http://www.stat.ubc.ca/~bouchard/courses/stat302-fa2014-15/

Intro to Probability

Instructor: Alexandre Bouchard

Fall 2014

Plan for today:

- Axioms of probability.
- Reliability, continued.
- Probability tree diagram.
- More examples.

Logistics

- Let me know if you cannot access website
- Last reminders:
  - Doodle under Contact tab.
  - Clickers (see links in Syllabus tab)
  - Piazza under Contact tab.
  - Pre-readings posted in Schedule tab
- Slides under Files tab
- Additional practice problems: Syllabus tab

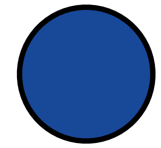
Logistics

- I will start using clickers today,
  - but I am not taking grade into account (yet)
- First problems will be released by Wednesday (more about that on Wed)
- We cannot have a larger room unfortunately, but keep monitoring the enrollment!

Disclaimer

- Workload and difficulty increases in the second half of semester (continuous probability)
- Make sure you have the time to stay on top of the material
- Good things to review from pre-requisite courses:
  - set theory notation
  - bivariate integration

Review

- <u>Outcome</u>: a scenario, s = 
- <u>Sample space</u> S: all the possible scenarios
- <u>Event</u>: a set of outcomes, e.g. $E = \{s \in S : s \text{ is red}\}$
- <u>Probability</u>: in the discrete, <u>equally weighted case</u>, $P(E) = |E| / |S|$
- Properties:
  - $P(E \cup F) = P(E) + P(F)$ when E & F are <u>disjoint</u> (non-overlapping)
  - $P(E^c) = 1 - P(E)$

Going beyond the equally weighted case

Def. 3

Discrete, equally weighted

Definition (1):

$P(E) = |E| \,/\, |S|$

Properties (2):

a) $0 \leq P(E) \leq 1$

b) $P(S) = 1$

c) $P(E \cup F) = P(E) + P(F)$
if E and F are disjoint

Def. 3

Discrete, equally weighted

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c) $P(E \cup F) = P(E) + P(F)$ if E and F are disjoint

Not equally weighted (or not discrete)

Example: redundancy problem (ex. 11)

Def. 3

General rules of probability

Assume:

a) $0 \leq P(E) \leq 1$

b) $P(S) = 1$

c) $P(E \cup F \cup ...) = P(E) + P(F) + ...$

if $E, F, ...$ are all disjoint

Discrete, equally weighted

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Not equally weighted (or not discrete)

Example: redundancy problem (ex. 11)

Def. 3

General rules of probability

These are called the **axioms of probability** ***

Assume:

a) $0 \leq P(E) \leq 1$

b) $P(S) = 1$

c) $P(E \cup F \cup \ldots) = P(E) + P(F) + \ldots$

if $E, F, \ldots$ are all disjoint

Discrete, equally weighted

Definition (1):

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Not equally weighted (or not discrete)

Example: redundancy problem (ex. 11)

Redundancy is not always stupid

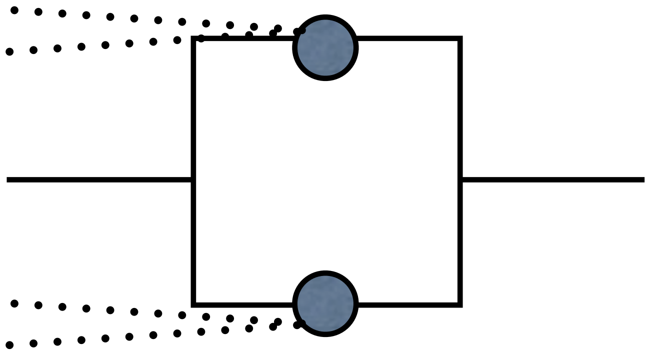
- Redundant systems:
  - create one or more duplicates of an important part of a machine
  - as long as one of the copies works, the machine works as a whole
  - the machine only fails when both copies break
- Many examples in biology (kidneys), engineering

Redundant server power supply

Ex. 11

Example of a typical reliability problem

- Known:
  - Power supply #1 works 60% of the time at delivery
  - Power supply #2 works 70% of the time at delivery
  - You also observed that both power supplies work at delivery 40% of the time
- Question: what is the probability that both power supply are broken at delivery?

