $L(n)$ of $\mathfrak{sl}_2$ $(n \geq 0)$. The pullback $L(n)^g$ is a representation of $U(\mathfrak{sl}_2)$ of dimension $n+1$, irreducible again, as g is surjective. The representation $L(n)^g$ is thus isomorphic to $L(n)$. Let x in $U(\mathfrak{sl}_2)$ such that $g(x) = 0$. The element x acts by zero on the pullback $L(n)^g$ for every $n \geq 0$. As a consequence, x acts by zero on $L(n)$ for every n, and is thus equal to zero (proposition 2.10). In other words g is injective. It implies that f is injective. □

3. Assertion (iii) implies assertion (i).

Proof. Let $U_h(\psi, 0)$ designate the $K[|h|]$-submodule of $U_h(\psi)$ formed by the elements x such that $[H, x] = 0$. We denote by P the subset of $U_h(\psi, 0)$ formed by the monomials $(X^-)^a(X^+)^a H^b$ $(a, b \geq 0)$. The $K[|h|]$-submodule $U_h(\psi, 0)$ is closed in $U_h(\psi)$ (for the h-adic topology). Hence, as $U_h(\psi)$ is topologically free (lemma 2.13), so then is $U_h(\psi, 0)$, and the inclusion map from P to $U_h(\psi, 0)$ induces a $K[|h|]$-linear map j from $(KP)[|h|]$ to $U_h(\psi, 0)$. We denote by f the inclusion map from $U_h(\psi, 0)$ to $U_h(\psi)$. The map $f_{h=0}$ is injective: $[H, hx] = 0$ implies $[H, x] = 0$ for any $x \in U_h(\psi)$. Let us suppose that $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$. Then, there exists an $\mathfrak{a}$-algebra isomorphism g_0 from $U_h(\psi)_{h=0}$ to $U(\mathfrak{sl}_2)$. The map $g_0 \circ f_{h=0}$ induces an injective map f_0 from $U_h(\psi, 0)_{h=0}$ to $U(0)$, where $U(0)$ designates the subspace of $U(\mathfrak{sl}_2)$ formed by the elements x such that $[H, x] = 0$. In view of the PBW basis of $U(\mathfrak{sl}_2)$, the monomials $(X^-)^a(X^+)^a H^b$ $(a, b \geq 0)$ form a basis of $U(0)$. This implies that $f_0 \circ j_{h=0}$ is surjective. As f_0 is injective, it follows that $j_{h=0}$ is surjective. Since $(KP)[|h|]$ and $U_h(\psi)$ are both topologically free, j is surjective as well. The element X^+X^- in $U_h(\psi, 0)$ therefore belongs to the image of j. In other words, there exists a regular sequence $\xi = (\xi^a)_{a \geq 0}$ in $\mathbb{X}^0_\infty$ such that the element X^+X^- is equal to $\sum_{a=0}^{\infty}(X^-)^a(X^+)^a\xi^a(H)$ in $U_h(\psi)$. This implies that ξ is a solution of $\psi \ltimes \xi = \psi(X^+X^-)$ (proposition 3.3). As $\psi(X^+X^-) = \psi[+1]$, this proves that the equation $\psi \ltimes \xi = \psi[+1]$ admits a regular solution in $\mathbb{X}^0_\infty$. Therefore, the colouring ψ is regular (proposition 3.4). $\square$

4.2. *Generators and relations*

We give here a Chevalley-Serre presentation for the coloured Kac-Moody algebra $U_h(\psi)$.

THEOREM 4.2. *Let ψ be a regular colouring. The $K[|h|]$-algebra $U_h(\psi)$ is topologically generated by H, X^-, X^+ and subject to the relations*

$$[H, X^\pm] = \pm 2X^\pm, \tag{4.1a}$$

$$X^+X^- = \sum_{a=0}^{\infty}(X^-)^a(X^+)^a\xi^a(H), \tag{4.1b}$$

where ξ designates the regular solution in $\mathbb{X}^0_\infty$ of the equation $\psi \ltimes \xi = \psi[+1]$ (proposition 3.4).

