**STAT547C: Topics in Probability** **Fall 2014-15**

# Assignment 1

*Due date: see course website (Under Schedule).*

## Instructions

*Academic integrity policy:* I encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself. For example, copying (either manually or electronically) part of a function or a latex equation is not permitted; and if you use online resources, you have to cite them. Also, please refrain from looking at answer keys from other schools, previous years, or the Math Stack Exchange. For more information, see:
`http://learningcommons.ubc.ca/guide-to-academic-integrity/`

1. The assignment due date is available from the course webpage (*before the lecture begins, as I may sometimes go over solutions in class*), see the Schedule section.
2. You only have to do the non-optional questions for full marks. While the other questions are optional, we encourage you to at least attempt them if you are motivated to learn the subject in depth. Effort to do so is one of the ways to get participation points (the main other ways are the exercises (shorter problems graded in class), and participation during the lecture and the office hours).
3. Hand-in a pdf typeset using latex (instructions on how to hand-in will be posted on the course webpage). Limit the total length of your answers for the non-optional questions to a maximum of three pages. You can use approximately one more page for each optional question. Code is excluded from these limits, i.e. code should not be included in the latex file, and should be handed-in separately (how to do this is explained in the above-mentioned instructions for handing-in the assignment).
4. *Justify your answers formally.*

## 1 Generated $\sigma$-algebra

(a) Prove: If $\mathcal{F}_i$, $i \in \mathcal{I}$ are $\sigma$-algebra, then $\bigcap_{i\in\mathcal{I}} \mathcal{F}_i$ is a $\sigma$-algebra. $\mathcal{I}$ may or may not be a countable set of indices.

(b) Show that given a set $\Omega$ and a collection $\mathcal{A}$ of subsets of $\Omega$, there is a smallest $\sigma$-algebra containing $\mathcal{A}$. This is called the $\sigma$-algebra generated by $\mathcal{A}$ and denoted $\sigma(\mathcal{A})$.

## 2 Expectation formula for densities

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and $f : \Omega \to [0, \infty)$ be such that $\int f \, \mathrm{d}\mu = 1$.

(a) Show that $\nu(E) := \int \mathbf{1}_E(\omega) f(\omega) \mu(\,\mathrm{d}\omega)$, $E \in \mathcal{F}$ defines a probability distribution on $(\Omega, \mathcal{F})$.

(b) If $g : \Omega \to \mathbb{R}$ is such that $\int |g| \,\mathrm{d}\nu < \infty$ (or $g \geq 0$), show that

$$\int g(\omega)\nu(\,\mathrm{d}\omega) = \int g(\omega)f(\omega)\mu(\,\mathrm{d}\omega).$$

Hint: start with a simple $g$...

(c) A real random variable $X$ has a density $f$ with respect to $\mu$ if for all $I \in \mathcal{F}_{\text{borel}}$,

$$\mathbb{P}(X \in I) = \int_I f(x)\mu(\,\mathrm{d}x).$$

Specialize (b) to justify the following formula (giving a set of condition(s), if any, where your result holds):

$$\mathbb{E}[X] = \int xf(x)\mu(\,\mathrm{d}x).$$

# 3 Generation of non-uniform random variables

Show how to transform a uniform random variable on [0,1] into a random variable $X$ with the following density:[1]

$$f(t) = \frac{\mathbf{1}[t > 0]\alpha\beta^\alpha}{(t+\beta)^{\alpha+1}},$$

for $\alpha > 0, \beta > 0$.

To test your idea, set $\alpha = \beta = 2$ and see if the following two methods for computing the mean of $X$ agree:

(a) The LLN (Law of Large Numbers): informally, $\mathbb{E}[X] \approx S_n/n$, where $S_n = X_1 + \cdots + X_n$ and $X_i$ are iid.

(b) The formula from the previous question: $\mathbb{E}[X] = \int_{-\infty}^{+\infty} xf(x)\,\mathrm{d}x$.

# 4 Inequalities

Prove the following statements (hint: use Jensen):

(a) Let $X$ be a non–degenerate random variable. Suppose $p$ and $q$ are two constants satisfying $0 < q < p$. Show that if $\mathbb{E}|X|^p < \infty$ then $\mathbb{E}|X|^q < \infty$. (Note: this is a frequently used result in probability and statistics: existence of higher–order moments implies the existence of lower–order ones.)

