Deep hedging methods for pricing and hedging financial derivatives

Alexandre Carbonneau

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Topic

- Topic: pricing and hedging derivatives with deep reinforcement learning.

$\Longrightarrow$ Main topic of my phd thesis.

Deep hedging

Quantitative Finance, 2019
https://doi.org/10.1080/14697688.2019.1571683

Deep hedging

H. BUEHLER†§, L. GONON‡*, J. TEICHMANN‡ and B. WOOD†§

†J.P. Morgan, London, UK
‡Eidgenössische Technische Hochschule Zürich, Zürich, Switzerland

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- I studied this class of algorithms for both pricing and optimal hedging financial derivatives throughout my thesis.

Outline

1) Hedging/notation:
   - Hedging in incomplete markets
   - Global hedging
   - General idea of deep hedging

2) Market setup

3) Two applications:
   - A) Global hedging financial derivatives
   - B) Pricing derivatives with deep hedging

Hedging in incomplete markets

In the complete market setting (e.g Black-Scholes market)

- Every derivative can be **perfectly hedged**, and are thus redundant assets.
- $\Longrightarrow$ Always exists a self-financing trading strategy which perfectly replicate the option.

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- Most options are **not attainable** (multiple risk factors, discrete-time trading, market frictions, etc.).
- **Residual hedging risk** cannot be completely hedged away at a reasonable cost.

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- Most options are **not attainable** (multiple risk factors, discrete-time trading, market frictions, etc.).
- **Residual hedging risk** cannot be completely hedged away at a reasonable cost.

$\Rightarrow$ Identification of optimal hedging policies is highly relevant!

Greek-based policy

One very popular approach for hedging in incomplete markets is the **greek-based policy**.

- Assets positions depend **on the sensitivities** of the option value to different risk factors (e.g. delta hedge, delta-rho hedge, delta-gamma hedge etc.)

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Issue:

- Suboptimal by design:
  - by-product of choice of pricing kernel (i.e. of the risk-neutral measure).
  - **not the result of an optimization procedure** over hedging decisions to minimize residual risk.

Variance-optimal hedging

For $\delta := \{\delta_n\}_{n=0}^{N}$ assets positions at each time-step, S_N the underlying stock price, European derivative of payoff $\Phi(S_N)$ and portfolio value V_N^{δ}:

$$\delta^\star = \arg\min_{\delta} \mathbb{E}\left[(\Phi(S_N) - {V_N}^{\delta})^2\right]. \tag{1}$$

- Seminal work of Schweizer (1995) introduced variance-optimal hedging in discrete-time.

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Can also generalize to optimize jointly for (V_0, δ):

$$(\delta^\star, V_0^\star) = \arg\min_{\delta, V_0} \mathbb{E}\left[(\Phi(S_N) - V_N^\delta)^2 \right],$$

and $V_0^\star$ can be viewed as the production cost of Φ (i.e. a price).

Benchmark 1: Black-Scholes discrete-time

Table: Hedge short position ATM put option of $T = 60/260$ with daily trades.

Statistics	Delta hedging	Global MSE
Mean	0.00	0.00
RMSE	0.404	-0.3%
Semi-RMSE	0.292	-1.7%
$\text{VaR}_{0.95}$	0.665	-0.3%
$\text{VaR}_{0.99}$	1.104	-0.7%
$\text{CVaR}_{0.95}$	0.937	-0.7%
$\text{CVaR}_{0.99}$	1.381	-1.7%

Notes: Hedging statistics under BSM with $\mu = 0.10, \sigma = 0.1898, r = 0.02$ and $S_0 = 100$. Global MSE presented as **% increase** of delta hedging.

Benchmark 2: Merton jump diffusion discrete-time

Table: Hedge short position ATM put option of $T = 60/260$ with daily trades.

