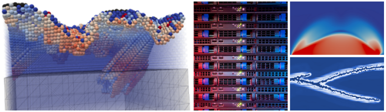
Absorbing Boundary Layers in Time Domain Elastodynamics

Stability Analysis

Alexandre Poulain
École Polytechnique Fédérale de Lausanne, Switzerland

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE



25 avril 2018

Outline

1. Introduction
   - Absorbing boundaries
   - Perfectly matched layers

2. Numerical Stability Analysis
   - Method
   - 1D stability
   - 2D stability

Absorbing boundaries

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- Common practice to solve numerically wave propagation on unbounded domains.
- Important topic for many research and engineering applications.
- Simulation of earthquake ground motion, for soil-structure, geophysical, subsurface sensing, waveguides problems.

- 2 kinds of method :
  - Absorbing boundary conditions : specific conditions at the model boundaries.
  - Absorbing boundary layers (ABL) : layer surrounding the domain of interest.

Absorbing boundaries

- Common practice to solve numerically wave propagation on unbounded domains.
- Important topic for many research and engineering applications.
- Simulation of earthquake ground motion, for soil-structure, geophysical, subsurface sensing, waveguides problems.

- 2 kinds of method :
  - Absorbing boundary conditions : specific conditions at the model boundaries
  - Absorbing boundary layers (ABL) : layer surrounding the domain of interest

Absorbing boundaries

- Common practice to solve numerically wave propagation on unbounded domains.
- Important topic for many research and engineering applications.
- Simulation of earthquake ground motion, for soil-structure, geophysical, subsurface sensing, waveguides problems.

- 2 kinds of method :

Absorbing boundaries

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- Common practice to solve numerically wave propagation on unbounded domains.
- Important topic for many research and engineering applications.
- Simulation of earthquake ground motion, for soil-structure, geophysical, subsurface sensing, waveguides problems.

- 2 kinds of method :
  - Absorbing boundary conditions : specific conditions on the domain boundaries
  - Absorbing boundary layers (ABL) : layer surrounding the domain of interest

Absorbing boundaries

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

- Common practice to solve numerically wave propagation on unbounded domains.
- Important topic for many research and engineering applications.
- Simulation of earthquake ground motion, for soil-structure, geophysical, subsurface sensing, waveguides problems.

- 2 kinds of method :
  - Absorbing boundary conditions : specific conditions at the model boundaries.
  - Absorbing boundary layers (ABL) : layer surrounding the domain of interest.

Absorbing boundaries

- Common practice to solve numerically wave propagation on unbounded domains.
- Important topic for many research and engineering applications.
- Simulation of earthquake ground motion, for soil-structure, geophysical, subsurface sensing, waveguides problems.

- 2 kinds of method :
  - Absorbing boundary conditions : specific conditions at the model boundaries.
  - Absorbing boundary layers (ABL) : layer surrounding the domain of interest.

Absorbing boundaries

- Common practice to solve numerically wave propagation on unbounded domains.
- Important topic for many research and engineering applications.
- Simulation of earthquake ground motion, for soil-structure, geophysical, subsurface sensing, waveguides problems.

- 2 kinds of method :
  - Absorbing boundary conditions : specific conditions at the model boundaries.
  - Absorbing boundary layers (ABL) : layer surrounding the domain of interest.

Absorbing boundaries

- Common practice to solve numerically wave propagation on unbounded domains.
- Important topic for many research and engineering applications.
- Simulation of earthquake ground motion, for soil-structure, geophysical, subsurface sensing, waveguides problems.

- 2 kinds of method :
  - Absorbing boundary conditions : specific conditions at the model boundaries.
  - Absorbing boundary layers (ABL) : layer surrounding the domain of interest.

Perfectly matched layers

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- split-field PML :
  - Introduced by Bérenger in the context of electromagnetics.
  - Extended by Hastings et al to elastodynamics.
  - Use on complex-valued coordinate stretching.
  - Field splitting : partition of the variables into two components parallel and perpendicular to the truncation boundary.
- unsplit-field PML :
  - Introduced by Wang in the context of elastodynamics (CPML) $\rightarrow$ Complexity
  - Basu and Chopra : unsplit-field PML for time-harmonic elastodynamics. $\rightarrow$ finite-element implementation

Perfectly matched layers

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- split-field PML :
  - Introduced by Bérenger in the context of electromagnetics.
  - Extended by Hastings et al to elastodynamics.
  - Use on complex-valued coordinate stretching.
  - Field splitting : partition of the variables into two components parallel and perpendicular to the truncation boundary.
- unsplit-field PML :
  - Introduced by Wang in the context of elastodynamics (CPML) $\rightarrow$ Complexity
  - Basu and Chopra : unsplit-field PML for time-harmonic elastodynamics. $\rightarrow$ finite-element implementation

