

Intro to Probability

Instructor: Alexandre Bouchard
Fall 2014

Plan for today:

- Definition of probability spaces.
- Examples of problems (discrete case).

Logistics

- Let me know if you cannot access website
- ‘Homeworks’ for next Monday:
 - **Doodle** under Contact tab.
 - Clickers (see links in Syllabus tab)
 - Piazza under Contact tab.
 - Pre-readings posted in Schedule tab
- Slides under Files tab
- Additional practice problems: Syllabus tab

Outline of the course

- **Discrete probability models**
- Conditioning and Bayes
- Expectation
- Continuous probability models
- Asymptotics

Discrete:

only a finite (not infinite)
number of scenarios

Probability:

extension of the notion of
proportion

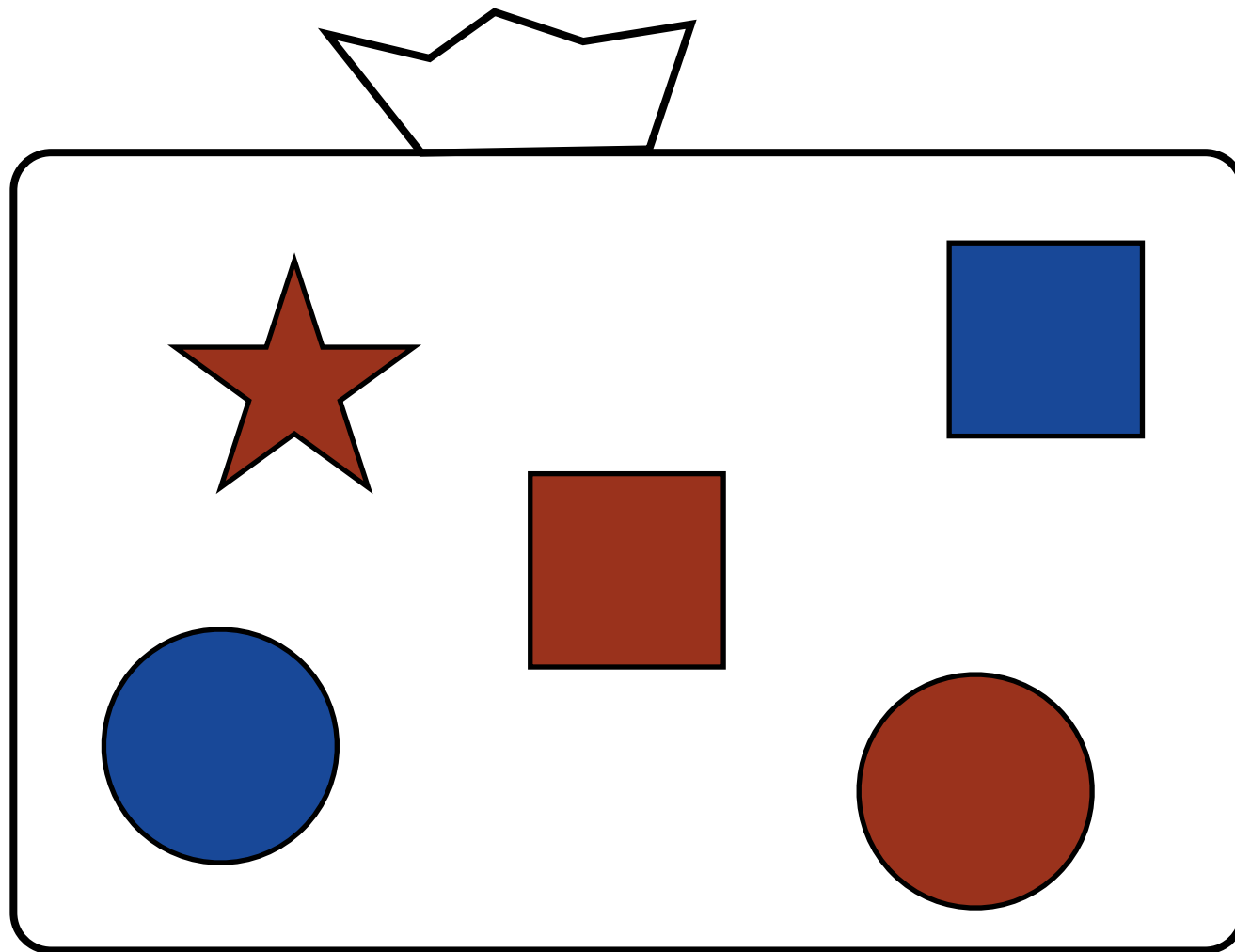
Model:


a simplification of reality
amenable to mathematical
investigation

Note: we will make this more precise today

Ex. 9

Example: bag of distinct objects of same size

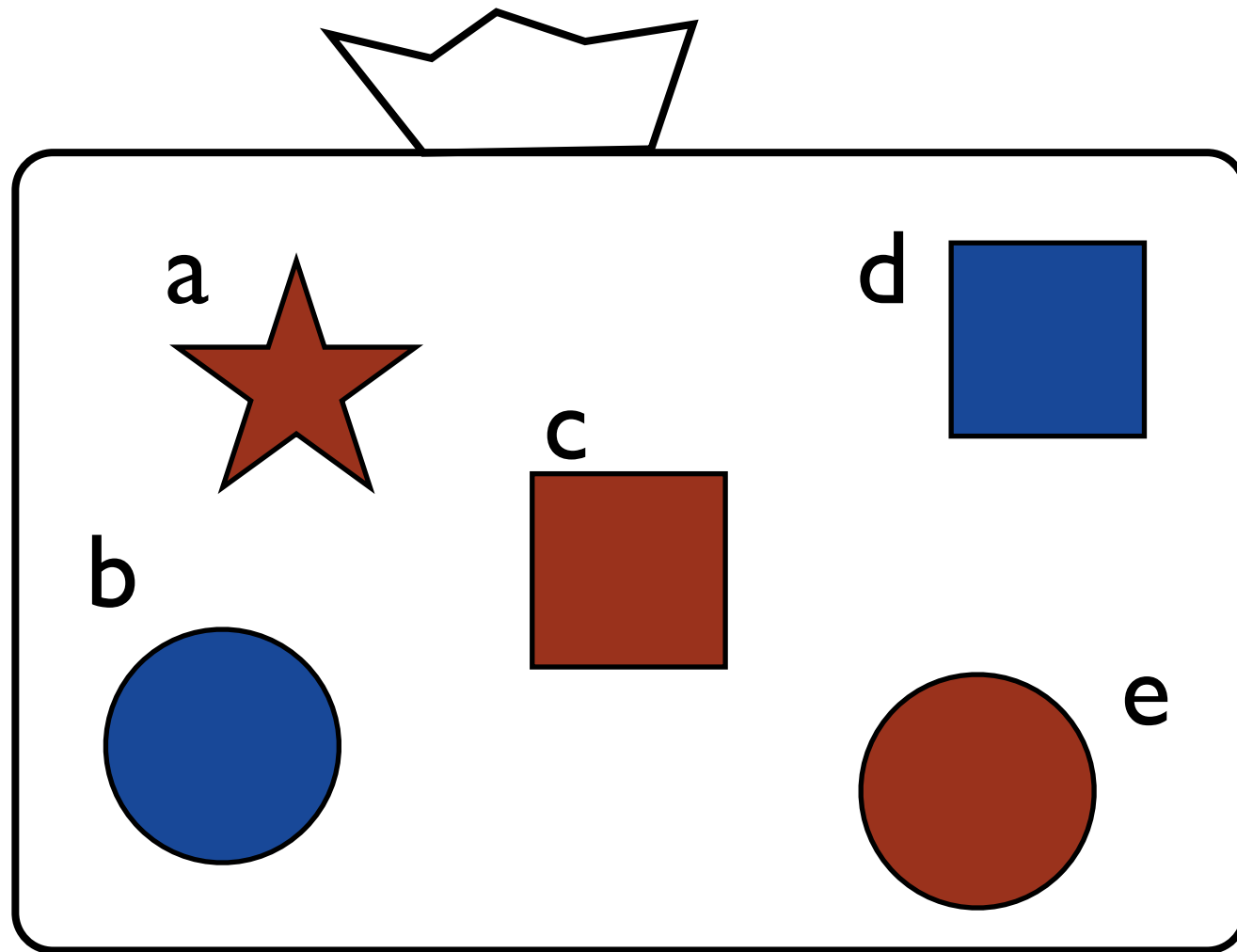


- Proportion of red shapes?
- Probability of drawing a red shape?
- *Outcome*: an individual object in the bag
ex.: $s =$ 