Statistics	Delta hedging	Global MSE
Mean	0.05	0.01
RMSE	0.68	-4%
Semi-RMSE	0.55	-10%
$\text{VaR}_{0.95}$	1.19	-6%
$\text{VaR}_{0.99}$	2.23	-13%
$\text{CVaR}_{0.95}$	1.83	-11%
$\text{CVaR}_{0.99}$	2.83	-13%

Notes: Hedging statistics under MJD with $[\mu, \sigma, \lambda, \mu_J, \sigma_J] = [0.0875, 0.1036, 92.3862, -0.0015, 0.0160]$, $r = 0.02$ and $S_0 = 100$. Global MSE presented as **% increase** of delta hedging.

Global risk minimization

Most general case we will consider:

$$\delta^\star = \arg\min_{\delta} \rho\left(\Phi(S_N) - V_N^\delta\right), \tag{2}$$

where ρ is a risk measure.

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1) ρ is an **expected penalty**: $\rho(X) = \mathbb{E}\left[\mathcal{L}(X)\right]$ where $\mathcal{L}$ is the loss function.

- Mean-square error (MSE) with $\mathcal{L}(x) = x^2$.
- Semi-L^p with $\mathcal{L}(x) = x^p \mathbb{1}_{\{x>0\}}$ with $p > 0$.

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2) ρ is a convex risk measure.

- Conditional Value-at-Risk (CVaR) at level α with

$$\rho(X) = \mathbb{E}[X | X \geq \mathrm{VaR}_\alpha(X)].$$

Benchmark 2: Merton jump diffusion discrete-time

Table: Hedge short position ATM put option of $T = 60/260$ with daily trades.

Statistics	Delta hedging	Global MSE	Global Semi-L^2
Mean	0.05	0.01	-0.01
RMSE	0.68	**-4%**	4%
Semi-RMSE	0.55	-10%	**-13%**
$\text{VaR}_{0.95}$	1.19	-6%	**-11%**
$\text{VaR}_{0.99}$	2.23	-13%	**-17%**
$\text{CVaR}_{0.95}$	1.83	-11%	**-15%**
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Notes: Hedging statistics under MJD with $[\mu, \sigma, \lambda, \mu_J, \sigma_J] = [0.0875, 0.1036, 92.3862, -0.0015, 0.0160]$, $r = 0.02$ and $S_0 = 100$.

Deep hedging

Buehler et al. (2019a) introduced a novel algorithm called **deep hedging** to optimize global hedging policies represented by neural networks.

- Idea: through many simulations of a synthetic market, optimize neural networks to approximate optimal trading strategies (complex trial-and-error approach).

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- Alleviate the curse of dimensionality:
  - Computational cost increases marginally with the dimension of state and action spaces (**feasible procedure for large-scale problems**).

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Key advantages over typical numerical schemes:

- Alleviate the curse of dimensionality:
  - Computational cost increases marginally with the dimension of state and action spaces (**feasible procedure for large-scale problems**).
- Theoretically founded:
  - Buehler et al. (2019a) show there exist a neural network which can **approximate arbitrarily well optimal hedging decisions** under very general conditions.

Neural network representing hedging policy

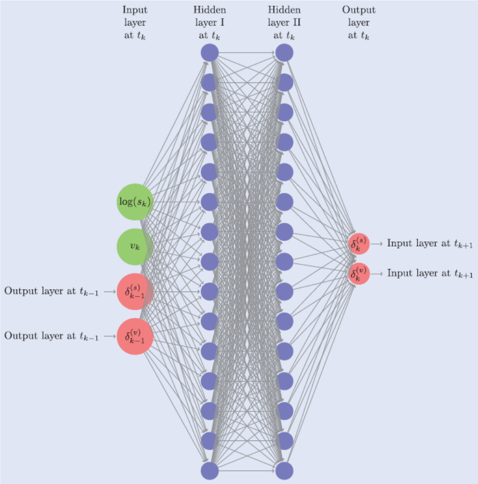

Figure: Figure 1 of Buehler et al. (2019a)

Literature review deep hedging

- Buehler et al. (2019a): introduce the deep hedging approach.
- Buehler et al. (2019b): hedge path-dependent contingent claims in the presence of transaction costs.
- Cao et al. (2020): deep hedging provides good approximation of optimal initial capital investments for variance-optimal hedging.
- Horvath et al. (2021): deep hedge in a non-Markovian framework with rough volatility models.
- Imaki et al. (2021): introduce novel neural network architecture for the 'no-transaction band strategy' in the presence of transaction costs.