Perfectly matched layers

- split-field PML :
  - Introduced by Bérenger in the context of electromagnetics.
  - Extended by Hastings et al to elastodynamics.
  - Use on complex-valued coordinate stretching.
  - Field splitting : partition of the variables into two components parallel and perpendicular to the truncation boundary.
- unsplit-field PML :
  - Introduced by Wang in the context of elastodynamics (CPML) $\rightarrow$ Complexity
  - Basu and Chopra : unsplit-field PML for time-harmonic elastodynamics. $\rightarrow$ finite-element implementation

Perfectly matched layers

- split-field PML :
  - Introduced by Bérenger in the context of electromagnetics.
  - Extended by Hastings et al to elastodynamics.
  - Use on complex-valued coordinate stretching.
  - Field splitting : partition of the variables into two components parallel and perpendicular to the truncation boundary.
- unsplit-field PML :
  - Introduced by Wang in the context of elastodynamics (CPML) → Complexity
  - Basu and Chopra : unsplit-field PML for time-harmonic elastodynamics. → finite-element implementation

Perfectly matched layers

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- split-field PML :
  - Introduced by Bérenger in the context of electromagnetics.
  - Extended by Hastings et al to elastodynamics.
  - Use on complex-valued coordinate stretching.
  - Field splitting : partition of the variables into two components parallel and perpendicular to the truncation boundary.
- unsplit-field PML :
  - Introduced by Wang in the context of elastodynamics (CPML) $\rightarrow$ Complexity
  - Basu and Chopra : unsplit-field PML for time-harmonic elastodynamics. $\rightarrow$ finite-element implementation

Stability of PML in the literature

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- split-field PML :
  - Stability analysis using Slowness diagrams and wave fronts.
  - Definitions of sufficiant and necessary conditions of stability [Becache 2006].
  - First order finite difference discretization : Prone to instability in anisotropic media.
  - Second order discretization : stable [Duru-2012]
- unsplit-field PML :
  - Stability proved for electromagnetics (Maxwell's equations) [Bécache 2004].
  - In the context of elastic wave propagation problems : no much publications.

Stability of PML in the literature

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- split-field PML :
  - Stability analysis using Slowness diagrams and wave fronts.
  - Definitions of sufficiant and necessary conditions of stability [Becache 2006].
  - First order finite difference discretization : Prone to instability in anisotropic media.
  - Second order discretization : stable [Duru-2012]
- unsplit-field PML :
  - Stability proved for electromagnetics (Maxwell's equations) [Bécache 2004].
  - In the context of elastic wave propagation problems : no much publications.

Stability of PML in the literature

- split-field PML :
  - Stability analysis using Slowness diagrams and wave fronts.
  - Definitions of sufficiant and necessary conditions of stability [Becache 2006].
  - First order finite difference discretization : Prone to instability in anisotropic media.
  - Second order discretization : stable [Duru-2012]
- unsplit-field PML :
  - Stability proved for electromagnetics (Maxwell's equations) [Bécache 2004].
  - In the context of elastic wave propagation problems : no much publications.

Stability of PML in the literature

- split-field PML :
  - Stability analysis using Slowness diagrams and wave fronts.
  - Definitions of sufficiant and necessary conditions of stability [Becache 2006].
  - First order finite difference discretization : Prone to instability in anisotropic media.
  - Second order discretization : stable [Duru-2012]
- unsplit-field PML :
  - Stability proved for electromagnetics (Maxwell's equations) [Bécache 2004].
  - In the context of elastic wave propagation problems : no much publications.

Stability of PML in the literature

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

- split-field PML :
  - Stability analysis using Slowness diagrams and wave fronts.
  - Definitions of sufficiant and necessary conditions of stability [Becache 2006].
  - First order finite difference discretization : Prone to instability in anisotropic media.
  - Second order discretization : stable [Duru-2012]
- unsplit-field PML :
  - Stability proved for electromagnetics (Maxwell's equations) [Bécache 2004].
  - In the context of elastic wave propagation problems : no much publications.

Principle

- Stable direct integration scheme :

$$\exists h_0 > 0 \quad \text{such as} \quad \forall h \in [0, h_0]$$

a finite perturbation of the state vector at t_n gives a non increasing variation of the state vector at a subsequent time t_{n+j}.

