http://www.stat.ubc.ca/~bouchard/courses/stat302-fa2014-15/

Intro to Probability

Instructor: Alexandre Bouchard

Fall 2014

Plan for today:

- Definitions and basic properties.
- Examples.
    - Introduction of the notion of *model*.
- Going beyond equally weighted outcomes.

Logistics

- Let me know if you cannot access website
- 'Homeworks' for next Monday:
  - Doodle under Contact tab.
  - Clickers (see links in Syllabus tab)
  - Piazza under Contact tab.
  - Pre-readings posted in Schedule tab
- Slides under Files tab
- Additional practice problems: Syllabus tab

Outline of the course

- **Discrete probability models**
- Conditioning and Bayes
- Expectation
- Continuous probability models
- Asymptotics

Discrete: only a finite (not infinite) number of scenarios

Probability: extension of the notion of proportion

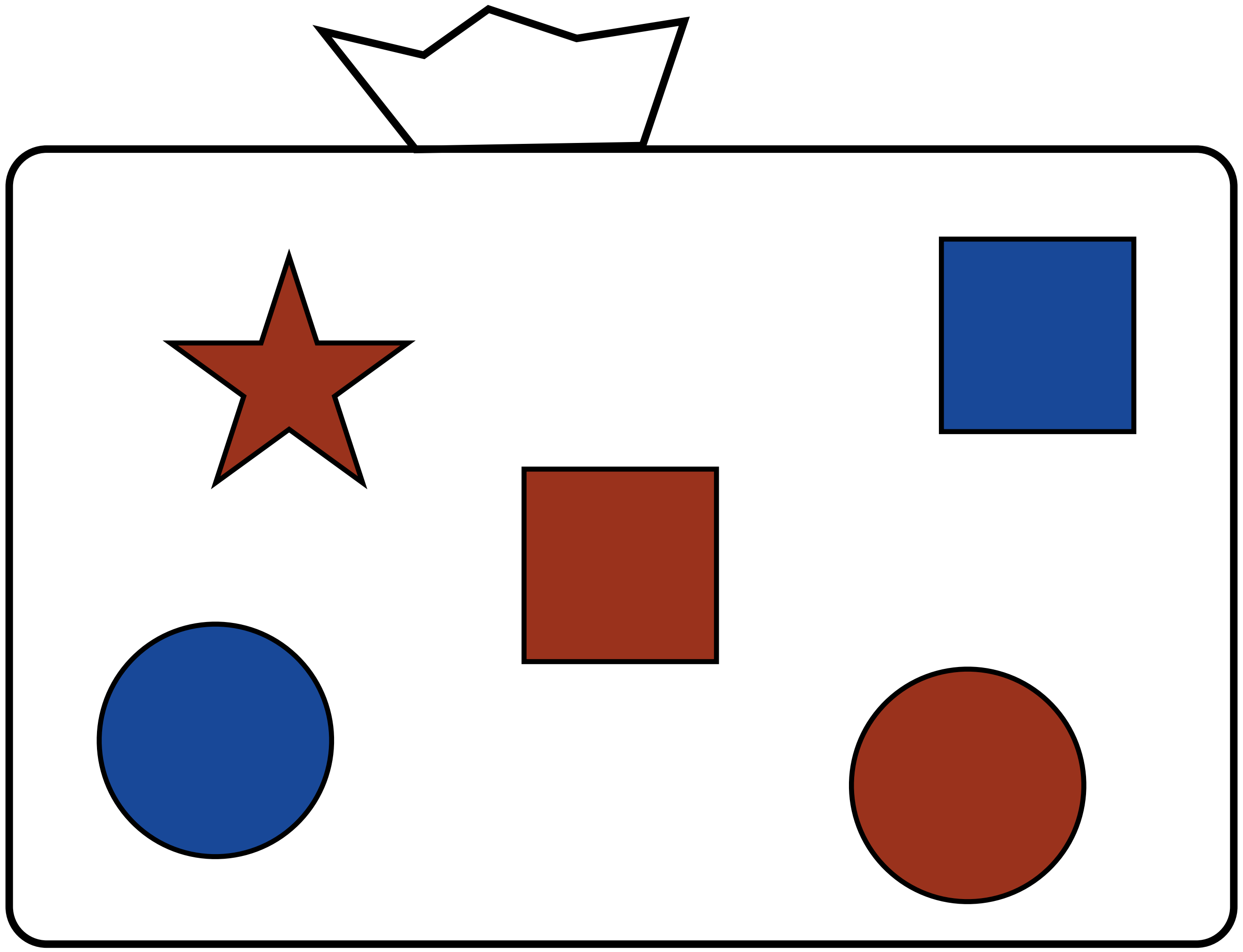
Model: a simplification of reality amenable to mathematical investigation

Note: we will make this more precise today

Definitions and basic properties

Ex. 9

Example: bag of distinct objects of same size



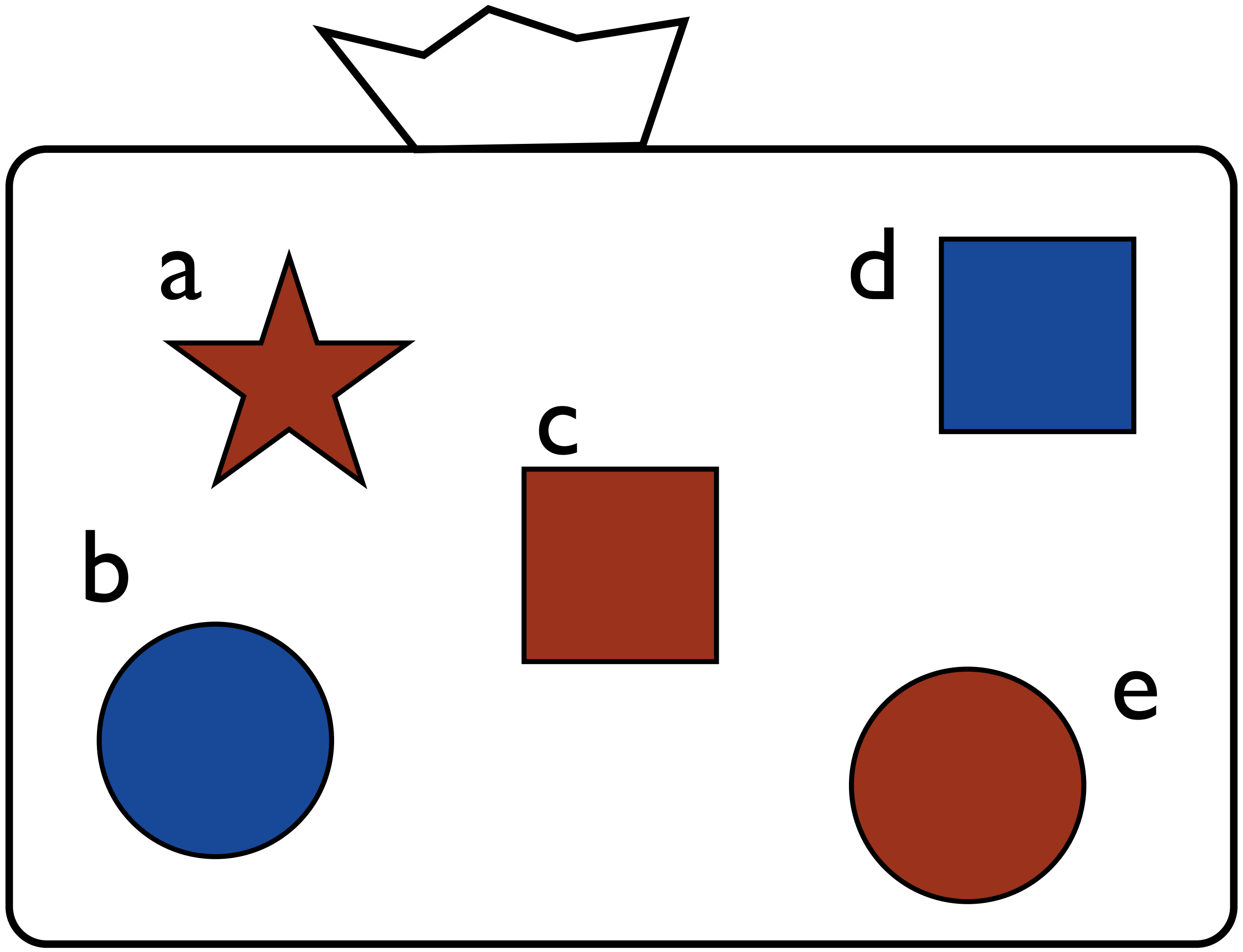
- Proportion of red shapes?
- Probability of drawing a red shape?
- *Outcome:* an individual object in the bag
  ex.: *s* = 

Probability <u>when outcomes are equally likely</u>:

$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

Def 1

Notation



- Proportion of red shapes?
- Probability of drawing a red shape?
- *Outcome:* an individual object in the bag