Proof. The element $\xi^a(H)$ tends to zero in $K[H][|h|]$ (for the h-adic topology), as a goes to infinity. This implies that the the right-hand side of (4.1b) converges to a unique element in the $K[|h|]$-algebra $K\langle H, X^-, X^+\rangle[|h|]$. Let then U be the quotient of $K\langle H, X^-, X^+\rangle[|h|]$ by the smallest closed (for the h-adic topology) two-sided ideal containing relations (4.1). The sequence ξ being a regular solution in $\mathbb{X}^0_\infty$ of the equation $\psi \ltimes \xi = \psi[+1]$, the series $\sum_{a \geq 0}(X^-)^a(X^+)^a\xi^a(H)$ converges to a unique element x in $U_h(\psi)$ (for the h-adic topology), such that $\psi(x) = \psi[+1]$ (proposition 3.3). As $\psi[+1] = \psi(X^+X^-)$, this means that for every $n \in \mathbb{Z}$, the elements X^+X^- and x act identically on $M_h(n, \psi)$, when viewed as a representation of $U_h(\psi)$. Hence, by definition of $U_h(\psi)$, relation (4.1b) holds in $U_h(\psi)$. It follows, as $U_h(\psi)$ is topologically free (lemma 2.13), that there is a canonical K-algebra homomorphism f from U to $U_h(\psi)$. On the other hand, relations (4.1) imply that $U_{h=0}$ is spanned by the monomials $(X^-)^a(X^+)^b H^c$ $(a, b, c \geq 0)$. These monomials form the PBW basis of $U(\mathfrak{sl}_2)$, and of $U_h(\psi)_{h=0}$ as well, since $U(\mathfrak{sl}_2)$ and $U_h(\psi)_{h=0}$ are isomorphic as $\mathfrak{a}$-algebras (theorem 4.1). In other words, the map $f_{h=0}$ sends a spanning subset to a basis. The map $f_{h=0}$ is therefore bijective. Since U is Hausdorff and complete (for the h-adic topology), and since $U_h(\psi)$ is topologically free (lemma 2.13), it follows that f is bijective. $\square$

4.3. *Classical and quantum realizations*

We prove here that the constant formal deformation $U(\mathfrak{sl}_2)[|h|]$ and the quantum algebra $U_h(\mathfrak{sl}_2)$ can both be realized as coloured Kac-Moody algebras.

We recall that the quantum algebra $U_h(\mathfrak{sl}_2)$ is the $U_h(\mathfrak{a})$-algebra topologically generated by H, X^-, X^+ and subject to the relation

$$[X^+, X^-] = \frac{q^H - q^{-H}}{q - q^{-1}} \quad \text{with } q = \exp(h) \text{ and } q^H = \exp(hH), \tag{4.2}$$

i.e. $U_h(\mathfrak{sl}_2)$ is the quotient of $U_h(\mathfrak{a})$ by the smallest closed (for the h-adic topology) two-sided ideal containing relation (4.2).

THEOREM 4.3. *As $U_h(\mathfrak{a})$-algebras, $U(\mathfrak{sl}_2)[|h|]$ and $U_h(\mathfrak{sl}_2)$ are isomorphic to the coloured Kac-Moody algebras $U_h(N)$ and $U_h(N_q)$, respectively.*

Proof. Let us recall that N and N_q are the colourings defined by $N^k(n) = k(n-k+1)$ and $N_q^k(n) = [k]_q[n-k+1]_q$ (for $k \geq 1$ and for $n \in \mathbb{Z}$).