- Initial disturbance :

$$\delta X_0 = X_0' - X_0 \tag{1}$$

- The non-perturbed solution :

$$X_{n+1} = HX_n + g_{n+1} \tag{2}$$

$$= H^2 X_{n-1} + Hg_n + g_{n+1} \tag{3}$$

$$\vdots \tag{4}$$

$$= H^{n+1} X_0 + \sum_{j=0}^{n+1} H^{n-j+1} g_j \tag{5}$$

- Perturbed solution :

$$X'_{n+1} = H^{n+1} X'_0 + \sum_{j=0}^{n+1} H^{n-j+1} g_j \tag{6}$$

- Effect of initial disturbance at time t_{n+1} :

$$\delta X_{n+1} = H^{n+1} \delta X_0 \tag{7}$$

- Eigenvalues of amplification matrix H :

$$det(H - \lambda I) = 0 \tag{8}$$

- $\lambda_r, x_{(r)}$ associated eigenvalues and eigenvectors.

- Modal expension :

$$\delta X_0 = \sum_{s=1}^{2N} a_s x_{(s)} \tag{9}$$

- Transform the recurrence relation to :

$$\delta X_{n+1} = H^{n+1} \sum_{s=1}^{2N} a_s x_{(s)} \tag{10}$$

$$= \sum_{s=1}^{2N} a_s \lambda_s^{n+1} x_{(s)} \tag{11}$$

- Disturbance will be amplified if eigenvalues are higher than unity.

- Equation of motion at time t_n and t_{n+1} :

$$\begin{cases} M\ddot{U}^n = -C\dot{U}^n - KU^n + P_{int}^n \\ M\ddot{U}^{n+1} = -C\dot{U}^{n+1} - KU^{n+1} + P_{int}^{n+1} \end{cases} \tag{12}$$

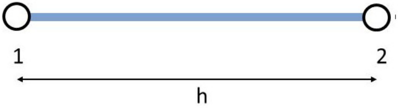
- And the recurrence relationships by Newmark method (it could be another method) :

$$\begin{cases} \dot{U}_{n+1} = \dot{U}_n + (1-\gamma)dt\ddot{U}_n + \gamma dt\ddot{U}_{n+1} \\ U_{n+1} = U_n + dt\dot{U}_n + dt^2\left(\frac{1}{2} - \beta\right)\ddot{U}_n + dt^2\beta\ddot{U}_{n+1} \end{cases} \tag{13}$$

- Check the stability for $h = [0.001, 10]$ by step of 0.001.

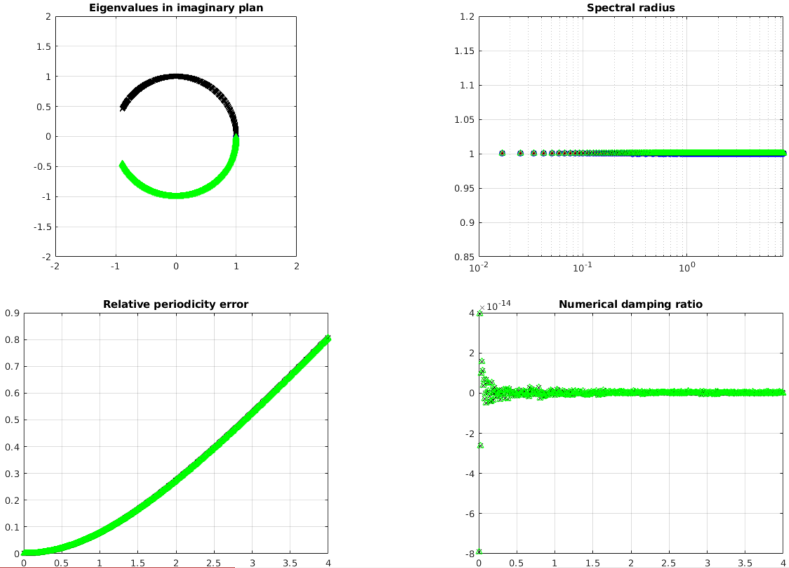
1D bar element

- 1D bar element :