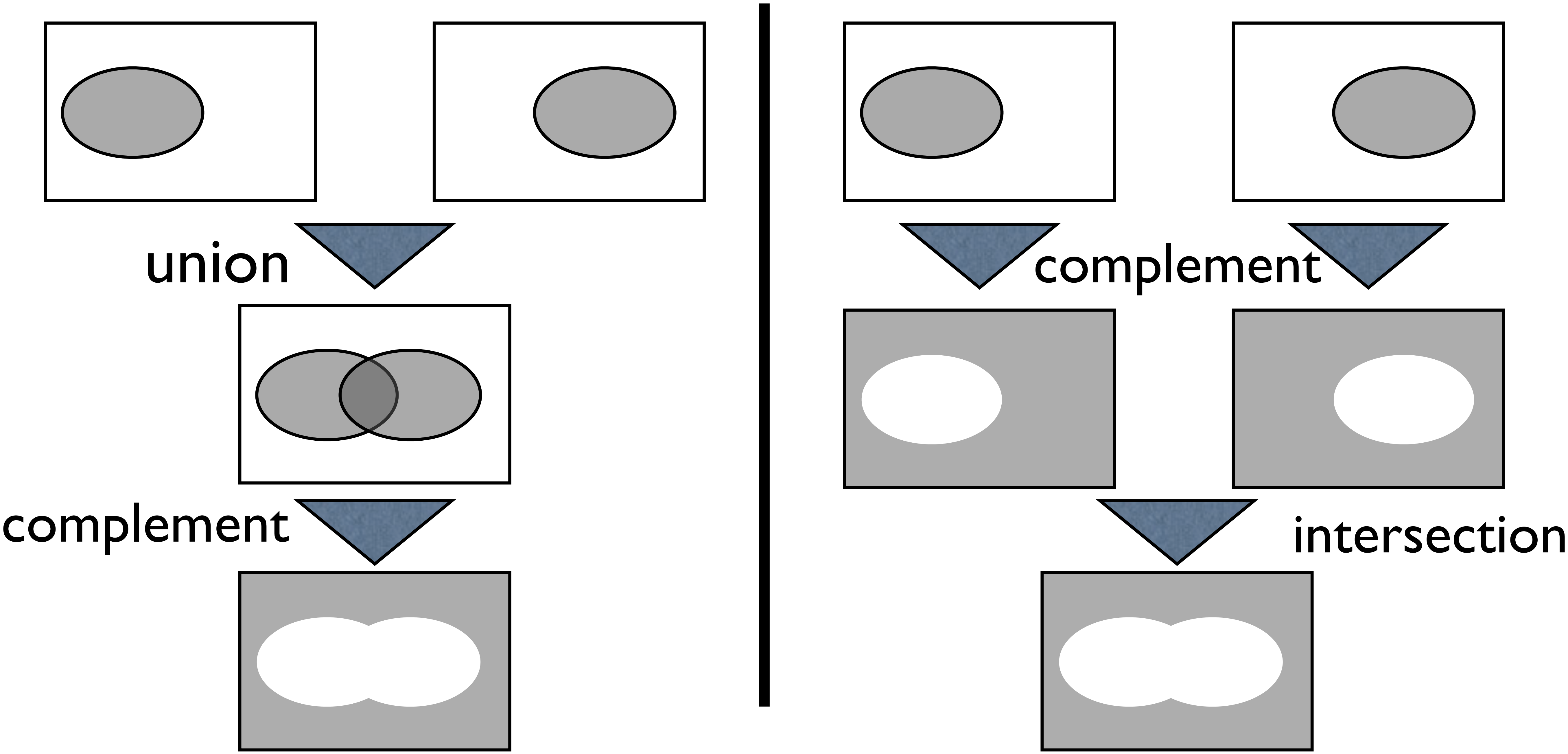


Strategy

1. Define a probability space S
2. Define the known information as events, ex.: W_1 = power supplies #1 works
3. Define the goal in terms of the probability of an event
4. Use properties 1 and 2 to find that probability (see next 2 slides as well to make your life easier)

Prop. 3

De Morgan's law: Distributing complements



$$\left(\cup_{i=1}^{n} E_i\right)^c \quad = \quad \cap_{i=1}^{n} E_i^c$$

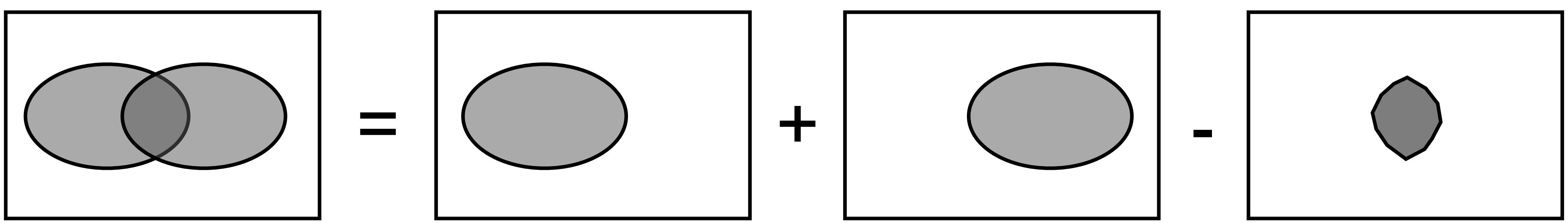
Prop. 4

Inclusion-exclusion: Going between unions and intersections

From earlier:

$P(E \cup F) = P(E) + P(F)$ if E and F are *disjoint*, i.e. $E \cap F = \varnothing$

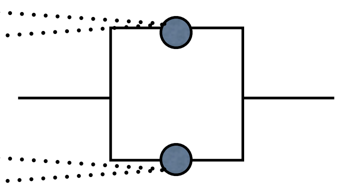
What if: we want the prob. of non-disjoint unions?



$P(E \cup F) \quad = \quad P(E) \quad + \quad P(F) \quad - \quad P(E \cap F)$

Ex. 11

- Known:
  - Power supply #1 works 60% of the time at delivery
  - Power supply #2 works 70% of the time at delivery
  - You also observed that both power supplies work at delivery 40% of the time
- Question: what is the probability that both power supply are broken at delivery?