(b) Let $X$ be a random variable with density $f : \mathbb{R} \to [0, \infty)$. Let $g : \mathbb{R} \to [0, \infty)$ be the density function of another probability distribution. Show that $\mathbb{E}[\log f(X)] \geq \mathbb{E}[\log g(X)]$.

[1] when the measure with respect to which a density is not specified, it assumed to be uniform by default.

## 5 Creating unbiased random variables (Optional)

1. Suppose you are given a sequence of approximations $Y_k$ with $\mathbb{E}Y_k \to a$, for some $a \in \mathbb{R}$, and where each $Y_k$ can be computed in finite time. Let $\Delta_0 = Y_0$, and $\Delta_k = Y_k - Y_{k-1}$ for $k \geq 1$ denote the successive differences, and assume

$$\mathbb{E}\left[\sum_{k=0}^{\infty} |\Delta_k|\right] < \infty.$$

   Find a random variable $Z$ such that:

   (a) $Z$ is unbiased, i.e. $\mathbb{E}Z = a$,

   (b) $Z$ can be computed in finite time (with probability one).

   Hint: introduce an independent, auxiliary geometric random variable $N$.

2. Can you replace the geometric distribution on $N$ by another one?

## 6 Properties of measure and good length (Optional)

Let $(X, \mathcal{F})$ be a measurable space. A *finitely additive* measure $\mu$ on $\mathcal{F}$ is a map $\mu : \mathcal{F} \to [0, \infty]$ such that,

i $\mu(\emptyset) = 0$

ii (Finite additivity) if $A_1, A_2, \ldots, A_m$ are disjoint sets, then $\mu\left(\coprod_{j=1}^{m} A_j\right) = \sum_{j=1}^{m} \mu(A_j)$ ($\coprod$ denotes union of disjoint sets).

(a) (Monotonicity) Show that a finitely additive measure is monotone, that is, $A \subset B \in \mathcal{F}$, $\mu(A) \leq \mu(B)$.

(b) Prove: $\mu$ is a countably additive measure $\Leftrightarrow$ for any sequence, $\{A_n\}_{n=1}^{\infty} \in \mathcal{F}$, if $A_n \uparrow A \in \mathcal{F}$, then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ (continuity).

Recall that a *countably additive* measure has the following property: if $\{A_n\}_{n=1}^{\infty}$ is a sequence of disoint sets in $\mathcal{F}$, then $\mu(\coprod_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

(c) A map $\mu : 2^{\mathbb{R}} \to [0, \infty]$ with the following properties is a *good length*,

i Finitely additive

ii Upper continuous ($A_n \uparrow A$)

iii Translation invariant: $\forall x \in \mathbb{R}$ and $E \subset \mathbb{R}$, $\mu(x + A) = \mu(A)$.

iv Normalization: $\mu([0, 1)) = 1$.

We will show that good length cannot exist for all subsets of $\mathbb{R}$. First, we define an *equivalence relation* on $\mathbb{R}$ as follows: $y \sim x \Leftrightarrow y - x \in \mathbb{Q}$. We denote by $E(x) = \{y : y - x \in \mathbb{Q}\} = x + \mathbb{Q}$, the equivalence class containing $x$. For all equivalence class $E$, choose $y_E \in [0, 1) \bigcap E$ and let $N = \{y_E : E \text{ is an equivalence class}\}$.

(I) Show that $\mathbb{R} = \coprod_{r\in\mathbb{Q}}(r + N)$.
Hint 1: To show the above, one must show that if $x \in \mathbb{R}$, then $x \in (r + N)$ for some $r \in \mathbb{Q}$ and show that $(r + N)$ is disjoint for $r \in \mathbb{Q}$.
Hint 2: Use without proof that $E \mapsto y_E$ is a one-to-one map.

(II) Show that $\mu(r+N)=0$ and hence by translation invariance, $\mu(N)=0$.

(III) Now, conclude that such $\mu$ cannot exist by showing,

$$1=\mu([0,1))\leq\mu(\mathbb{R})=\mu\left(\coprod_{r\in\mathbb{Q}}(r+N)\right)=\sum_{r\in\mathbb{Q}}\mu(r+N)=0$$

# 7 Measurability (Optional)

Show that the class of real-valued random variables (i.e. $\mathcal{F}$-measurable functions) is the smallest class containing the simple functions and closed under pointwise limits. Use this result to conclude that $Y\in\sigma(X)$ if and only if $Y=f(X)$ for some $f$.