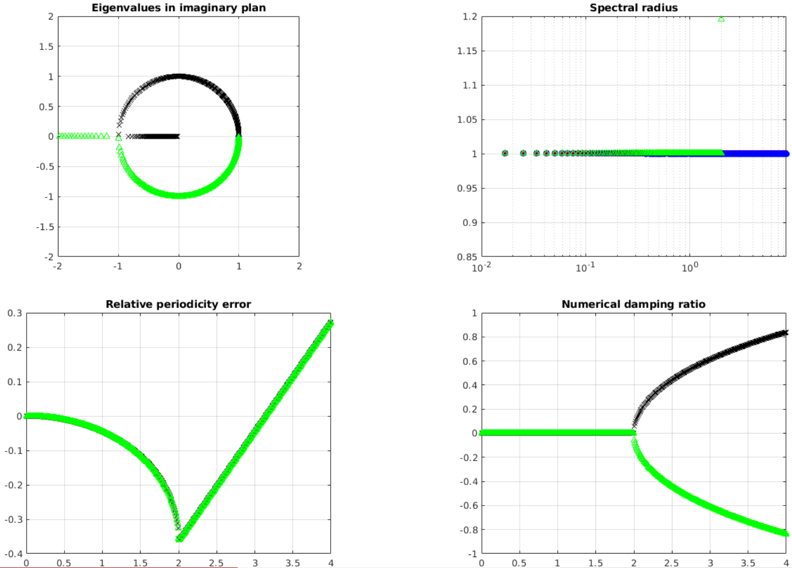
- 4 eigenpairs : corresponding to 2 rigid body motions and traction-elongation modes.

1D bar element : Implicit



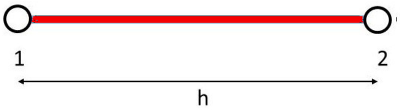
1D bar element : Explicit

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE



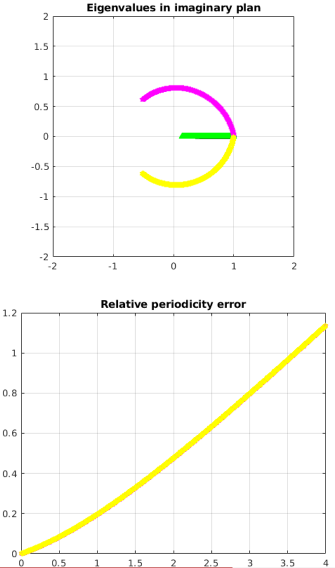
1D PML bar element

- 1D PML bar element :

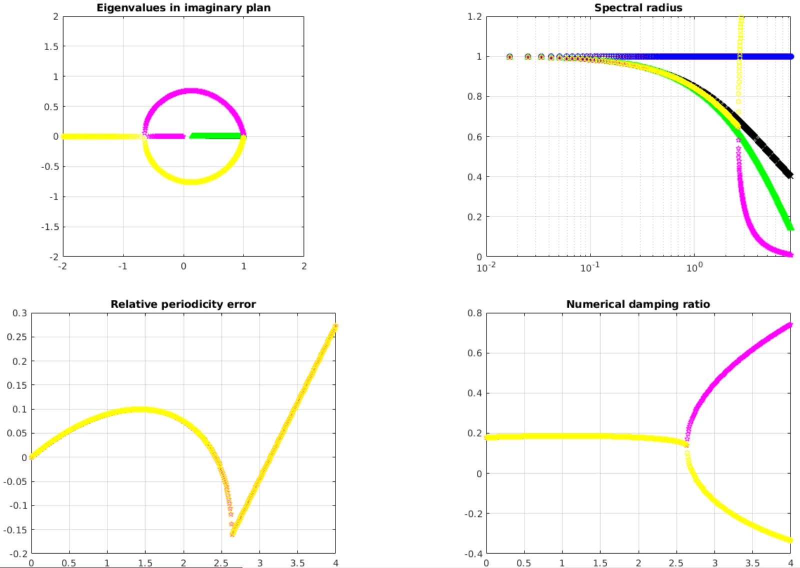


- 6 eigenpairs (depending on the order of numerical quadrature) : 2 rigid body motions, traction-elongation and 2 spurious eigenvalues.