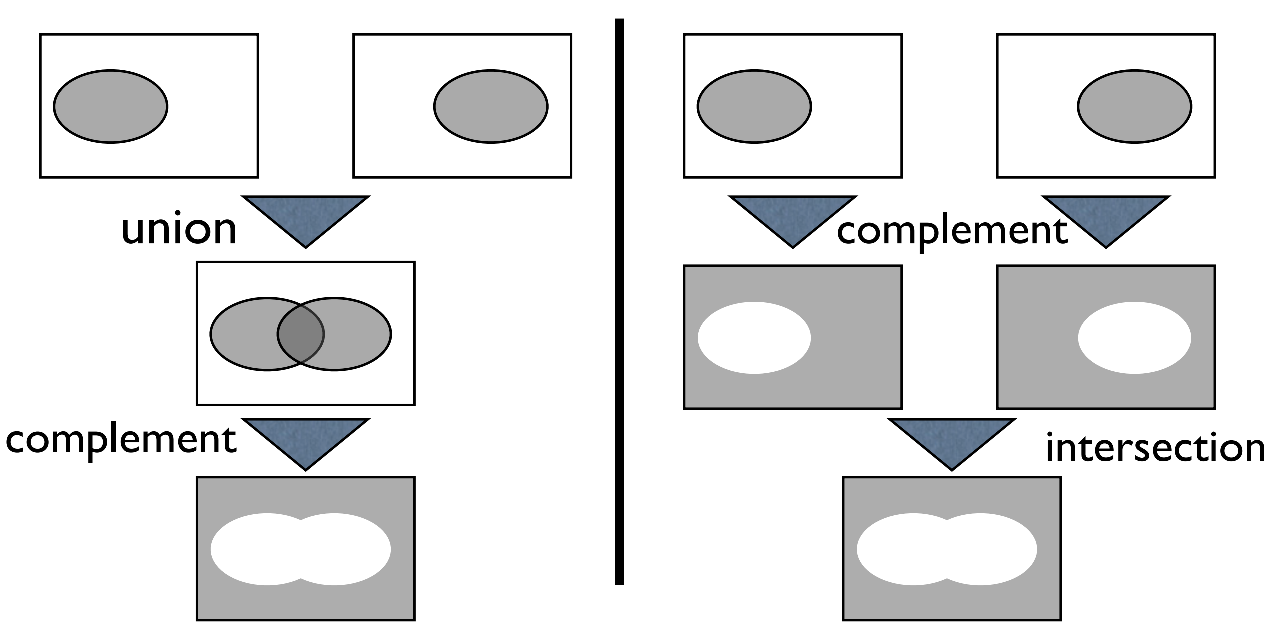
A) 6%

B) 10%

C) 12%

D) 20%

De Morgan's law: Distributing complements



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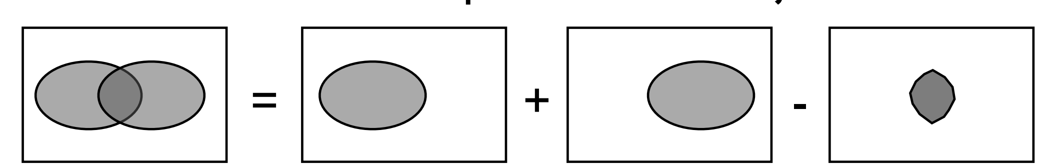
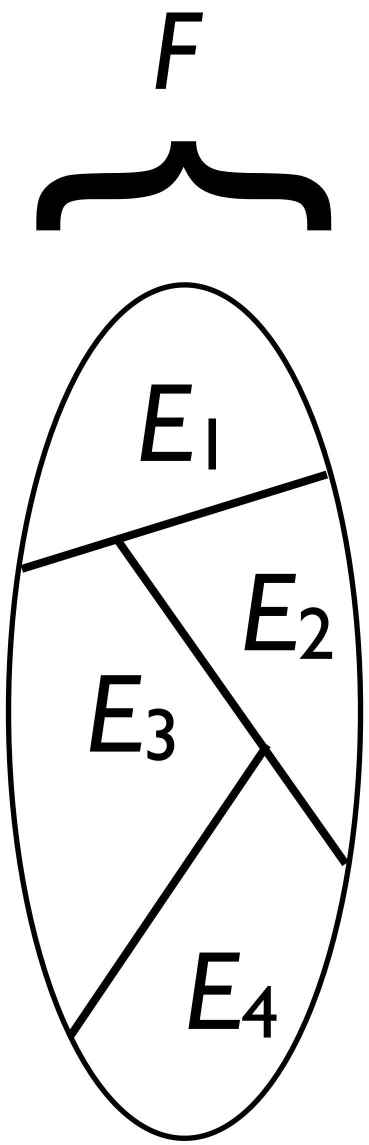


$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Partitions
and
probability tree diagrams

Def. 4

Partition of an event



The events E_i form a <u>partition</u> of the event F if:

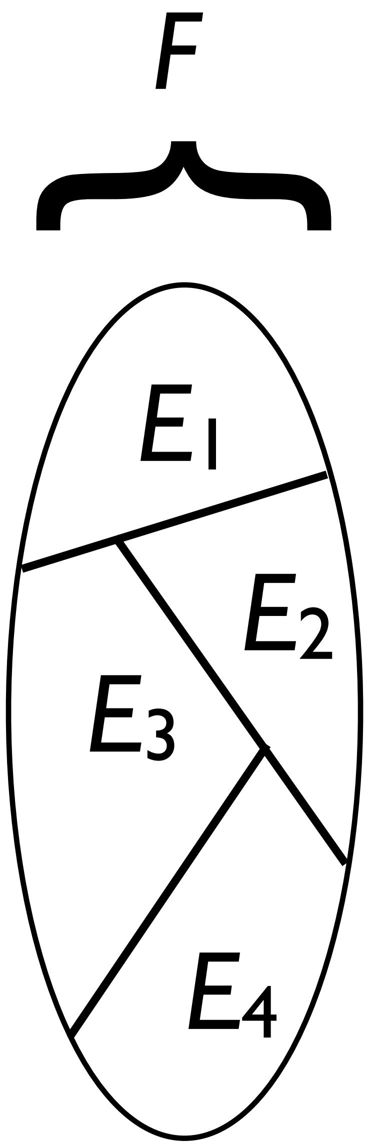
1. The union of the E_i's is equal to F:
   $\cup_i E_i = F$

2. The E_i's are disjoint:
   if $i \neq j$, then $E_i \cap E_j = \varnothing$

Why is this useful?