The relation $[X^+, X^-] = H$ holds in the representation $M_h(n, N)$ for all $n \in \mathbb{Z}$. It follows, in view of the Chevalley-Serre presentation of $U(\mathfrak{sl}_2)$, and as $M_h(n, N)$ is Hausdorff (for the h-adic topology), that the action of $U_h(\mathfrak{a})$ on $M_h(n, N)$ factorises through $U(\mathfrak{sl}_2)[|h|]$ for all $n \in \mathbb{Z}$. This implies, by the universal property of $U_h(N)$ (proposition 2.12), that there exists a surjective $U_h(\mathfrak{a})$-algebra homomorphism f from $U(\mathfrak{sl}_2)[|h|]$ to $U_h(N)$. On the other hand, the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)[|h|]_{h=0}$ is isomorphic to $U(\mathfrak{sl}_2)$. Hence, there is an $\mathfrak{a}$-algebra homomorphism g_0 from $U_h(N)_{h=0}$ to $U(\mathfrak{sl}_2)[|h|]_{h=0}$ (lemma 2.14). Let us then consider the map $g_0 \circ f_{h=0}$. It is an $\mathfrak{a}$-algebra endomorphism of $U(\mathfrak{sl}_2)[|h|]_{h=0}$. Therefore, $g_0 \circ f_{h=0}$ is equal to the identity map. This implies that $f_{h=0}$ is injective. Since $U_h(N)$ is torsion-free (lemma 2.13), and since $U(\mathfrak{sl}_2)[|h|]$ is Hausdorff (for the h-adic topology), it follows that f is injective, and thus bijective.

The proof for $U_h(\mathfrak{sl}_2)$ is similar. Namely, relation (4.2) holds in the representation $M_h(n, N_q)$ for all $n \in \mathbb{Z}$. It follows, as $M_h(n, N_q)$ is Hausdorff (for the h-adic topology), that the action of $U_h(\mathfrak{a})$ on $M_h(n, N_q)$ factorises through $U_h(\mathfrak{sl}_2)$ for all $n \in \mathbb{Z}$. This implies, by the universal property of $U_h(N_q)$ (proposition 2.12), that there exists a surjective $U_h(\mathfrak{a})$-algebra homomorphism f from $U_h(\mathfrak{sl}_2)$ to $U_h(N_q)$. On the other hand, the $\mathfrak{a}$-algebra $U_h(\mathfrak{sl}_2)_{h=0}$ is isomorphic to $U(\mathfrak{sl}_2)$ (the functor $(\bullet)_{h=0}$ from the category of $K[|h|]$-modules to the category of K-vector spaces is a right-exact functor, and relation (4.2) is $[X^+, X^-] = H$ modulo h). Hence, there is an $\mathfrak{a}$-algebra homomorphism g_0 from $U_h(N_q)_{h=0}$ to $U_h(\mathfrak{sl}_2)_{h=0}$ (lemma 2.14). Let us then consider the map $g_0 \circ f_{h=0}$. It is an $\mathfrak{a}$-algebra endomorphism of $U_h(\mathfrak{sl}_2)_{h=0}$. Therefore, $g_0 \circ f_{h=0}$ is equal to the identity map. This implies that $f_{h=0}$ is injective. Since $U_h(N_q)$ is torsion-free (lemma 2.13), and since $U_h(\mathfrak{sl}_2)$ is Hausdorff (for the h-adic topology), it follows that f is injective, and thus bijective. □

4.4. $\mathfrak{b}$-*triviality*

Let ψ be a regular colouring. We know that the coloured Kac-Moody algebra $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ (theorem 4.1). We say that the formal deformation $U_h(\psi)$ is $\mathfrak{b}$-*trivial* if there is a $K[|h|]$-algebra isomorphism g from $U(\mathfrak{sl}_2)[|h|]$ to $U_h(\psi)$, such that $g(H) = H$, $g(X^-) = X^-$, and such that $g_{h=0}$ is an $\mathfrak{a}$-algebra isomorphism. The isomorphism g is called a $\mathfrak{b}$-*trivialization* of $U_h(\psi)$.

THEOREM 4.4. *Let ψ be a regular colouring. The coloured Kac-Moody algebra $U_h(\psi)$ is $\mathfrak{b}$-trivial, and it admits a unique $\mathfrak{b}$-trivialization.*

Proof of theorem 4.4. The proof of the theorem is in two steps.

STEP 1. There exists a surjective $K[|h|]$-algebra homomorphism g from $U(\mathfrak{sl}_2)[|h|]$ to $U_h(\psi)$ such that $g(H) = H$ and $g(X^-) = X^-$.