**Probability when
outcomes are
equally likely:**

$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

Def I Notation



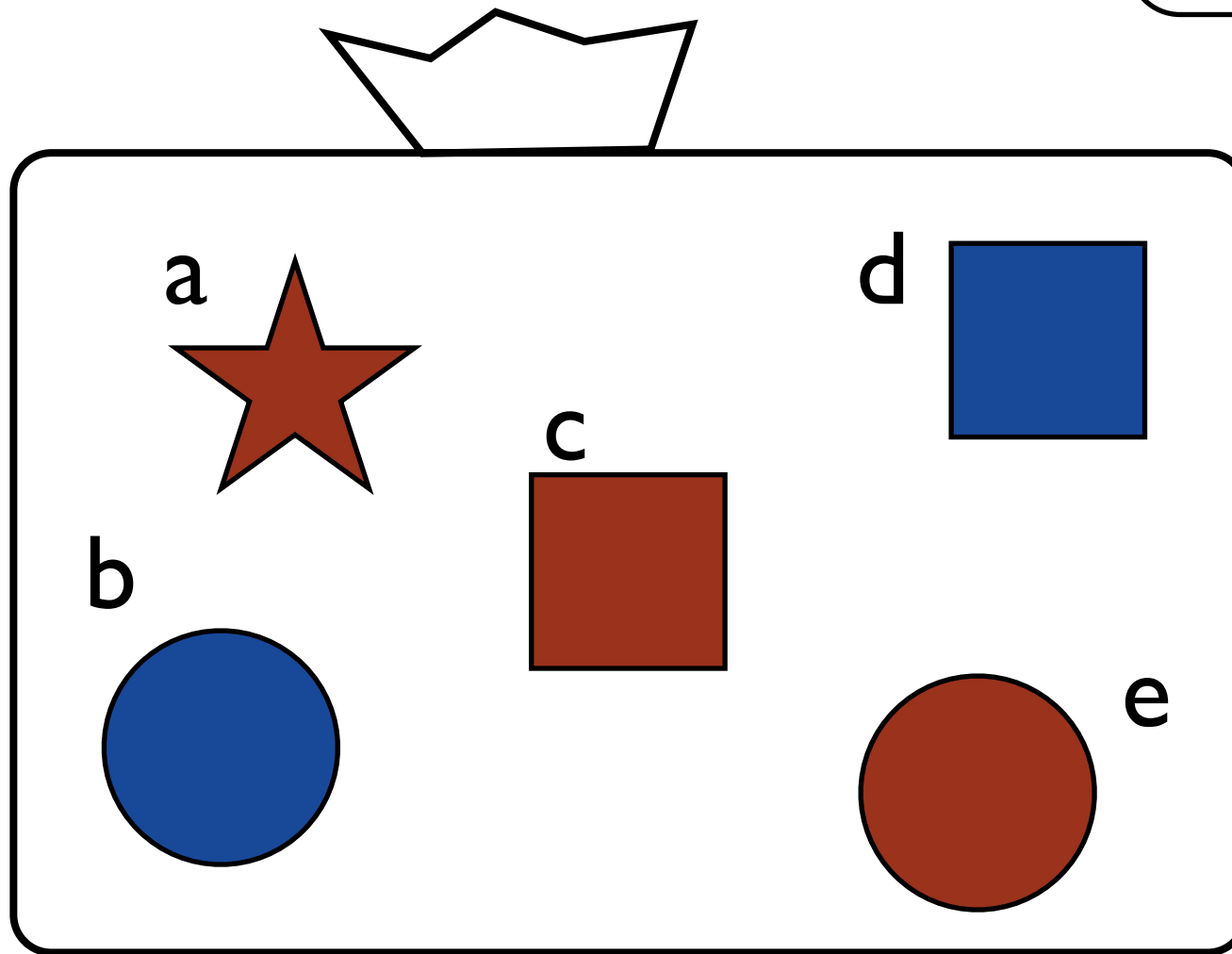
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Typical notation: red, E , blue, F , ..
 $E = \{a, c, e\}$



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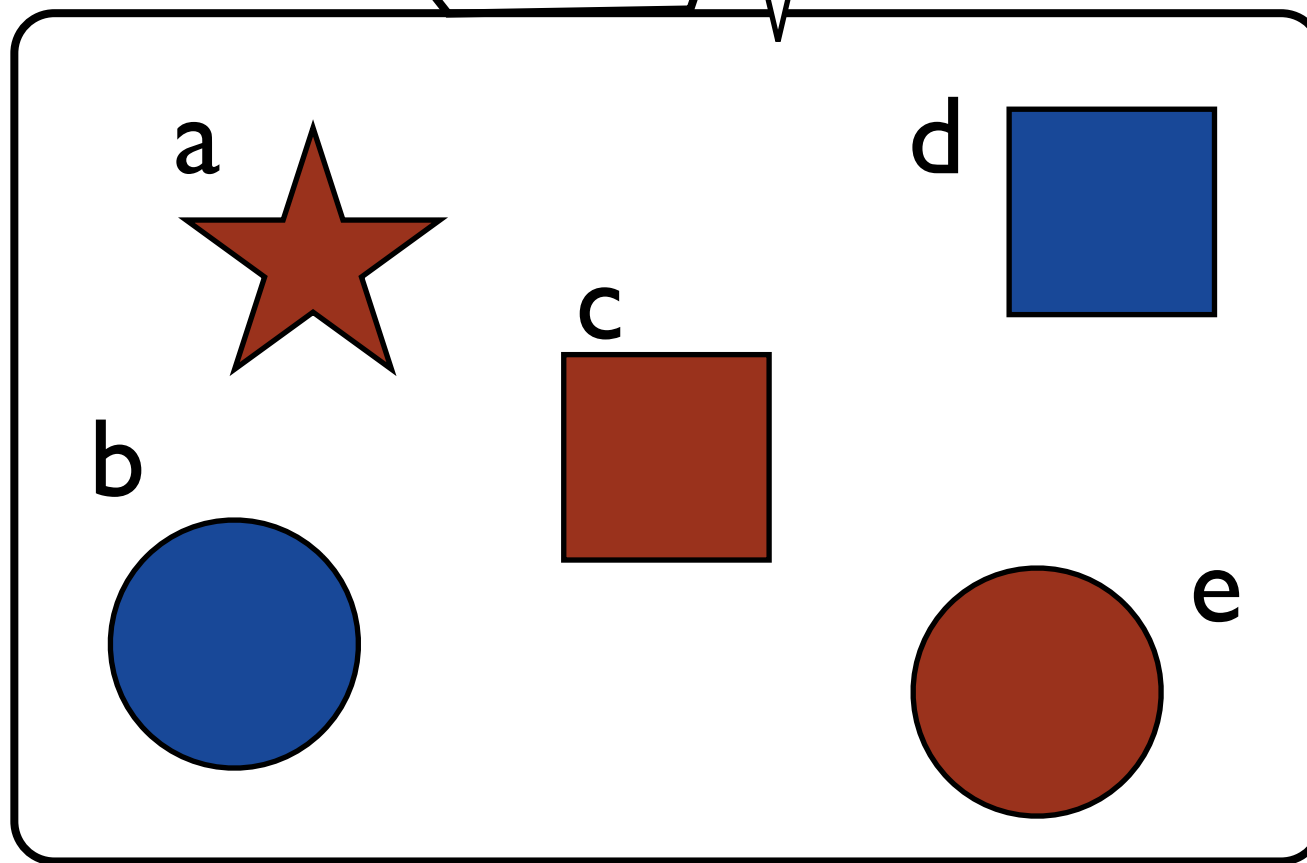
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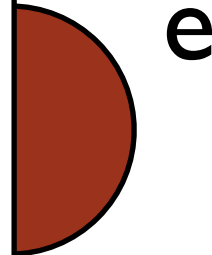
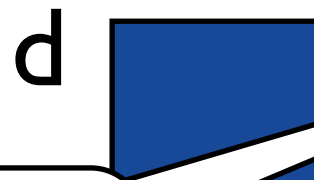
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Notation: $P, Q, ..$

