QIC 710 Project Report: The Complexity Classes PostBQP and PP

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Contents

1 Definitions: P, NP, PP, BQP, PostBQP

P is the set of problems solvable by a deterministic algorithm in polynomial time.

NP is the set of problems for which a deterministic polynomial time algorithm exists which can verify a proposed solution to the problem.

PP is the set of problems which can be solved by a probabilistic polynomial time algorithm with probability $> \frac{1}{2}$. Note that this is not considered to be a feasible set of problems, because the algorithm may only determine an answer with probability $\frac{1}{2} + \frac{1}{2^n}$ where n is the input size. Thus, in order to be sure of an answer, we would need to repeat the algorithm an exponential number of times.

A dramatic demonstration of the size of PP is given by Toda's theorem. Complexity theorists are reasonably certain that P $\neq$ NP. They are also reasonably certain of a generalization of this, namely that $P \neq NP \neq NP^{NP} \neq NP^{NP^{NP}} \neq \cdots$. Here A^B means the complexity class A with an oracle for solving problems in the complexity class B in $O(1)$ time. If we define the complexity

class PH = P$\cup$NP$\cup$NP$^{\text{NP}}\cup\cdots$, clearly PH (the polynomial hierarchy) is a huge class of problems. Toda's theorem states that PH $\subseteq$ P$^{\text{PP}}$. In other words, a polynomial time deterministic computer that can solve problems in PP can solve any problem in the entire polynomial hierarchy!

BQP is the class of problems for which a probabilistic polynomial time quantum algorithm can determine the solution with probability $\geq \frac{2}{3}$.

PostBQP is the class of problems for which a probabilistic polynomial time quantum algorithm with postselection can determine the solution with probability $\geq \frac{2}{3}$.

In the previous definitions I have not been very precise, using the intuitive definitions of "solve" and "problem". Let us be more precise for PostBQP

PostBQP is the set of languages $L \subseteq \{0,1\}^*$ for which a uniform family of polynomial size quantum circuits $\{G_n\}$ for $n \geq 1$ exists such that when $|\psi\rangle = G_n |x\rangle \otimes |0\cdots 0\rangle$
(i) The first qubit in $|\psi\rangle$ has > 0 chance of being in the state $|1\rangle$
And when the first qubit is in state $|1\rangle$,
(ii) If $x \in L$ the second qubit in $|\psi\rangle$ has a $\geq \frac{2}{3}$ chance of being in state $|1\rangle$
(iii) If $x \notin L$ the second qubit in $|\psi\rangle$ has a $\leq \frac{1}{3}$ chance of being in state $|1\rangle$

Throughout this report I'll refer to the first qubit as P and the second qubit as "the output", or Q.

2 NP $\subseteq$ PostBQP

This section is not important to the rest of the results in the paper. In fact, it is a trivial consequence of Section 5: PP $\subseteq$ PostBQP. I include it only because it is simple and demonstrative of a PostBQP algorithm.

We wish to show that any problem in NP can be solved with a PostBQP algorithm. To do this, we assume that we have implemented the verification algorithm $\varphi(x)$ which outputs 1 when $x \in L$ and 0 when $x \notin L$, for an NP language L. φ is guarenteed to exist by the definition of NP.

The first step in the algorithm is to prepare the following state (I'll show how soon):
$\sqrt{\varepsilon}\,|0\rangle\,|0\rangle\,|1\rangle + \sqrt{1-\varepsilon}\left(\sum_{x\in\{0,1\}^n} \frac{1}{\sqrt{2^n}}\,|1\rangle\,|x\rangle\,|\varphi(x)\rangle\right)$
Once we have this state, we proceed in this way:
Measure the last register.
If you get 0, kill yourself.
Measure the first register.
If you get 1, you know that the second register contains an x for which $\varphi(x) = 1$.
If you get 0, then either no solution exists, so you had to get 0,
or a solution existed, but you got unlucky and chose $|0\rangle\,|0\rangle\,|1\rangle$ anyway.

Now, we just need to set ε to be small enough that we can be confident that measuring 0 means there was no choice. $\varepsilon = \frac{1}{2^{2n}}$ does the trick.

We now show how to prepare this initial state.

Start with

$|0\rangle |0\rangle |0\rangle$

Where the second register contains n qubits. Rotate first qubit to put $\frac{1}{2^n}$ weight on $|0\rangle$ and $\sqrt{1 - \frac{1}{2^{2n}}}$ on $|1\rangle$.