Literature review deep hedging

- Gong et al. (2021): reduce trading frequency with a so-called 'price change threshold'.
- Lutkebohmert et al. (2021): deep hedging in the presence of parameter uncertainty for a class of Markov processes.
- Carbonneau and Godin (2021a, 2021b, 2021c): study an option pricing scheme consistent with global hedging strategies obtained with deep hedging.
- Carbonneau (2021): deep hedge very long-term European financial derivatives analoguous to variable annuity guarantees.

Outline

1) Deep hedging class of algorithms:
   - Hedging in incomplete markets
   - General idea of deep hedging
   - Literature review

2) **Market setup**

3) Two applications:
   - A) Pricing derivatives with deep hedging
   - B) Global hedging financial derivatives

Market setup

Discrete-time market: $\mathcal{T} = \{0 = t_0 < t_1 < \ldots < t_N = T\}$.

Traded assets: time-t_n price:

- Risk-free asset: $B_n = \exp(rt_n)$,
- D risky assets: $S_n = [S_n^{(1)}, \ldots, S_n^{(D)}]$.

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We want to **hedge/price** a European derivative.

- Payoff function is $\Phi(S_N^{(1)})$ or $\Phi(S_N^{(1)}, Z_N)$ if path-dependent.

Trading strategy

Let $\delta := \{\delta_n\}_{n=0}^{N}$ with $\delta_n := [\delta_n^{(B)}, \delta_n^{(1)}, \ldots, \delta_n^{(D)}]^\top$ be a **trading strategy**.

- δ_n is the number of shares of each asset in the portfolio during $(t_{n-1}, t_n]$.

Trading strategies we consider are **self-financing**.

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Trading strategies we consider are **self-financing**.

- No cash flow injection/withdrawal at intermediate times.

$$\delta_{n+1}^{(1:D)} \cdot S_n + \delta_{n+1}^{(0)} B_n = \delta_n^{(1:D)} \cdot S_n + \delta_n^{(0)} B_n, \quad n = 0, 1, \ldots, N-1.$$

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- Time-t_n portfolio value is denoted as V_n^δ:

$$V_n^\delta := B_n(V_0 + G_n^\delta)$$

where V_0 is initial capital investment and G_n^δ is the discounted cumulative gains.

Hedging optimization problem

Optimization problem: minimize hedging shortfall for a short position in Φ:

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- Closed-form solution unknown for (3).
- Need a numerical scheme.

Policy approximation through a neural network

For several popular (Markov) dynamics of the traded assets, the optimal policy has the form

$$\delta^{\star}_{n+1} = f(n, V_n, S_n, Z_n, \mathcal{I}_n)$$

for some function f and $\mathcal{I}_n$ a vector of variables containing relevant information.

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- market regimes (regime-switching): $\mathcal{I}_n = \eta_n$ where

$$\eta_n := [\mathbb{P}(h_n = 1|\mathcal{F}_n), \ldots, \mathbb{P}(h_n = K|\mathcal{F}_n)]$$

for h_n the regime during $[t_n, t_{n+1})$ with K regimes.

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$\Longrightarrow$ And they add up: regime-swithing + lookback option + transaction costs + liquidity impact:

$$X_n = [n, V_n, S_n, Z_n, \eta_n, \delta_n, A_n, B_n].$$

Policy approximation through a neural network

We approximate the optimal policy function f with a neural network F_θ with parameters θ.

- $f(n, V_n, S_n, Z_n, \mathcal{I}_n) \approx F_\theta(n, V_n, S_n, Z_n, \mathcal{I}_n)$.

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The optimization problem **boils down to optimizing** θ:

$$\min_{\delta} \rho \left(\Phi(S_N^{(1)}, Z_N) - V_N^{\delta} \right) \tag{4}$$

$$\approx \min_{\theta} \rho \left(\Phi(S_N^{(1)}, Z_N) - V_N^{\delta^\theta} \right), \tag{5}$$

where δ^θ is to be understood as the output of F_θ.