P is a function:

- input: an event
- output: a number in $[0, 1]$

$$P : 2^S \rightarrow [0, 1]$$

Example: $P(E) = 3/5$

- Proportion of red shapes?
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- *Outcome*: an individual object in the bag

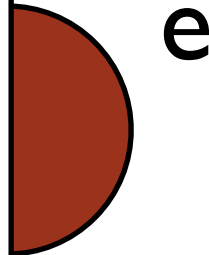
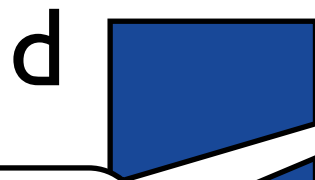
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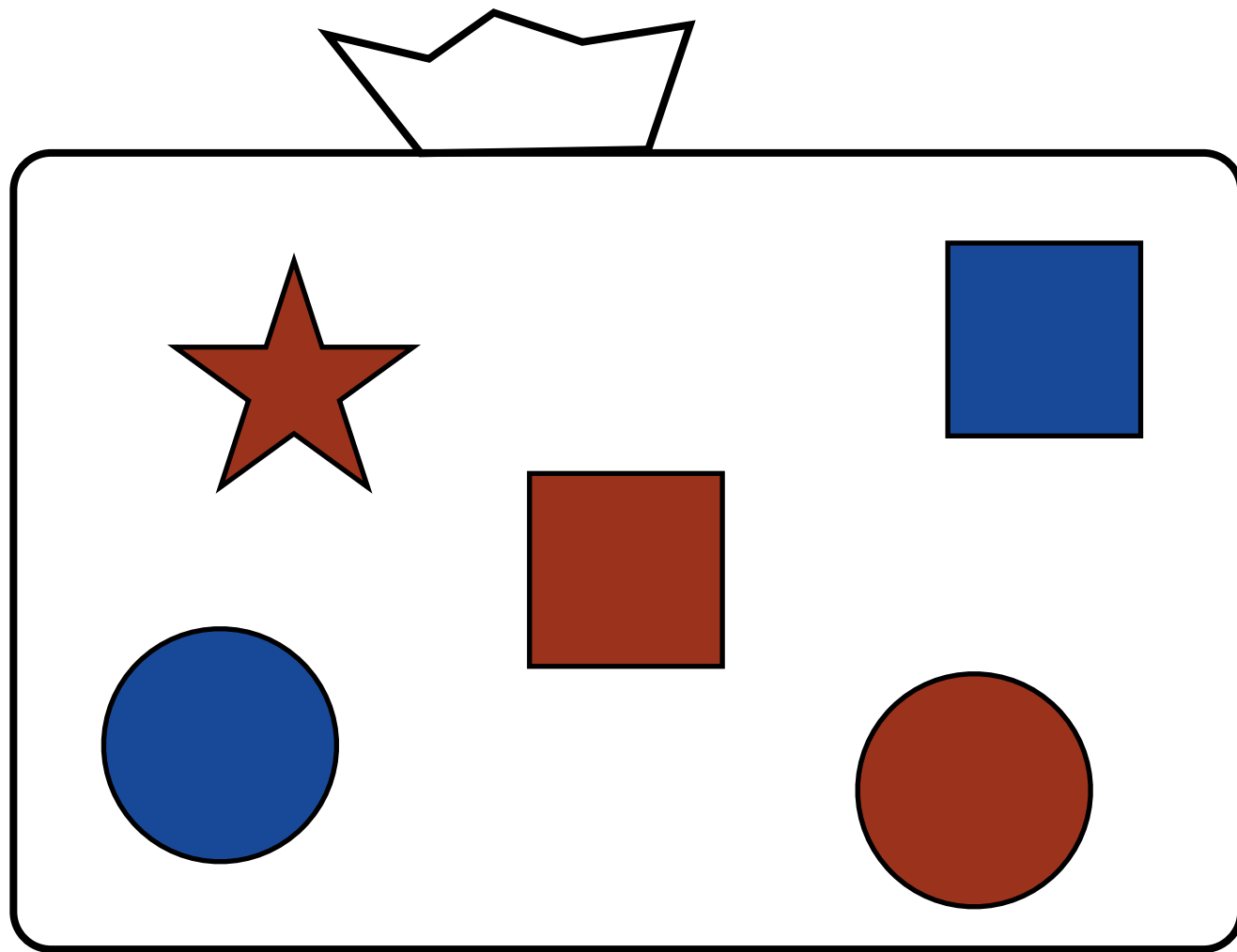
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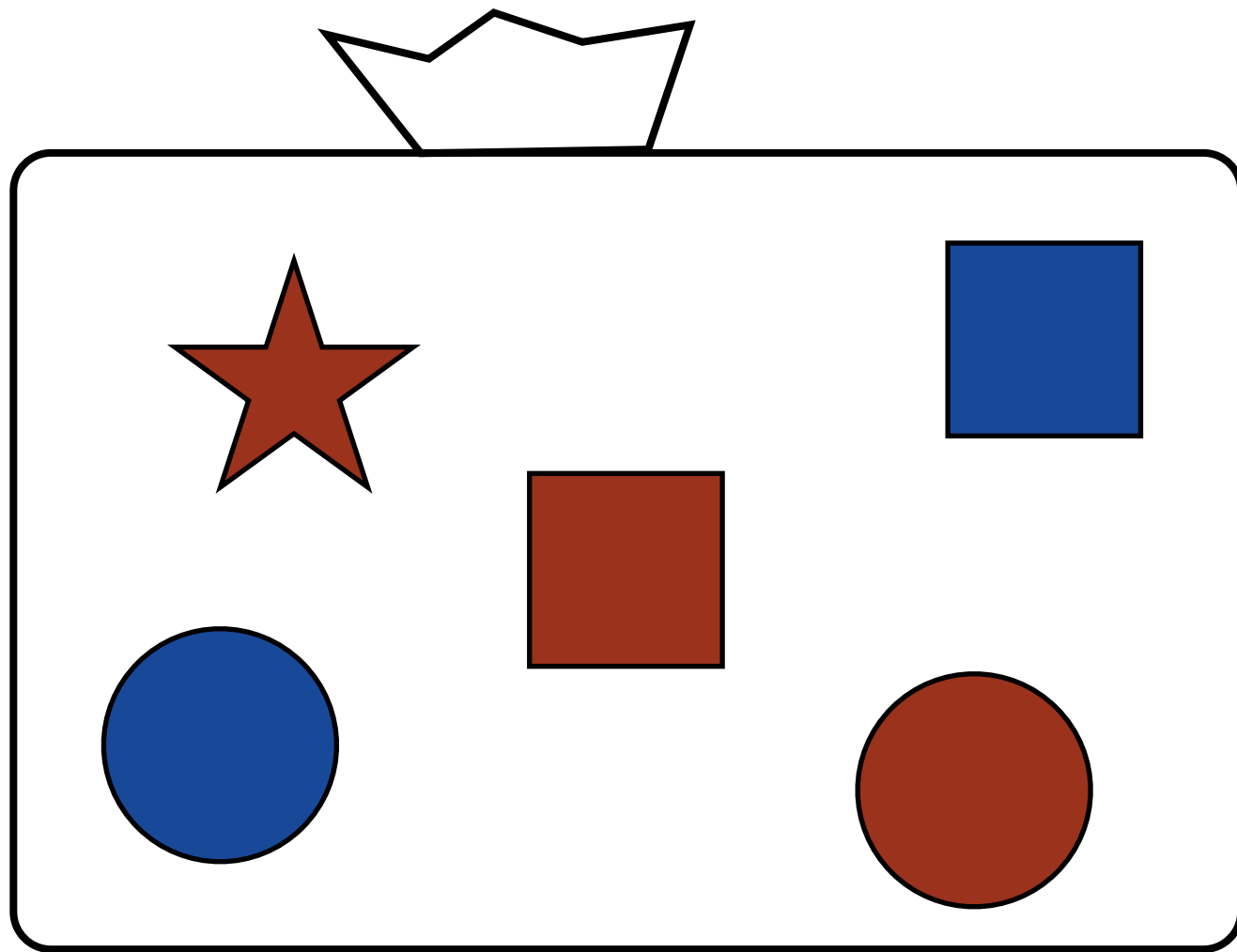
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Basic properties