Proof. Let us consider the natural colouring N, defined by $N^k(n) = k(n-k+1)$ for $k \geq 1$ and for $n \in \mathbb{Z}$. As N and ψ are regular, the equation $N \ltimes \xi = \psi$ admits a regular solution ξ in $\mathbb{X}^1_\infty$ (proposition 3.4). The series $\sum_{a\geq 1}(X^-)^{a-1}(X^+)^a\xi^a(H)$ converges to a unique element x in $U_h(N)$ (for the h-adic topology), such that $N(x) = \psi$ (proposition 3.3). We denote by f be the $K[|h|]$-algebra homomorphism from $U_h(\mathfrak{a})$ to $U_h(N)$ defined by $f(H) = H$, $f(X^-) = X^-$, $f(X^+) = x$. In view of the first colouring axiom, $\xi_0 = (\xi_0^k)_{k\geq 1}$ is a solution of the equation $N \ltimes \xi_0 = N$. Therefore, $\xi_0^1 = 1$, and $\xi_0^k = 0$ for all $k \geq 2$, since the equation admits a unique regular solution (proposition 3.4). This proves that the images of x and X^+ in $U_h(N)_{h=0}$ are equal, and thus that $f_{h=0}$ is surjective. Since $U_h(\mathfrak{a})$ is by definition topologically free, and since $U_h(N)$ is as well (lemma 2.13), f is also surjective. It follows from $N(x) = \psi$ that for all $n \in \mathbb{Z}$, the pullback by f of the representation $M_h(n, N)$, when viewed as a representation of $U_h(N)$, is equal to the representation $M_h(n, \psi)$. In other words, the action of $U_h(\mathfrak{a})$ on $M_h(n, \psi)$ factorises through f for all $n \in \mathbb{Z}$. It then follows from the universal property of $U_h(\psi)$ (proposition 2.12) that there exists a surjective $U_h(\mathfrak{a})$-algebra homomorphism g from $U_h(N)$ to $U_h(\psi)$, where $U_h(N)$ is endowed with the $U_h(\mathfrak{a})$-algebra structure defined by f. In particular, g satisfies $g(H) = H$ and $g(X^-) = X^-$. As $U(\mathfrak{sl}_2)[|h|]$ and $U_h(N)$ are isomorphic as $U_h(\mathfrak{a})$-algebras (theorem 4.3), this concludes the proof of step 1. □

STEP 2. The identity map is the unique $K[|h|]$-algebra endomorphism of $U(\mathfrak{sl}_2)[|h|]$ which fixes both H and X^-.

Proof. Let g be a $K[|h|]$-algebra endomorphism of $U(\mathfrak{sl}_2)[|h|]$ which fixes both H and X^-. Let then x^+ be the image of X^+ by g. The relations $[H, X^+] = 2X^+$ and $[X^+, X^-] = H$ hold in $U(\mathfrak{sl}_2)[|h|]$, hence so do the relations $[H, x^+] = 2x^+$ and $[x^+, X^-] = H$. Let $n \in \mathbb{Z}$, and let us consider the representation $M(n)[|h|]$ of $U(\mathfrak{sl}_2)[|h|]$. The relation $[H, x^+] = 2x^+$ implies $H.(x^+.b_0) = (n+2)(x^+.b_0)$, and thus $x^+.b_0 = 0$. It then follows, by induction k, and using the relation $[x^+, X^-] = H$, that $x^+.b_k = k(n-k+1)b_{k-1}$ for all $k \geq 1$. Therefore, $x^+ - X^+$ acts by zero on $M(n)[|h|]$ for all $n \in \mathbb{Z}$. It follows from proposition 2.10 that $x^+ - X^+$ is zero. □