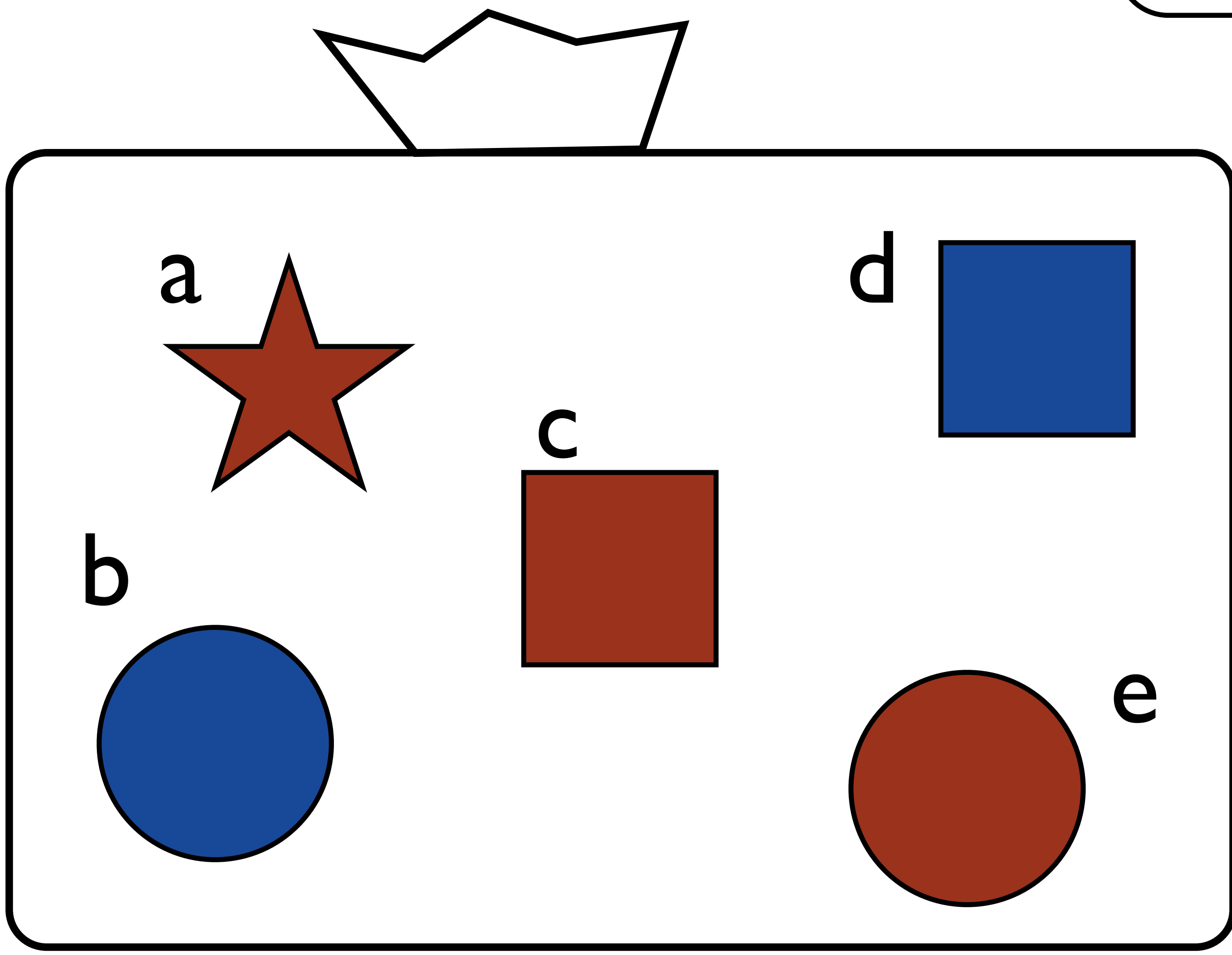
Probability <u>when outcomes are equally likely</u>:

$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

Def 1

Notation

This is a *set of outcomes*
Nickname: *event*
Typical notation: red, *E*, blue, *F*, ..
E = {a, c, e}



- Proportion of red shapes?
- Probability of drawing a red shape?
- *Outcome:* an individual object in the bag

Probability <u>when outcomes are equally likely</u>:

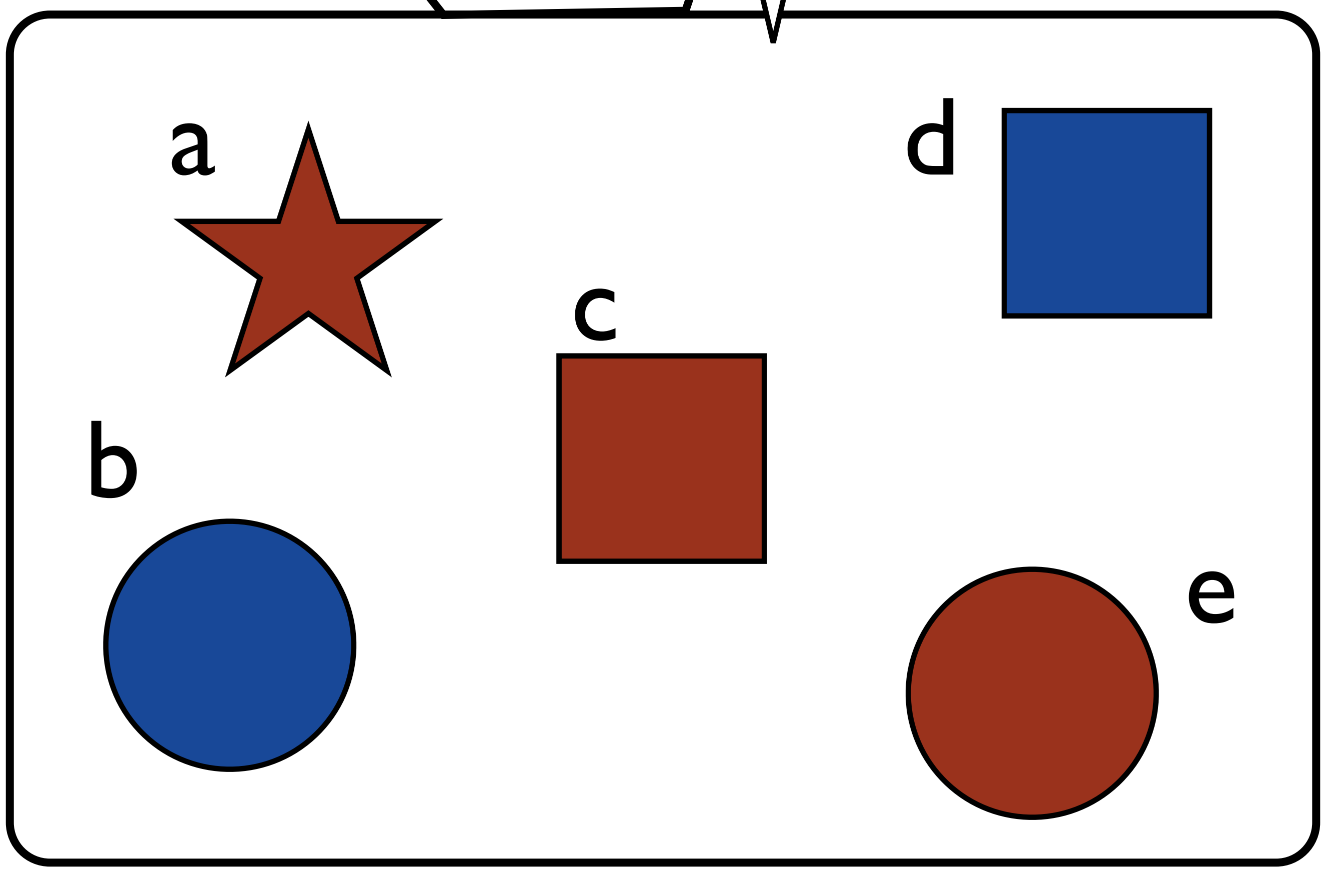
$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

Def 1

Notation

Sample space
Typical notation: *S*

This is a *set of outcomes*
Nickname: *event*
Typical notation: red, *E*, blue, *F*, ..
$E = \{a, c, e\}$



- Proportion of red shapes?
- Probability of drawing a red shape?
- *Outcome:* an individual object in the bag

Probability <u>when outcomes are equally likely</u>:

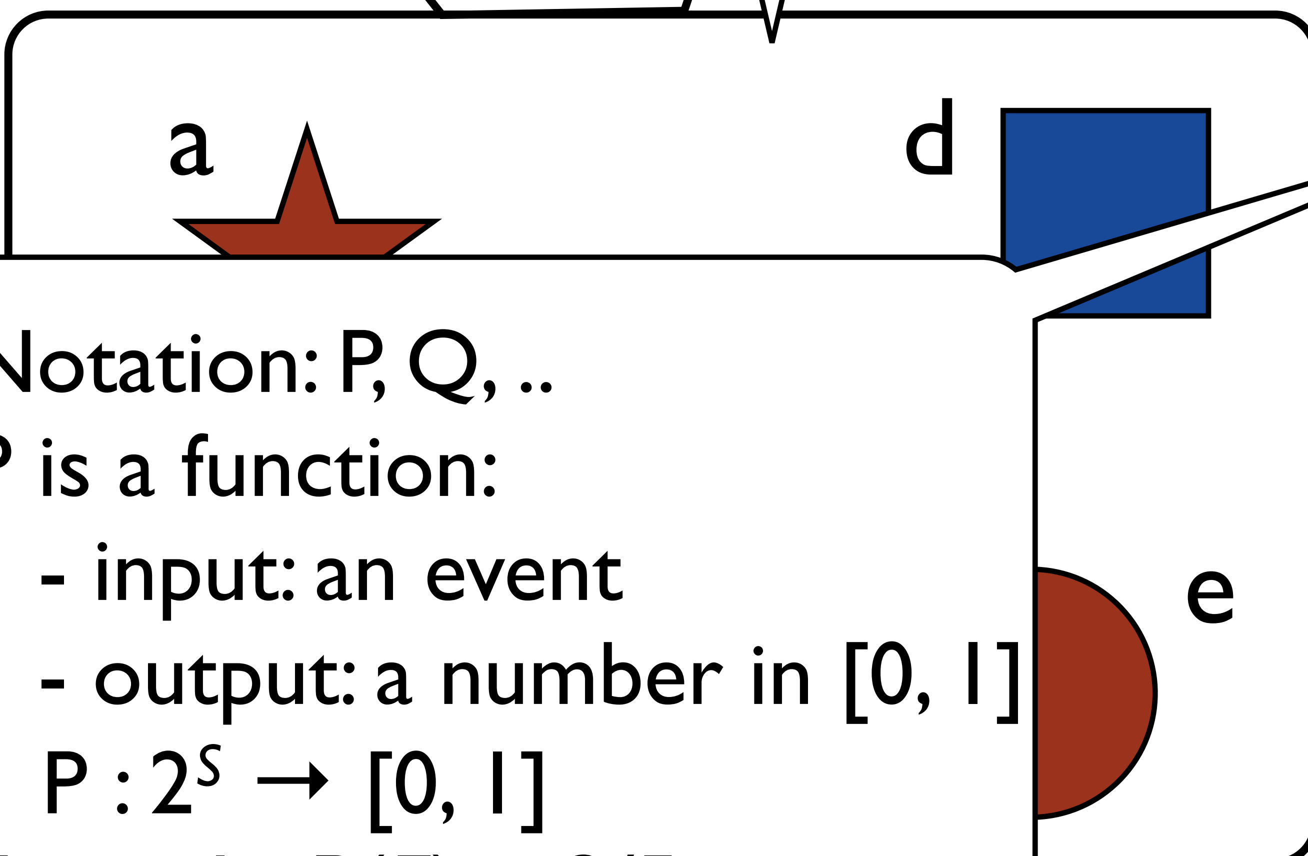
$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

Def 1

Notation

Sample space
Typical notation: S

This is a *set of outcomes*
Nickname: *event*
Typical notation: red, E, blue, F, ..
E = {a, c, e}



Notation: P, Q, ..
P is a function:
- input: an event
- output: a number in [0, 1]

$P : 2^S \rightarrow [0, 1]$
Example: P(E) = 3/5

- Proportion of red shapes?
- Probability of drawing a red shape?
- *Outcome:* an individual object in the bag

Probability <u>when outcomes are equally likely</u>:

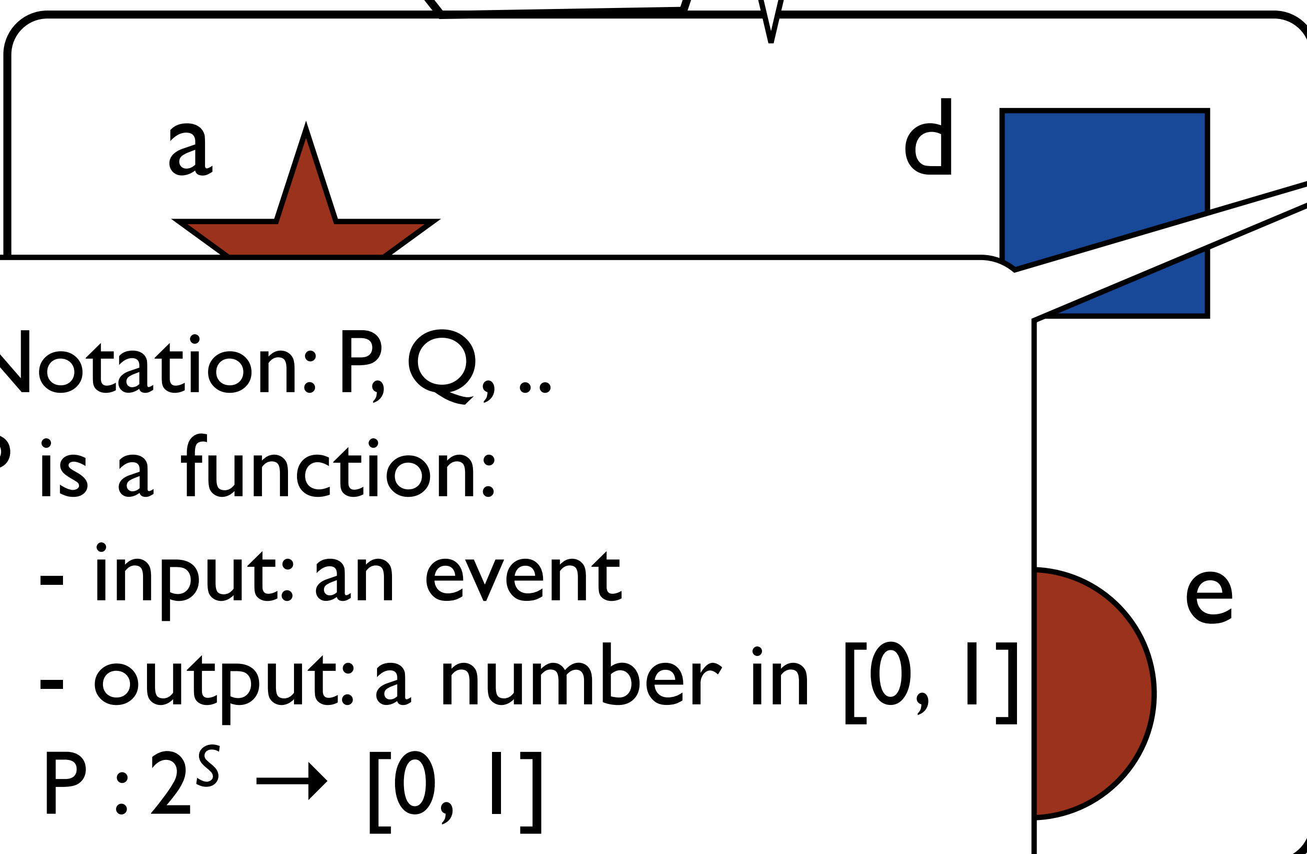
$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

Def 1

Notation

Sample space
Typical notation: *S*

This is a *set of outcomes*
Nickname: *event*
Typical notation: red, *E*, blue, *F*, ..
E = {a, c, e}



Notation: P, Q, ..
P is a function:
- input: an event
- output: a number in [0, 1]