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where δ^θ is to be understood as the output of F_θ.

Last step: SGD procedure for (5) with $\hat{\rho}$ as the empirical estimator of ρ estimated with Monte Carlo sampling.

Pseudo-code

Algorithm 1 Pseudo-code deep hedging

Input: θ_j

Output: θ_{j+1}

for $i = 1, \ldots, N_{\text{batch}}$ **do** ▷ Loop over each path of minibatch
  $X_{0,i} = [t_0, S_{0,i}, V_{0,i}, \mathcal{I}_{0,i}]$ ▷ Time-0 feature vector
  for $n = 0, \ldots, N-1$ **do**
    $\delta_{n+1,i} \leftarrow$ time-t_n output of F_θ with $\theta = \theta_j$
    $\{S_{n+1,i}, V_{n+1,i}, \mathcal{I}_{n+1,i}\} \leftarrow \{S_{n,i}, V_{n,i}, \mathcal{I}_{n,i}\}$. ▷ Update feature variables
    $X_{n+1,i} = [t_{n+1}, S_{n+1,i}, V_{n+1,i}, \mathcal{I}_{n+1,i}]$ ▷ Time-t_{n+1} feature vector
  end for
  $\pi_{i,j} = \Phi(S_{N,i} - V_{N,i})$ ▷ i^{th} hedging error if $\theta = \theta_j$
end for
$\eta_j \leftarrow$ Adam algorithm
$\theta_{j+1} = \theta_j - \eta_j \nabla_\theta \widehat{\rho}$ ▷ $\widehat{\rho}$ is estimator of $\rho(\pi)$ computed with $\{\pi_{i,j}\}_{i=1}^{N_{\text{batch}}}$.

Note: minibatches computation is done in parallel.

First application: Hedging long-term European derivatives

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journal homepage: www.elsevier.com/locate/ime

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Deep hedging of long-term financial derivatives☆

Alexandre Carbonneau

Concordia University, Department of Mathematics and Statistics, Montréal, Canada

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- Topic: Optimal hedging of **long-term** European financial derivatives.

Motivation

This paper studies the problem of global hedging **very long-term European derivatives** (many years) with dynamic hedging.

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This paper studies the problem of global hedging **very long-term European derivatives** (many years) with dynamic hedging.

Such long maturity derivatives are analogous, under some assumptions, to financial guarantees sold with **equity-linked insurance products**.

- Enable investors to gain exposure to the market through **cash flows that depend on equity performance**.
- Sold with **various guarantees** to protect against equity market risk (e.g. minimum guaranteed return).
- Retirement vehicule by having **stock market participation** and **protections** for the policyholder.

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- Retirement vehicule by having **stock market participation** and **protections** for the policyholder.

This study examines exclusively the mitigation of **financial risk exposure**.

Contributions

1) **Broad numerical experiments** of global hedging long-term lookback options.

- Multiple hedging instruments + presence of jump risk + different objective functions.
- To the best of my knowledge, at the time, was the first paper to consider global hedging methods for very long-term European derivatives.

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1) **Broad numerical experiments** of global hedging long-term lookback options.

- Multiple hedging instruments + presence of jump risk + different objective functions.
- To the best of my knowledge, at the time, was the first paper to consider global hedging methods for very long-term European derivatives.

2) Provide **novel qualitative insights** into long-term global hedging policies.

Market setup

- **Lookback option** to hedge with time-to-maturity of 10 years (i.e. $T = 10$).
  - $\Phi(S_N^{(1)}, Z_N) = \max(Z_N - S_N^{(1)}, 0)$ where Z_N is the maximum yearly value of the underlying stock for years $0, 1, \ldots, 9$.

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- $r = 0.03$ and $S_0^{(1)} = 100$.
- **Merton Jump-Diffusion** for underlying (jump risk).
- **Hedging statistics** used: mean, RMSE, semi-RMSE, VaR_α and CVaR_α.