$R_\theta \otimes I \otimes I \, |0\rangle |0\rangle |0\rangle$
$= \frac{1}{2^n} |0\rangle |0\rangle |0\rangle + \sqrt{1 - \frac{1}{2^{2n}}} |1\rangle |0\rangle |0\rangle$

Apply controlled $H^{\otimes n}$ gate to second register

$$\frac{1}{2^n} |0\rangle |0\rangle |0\rangle + \sqrt{1 - \frac{1}{2^{2n}}} \left(\sum_{x \in \{0,1\}^n} \frac{1}{\sqrt{2^n}} |1\rangle |x\rangle |0\rangle \right)$$

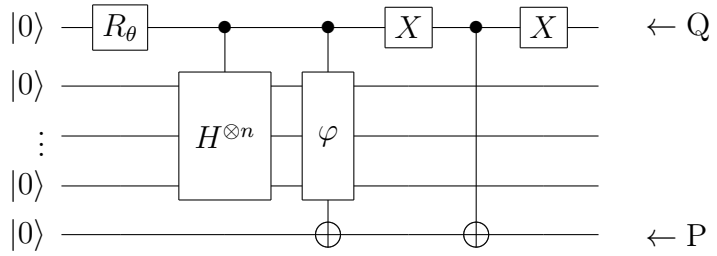
Apply the conditional verification algorithm for the NP problem:
$|1\rangle |x\rangle |0\rangle \to |1\rangle |x\rangle |\varphi(x)\rangle$
$|0\rangle |x\rangle |0\rangle \to |0\rangle |x\rangle |0\rangle$

$$\frac{1}{2^n} |0\rangle |0\rangle |0\rangle + \frac{\sqrt{1 - \frac{1}{2^{2n}}}}{2^{n/2}} \left(\sum_{x \in \{0,1\}^n} |1\rangle |x\rangle |\varphi(x)\rangle \right)$$

Make the third register a 1 if the first register is 0 (X the first register, apply CNOT with the first as control and the last as target, X the first register again)

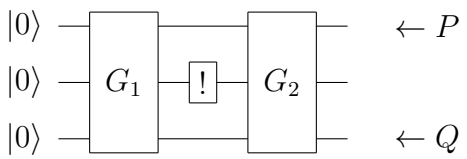
$$\frac{1}{2^n} |0\rangle |0\rangle |1\rangle + \frac{\sqrt{1 - \frac{1}{2^{2n}}}}{2^{n/2}} \left(\sum_{x \in \{0,1\}^n} |1\rangle |x\rangle |\varphi(x)\rangle \right)$$

The following diagram shows a circuit which follows the steps outlined above. It produces the initial state for our PostBQP algorithm.

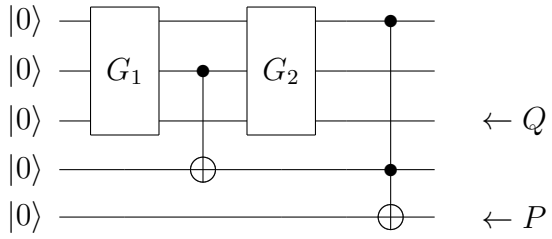


3 Properties of PostBQP

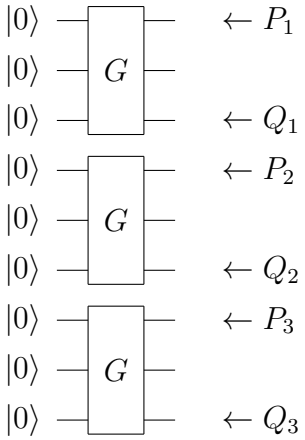
In the definition of PostBQP we've restricted ourselves to postselecting on a single qubit at the very end of the circuit. What if we want to postselect on qubits at intermediate stages in the computation?