Def. 4

Partition of an event



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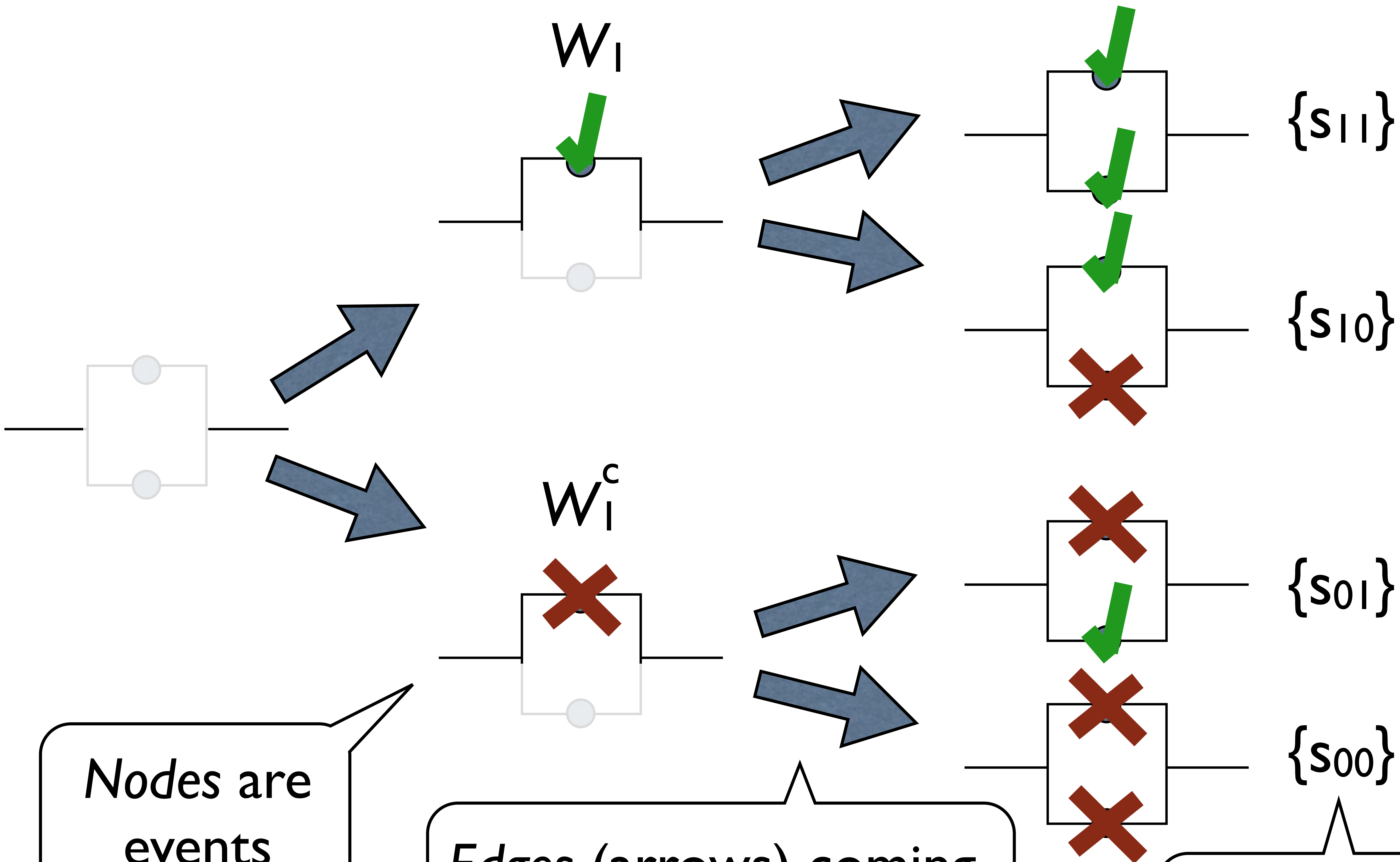
Why is this useful?

Note: as consequence of 1, 2 and the axioms of probability, $P(F) = P(E_1) + P(E_2) + P(E_3) + P(E_4)$

Def. 5

Probability tree diagram



Nodes are <u>events</u>

Edges (arrows) coming out of an event are <u>partitions</u> of that event

Leaves contain a single <u>outcome</u>

Examples of solving discrete probability problems with tree diagrams

Ex. 12

Probability that n coins are all tails

Recall:

$P(E) = |E| / |S|$

Probability <u>when outcomes are equally likely</u>:

$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

Here: $|E| = ?$
$|S| = ?$

Ex. 12

Probability that n coins are all tails

Recall:

$$P(E) = |E| \;/\; |S|$$

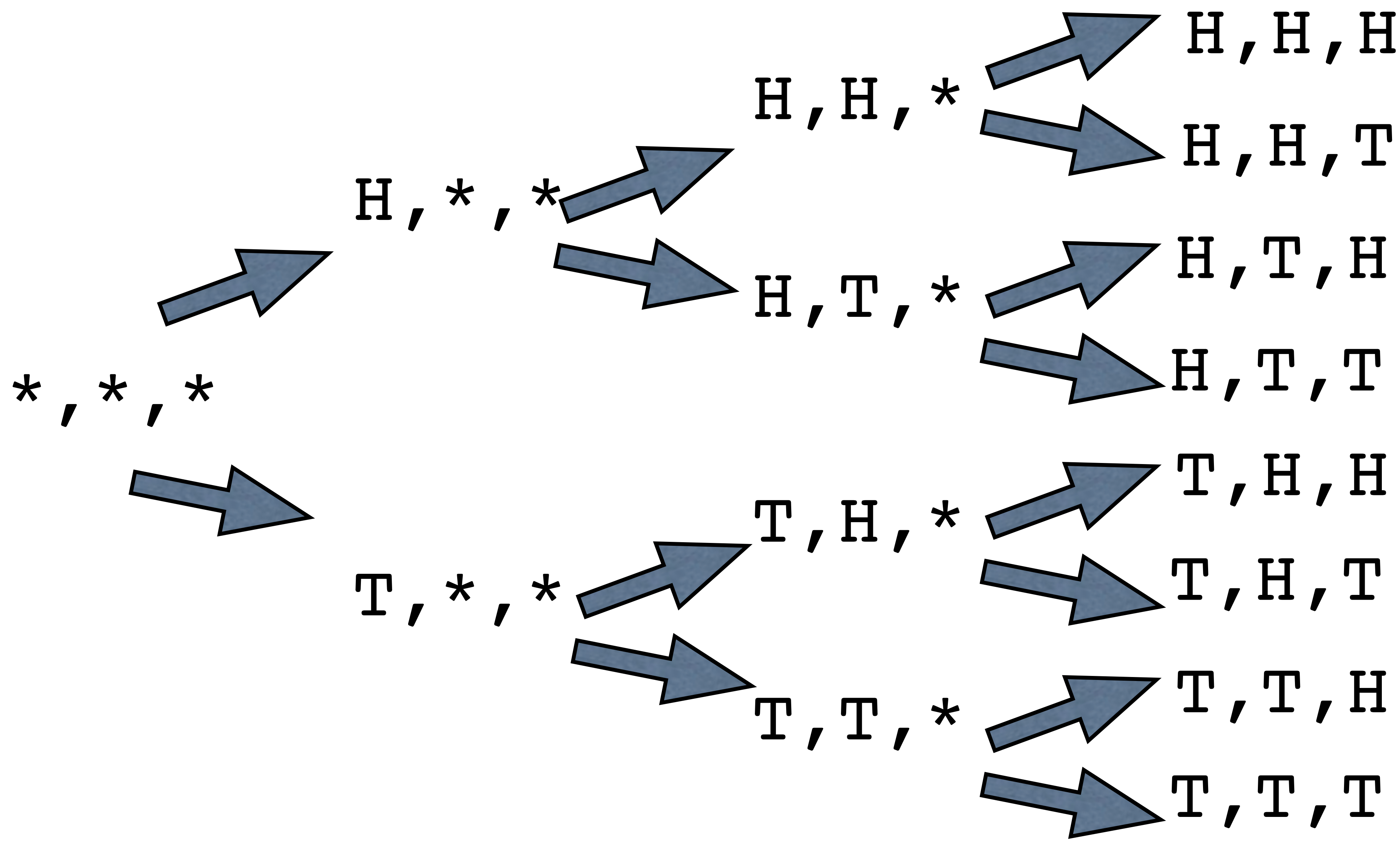
Probability <u>when outcomes are equally likely</u>:

$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

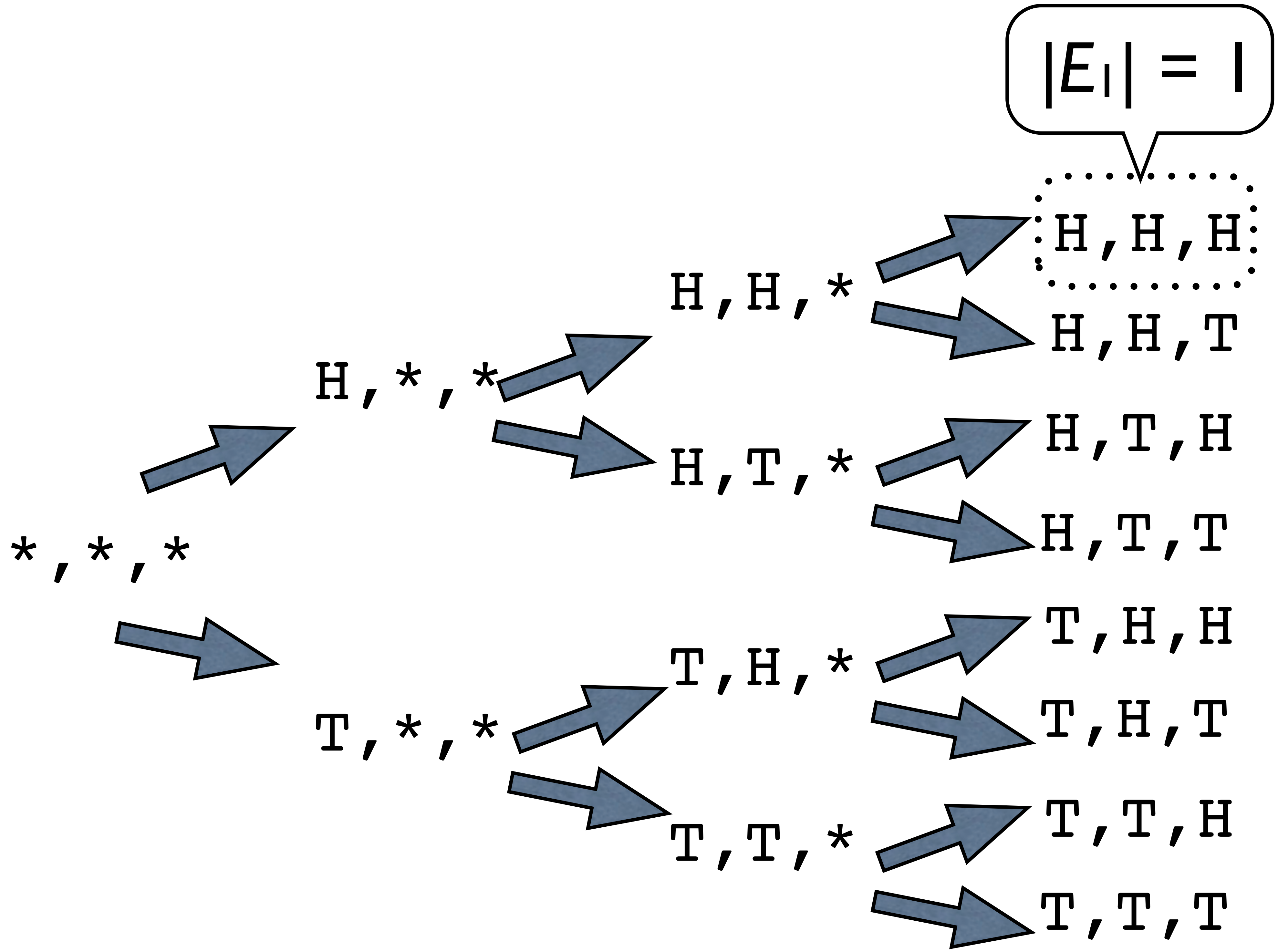
Here: $|E| = 1$

$|S| = ?$

Ex. 12 Probability that *n* coins are all tails: finding $|S|$



Ex. 12 Probability that *n* coins are all tails: finding $|S|$



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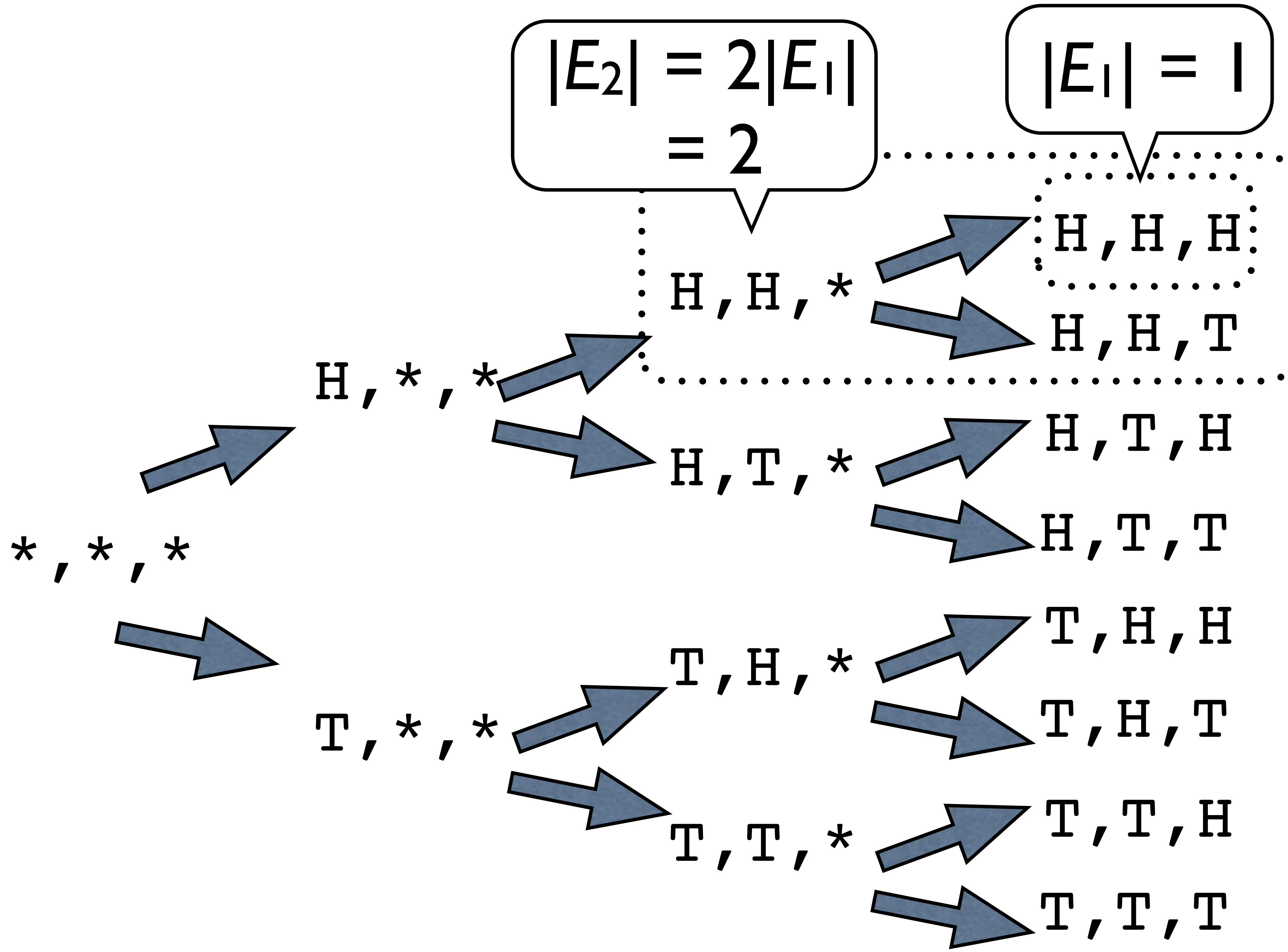
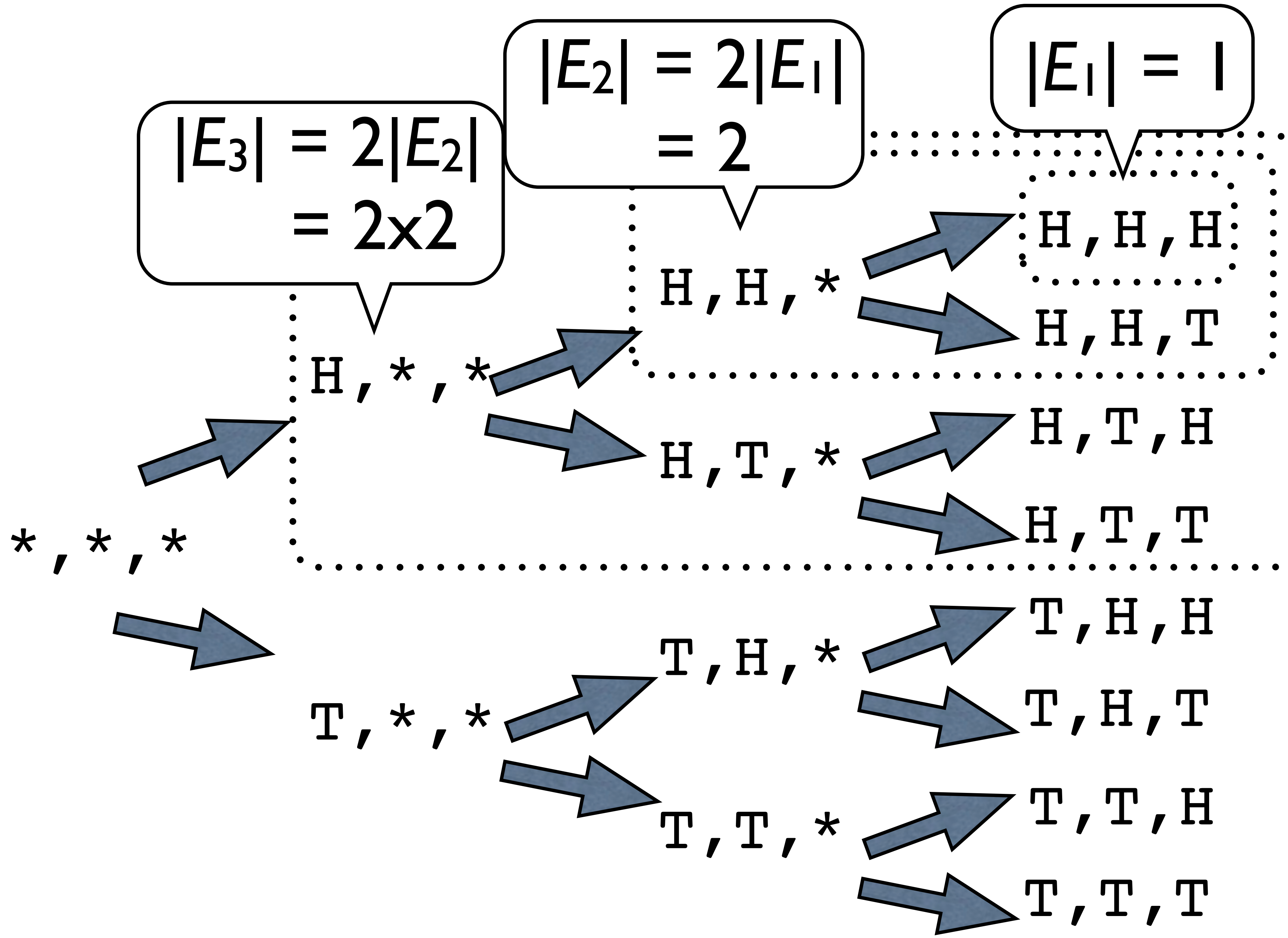