- We know $P(E)$
- Example: $P(\text{square}) = 2/5$
- What is $P(E^c)$?
- Example: $P(\text{not square})$
- E^c means:
 - the *complement* of E
 - the outcomes not in E
 - $= S \setminus E$ (minus, for sets)

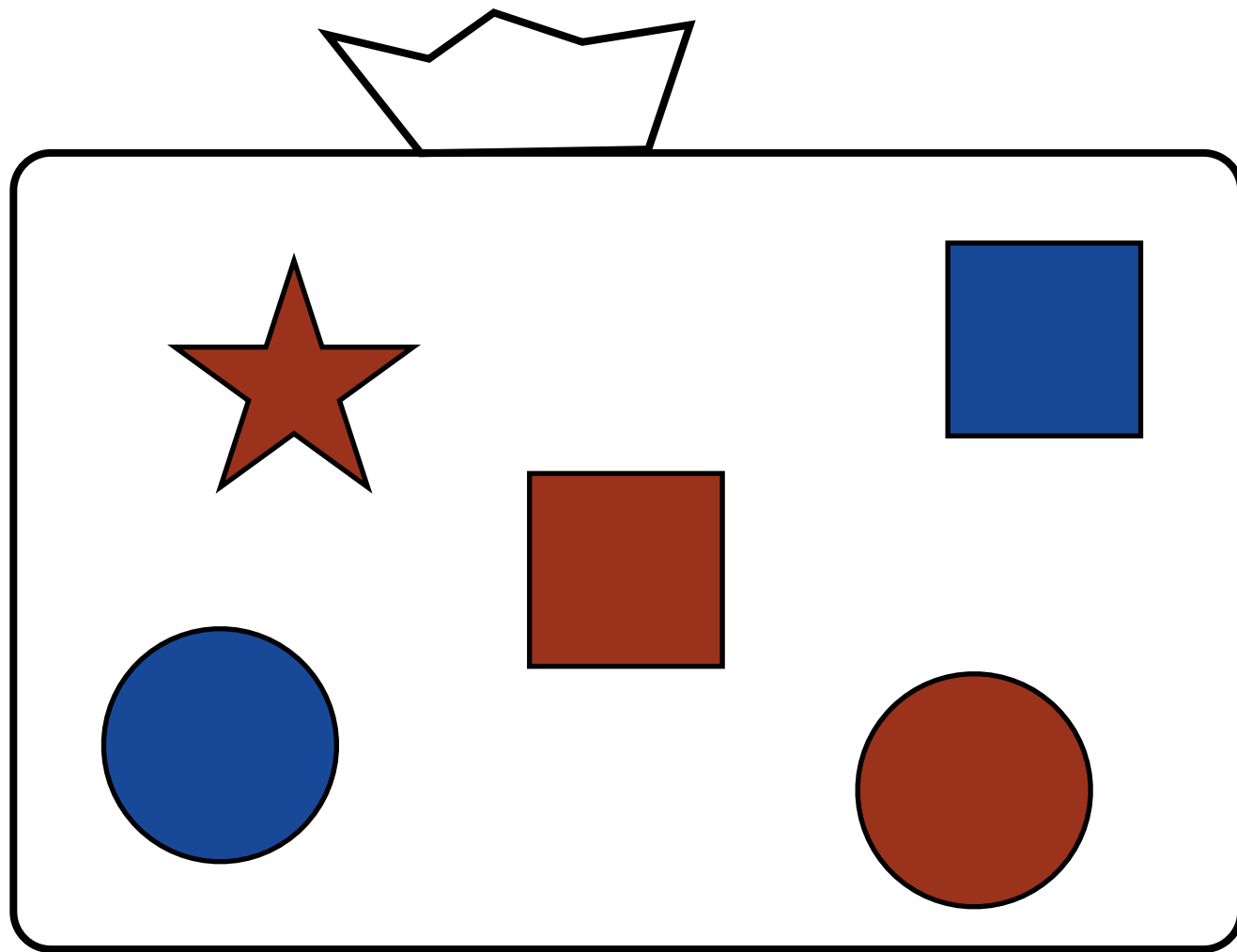
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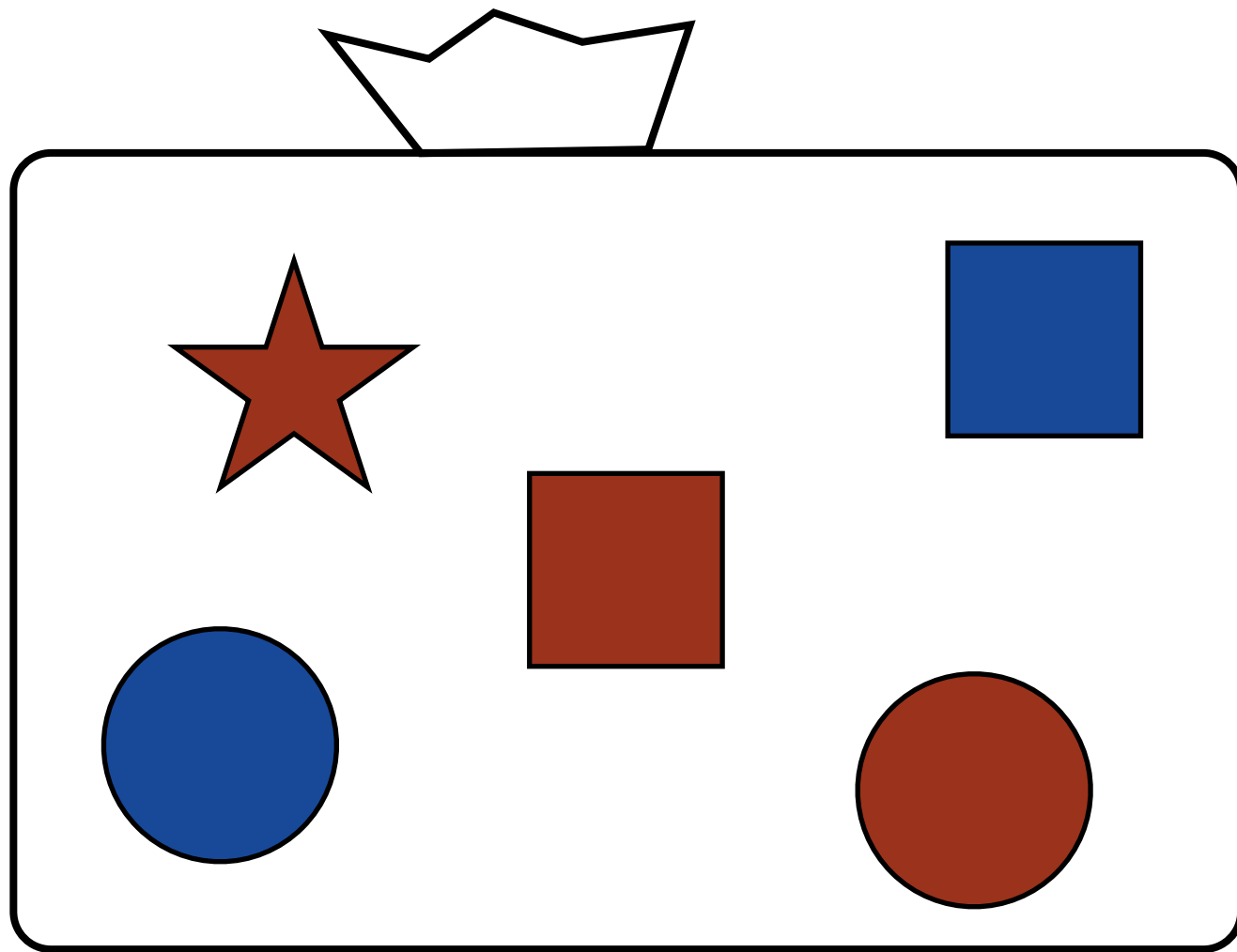
$$P(E^c) = 1 - P(E)$$