$P : 2^S \rightarrow [0, 1]$

Example: P(*E*) = 3/5

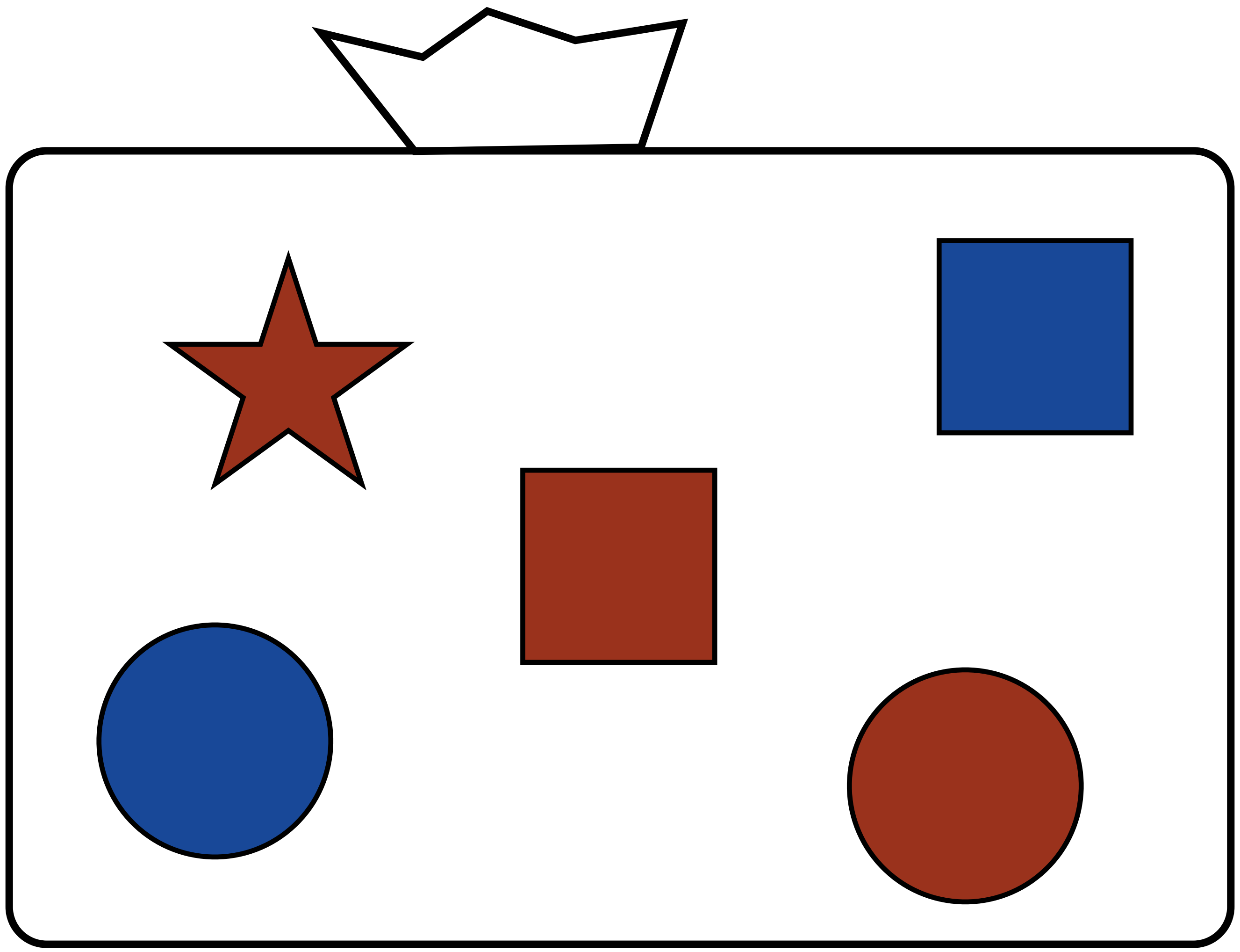
- Proportion of red shapes?
- Probability of drawing a red shape?
- *Outcome:* an individual object in the bag

$$P(E) = |E| \,/\, |S|$$

Probability <u>when outcomes are equally likely</u>:

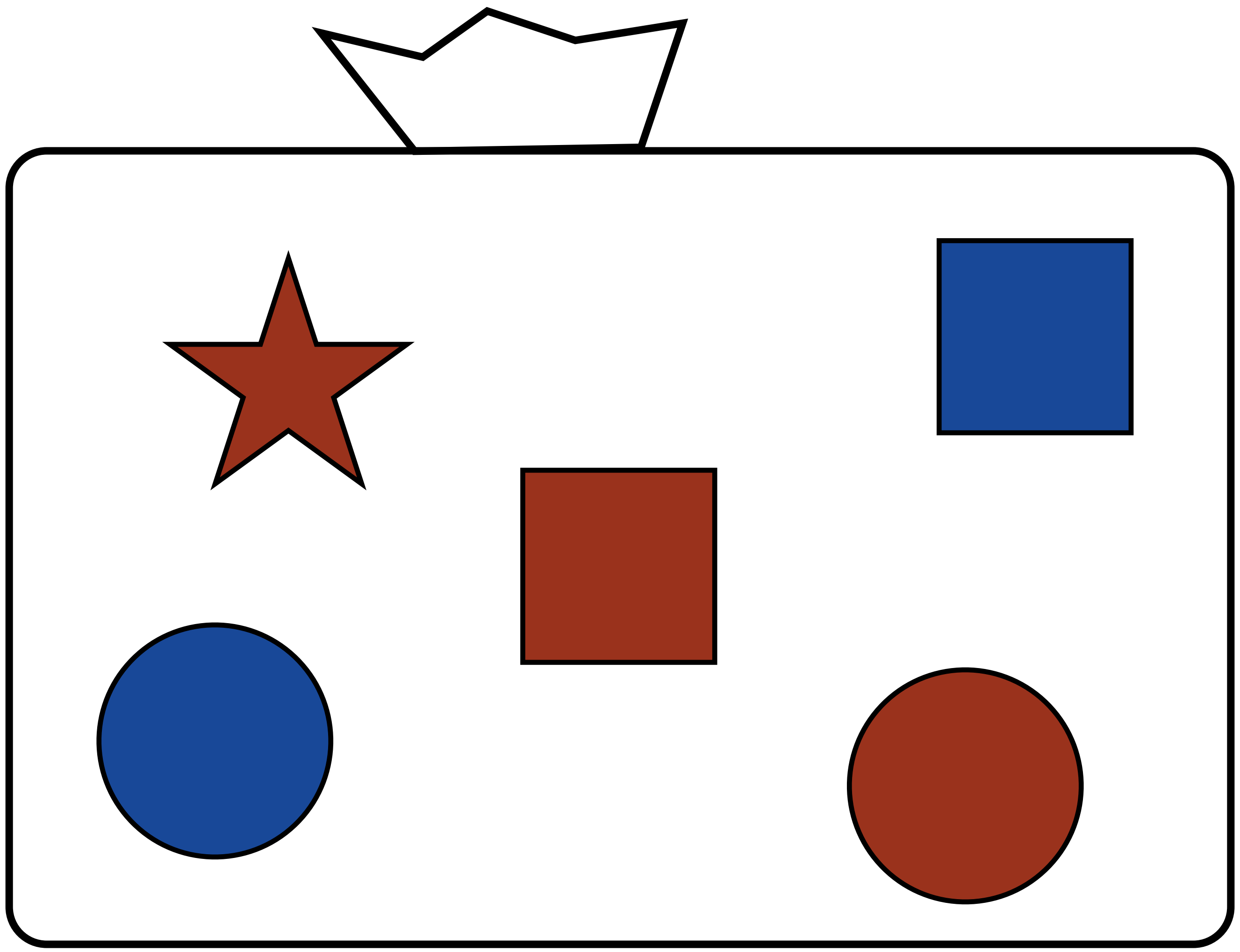
$$\frac{\text{\# of outcomes of interest}}{\text{\# of outcomes}}$$

Basic properties



- We know P(*E*)
  - Example: P(square) = 2/5
- What is P(E^c) ?
  - Example: P(not square)
- E^c means:
  - the *complement* of *E*
  - the outcomes not in *E*
  - = $S \setminus E$ (minus, for sets)