Ex. 12 Probability that n coins are all tails: finding $|S|$



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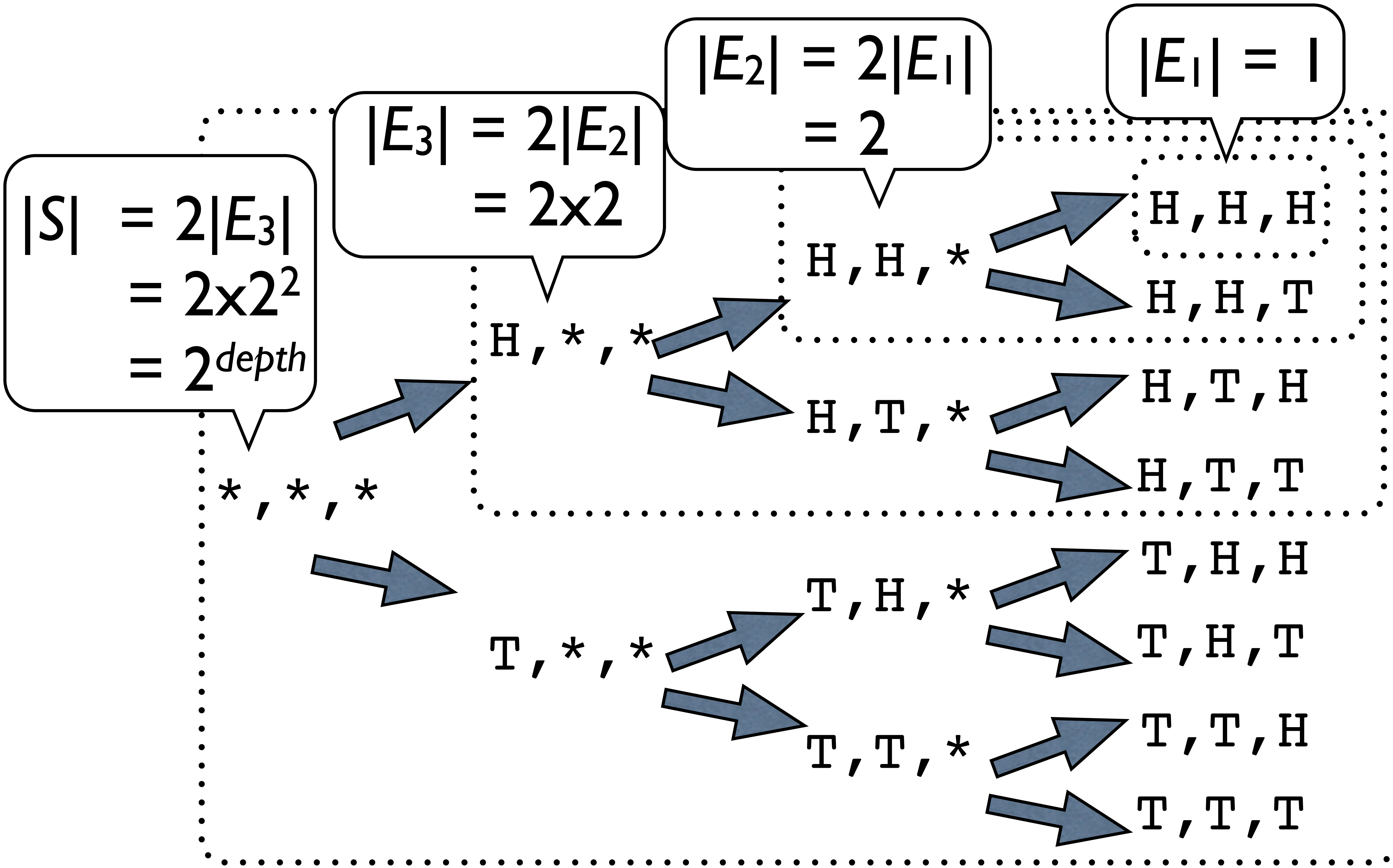