Basic properties



- We know $P(E)$, $P(F)$
- Example: $P(\text{blue}) = 2/5$
 $P(\text{star}) = 1/5$
- What is $P(E \cup F)$?
- Example: $P(\text{blue or star})$

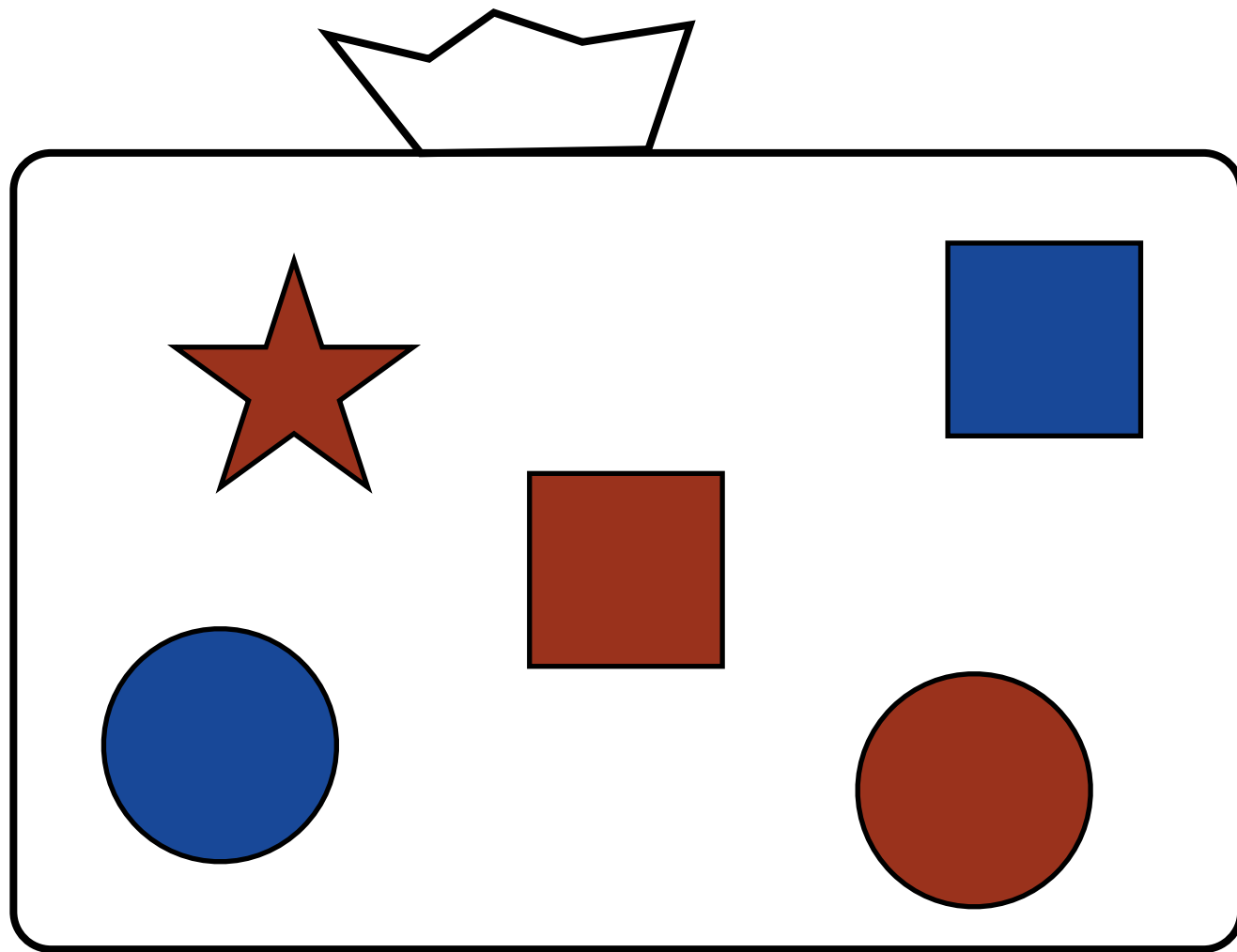
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$$P(E \cup F) = P(E) + P(F)$$