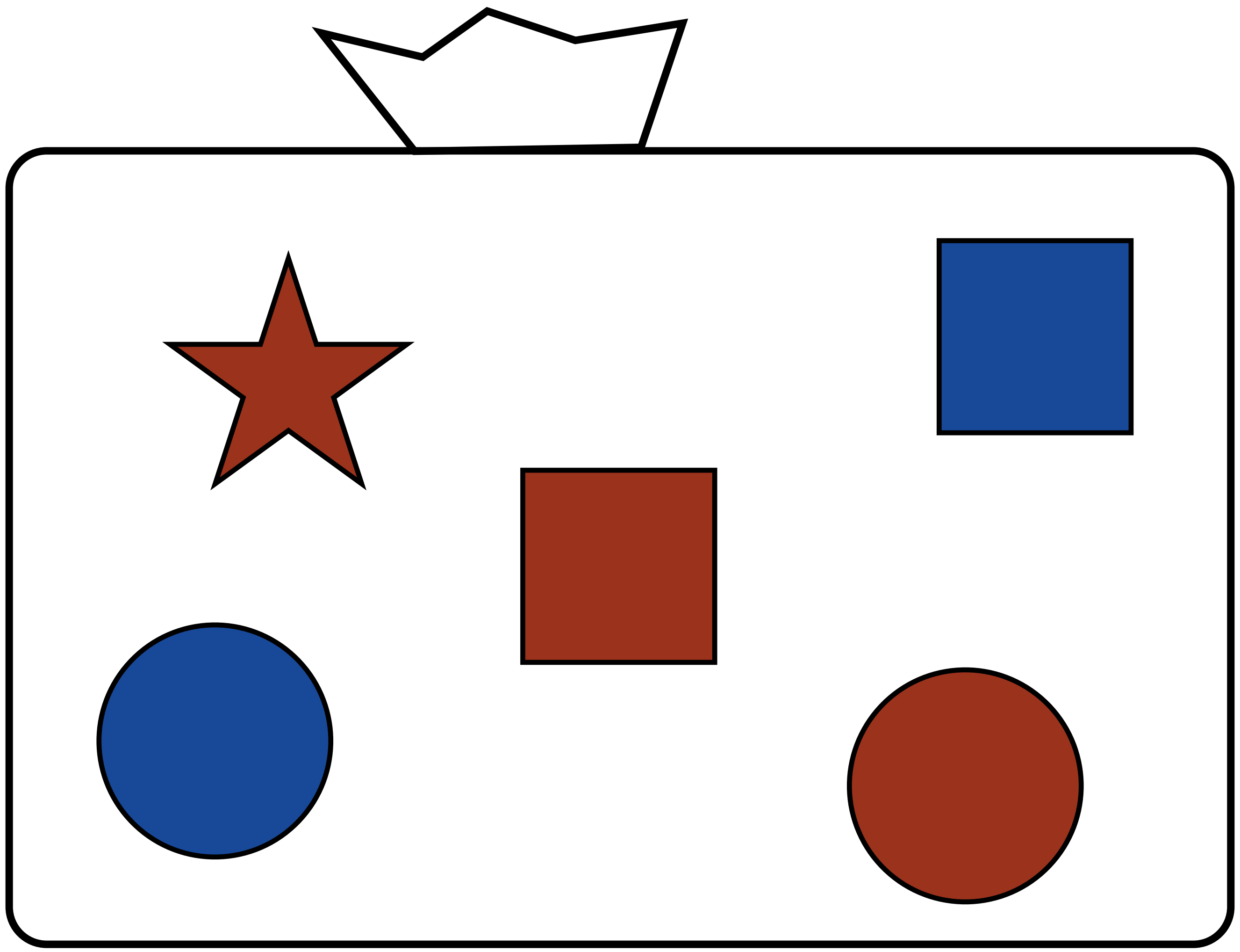
Basic properties



- We know $P(E)$
  - Example: P(square) = 2/5
- What is $P(E^c)$?
  - Example: P(not square)
- E^c means:
  - the *complement* of E
  - the outcomes not in E
  - $= S \setminus E$ (minus, for sets)

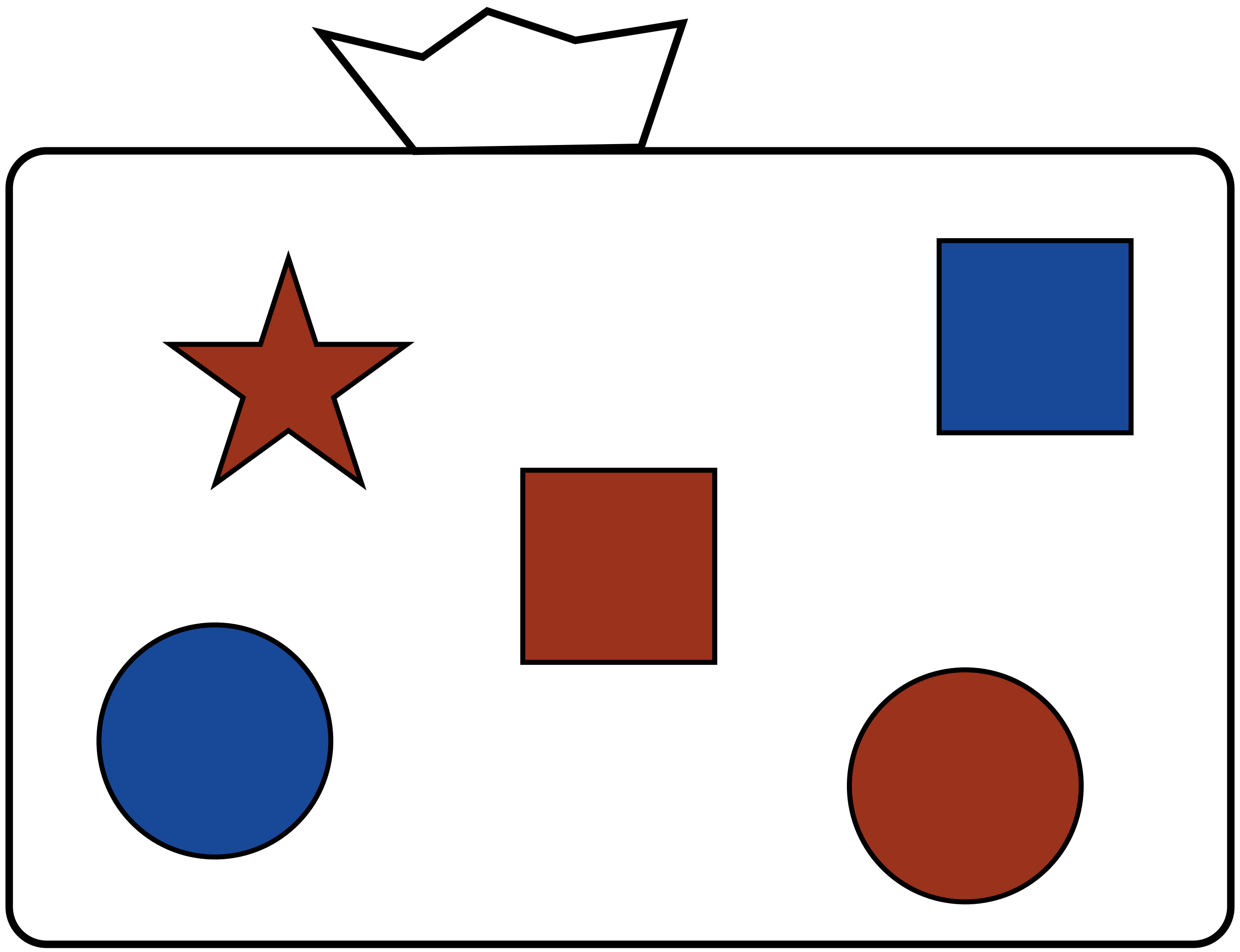
$$\boxed{P(E^c) = 1 - P(E)}$$

Basic properties



- We know P(*E*), P(*F*)
  - Example: P(blue) = 2/5
    P(star) = 1/5
- What is P(*E* ∪ *F*) ?
  - Example: P(blue or star)

Basic properties

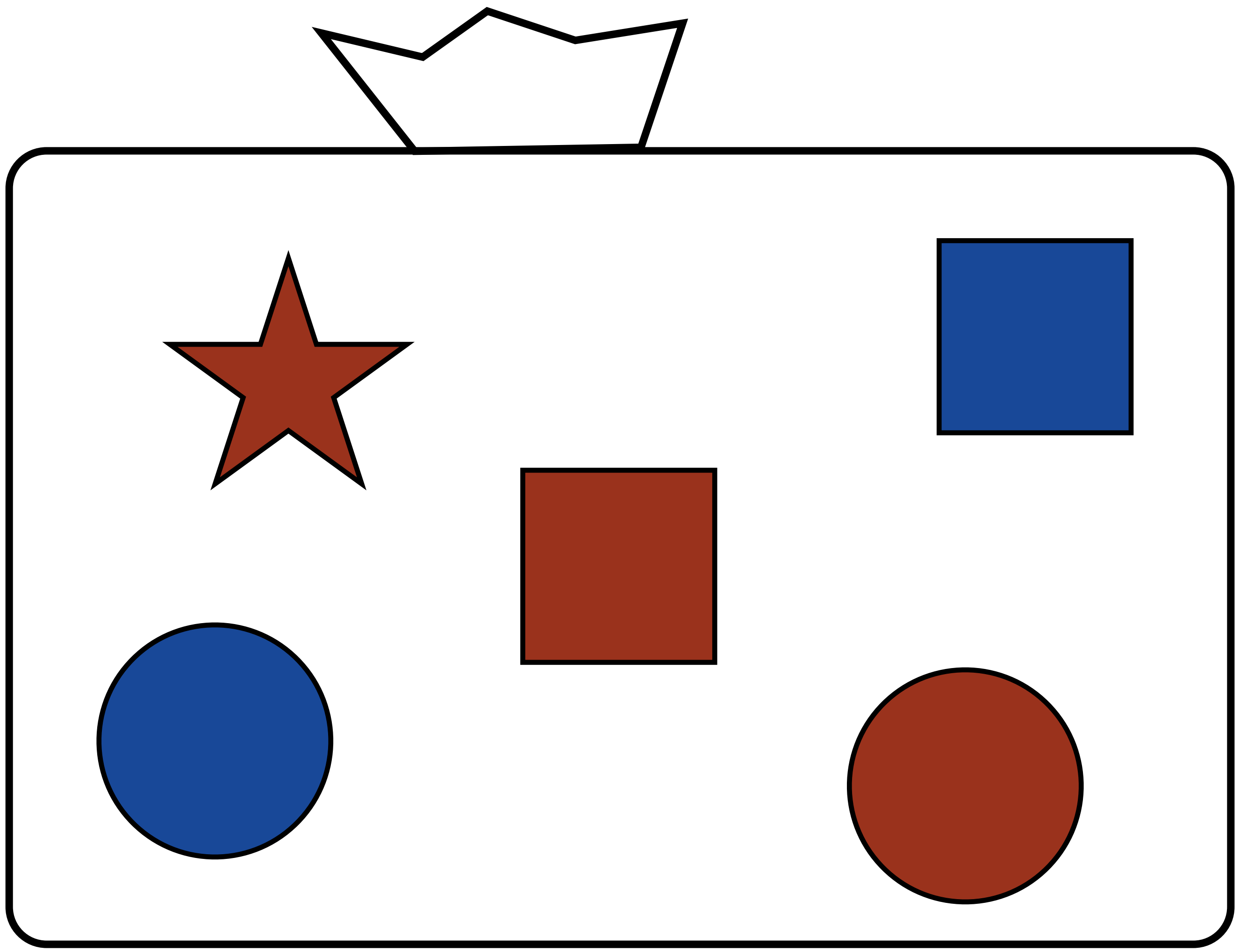


- We know P(E), P(F)
  - Example: P(blue) = 2/5
    P(star) = 1/5
- What is P($E \cup F$) ?
  - Example: P(blue or star)