Conclusion. The unicity of a $\mathfrak{b}$-trivialization for $U_h(\psi)$ follows from steps 1 and 2. It remains to prove that $U_h(\psi)$ is $\mathfrak{b}$-trivial. Let g be a $K[|h|]$-algebra homomorphism from $U(\mathfrak{sl}_2)[|h|]$ to $U_h(\psi)$, such that $g(H) = H$ and $g(X^-) = X^-$ (step 1). The map $g_{h=0}$ satisfies in particular $g_{h=0}(H) = H$ and $g_{h=0}(X^-) = X^-$. We denote by $\tilde{g}$ the $K[|h|]$-algebra homomorphism induced by $g_{h=0}$ from $(U(\mathfrak{sl}_2)[|h|]_{h=0})[|h|]$ to $(U_h(\psi)_{h=0})[|h|]$. The $U_h(\mathfrak{a})$-algebra $(U(\mathfrak{sl}_2)[|h|]_{h=0})[|h|]$ is canonically isomorphic to $U(\mathfrak{sl}_2)[|h|]$. The $U_h(\mathfrak{a})$-algebra $(U_h(\psi)_{h=0})[|h|]$ is also isomorphic to $U(\mathfrak{sl}_2)[|h|]$, since $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ (theorem 4.1). Step 2 therefore implies that $\tilde{g}$ is a $U_h(\mathfrak{a})$-algebra isomorphism. This proves that $g_{h=0}$ is an $\mathfrak{a}$-algebra isomorphism. In particular, $g_{h=0}$ is bijective. Since $U_h(\psi)$ is topologically free (lemma 2.13), it follows that g is bijective as well. □

4.5. *Classification*

Let A be a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$. We designate again by H, X^-, X^+ the images of H, X^-, X^+, by the structural homomorphism from $U_h(\mathfrak{a})$ to A. We say that the formal deformation A is $\mathfrak{h}$-*trivial* if there is a $K[|h|]$-algebra isomorphism g from $U(\mathfrak{sl}_2)[|h|]$ to

A, such that $g(H) = H$, and such that $g_{h=0}$ is an $\mathfrak{a}$-algebra isomorphism. The isomorphism g is called a $\mathfrak{h}$-*trivialization* of A.

It follows from theorem 4.4 that coloured Kac-Moody algebras are in particular $\mathfrak{h}$-trivial formal deformations of $U(\mathfrak{sl}_2)$. We establish in the following theorem that up to $\mathfrak{a}$-algebra isomorphism, there is no other $\mathfrak{h}$-trivial formal deformations of $U(\mathfrak{sl}_2)$, and that regular colourings classify such deformations.

THEOREM 4.5. *For every $\mathfrak{h}$-trivial formal deformation A of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$, there is a unique regular colouring ψ such that A and $U_h(\psi)$ are isomorphic as $U_h(\mathfrak{a})$-algebras.*

We thus obtain, in view of theorem 4.4, the following rigidity result for $U(\mathfrak{sl}_2)$.

COROLLARY 4.6. *A $\mathfrak{h}$-trivial formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ is also $\mathfrak{b}$-trivial, and admits a unique $\mathfrak{b}$-trivialization.*

Proof of theorem 4.5. Let us adapt the definition of the representation $M_h(n, \psi)$. Namely, for $n \in \mathbb{Z}$ and for $\varphi = (\varphi^k)_{k\geq 1}$ any sequence with values in $K[|h|]$, we denote by $M_h(n, \varphi)$ the representation of $U_h(\mathfrak{a})$, whose underlying $K[|h|]$-module is $(\bigoplus_{k\geq 0} Kb_k)[|h|]$, and where the action of $U_h(\mathfrak{a})$ is given by

$$\begin{aligned} H.b_k &= (n-2k)b_k, \\ X^-.b_k &= b_{k+1}, \\ X^+.b_k &= \begin{cases} 0 & \text{if } k=0, \\ \varphi^k b_{k-1} & \text{if } k \geq 1. \end{cases} \end{aligned}$$

The proof of the theorem is in three steps.

STEP 1. Let A be a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$, and let $n \in \mathbb{Z}$. There is at most one sequence φ with values in $K[|h|]$ such that the action of $U_h(\mathfrak{a})$ on $M_h(n, \varphi)$ factorises through A.