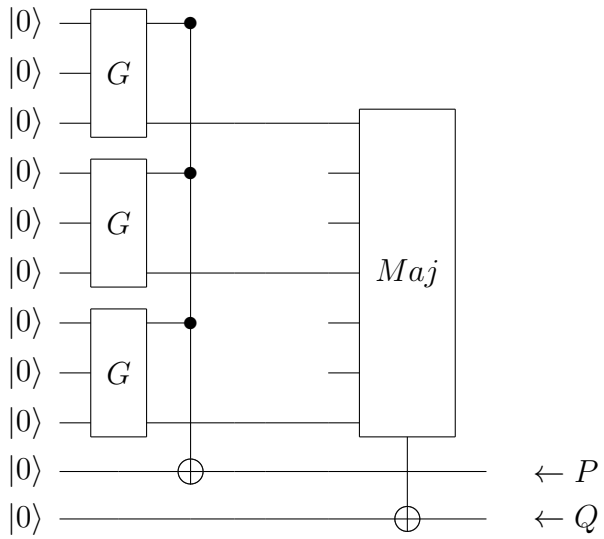
What if we need to ensure that the register is in state $|1\rangle$ at the "!" sign? We simply CNOT the register onto an ancilla qubit, and we take the AND of the qubit with the postselected qubit right before our postselection.



If we want to amplify the success probability from 2/3 to an arbitrary amount of precision, we can run many copies of the circuit.



We can use the trick from before to turn this into one run of a PostBQP algorithm with higher success probability.

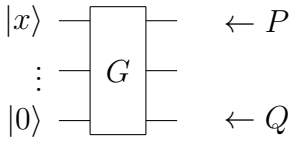


This hideous circuit diagram shows the general process. We run the algorithm many times, we postselect one qubit to make sure all the circuits worked, and we take the majority output of the output (Q) qubits. Using a polynomial number of runs, we can increase the success probability to $1 - 2^{p(n)}$ for any polynomial $p(n)$.

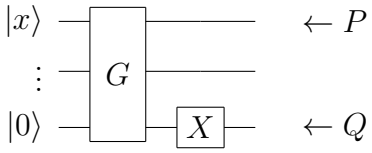
Closure properties:

Is PostBQP = co-PostBQP?

Yes. If we want to know if $x \notin L$, we simply run the algorithm which tests the membership $x \in L$ (call it G) and output the opposite answer. So PostBQP is closed under complement.



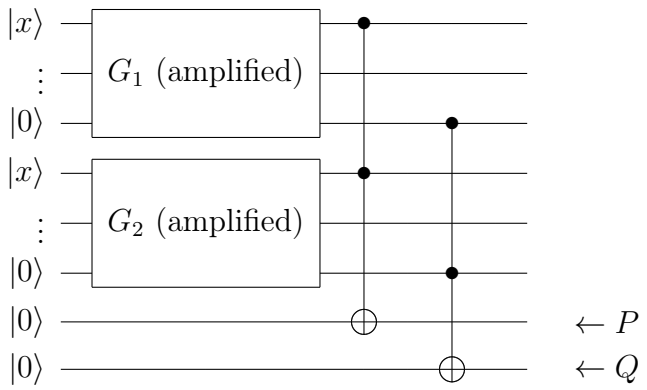
Becomes



Is PostBQP closed under intersection?

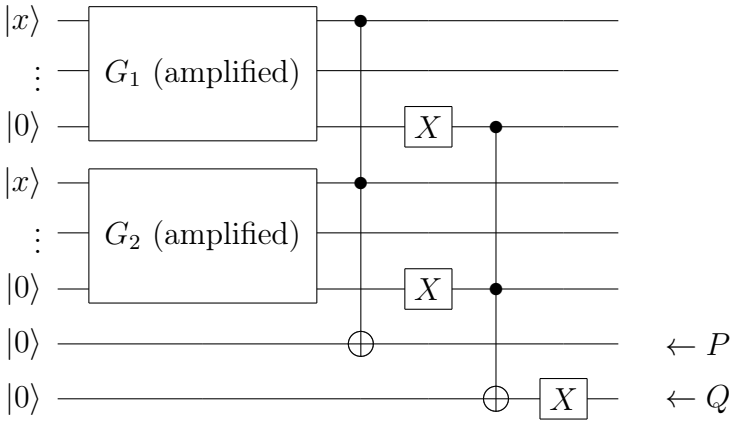
If we want to know if $x \in L_1 \cap L_2$, we amplify the $x \in L_1$ and $x \in L_2$ algorithms (G_1 and G_2 resp.) to each determine with error at most 1/6. Then (using our trick from earlier to perform a

single postselection) we take the logical "AND" of the two output qubits as our new output qubit. This way we have a PostBQP algorithm to determine if $x \in L_1 \cap L_2$ with error at most 1/3.