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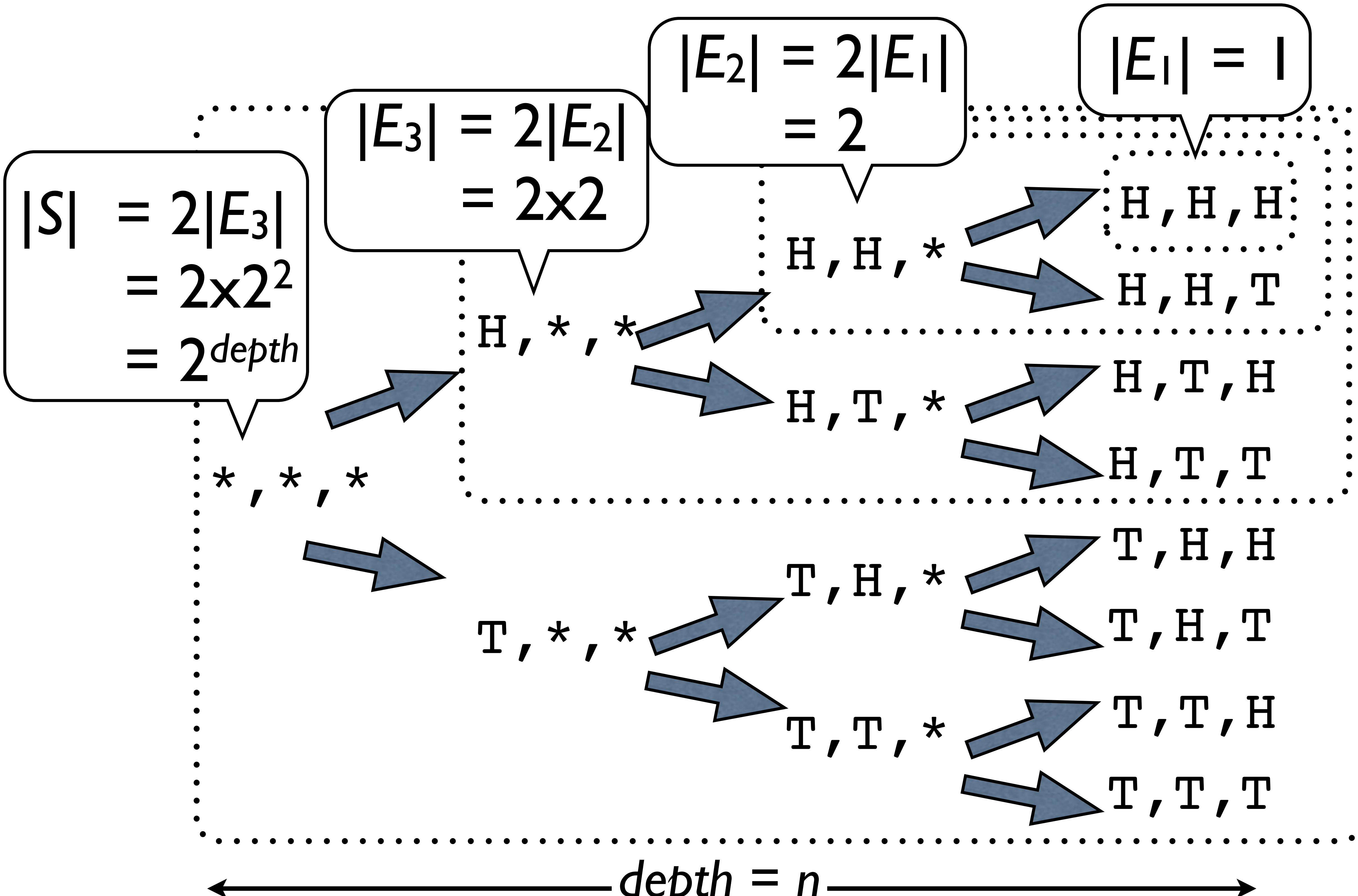

Ex. 13

Probability of winning the lottery

- You pick your number when you buy a ticket
- The lottery company draws at random from an urn containing n numbered balls $\{1, 2, ..., n\}$ [example: n=5]
  - k times [example: k=3]
  - without replacement (each number is either picked 0 or 1 time, not more)

Probability of winning the lottery (without replacement)

- You win if the numbers you picked *match* those from the draw.
- See example on the right, do you win in this case?

Draw result:

Your picks:
12, 31, 10, 23, 8, 45

Ex. 13

Probability of winning the lottery (without replacement)

- You win if the numbers you picked *match* those from the draw.
- See example on the right, do you win in this case?

a) if order matters, NO
b) if order does not matter, YES

Draw result:

Your picks:
12, 31, 10, 23, 8, 45

Note: most lottery use (b), but let's do (a) first---it is simpler

Ex. 13a

Probability of winning the lottery (order matters, without replacement)

Recall:

$P(E) = |E| \,/\, |S|$

Probability <u>when outcomes are equally likely</u>:

$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

Here: $|E| = 1$

$|S| = ?$

Ex. 13a

Find |*S*|:

A) 243
B) 125
C) 60
D) 15

Hint: the probability tree looks like:

k = number of draws = 3
n = number of balls in urn = 5