1D PML bar element : Implicit

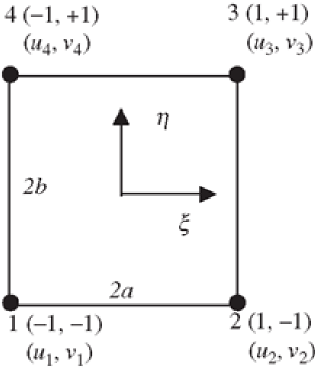


1D PML bar element : Explicit



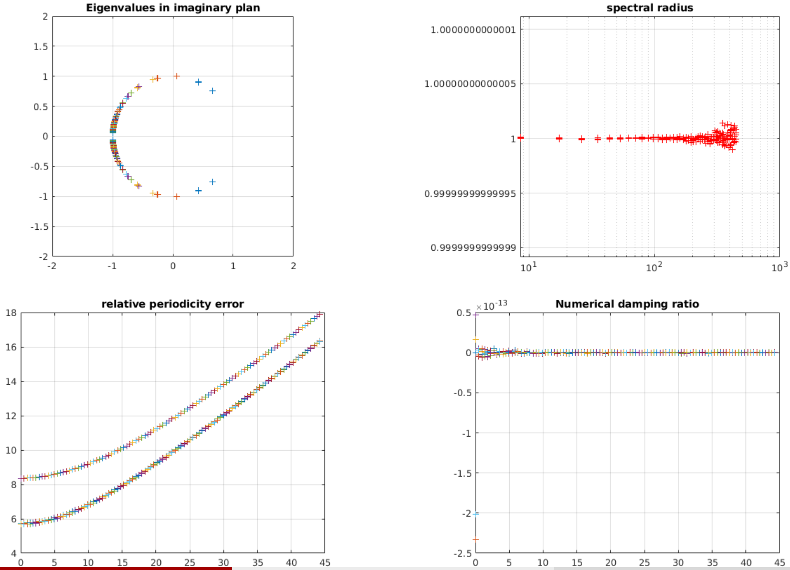
2D element stability

- 2D linear 4-noded element :

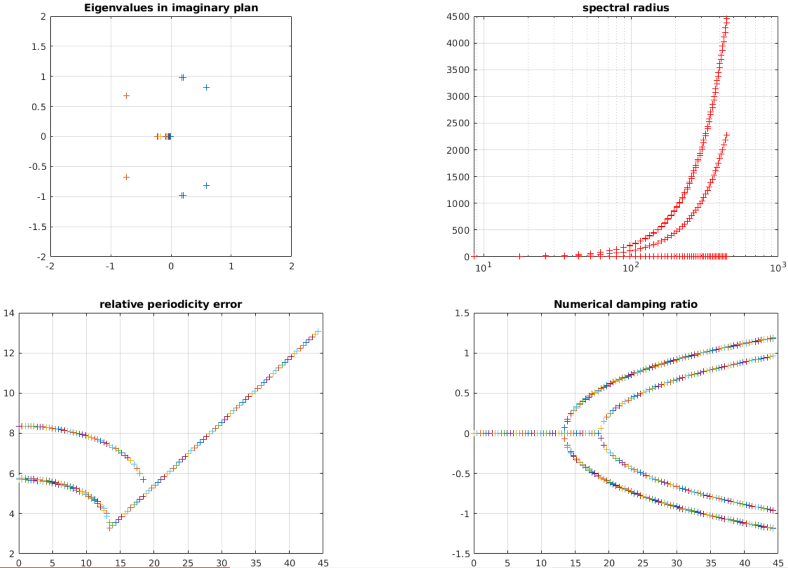


- 8 eigenpairs for the 2D element.
- At least 52 eigenpairs for the 2D PML element (depending on the order of the numerical quadrature used).

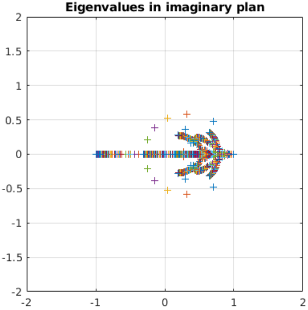
2D element : Implicit

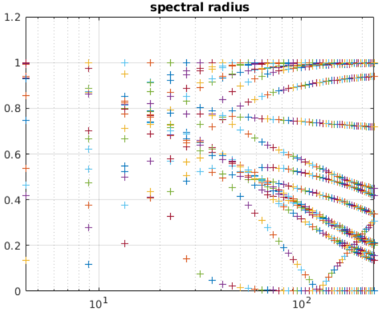


2D element : Explicit

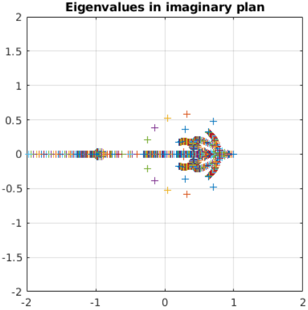


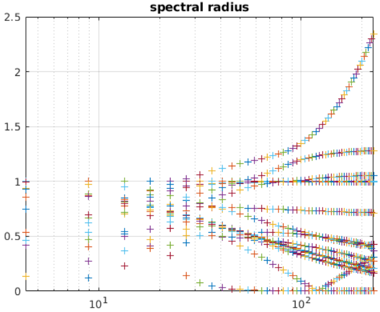
2D PML element : Implicit





2D PML element : Explicit





Conclusion

- Overview of the analysis of stability for integration method.
- Stability of 1D and 2D perfectly matched layer (for implicit time integration method).
- Properties of attenuation and delay of unstability for 1D PML.
- Further work :
  - Same analysis of relative periodicity error and numerical damping for 2D PML.
  - Implementation of 2D Perfectly matched layer within Akantu.