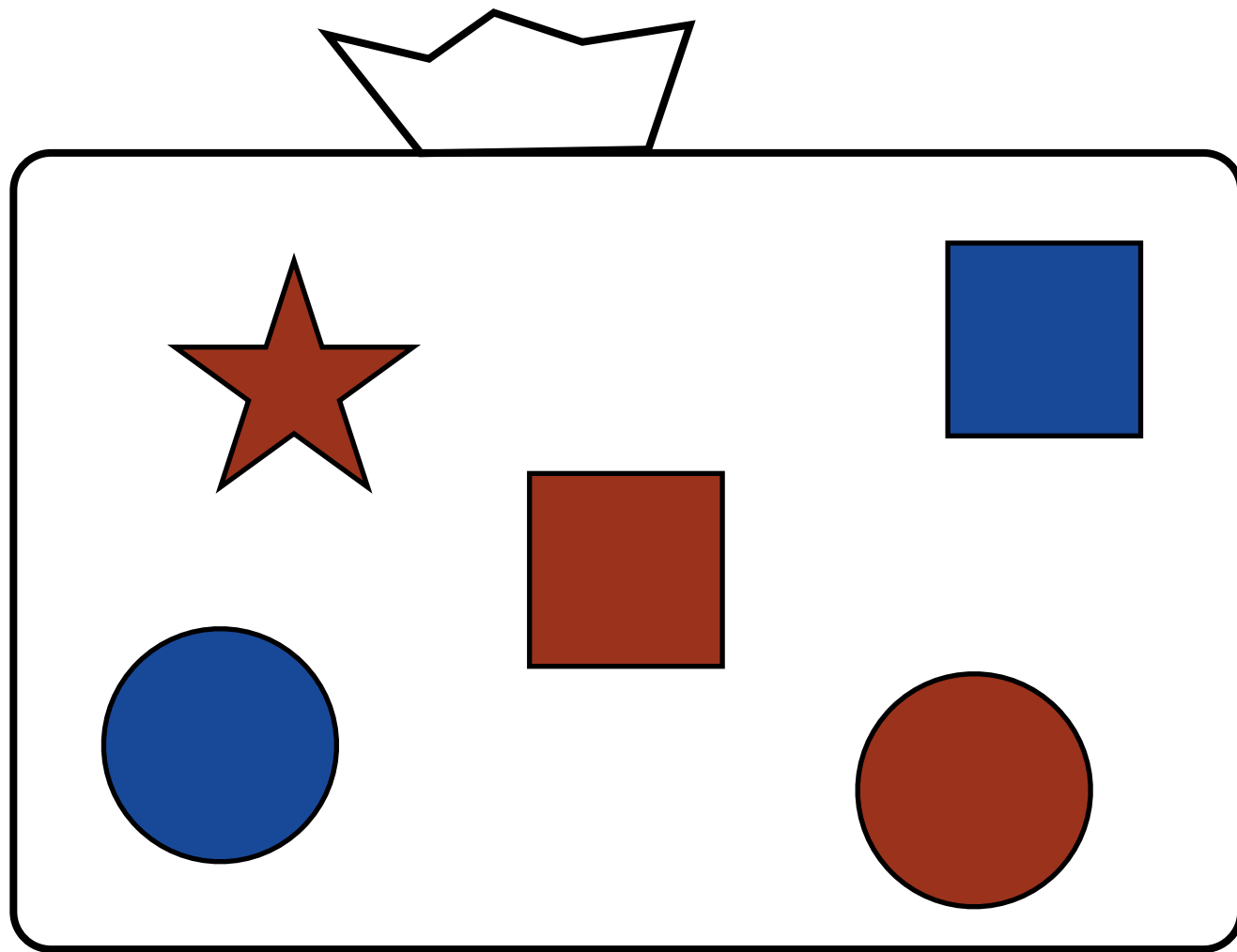
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- Try now on:
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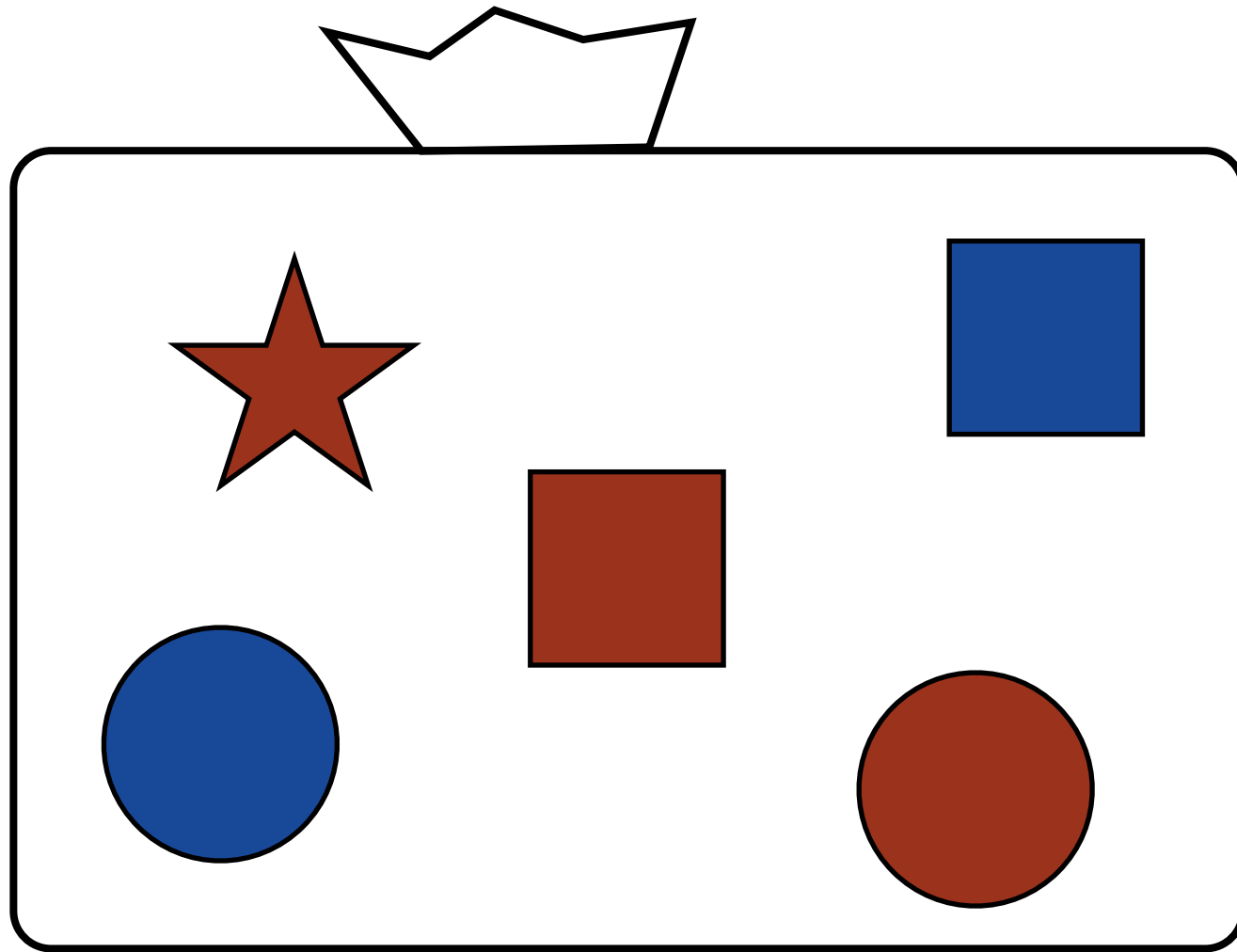
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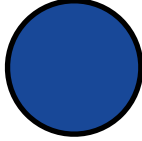
$$P(E \cup F) = P(E) + P(F) \quad \text{if } E \text{ and } F \text{ are disjoint,} \\ \text{i.e. } E \cap F = \emptyset$$

Odds



- Equivalent, but less used terminology
- The odds of red shapes is $\#(\text{red}) / \#(\text{not red}) = 3/2$
- $\text{odds}(E) = P(E) / P(E^c)$
 $= P(E) / (1 - P(E))$

Summary so far

- Outcome: a scenario, $s =$ 
- Sample space S : all the possible scenarios
- Event: a set of outcomes, e.g. $E = \{s \in S : s \text{ is red}\}$
- Probability: in the discrete, equally weighted case,
 $P(E) = |E| / |S|$
- Properties:
 - $P(E \cup F) = P(E) + P(F)$ when E & F are disjoint (non-overlapping)
 - $P(E^c) = 1 - P(E)$

Another example: two coins

- You flip 2 coins (pennies). What is the probability that the 2 coins both show heads?
 - A. $1/2$
 - B. $1/3$
 - C. $1/4$
 - D. none of above

Probability that 2 coins show heads

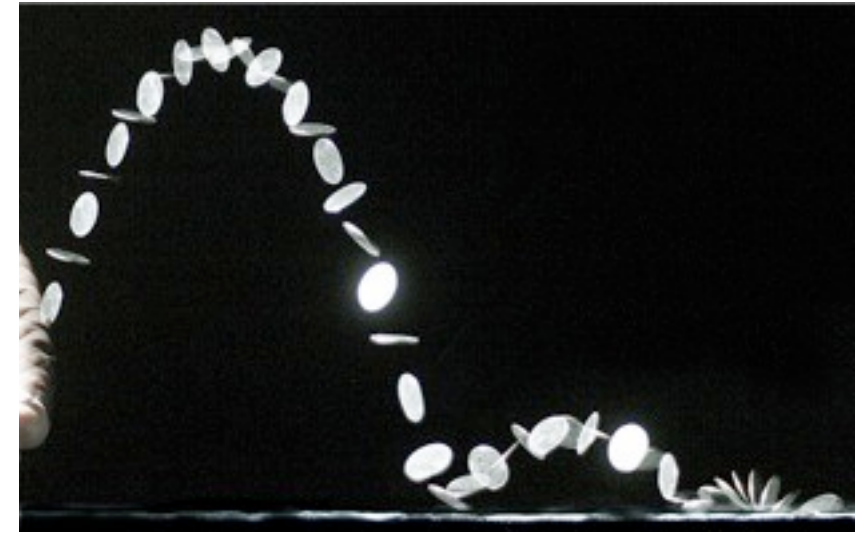
- 1/2? Either they do, or they don't.
- 1/3? Either both heads, both tails, or head-tail.
- 1/4? Imagine one coin is painted red, one is painted blue. There are then 4 possibilities:
 - red: head, blue: head
 - red: head, blue: tail
 - red: tail, blue: head
 - red: tail, blue: tail