$$P(E \cup F) = P(E) + P(F)$$

Prop. 2

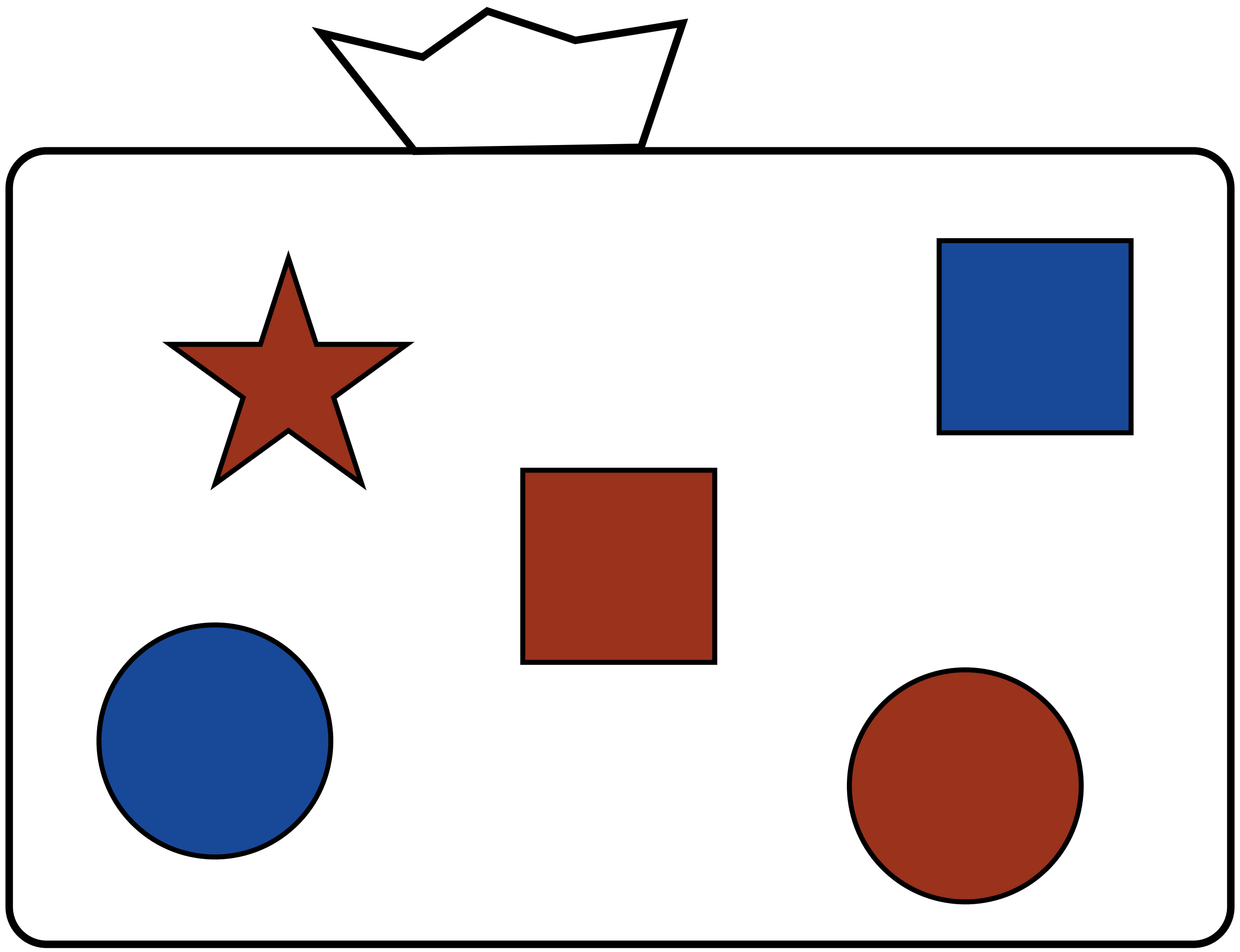
Basic properties



- We know P(*E*), P(*F*)
  - Example: P(blue) = 2/5
    P(star) = 1/5
- What is P(*E* ∪ *F*) ?
  - Example: P(blue or star)
- Try now on:
  - Example: P(red) = 3/5
    P(square) = 2/5

$$P(E \cup F) = P(E) + P(F)$$

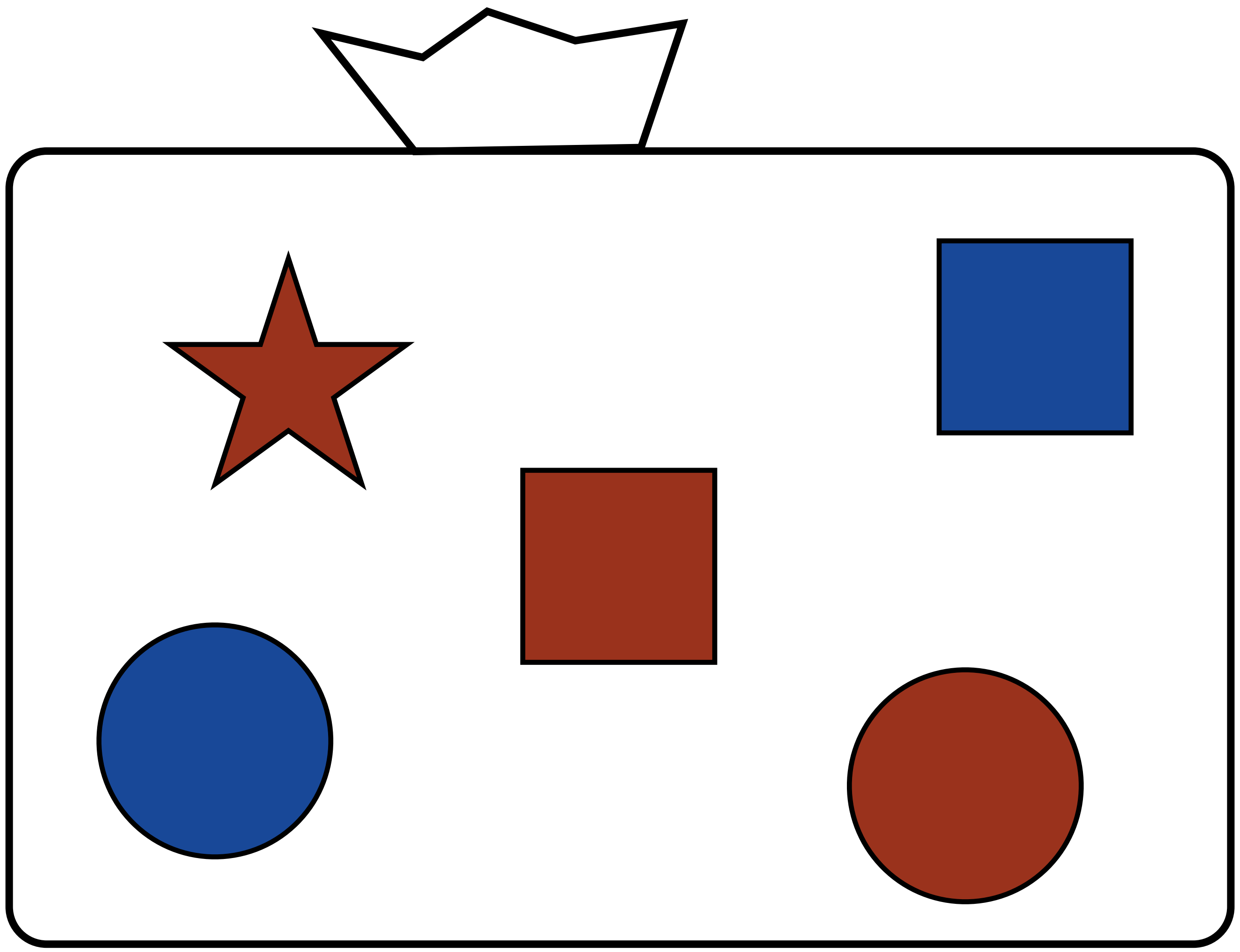
Basic properties



- We know P(E), P(F)
  - Example: P(blue) = 2/5
    P(star) = 1/5
- What is P($E \cup F$) ?
  - Example: P(blue or star)
- Try now on:
  - Example: P(red) = 3/5
    P(square) = 2/5

