Absorbing Boundary Layers in Time Domain Elastodynamics
Stability Analysis

Alexandre Poulain

École Polytechnique Fédérale de Lausanne, Switzerland

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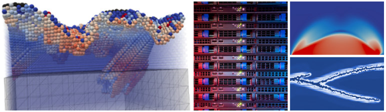

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Outline

1. Introduction
   - Absorbing boundaries
   - Perfectly matched layers
2. Formulation
   - Propagation of Elastic Waves in Solids
3. Stability
   - Method
4. 2D stability

Absorbing boundaries

- Common practice to solve numerically wave propagation on unbounded domains.
- Important topic for many research and engineering applications.
- Simulation of earthquake ground motion, for soil-structure, geophysical, subsurface sensing, waveguides problems.

- 2 kinds of method :

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  - Absorbing boundary conditions : specific conditions at the model boundaries.
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Perfectly matched layers

- split-field PML :
  - Introduced by Bérenger in the context of electromagnetics.
  - Extended by Hastings et al to elastodynamics.
  - Use on complex-valued coordinate stretching.
  - To avoid convolutional operations in the time domain.
  - Field splitting : partition of the variables into two components parallel and perpendicular to the truncation boundary.
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Stability of PML in the literature

- split-field PML :
  - Stability analysis using Slowness diagrams and wave fronts.
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Equations

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- System of equations :

$$\begin{cases} \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \\ \sigma_{ij} = \sum_{k,l} C_{ijkl} \epsilon_{ij} \\ \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \end{cases} \tag{1}$$

- Complex coordinates : $x_i \to \tilde{x}_i : \mathbb{R} \to \mathbb{C}$

$$\frac{\partial \tilde{x}_i}{\partial x_i} = \lambda_i(x_i) = 1 + f_i^e(x_i) - i \frac{f_i^p(x_i)}{b k_s} \tag{2}$$

b : characteristic length of the physical problem.
$k_s = \frac{\omega}{c_s}$: wavenumber.
c_s : shear wave velocity.

- System of equations in frequency domain :

$$\begin{cases} \sum_j \frac{1}{\lambda_j(x_j)} \frac{\partial \sigma_{ij}}{\partial x_j} = -\rho\omega^2 u_j \\ \sigma_{ij} = \sum_{k,l} C_{ijkl}\epsilon_{ij} \\ \epsilon_{ij} = \frac{1}{2}\left(\frac{1}{\lambda_j(x_j)}\frac{\partial u_i}{\partial x_j} + \frac{1}{\lambda_i(x_i)}\frac{\partial u_j}{\partial x_i}\right) \end{cases} \tag{3}$$

Strong Form of PML

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- Inverse Fourier Transform :

$$\begin{cases} div(\underline{\underline{\sigma}}\tilde{F}^e + \underline{\underline{\Sigma}}\tilde{F}^p) = \rho f_m \underline{\ddot{u}} + \rho \frac{c_s}{b} f_c \underline{\dot{u}} + \frac{\mu}{b^2} f_k \underline{u}, & \text{In } \Omega_{PML} \times J \\ \underline{\underline{\sigma}} = C : \underline{\underline{\epsilon}}, & \text{In } \Omega_{PML} \times J \\ F^{eT} \underline{\underline{\dot{\epsilon}}} F^e + F^{pT} \underline{\underline{\epsilon}} F^e + F^{eT} \underline{\underline{\epsilon}} F^p + F^{pT} \underline{\underline{E}} F^p = \dots & \\ \frac{1}{2}(\nabla \underline{\dot{u}}^T F^e + F^{eT} \nabla \underline{\dot{u}}) + \frac{1}{2}(\nabla \underline{u}^T F^p + F^{pT} \nabla \underline{u}), & \text{In } \Omega_{PML} \times J \end{cases} \quad (4)$$

With homogeneous initial conditions :

$$\begin{cases} \underline{u} = 0, & \text{on } \Gamma^D_{PML} \\ (\underline{\underline{\sigma}}\tilde{F}^e + \underline{\underline{\Sigma}}\tilde{F}^p).n, & \text{on } \Gamma^N_{PML} \end{cases} \quad (5)$$

Descriptions of the components

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$$F^e = \begin{bmatrix} 1 + f_1^e(x1) & 0 \\ 0 & 1 + f_2^e(x2) \end{bmatrix}, F^p = \begin{bmatrix} \frac{c_s}{b} f_1^p(x1) & 0 \\ 0 & \frac{c_s}{b} f_2^p(x2) \end{bmatrix}$$

$$\tilde{F}^e = \begin{bmatrix} 1 + f_2^e(x2) & 0 \\ 0 & 1 + f_1^e(x1) \end{bmatrix}, \tilde{F}^p = \begin{bmatrix} \frac{c_s}{b} f_2^p(x2) & 0 \\ 0 & \frac{c_s}{b} f_1^p(x1) \end{bmatrix}$$

$$\begin{cases} f_m = (1 + f_1^e(x1))(1 + f_2^e(x2)) \\ f_c = (1 + f_1^e(x1))f_2^p(x2) + (1 + f_2^e(x2))f_1^p(x1) \\ f_k = f_1^p(x_1)f_2^p(x_2) \end{cases}$$