Proof. Let $V(n)$ be the representation of A topologically generated by v and subject to the relations $H.v = nv$, $X^+.v = 0$, i.e. the representation $V(n)$ is the quotient of the left regular representation A by the smallest closed (for the h-adic topology) subrepresentation containing $H - n$ and X^+. Let φ be a sequence with values in $K[|h|]$ such that the action of $U_h(\mathfrak{a})$ on $M_h(n, \varphi)$ factorises through A. We regard from now $M_h(n, \varphi)$ as a representation of A. By definition of $V(n)$, and since $M_h(n, \varphi)$ is Hausdorff (for the h-adic topology), there exists an A-morphism f from $V(n)$ to $M_h(n, \varphi)$ such that $f(v) = b_0$. Let v_0 be the image of v in $V(n)_{h=0}$. The representation $V(n)_{h=0}$ of $A_{h=0}$ is generated by v_0, and verifies $H.v_0 = nv_0$, $X^+.v_0 = 0$. On the other hand, the algebra A being by assumption a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$, the monomials $(X^-)^a(X^+)^bH^c$ $(a, b, c \geq 0)$ span $A_{h=0}$. It then follows that the vectors $(X^-)^a.v_0$ $(a \geq 0)$ span $V_{h=0}$. As a consequence, $f_{h=0}$ sends a spanning subset to a basis. The map $f_{h=0}$ is therefore bijective. As $V(n)$ is Hausdorff and complete (for the h-adic topology), and since $M_h(n, \varphi)$ is topologically free, it follows that f is bijective. Let us now suppose that there is another sequence φ' with values in $K[|h|]$ such that the action of $U_h(\mathfrak{a})$ on $M_h(n, \varphi')$ factorises through A. Then, as for φ, there is an A-isomorphism f' from $V(n)$ to $M_h(n, \varphi')$ such that $f'(v) = b_0$. It follows that there is an A-isomorphism g from $M_h(n, \varphi)$ to $M_h(n, \varphi')$ such that $g(b_0) = b_0$. As g commutes with the action of X^-, $g(b_k) = b_k$ for all $k \geq 0$. Since g also commutes with the action of X^+, it follows that φ are φ' are equal. $\square$

STEP 2. Let A be a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$. If ψ is a colouring such that the action of $U_h(\mathfrak{a})$ on $M_h(n,\psi)$ factorises through A for all $n \in \mathbb{Z}$, then A and $U_h(\psi)$ are isomorphic as $U_h(\mathfrak{a})$-algebras.

Proof. Let ψ be a colouring such that the action of $U_h(\mathfrak{a})$ on $M_h(n,\psi)$ factorises through A for all $n \in \mathbb{Z}$. Then, by the universal property of $U_h(\psi)$ (proposition 2.12), there exists a surjective $U_h(\mathfrak{a})$-algebra homomorphism f from A to $U_h(\psi)$ (we recall that as A is by assumption a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$, the structural homomorphism from $U_h(\mathfrak{a})$ to A is surjective). On the other hand, $A_{h=0}$ and $U(\mathfrak{sl}_2)$ being isomorphic as $\mathfrak{a}$-algebras, there is an $\mathfrak{a}$-algebra homomorphism g_0 from $U_h(\psi)_{h=0}$ to $A_{h=0}$ (lemma 2.14). Let us then consider the map $g_0 \circ f_{h=0}$. It is an $\mathfrak{a}$-algebra endomorphism of $A_{h=0}$. Therefore, $g_0 \circ f_{h=0}$ is equal to the identity map. This implies that $f_{h=0}$ is injective. Since A is by assumption topologically free, and since $U_h(\psi)$ is as well (lemma 2.13), it follows that f is injective, and thus bijective. □

STEP 3. For every $\mathfrak{h}$-trivial formal deformation A of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$, there exists a regular colouring ψ such that A and $U_h(\psi)$ are isomorphic as $U_h(\mathfrak{a})$-algebras.