Market setup

Four categories of hedging instruments considered:

(1-2) **Underlying stock** on a monthly and yearly basis.

(3) **Two options**: at the beginning of each year, hedging instruments are ATM calls and puts of 1 year maturity.

(4) **Six options**: at the beginning of each year t_n, three calls of moneynesses $K \in \{S_n^{(1)}, 1.1S_n^{(1)}, 1.2S_n^{(1)}\}$ and three puts of moneynesses $K \in \{S_n^{(1)}, 0.9S_n^{(1)}, 0.8S_n^{(1)}\}$.

Note: Methodological approach is in no way constraint to this choice of hedging instruments.

Numerical experiments

Numerical experiments conducted in the paper demonstrate (not shown in this presentation):

- Deep hedging with the mean-square-error penalty, $\mathcal{L}^{\text{MSE}}(x) = x^2$, **is superior** to **local risk minimization** and **greek-based hedging**.

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Numerical experiments conducted in the paper demonstrate (not shown in this presentation):

- Deep hedging with the mean-square-error penalty, $\mathcal{L}^{\text{MSE}}(x) = x^2$, **is superior** to **local risk minimization** and **greek-based hedging**.

I will now present benchmarking results of global hedging obtained with

- $\mathcal{L}^{\text{MSE}}(x) = x^2$ and
- semi-mean-squared-error penalty (SMSE): $\mathcal{L}^{\text{SMSE}}(x) = x^2 \mathbb{1}_{\{x>0\}}$.

Global quadratic vs semi-quadratic - MJD

Table 6: Benchmarking of quadratic deep hedging (QDH) and semi-quadratic deep hedging (SQDH) to hedge the lookback option of $T = 10$ years under the MJD model.

Statistics	Mean	RMSE	semi-RMSE	$\text{VaR}_{0.95}$	$\text{VaR}_{0.99}$	$\text{CVaR}_{0.95}$	$\text{CVaR}_{0.99}$	Skew
L^{MSE}								
Stock (year)	−1.6	19.8	15.6	32.3	66.4	54.5	95.4	2.1
Stock (month)	0.2	11.2	9.4	15.7	42.8	32.6	64.6	3.2
Two options	0.0	5.2	3.8	6.7	15.4	12.7	25.1	1.6
Six options	−0.1	1.3	0.9	1.4	3.6	2.9	6.2	2.3
L^{SMSE}								
Stock (year)	−35.2	49.7	6.7	11.4	31.7	24.6	47.7	−0.8
Stock (month)	−22.8	33.8	4.2	6.5	18.3	14.3	29.6	−1.1
Two options	−5.9	11.2	1.7	2.2	7.1	5.5	12.2	−2.5
Six options	−1.3	3.1	0.5	0.3	1.4	1.1	2.9	−4.8

Global quadratic vs semi-quadratic - MJD

Table 6: Benchmarking of quadratic deep hedging (QDH) and semi-quadratic deep hedging (SQDH) to hedge the lookback option of $T = 10$ years under the MJD model.

Statistics	Mean	RMSE	semi-RMSE	$\text{VaR}_{0.95}$	$\text{VaR}_{0.99}$	$\text{CVaR}_{0.95}$	$\text{CVaR}_{0.99}$	Skew
L^{MSE}								
Stock (year)	-1.6	19.8	15.6	32.3	66.4	54.5	95.4	2.1
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- Downside risk reduction improvement with $\mathcal{L}^{\text{SMSE}}$ over $\mathcal{L}^{\text{MSE}}$ ranges between 45% to 76%.
- Hedging gains across all hedging instruments with $\mathcal{L}^{\text{SMSE}}$.

Benchmarking takeaway

With semi-quadratic global hedging $\mathcal{L}^{\mathsf{SMSE}}$:

- **Tailor-made** to match the financial objectives of hedgers.
  - Smallest downside risk metrics + significant hedging gains across all benchmarks.

- Conclusion: **should be prioritized** (when possible) over other dynamic hedging procedures considered in this study.