Is PostBQP closed under union?

In exactly the same way, we can design a PostBQP circuit to determine $x \in L_1 \cup L_2$, Using the logical "OR".



4 PostBQP $\subseteq$ PP

Any PostBQP algorithm can be seen as a polynomial number $G(n)$ of Toffoli and Hadamard gates acting on n qubits, followed by a one qubit postselection, and a one qubit measurement.

Applying an algorithm A to an input $|x\rangle$:
$A^G A^{G-1} \cdots A^2 A^1 |x\rangle$

π_1: probability that the output has $P = 1$, $Q = 1$
π_0: probability that the output has $P = 1$, $Q = 0$

Can we make a classical algorithm which accepts with probability $> \frac{1}{2}$ if $\pi_1 > \pi_0$, in polynomial time? (i.e. is PostBQP $\subseteq$ PP ?)

The probability that $P = 1$ and $Q = 1$ is equal to the sum of the squares of the amplitudes of the basis states of $A^G A^{G-1} \cdots A^2 A^1 |x\rangle$ for which $P = 1$ and $Q = 1$. For example: If

$$A^G A^{G-1} \cdots A^2 A^1 |x\rangle = \begin{pmatrix} 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \end{pmatrix} \begin{matrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{matrix} \begin{matrix} \\ \\ \\ \leftarrow \\ \\ \\ \\ \leftarrow \end{matrix}$$

The two positions where the arrows point correspond to $P = 1$ and $Q = 1$ when P and Q are the last 2 qubits.

Let Ψ_ω be the amplitude on the basis state $|\omega\rangle$. In other words,
$A^G A^{G-1} \cdots A^2 A^1 |x\rangle = \sum_{\omega=0}^{2^n-1} \Psi_\omega |\omega\rangle$ This is our quantum state before the postselcection and measurement at the end of the algorithm.

The probability that the output has $P = 1$, $Q = 1$:
$\pi_1 = \sum_{\omega \in S_1} |\Psi_\omega|^2$
The probability that the output has $P = 1$, $Q = 0$:
$\pi_0 = \sum_{\omega \in S_0} |\Psi_\omega|^2$
Where S_1 is the set of basis states for which $P = 1$, $Q = 1$
and S_0 is the set of basis states for which $P = 1$, $Q = 0$.

Assume (without loss of generality) that Ψ_ω is a real number. (There are universal gate sets for which this is so.)

The probability that the output has $P = 1$, $Q = 1$:
$\pi_1 = \sum_{\omega \in S_1} \Psi_\omega \Psi_\omega$
The probability that the output has $P = 1$, $Q = 0$:
$\pi_0 = \sum_{\omega \in S_0} \Psi_\omega \Psi_\omega$

Let's look at what the amplitudes Ψ_ω look like by considering an example:

$$A^G A^{G-1} \cdots A^2 A^1 |x\rangle = \begin{pmatrix} A^2_{11} & A^2_{12} & A^2_{13} \\ A^2_{21} & A^2_{22} & A^2_{23} \\ A^2_{31} & A^2_{32} & A^2_{33} \end{pmatrix} \begin{pmatrix} A^1_{11} & A^1_{12} & A^1_{13} \\ A^1_{21} & A^1_{22} & A^1_{23} \\ A^1_{31} & A^1_{32} & A^1_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} A^2_{11} & A^2_{12} & A^2_{13} \\ A^2_{21} & A^2_{22} & A^2_{23} \\ A^2_{31} & A^2_{32} & A^2_{33} \end{pmatrix} \begin{pmatrix} \sum_{a=1}^3 A^1_{1a} x_a \\ \sum_{a=1}^3 A^1_{2a} x_a \\ \sum_{a=1}^3 A^1_{3a} x_a \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{b=1}^3 A^2_{1b} \left(\sum_{a=1}^3 A^1_{ba} x_a \right) \\ \sum_{b=1}^3 A^2_{2b} \left(\sum_{a=1}^3 A^1_{ba} x_a \right) \\ \sum_{b=1}^3 A^2_{3b} \left(\sum_{a=1}^3 A^1_{ba} x_a \right) \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{b=1}^3 \sum_{a=1}^3 A^2_{1b} A^1_{ba} x_a \\ \sum_{b=1}^3 \sum_{a=1}^3 A^2_{2b} A^1_{ba} x_a \\ \sum_{b=1}^3 \sum_{a=1}^3 A^2_{3b} A^1_{ba} x_a \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{\forall a,b} A^2_{1b} A^1_{ba} x_a \\ \sum_{\forall a,b} A^2_{2b} A^1_{ba} x_a \\ \sum_{\forall a,b} A^2_{3b} A^1_{ba} x_a \end{pmatrix}$$