Proof. Let $f : U(\mathfrak{sl}_2)[|h|] \to A$ be a $\mathfrak{h}$-trivialization of A. We denote by $V(n)$ $(n \in \mathbb{Z})$ the pullback of the representation $M(n)[|h|]$ by f^{-1}. We denote by $U(\pm 1)$ the subspace of $U(\mathfrak{sl}_2)$ formed by the elements x such that $[H,x] = \pm 2x$. As $f(H) = H$, and in view of the relation $[H, X^{\pm}] = \pm 2X^{\pm}$ in A, $f^{-1}(X^{\pm})$ belongs to $U(\pm 1)[|h|]$. This implies that in $V(n)$, $X^+.b_0 = 0$, $X^-.b_k = \alpha^k(n)b_{k+1}$ and $X^+.b_{k+1} = \beta^k(n)b_k$ for some $\alpha^k(n), \beta^k(n) \in K[|h|]$ $(k \geq 0)$. Let us denote by b'_k $(k \geq 0)$ the vector $(X^-)^k.b_0$ in $V(n)$. As $f_{h=0}$ is by assumption an $\mathfrak{a}$-algebra homomorphism, and in view of definition (2.1) of $M(n)$, $\alpha^k(n)$ and $\beta^k(n)$ are equal to 1 and to $(k+1)(n-k)$, respectively, modulo h. This proves first that b'_k is a non-zero scalar multiple of b_k for all $k \geq 0$, and then that $V(n) = (\bigoplus_{k \geq 0} K b'_k)[|h|]$ as a $K[|h|]$-module, with

$$\begin{aligned} H.b_k &= (n-2k)b_k \quad (\text{since } f(H) = H), \\ X^-.b'_k &= b'_{k+1}, \\ X^+.b'_k &= \begin{cases} 0 & \text{if } k = 0, \\ \psi^k(n) b'_{k-1} & \text{if } k \geq 1, \end{cases} \end{aligned}$$

for $\psi^k(n) \in K[|h|]$ such that $\psi^k(n) = k(n-k+1)$ modulo h. If $n \geq 0$, then $\bigoplus_{k \geq n+1} K b_k$ is a subrepresentation of $M(n)$, therefore $(\bigoplus_{k \geq n+1} K b'_k)[|h|]$ is a subrepresentation of $V(n)$. This implies that $\psi^{n+1}(n)$ has to be zero for every $n \geq 0$. It then follows from step 1 that $\psi^{n+k+1}(n) = \psi^k(-n-2)$ for all $k \geq 1$ and all $n \geq 0$. We thus have proved that the values $\psi^k(n)$ $(k \geq 1,\ n \in \mathbb{Z})$ define a colouring ψ, such that for all $n \in \mathbb{Z}$ the action of $U_h(\mathfrak{a})$ on $M_h(n,\psi)$ factorises through A. It follows from step 2 that A and $U_h(\psi)$ are isomorphic as $U_h(\mathfrak{a})$-algebras. In particular, $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$. Hence, the colouring ψ is regular (theorem 4.1). □

Conclusion. Step 3 proves that every $\mathfrak{h}$-trivial formal deformation of $U(\mathfrak{sl}_2)$ is isomorphic as a $U_h(\mathfrak{a})$-algebra to $U_h(\psi)$, for some regular colouring ψ. It remains to prove that for two regular colourings ψ and ψ', if $U_h(\psi)$ and $U_h(\psi')$ are isomorphic as $U_h(\mathfrak{a})$-algebras, then $\psi = \psi'$. Let $n \in \mathbb{Z}$. By definition, the actions of $U_h(\mathfrak{a})$ on the representations $M_h(n,\psi)$ and $M_h(n,\psi')$ factorise through $U_h(\psi)$ and $U_h(\psi')$, respectively. Let us suppose that $U_h(\psi)$ and $U_h(\psi')$ are isomorphic as $U_h(\mathfrak{a})$-algebras. Then, the action of $U_h(\mathfrak{a})$ on $M_h(n,\psi')$ also factorises through $U_h(\psi)$. As $U_h(\psi)$ is a formal deformation of the $\mathfrak{a}$-algebra $U(\mathfrak{sl}_2)$ (theorem 4.1), it follows from step 1 that $\psi^k(n) = (\psi')^k(n)$ for all $k \geq 1$. □

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