Second application: Equal risk pricing

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Derivative price set as the **premium needed to make residual risk of both hedgers equal**.

- Residual risk quantified by a risk measure.
- Consider in Carbonneau and Godin (2021a-2021b) the ERP framework under **convex risk measures**.
- Consider in Carbonneau and Godin (2021c) the ERP framework under the class **of semi-$\mathcal{L}^p$ risk measures**, i.e. $\rho(X) = \mathbb{E}[X^p \mathbb{1}_{\{x>0\}}]$ for $p > 0$.

Hedging optimization problem

For ρ a convex risk measure (e.g. CVaR), two distinct problems for the short and long positions:

$$\epsilon^{(S)}(V_0) := \min_{\delta} \rho\left(\Phi(S_N^{(1)}) - V_N^{\delta}\right),$$
$$\epsilon^{(L)}(V_0) := \min_{\delta} \rho\left(-\Phi(S_N^{(1)}) - V_N^{\delta}\right).$$

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The **equal risk price** C_0^* is the value of V_0 for which optimal residual risks are equal:

$$\epsilon^{(L)}(-V_0) = \epsilon^{(S)}(V_0).$$

Equal risk price

Under some technical conditions, the **equal risk price** exists, is unique, is arbitrage-free and is given by

$$C_0^\star = \frac{\epsilon^{(S)}(0) - \epsilon^{(L)}(0)}{2B_N}.$$

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$$C_0^{\star} = \frac{\epsilon^{(S)}(0) - \epsilon^{(L)}(0)}{2B_N}.$$

Difficulty is to evaluate $\epsilon^{(S)}(0)$ and $\epsilon^{(L)}(0)$.

- In our paper: use the deep hedging algorithm.

Contributions of my papers on ERP

1) Provide a tractable methodology to implement the ERP framework:
    - Two distinct neural networks, one for the long and one for the short optimal hedging policy.

2) Introduce asymmetric ϵ-completeness measure to quantify the level of market incompleteness.

3) Perform various Monte Carlo experiments to study equal risk prices generated under the ERP framework paired with convex risk measures.

Numerical results

- Pricing 2-months maturity **European put option**.
- Daily rebalancing.
- Single traded risky asset (the underlying) of initial price $S_0^{(1)} = 100$.
- Various moneynesses: OTM ($K = 90$), ATM ($K = 100$) and ITM ($K = 110$).
- **Regime-switching model** for the underlying asset.
- **Conditional Value-at-Risk** (CVaR) as the convex risk measure.

Sensitivity of prices to the risk measure

Table: Sensitivity analysis of equal risk prices $C_0^\star$.

Moneyness	OTM	ATM	ITM
$\mathrm{CVaR}_{0.90}$	1.40	4.19	11.14
$\mathrm{CVaR}_{0.95}$	32%	4%	2%
$\mathrm{CVaR}_{0.99}$	91%	42%	14%

Notes: Values for the $\mathrm{CVaR}_{0.95}$ and $\mathrm{CVaR}_{0.99}$ risk measures are expressed relative to $\mathrm{CVaR}_{0.90}$ (% increase).

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Notes: Values for the $\mathrm{CVaR}_{0.95}$ and $\mathrm{CVaR}_{0.99}$ risk measures are expressed relative to $\mathrm{CVaR}_{0.90}$ (% increase).

The **choice of the risk measure is material**.

- Hedging shortfall for short position has thicker right tail than the long position hedging shortfall.
- α increase $\rightarrow$ more risk for short position $\rightarrow$ $C_0^\star$ increase.

Future potential avenue

2) Move **beyond static portfolio** of options (e.g. pov of market maker).

- portfolio of different options characteristics (maturity, moneyness etc.) with different initial dates (i.e. time-series).
- Difficulties:
  - global trading strategies **are not additive** across different option contracts (unlike the greeks).
  - objective function is not trivial to specify.
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1) **Data-driven market simulation**:

- Some work consider the use of deep learning generative models: e.g. Wiese et al. (2019), Wiese et al. (2020) and Buehler et al. (2020).

Thank you!

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