So the amplitude of $|\omega\rangle$ is $\sum_{\forall a,b} A^2_{\omega b} A^1_{ba} x_a$

More generally:
$\Psi_\omega = \sum_{\forall \alpha_1, \cdots \alpha_G} A^G_{\omega,\alpha_G} A^{G-1}_{\alpha_G,\alpha_{G-1}} \cdots A^2_{\alpha_3,\alpha_2} A^1_{\alpha_2,\alpha_1} x_{\alpha_1}$

Notes:
This is a sum over exponentially many terms.
However, every specific term in this sum is computable in polynomial time. (It is just a product of polynomially many numbers)

Let's call $\psi_{\omega,\alpha} = \left(A^G_{\omega,\alpha_G} A^{G-1}_{\alpha_G,\alpha_{G-1}} \cdots A^2_{\alpha_3,\alpha_2} A^1_{\alpha_2,\alpha_1} x_{\alpha_1} \right)$. Here α is shorthand for $\alpha_1, \cdots, \alpha_G$. This corresponds to a specific term in the sum. So, $\Psi_\omega = \sum_\alpha \psi_{\omega,\alpha}$.

Now, let's call $X_{\omega,\alpha,\alpha'} = \psi_{\omega,\alpha} \psi_{\omega,\alpha'}$.
So, $\Psi^2_\omega = \left(\sum_\alpha \psi_{\omega,\alpha} \right) \left(\sum_\alpha \psi_{\omega,\alpha} \right) = \sum_{\alpha,\alpha'} \psi_{\omega,\alpha} \psi_{\omega,\alpha'} = \sum_{\alpha,\alpha'} X_{\omega,\alpha,\alpha'}$

Notice again, $X_{\omega,\alpha,\alpha'}$ is computable in polynomial time for any specific ω, $\alpha_1, \cdots \alpha_G$, and $\alpha'_1, \cdots \alpha'_G$.

We can revise our definitions of π_1 and π_0,
The probability that the output has $P = 1$, $Q = 1$:
$\pi_1 = \sum_{\omega \in S_1} \sum_{\alpha,\alpha'} X_{\omega,\alpha,\alpha'}$
The probability that the output has $P = 1$, $Q = 0$:
$\pi_0 = \sum_{\omega \in S_0} \sum_{\alpha,\alpha'} X_{\omega,\alpha,\alpha'}$

Finally, let's construct a classical polynomial time algorithm which accepts with probability $> \frac{1}{2}$ when $\pi_1 > \pi_0$.

Step 1: Choose a random ω, α, and α'.

Step 2: If $\omega \notin S_1 \cup S_2$, accept with probability $\frac{1}{2}$.

Step 3: If $\omega \in S_1$, accept with probability $\frac{1}{2} + \frac{X_{\omega,\alpha,\alpha'}}{2}$. (we just want this normalized, and apparently X has to be between -1 and 1 for some reason).

Step 4: If $\omega \in S_0$, accept with probability $\frac{1}{2} - \frac{X_{\omega,\alpha,\alpha'}}{2}$.

That's it.

What is the probability that this algorithm accepts?

$$= \frac{1}{2} + \frac{\sum_{\omega \in S_1,} \sum_{\alpha,\alpha'} \frac{X_{\omega,\alpha,\alpha'}}{2} - \sum_{\omega \in S_0,} \sum_{\alpha,\alpha'} \frac{X_{\omega,\alpha,\alpha'}}{2}}{\text{number of X's}}$$