Probability that 2 coins show heads

- 1/2? Either they do, or they don't.
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 - **red**: head, **blue**: head
 - **red**: head, **blue**: tail
 - **red**: tail, **blue**: head
 - **red**: tail, **blue**: tail
- These correspond to 3 different models
 - None is 'true'
 - But the third is more useful (accurate at doing predictions)

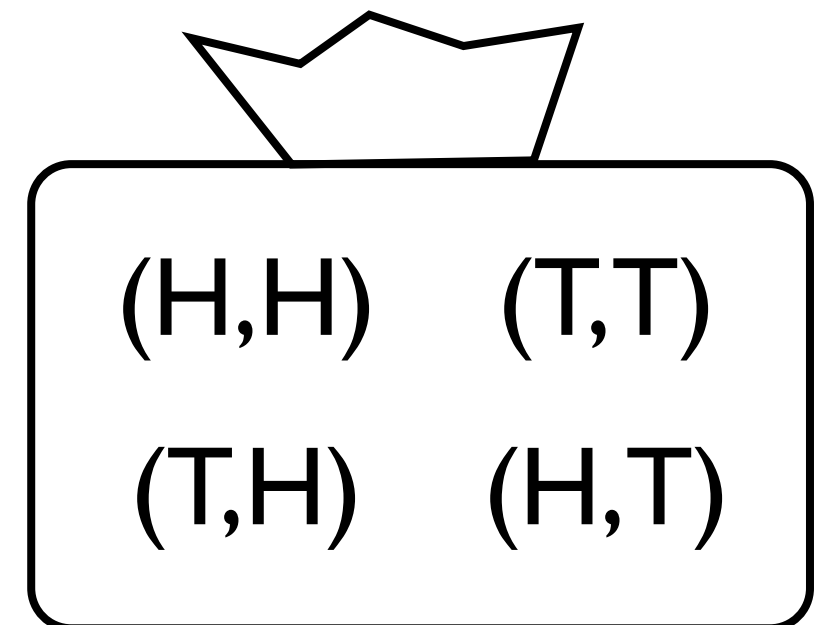
Why is this a model?

- Reality: a complex dynamical system



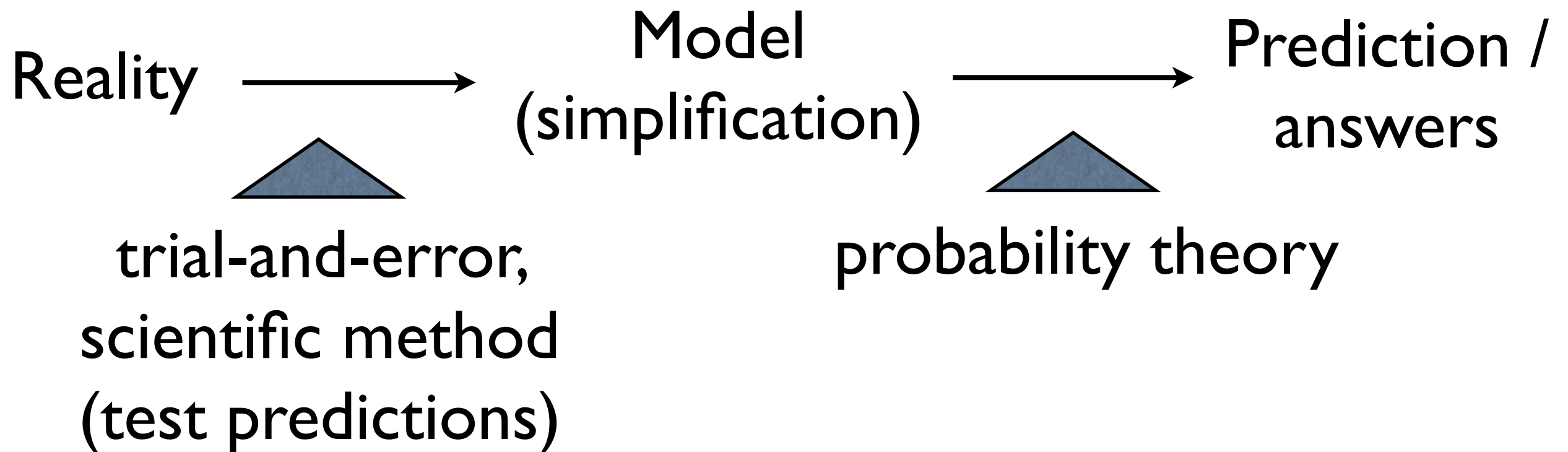
Stroboscopic image of a coin flip by [Andrew Davidhazy](#)

- Model: a 'bag' with 4 'objects' in it



How to decide which model?

- Probability theory does not have the answer!