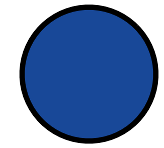
$$P(E \cup F) = P(E) + P(F)$$ if E and F are *disjoint*, i.e. $E \cap F = \varnothing$

Odds



- Equivalent, but less used terminology
- The odds of red shapes is #(red) / #(not red) = 3/2
- odds(E) = P(E) / P(E^c)
  = P(E) / (1 - P(E))

Summary so far

- Outcome: a scenario, s = 
- Sample space S: all the possible scenarios
- Event: a set of outcomes, e.g. $E = \{s \in S : s \text{ is red}\}$
- Probability: in the discrete, equally weighted case, $P(E) = |E| / |S|$
- Properties:
  - $P(E \cup F) = P(E) + P(F)$ when E & F are disjoint (non-overlapping)
  - $P(E^c) = 1 - P(E)$

Another example
where we build a *model*

Another example: two coins

- You flip 2 coins (pennies). What is the probability that the 2 coins both show heads?
  - A. 1/2
  - B. 1/3
  - C. 1/4
  - D. none of above

Probability that 2 coins show heads

- 1/2? Either they do, or they don't.
- 1/3? Either both heads, both tails, or head-tail.
- 1/4? Imagine one coin is painted red, one is painted blue. There are then 4 possibilities:
  - red: <u>head</u>, blue: <u>head</u>
  - red: <u>head</u>, blue: tail
  - red: tail, blue: <u>head</u>
  - red: tail, blue: tail

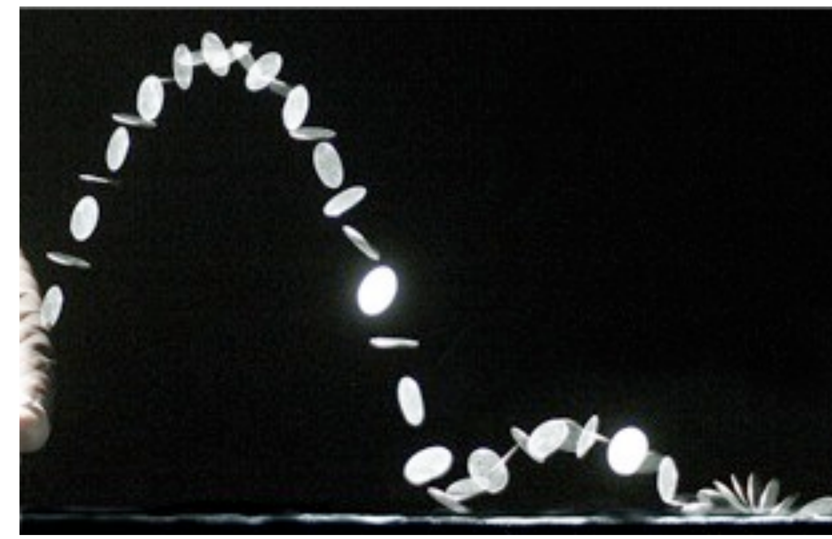
Probability that 2 coins show heads

- 1/2? Either they do, or they don't.
- 1/3? Either both heads, both tails, or head-tail.
- 1/4? Imagine one coin is painted red, one is painted blue. There are then 4 possibilities:
  - red: <u>head</u>, blue: <u>head</u>
  - red: <u>head</u>, blue: tail
  - red: tail, blue: <u>head</u>
  - red: tail, blue: tail

- These correspond to 3 different <u>models</u>
- None is 'true'
- But the third is more useful (accurate at doing predictions)

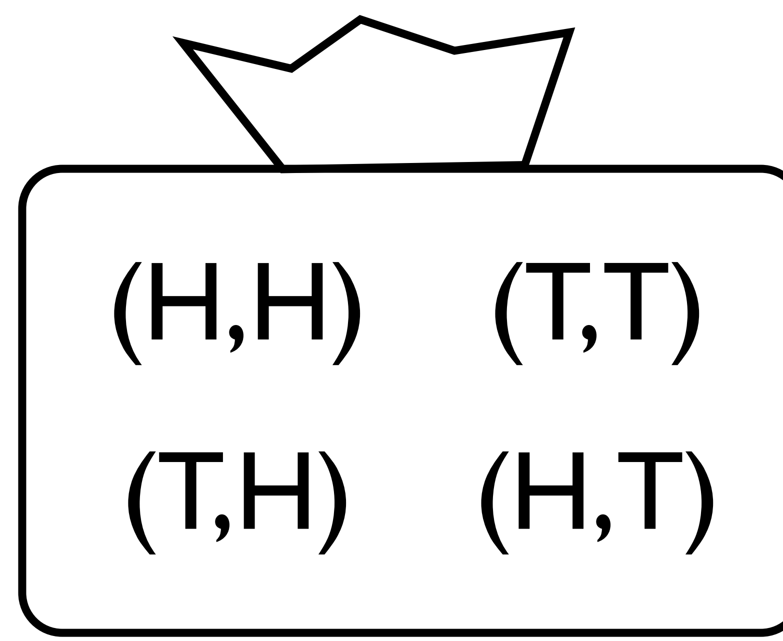
Why is this a model?

- Reality: a complex dynamical system



Stroboscopic image of a coin flip by Andrew Davidhazy

- Model: a ‘bag’ with 4 ‘objects’ in it



How to decide which model?

- Probability theory does not have the answer!

Reality $\longrightarrow$ Model (simplification) $\longrightarrow$ Prediction / answers

trial-and-error, scientific method (test predictions)

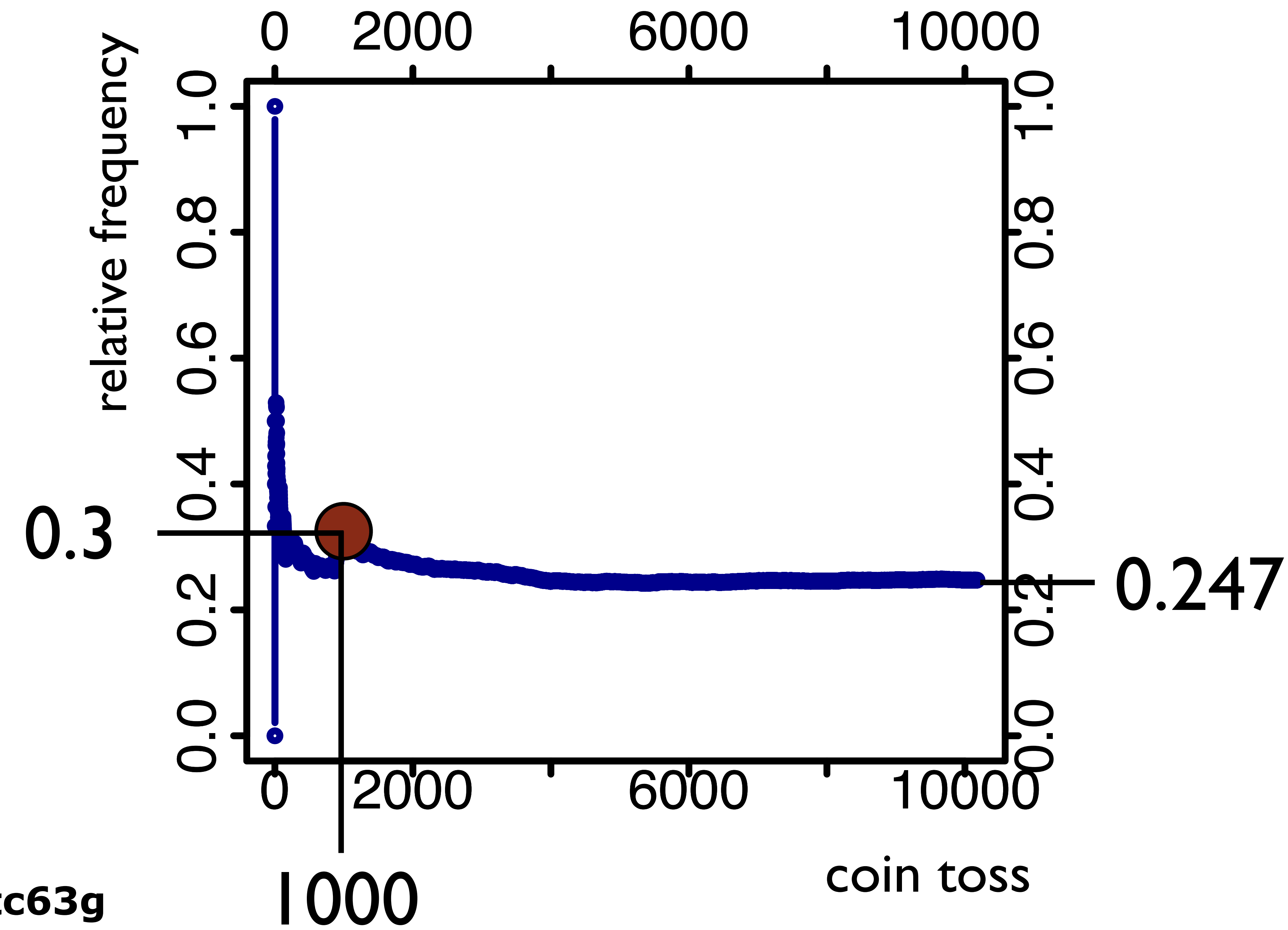
probability theory

But..