From here we see that the probability of accepting is $> \frac{1}{2}$ if and only if

$$\frac{\sum_{\omega \in S_1,} \sum_{\alpha,\alpha'} \frac{X_{\omega,\alpha,\alpha'}}{2} - \sum_{\omega \in S_0,} \sum_{\alpha,\alpha'} \frac{X_{\omega,\alpha,\alpha'}}{2}}{\text{number of X's}} > 0$$

$$\sum_{\omega \in S_1,} \sum_{\alpha,\alpha'} \frac{X_{\omega,\alpha,\alpha'}}{2} - \sum_{\omega \in S_0,} \sum_{\alpha,\alpha'} \frac{X_{\omega,\alpha,\alpha'}}{2} > 0$$

$$\sum_{\omega \in S_1,} \sum_{\alpha,\alpha'} X_{\omega,\alpha,\alpha'} > \sum_{\omega \in S_0,} \sum_{\alpha,\alpha'} X_{\omega,\alpha,\alpha'}$$

$\pi_1 > \pi_0$.

5 PP $\subseteq$ PostBQP

If we have a function $f : \{0,1\}^n \to \{0,1\}$, deciding whether most outputs are 1 is PP-Complete. If $s = |\{x : f(x) = 1\}|$ is the number of outputs that are 1, and 2^n is the number of possible inputs, we want to know whether $s > \frac{2^n}{2}$.

Here we present a PostBQP algorithm which solves this problem. Note the problem is still PP-Complete if we assume that $s \neq \left\{0, \frac{2^n}{2}, 2^n\right\}$, and we shall.

Step 1: Prepare the state (we'll see how soon)

$$|\psi\rangle = \frac{(2^n - s)\,|0\rangle + s\,|1\rangle}{\sqrt{(2^n - s)^2 + s^2}}$$

Step 2: Now prepare the state

$$\alpha\,|0\rangle\,|\psi\rangle + \beta\,|1\rangle\,H\,|\psi\rangle$$

Where α and β are positive real numbers to be determined later.

Note that

$$H|\psi\rangle = \frac{(2^n - s)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + s\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}{\sqrt{(2^n - s)^2 + s^2}}$$
$$H|\psi\rangle = \frac{(2^n - s)|0\rangle + (2^n - s)|1\rangle + s|0\rangle - s|1\rangle}{\sqrt{2}\sqrt{(2^n - s)^2 + s^2}}$$
$$H|\psi\rangle = \frac{2^n|0\rangle + (2^n - 2s)|1\rangle}{\sqrt{2}\sqrt{(2^n - s)^2 + s^2}}$$

So in total our state is

$$\alpha|0\rangle \frac{(2^n - s)|0\rangle + s|1\rangle}{\sqrt{(2^n - s)^2 + s^2}} + \beta|1\rangle \frac{2^n|0\rangle + (2^n - 2s)|1\rangle}{\sqrt{2}\sqrt{(2^n - s)^2 + s^2}}$$

Step 3: Postselect on the second qubit being 1. Here we have just measured the second qubit's state as 1, so we erase the 0s and renormalize.

$$\alpha|0\rangle \frac{s|1\rangle}{\sqrt{(2^n - s)^2 + s^2}} + \beta|1\rangle \frac{(2^n - 2s)|1\rangle}{\sqrt{2}\sqrt{(2^n - s)^2 + s^2}} \text{ (erase 0's)}$$
$$\frac{\alpha s|0\rangle}{\sqrt{(2^n - s)^2 + s^2}} + \frac{\beta(2^n - 2s)\frac{1}{\sqrt{2}}|1\rangle}{\sqrt{(2^n - s)^2 + s^2}} \text{ (ignore the second qubit)}$$
$$\left|\varphi_{\beta/\alpha}\right\rangle = \frac{\alpha s|0\rangle + \beta(2^n - 2s)\frac{1}{\sqrt{2}}|1\rangle}{\sqrt{\alpha^2 s^2 + (\beta^2/2)(2^n - 2s)^2}} \text{ (renormalize)}$$

We call this new state $\left|\varphi_{\beta/\alpha}\right\rangle$.

If $s < \frac{2^n}{2}$, then $(2^n - 2s) > 0$. This means the amplitude of $|0\rangle$ and the amplitude of $|1\rangle$ are both positive, whatever values we choose for α and β.

If $s > \frac{2^n}{2}$, then $(2^n - 2s) < 0$ for any α and β.

When α and β are real positive quantities, the vector $\left|\varphi_{\beta/\alpha}\right\rangle$ lies in the unit circle spanned by $|0\rangle$ and $|1\rangle$. When $s < \frac{2^n}{2}$, $\left|\varphi_{\beta/\alpha}\right\rangle$ is always in the top right quadrant. When $s > \frac{2^n}{2}$, $\left|\varphi_{\beta/\alpha}\right\rangle$ is always in the bottom right quadrant.