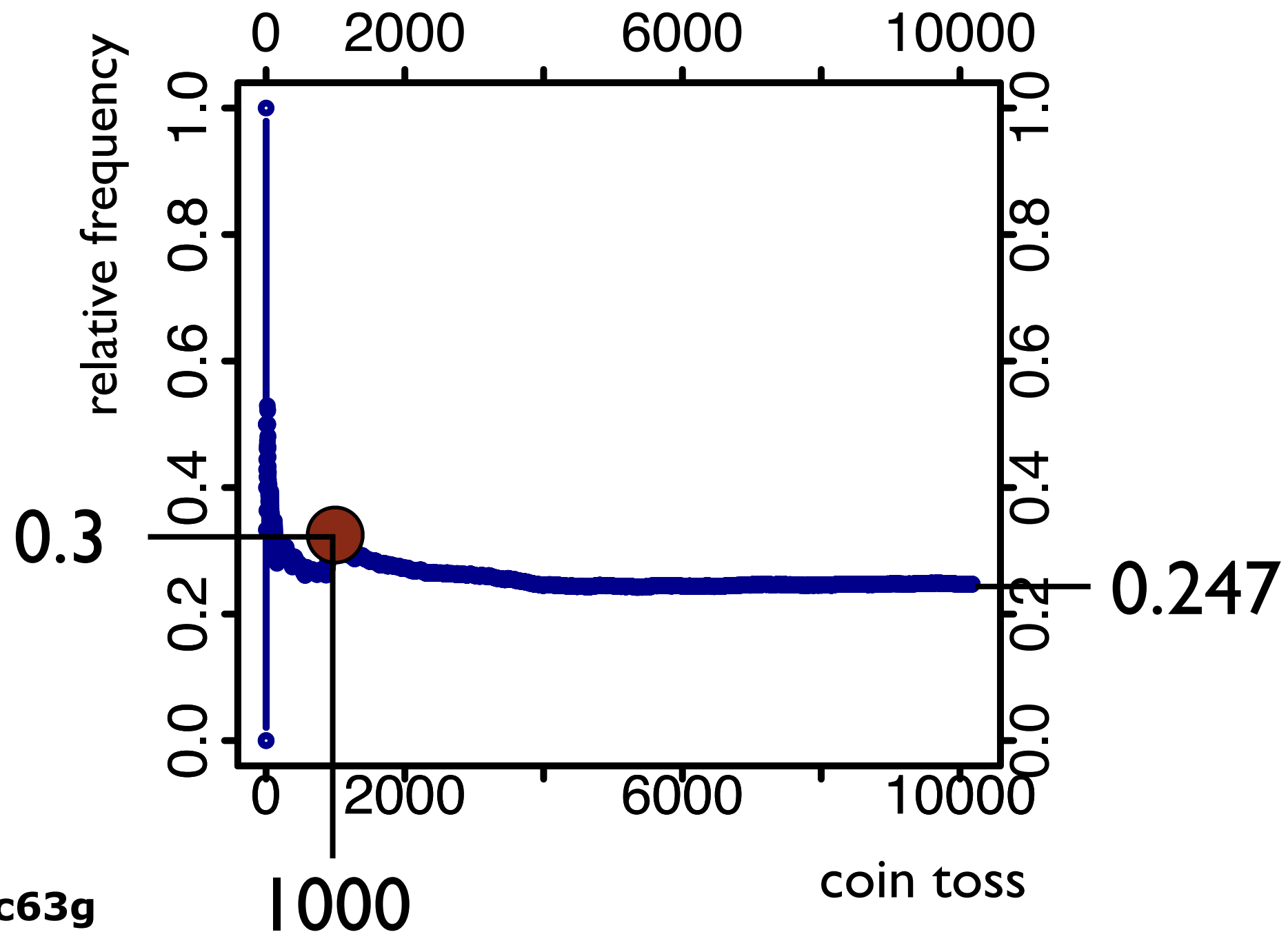


But..

- Probability theory still useful:
 - Given a model, it makes certain predictions
 - We can then test those predictions
- Example: *law of large numbers*
 - Relates *probability* to *frequency* in repeated experiments

Reality check

- Dataset of 10,000 actual tosses of two coins available on the web (!)



<http://tinyurl.com/ljtc63g>

All models are wrong

- Given the large number of throws, 0.247 may seem a bit far from 0.25 (we will formalize this idea later)
- is there something fishy?
- after reading the fine prints of the data, realized all throws started with same face in hand of thrower
- see P. Diaconis' paper, <http://tinyurl.com/yked5fk>
- Essentially, all models are wrong, but some are useful. -- G. Box



Redundancy is not always stupid

- Redundant systems:
 - create one or more duplicates of an important part of a machine
 - as long as one of the copies works, the machine works as a whole
 - the machine only fails when both copies break
- Many examples in biology (kidneys), engineering

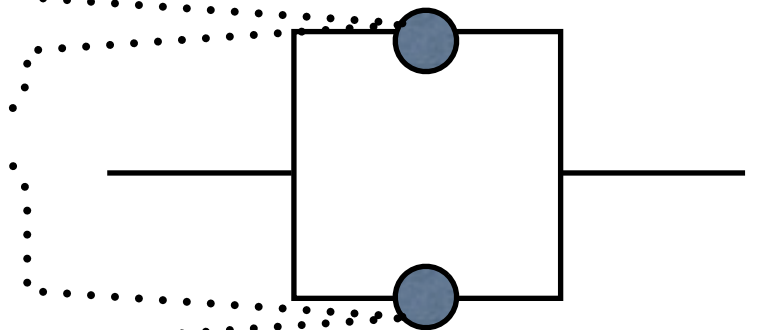


Redundant
server power
supply

Example of a typical reliability problem

- Known:

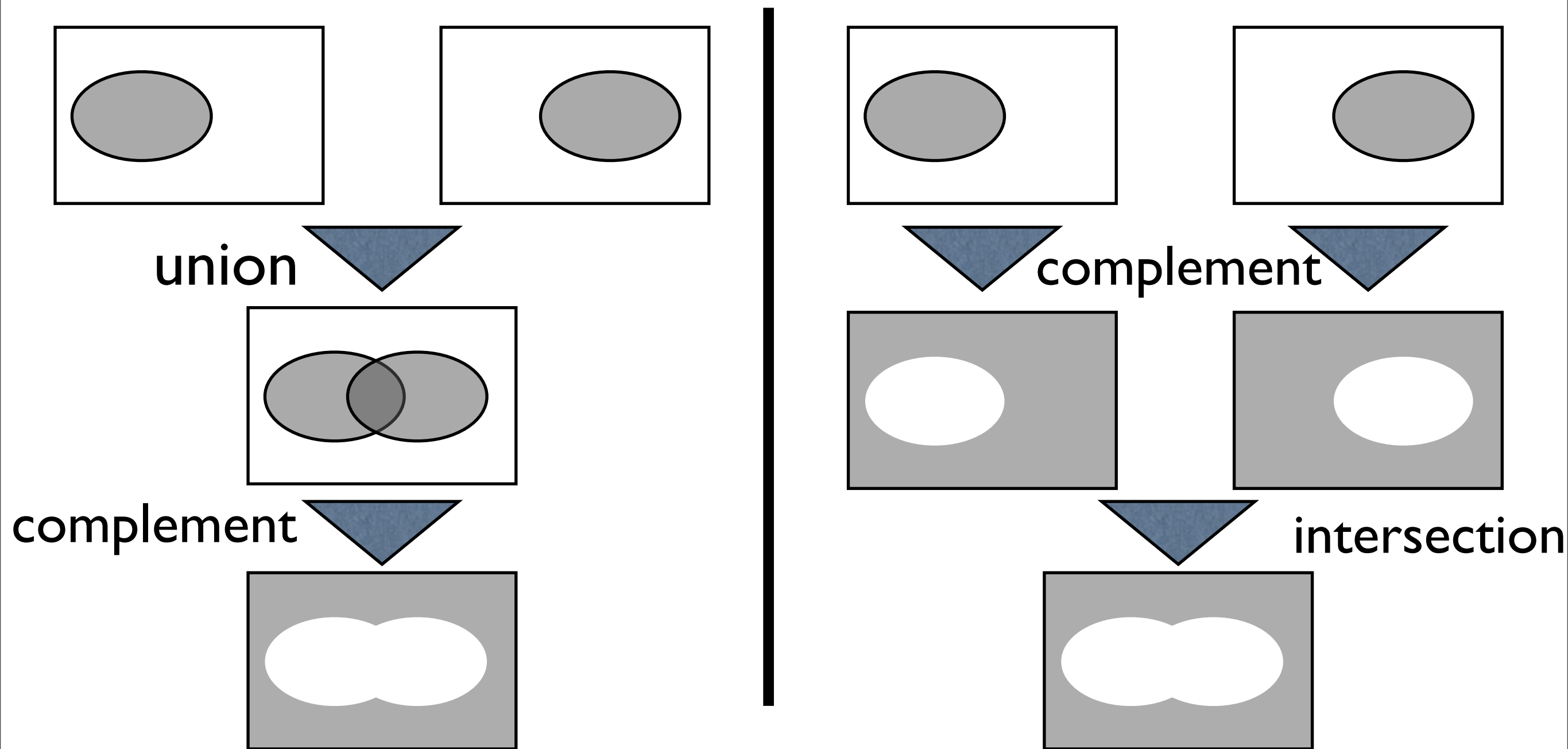
- Power supply #1 works 60% of the time at delivery
- Power supply #2 works 70% of the time at delivery
- You also observed that both power supplies work at delivery 40% of the time
- Question: what is the probability that both power supply are broken at delivery?



Strategy

1. Define a probability space S
2. Define the known information as events,
ex.: $W_1 = \text{power supplies \#1 works}$
3. Define the goal in terms of the probability
of an event
4. Use properties 1 and 2 to find that
probability (see next 2 slides as well to
make your life easier)

De Morgan's law: Distributing complements



$$\left(\bigcup_{i=1}^n E_i\right)^c = \bigcap_{i=1}^n E_i^c$$

Inclusion-exclusion:

Going between unions and intersections

From earlier:

$$P(E \cup F) = P(E) + P(F) \quad \text{if } E \text{ and } F \text{ are disjoint,} \\ \text{i.e. } E \cap F = \emptyset$$

What if: we want the prob. of non-disjoint unions?