- Probability theory still useful:
  - Given a model, it makes certain predictions
  - We can then test those predictions
- Example: *law of large numbers*
  - Relates *probability* to *frequency* in repeated experiments

Reality check

- Dataset of 10,000 actual tosses of two coins available on the web (!)



http://tinyurl.com/ljtc63g

All models are wrong

- Given the large number of throws, 0.247 may seem a bit far from 0.25 (we will formalize this idea later)
  - is there something fishy?
  - after reading the fine prints of the data, realized all throws started with same face in hand of thrower
  - see P. Diaconis' paper, http://tinyurl.com/yked5fk
- Essentially, all models are wrong, but some are useful. -- G. Box

Going beyond the equally weighted case

General rules of probability

Assume:

a) $0 \leq P(E) \leq 1$

b) $P(S) = 1$

c) $P(E \cup F \cup \ldots) = P(E) + P(F) + \ldots$

if E, F, ... are all disjoint

Discrete, equally weighted

Definition (1):

$$P(E) = |E| / |S|$$

Properties (2):

a) $0 \leq P(E) \leq 1$

b) $P(S) = 1$

c) $P(E \cup F) = P(E) + P(F)$

if E and F are disjoint

Not equally weighted (or not discrete)

Example: redundancy problem (ex. 11)

Redundancy is not always stupid

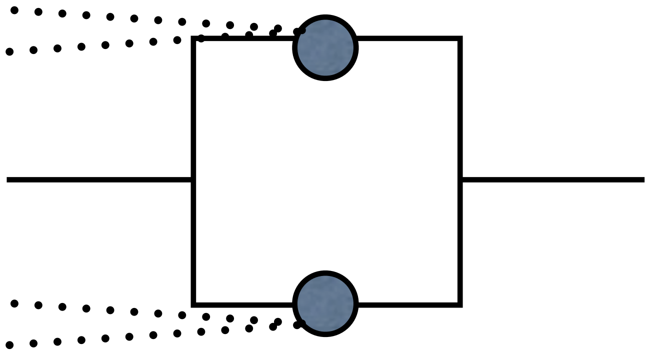
- Redundant systems:
  - create one or more duplicates of an important part of a machine
  - as long as one of the copies works, the machine works as a whole
  - the machine only fails when both copies break
- Many examples in biology (kidneys), engineering

Redundant server power supply

Ex. 11

Example of a typical reliability problem

- Known:
  - Power supply #1 works 60% of the time at delivery
  - Power supply #2 works 70% of the time at delivery
  - You also observed that both power supplies work at delivery 40% of the time
- Question: what is the probability that both power supply are broken at delivery?

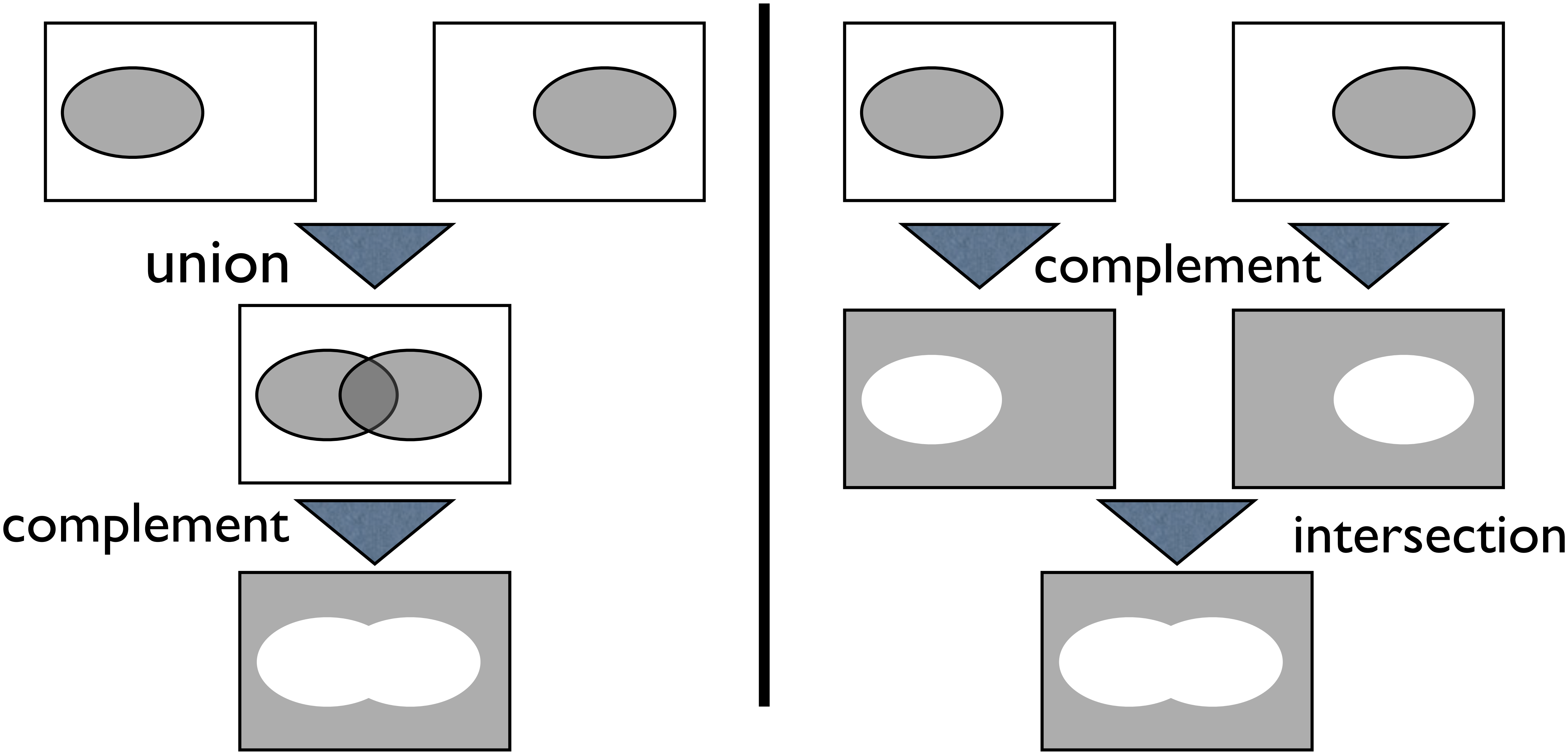


Strategy

1. Define a probability space S
2. Define the known information as events, ex.: W_1 = power supplies #1 works
3. Define the goal in terms of the probability of an event
4. Use properties 1 and 2 to find that probability (see next 2 slides as well to make your life easier)

Prop. 3

De Morgan's law: Distributing complements



$$\left(\cup_{i=1}^{n} E_i\right)^c \quad = \quad \cap_{i=1}^{n} E_i^c$$

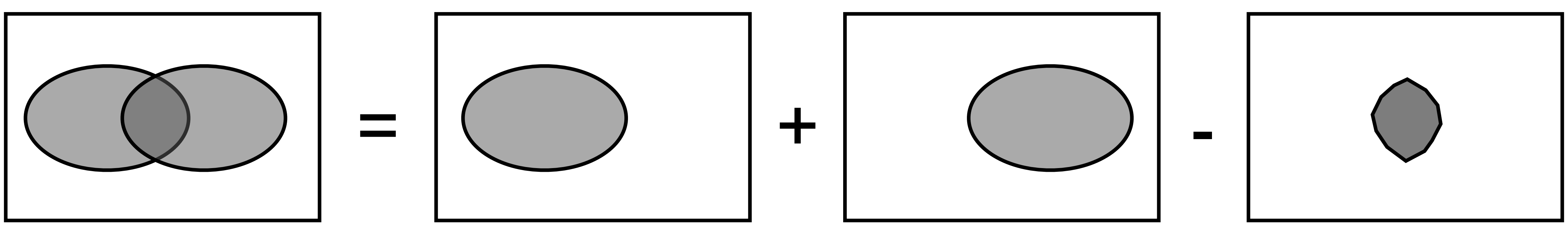
Prop. 4

Inclusion-exclusion: Going between unions and intersections

From earlier:

$$P(E \cup F) = P(E) + P(F)$$ if E and F are *disjoint*, i.e. $E \cap F = \varnothing$

What if: we want the prob. of non-disjoint unions?



$P(E \cup F)$ = $P(E)$ + $P(F)$ - $P(E \cap F)$