We now give a method of distinguishing these two cases with high probability.

We note that if $s > 2^{n-1}$, then measuring $\left|\varphi_{\beta/\alpha}\right\rangle$ in the $\{|+\rangle, |-\rangle\}$ basis always has a less than 50% chance of giving outcome$|+\rangle$. Whereas if $s < 2^{n-1}$, there is a value of β/α for which $\left|\varphi_{\beta/\alpha}\right\rangle \approx |+\rangle$, and so almost always gives $|+\rangle$. (This assumes that $s \notin \{0, 2^{n-1}, 2^n\}$) The distinguishing procudure basically consists of a witchhunt through many values of β/α to see if any of them always cause us to measure $\left|\varphi_{\beta/\alpha}\right\rangle$ as $|+\rangle$. If none do, we can be confident that $s \geq \frac{2^n}{2}$.

Thus, we prepare the state $\left|\varphi_{\beta/\alpha}\right\rangle$ $O(\log n)$ times for every $\beta/\alpha = 2^j$ between $-n \leq j \leq n$. Whatever s is, (as long as it isn't 0, 2^{n-1}, or 2^n) there will be at least one value of β/α for which $\left|\varphi_{\beta/\alpha}\right\rangle$ is between $|1\rangle$ and $|+\rangle$, and one value for which it is between $|+\rangle$ and $|0\rangle$. In the

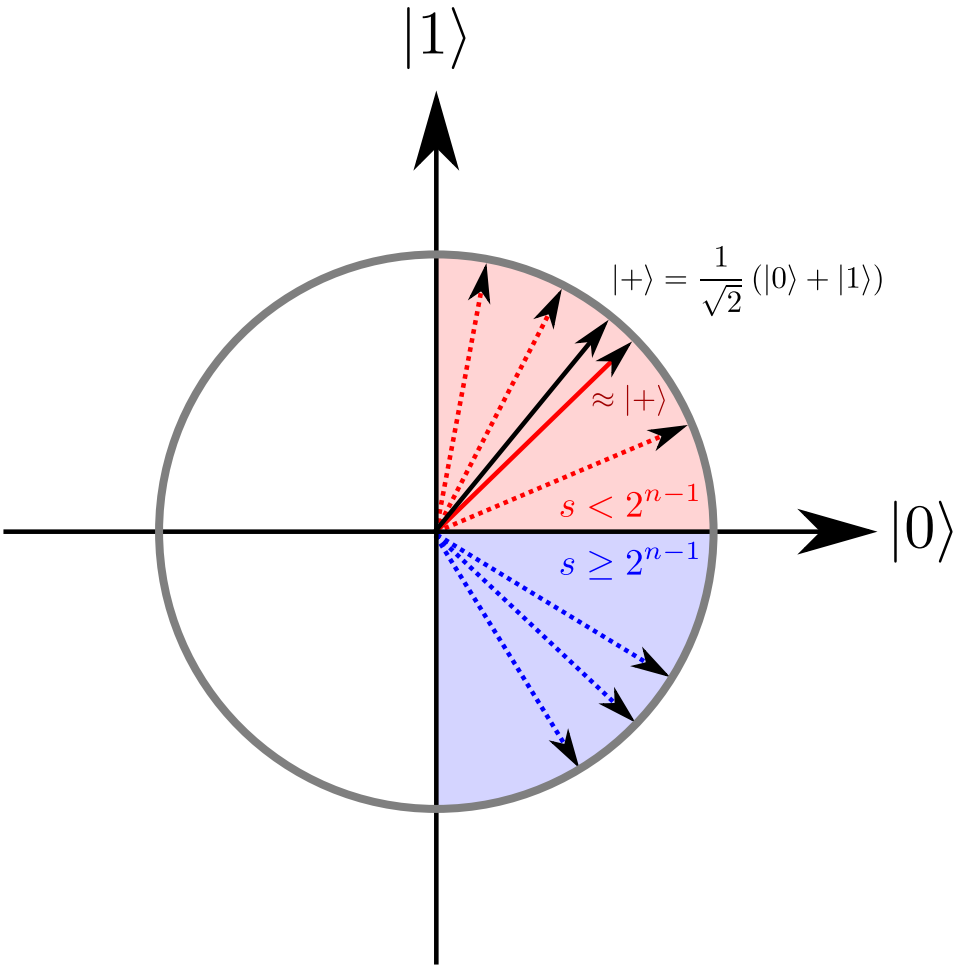


Figure 1: When $s < 2^{n-1}$, $\left|\varphi_{\beta/\alpha}\right\rangle$ is always in the top right quadrant. When $s > 2^{n-1}$, $\left|\varphi_{\beta/\alpha}\right\rangle$ is always in the bottom right quadrant. Image taken from http://www.scottaaronson.com/democritus/lec17.html

worst case (i.e. the case where s is such that the closest value of $\left|\varphi_{\beta/\alpha}\right\rangle$ to $|+\rangle$ is as far away as possible from $|+\rangle$) this distance is the same. This happens when $|\varphi_{2^i}\rangle = \sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle$, $|\varphi_{2^{i+1}}\rangle = \sqrt{\frac{1}{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$. Even in this case, $|\langle + \varphi_{2^i}|\rangle|^2 = \left|\frac{1}{\sqrt{2}}\sqrt{\frac{2}{3}} + \frac{1}{\sqrt{2}}\sqrt{\frac{1}{3}}\right|^2 = \left(\frac{\sqrt{2}+1}{\sqrt{6}}\right)^2 > 0.971$.

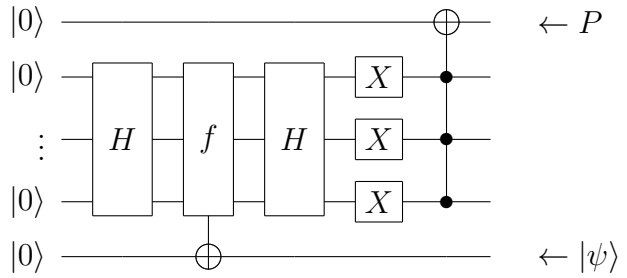
So we choose a number $\frac{1}{2} < k < 0.971$. If our $O(\log n)$ measurements of $|\varphi_{2^j}\rangle$ gives us at least k measurements of $|+\rangle$, we say $s < 2^{n-1}$. We do this for all 2^j's. If we don't reject for any 2^j, we say $s > 2^{n-1}$. Without going into the (even) messier Chernoff bound calculations, it should be clear that repeating this entire process a polynomial number of times and taking the majority vote will give us a constant probability of error.

All that is left now is to show how to obtain the initial state $|\psi\rangle = \dfrac{(2^n - s)|0\rangle + s|1\rangle}{\sqrt{(2^n - s)^2 + s^2}}$.

First prepare the state

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle$$

Then apply $H^{\otimes n}$ on the first register. And postselect on the first register being in the state $|0^n\rangle$.



This produces the initial state $|\psi\rangle$ for this PostBQP algorithm.

Thus, we have a PostBQP algorithm for solving a PP-complete problem. So, PP $\subseteq$ PostBQP.

6 Conclusions

By combining the closure properties of PostBQP with the result that PP = PostBQP, we have shown that PP is closed under union, intersection, and complement.

Also, since $\text{PostBQP}^{\text{BQP}} = \text{PostBQP}$, we showed that $\text{PP}^{\text{BQP}} = \text{PP}$.

The fact that PP is closed under complementation was first proved in the same paper in which PP was first defined, by John Gill in 1972. The fact that PP is closed under union and intersection was an open problem for 19 years and was finally proved by Beigel, Reingold, and Spielman in 1991 using something called "threshold polynomials". The fact that $\text{PP}^{\text{BQP}} = \text{PP}$ was first proved

by Fortnow and Rogers in 1999.

There are many other results which become more simple when we notice that PP = PostBQP, such as QMA $\subseteq$ PP, but I won't go into it any more than that.

We can also show that non-unitary gates, as well as measurement rules where the probability of a state is any other power of the amplitude of that state than 2 (i.e. $|\psi|^p$ with $p \neq 2$ instead of $|\psi|^2$) both allow us to simulate postselection. This is strong evidence that both of these things are not possible, otherwise quantum computers could not only to solve NP-Complete problems, but by Toda's theorem, any problem in the polynomial hierarchy!

7 References

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