Integral of stess and strain :

$$\underline{\underline{\Sigma}} = \int_0^t \underline{\underline{\sigma}} dt, \underline{\underline{E}} = \int_0^t \underline{\underline{\epsilon}} dt$$

Attenuation function :

$$f_i^\alpha = a_\alpha \left(\frac{x_i - x_0}{L_p} \right)^n$$

Discrete form of PML

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After :

- Multiplying 5 test functions v belonging to an appropriate space.
- Integrating over the computational domain.
- Discretization in time and space.

$$M\ddot{U}_{n+1} + (C + \tilde{C})\dot{U}_{n+1} + (K + \tilde{K})U_{n+1} + P(\epsilon_n, E_n, \Sigma_n) = F_{ext} \qquad (6)$$

with

$$m^e = \int_{\Omega_e} \rho f_m N_I N_J d\Omega_e I_d \qquad \tilde{c}^e = \frac{1}{dt}\int_{\Omega_e} \tilde{B}^T D B^{\epsilon} d\Omega_e$$

$$c^e = \int_{\Omega_e} \rho f_c \frac{c_s}{b} N_I N_J d\Omega_e I_d \qquad \tilde{k}^e = \frac{1}{dt}\int_{\Omega_e} \tilde{B}^T D B^{Q} d\Omega_e$$

$$k^e = \int_{\Omega_e} \frac{\mu}{b^2} f_k N_I N_J d\Omega_e I_d$$

Internal Forces

The last point is the Internal Forces term $P(\epsilon_n, E_n, \Sigma_n)$

$$P^e(\epsilon_n, E_n, \Sigma_n) = \int_{\Omega_e} \tilde{B}^T \frac{D}{dt}\left[\frac{1}{dt}\hat{F}^\epsilon \hat{\epsilon} - \hat{F}^Q \hat{E}_n\right] + \tilde{B}^p \hat{\Sigma}_n d\Omega_e \tag{7}$$

with

$$\tilde{B}^e_I = \begin{bmatrix} \tilde{N}^e_{I1} & 0 \\ 0 & \tilde{N}^e_{I2} \\ \tilde{N}^e_{I2} & \tilde{N}^e_{I1} \end{bmatrix}, \tilde{B}^p_I = \begin{bmatrix} \tilde{N}^p_{I1} & 0 \\ 0 & \tilde{N}^p_{I2} \\ \tilde{N}^p_{I2} & \tilde{N}^p_{I1} \end{bmatrix}$$

$$\tilde{N}^e_{Ii} = \tilde{F}^e_{ji} N_{I,j}, \tilde{N}^p_{Ii} = \tilde{F}^p_{ji} N_{I,j}$$

And

$$\tilde{B}^T = \tilde{B}^{eT} + dt\tilde{B}^{pT}$$

$$B^\epsilon_I = \begin{bmatrix} F^\epsilon_{11} N^I_{I1} & F^\epsilon_{21} N^I_{I1} \\ F^\epsilon_{12} N^I_{I2} & F^\epsilon_{22} N^I_{I2} \\ F^\epsilon_{11} N^I_{I2} + F^\epsilon_{12} N^I_{I1} & F^\epsilon_{21} N^I_{I2} + F^\epsilon_{22} N^I_{I1} \end{bmatrix}$$

following

with

$$F^I = \left[F^p + \frac{F^e}{dt}\right]^{-1}, F^\epsilon = F^e F^I, F^Q = F^p F^I$$

and

$$N^I_{Ii} = F^I_{ij} N_{I,j}$$

and

$$\hat{F}^\epsilon_I = \begin{bmatrix} (F^\epsilon_{11})^2 & (F^\epsilon_{21})^2 & F^\epsilon_{11} F^\epsilon_{21} \\ (F^\epsilon_{12})^2 & (F^\epsilon_{22})^2 & F^\epsilon_{12} F^\epsilon_{22} \\ 2F^\epsilon_{11} F^\epsilon_{12} & 2F^\epsilon_{21} F^\epsilon_{22} & F^\epsilon_{11} F^\epsilon_{22} + F^\epsilon_{12} F^\epsilon_{21} \end{bmatrix}$$

Principle

- Stable direct integration scheme :

$$\exists h_0 > 0 \quad \text{such as} \quad \forall h \in [0, h_0]$$

a finite perturbation of the state vector at t_n gives a non increasing variation of the state vector at a subsequent time t_{n+j}.

- Initial distrubance :

$$\delta X_0 = X_0' - X_0 \tag{8}$$

- The non-perturbed solution :

$$\begin{aligned} X_{n+1} &= HX_n + g_{n+1} && (9) \\ &= H^2 X_{n-1} + Hg_n + g_{n+1} && (10) \\ &\vdots && (11) \\ &= H^{n+1} X_0 + \sum_{j=0}^{n+1} H^{n-j+1} g_j && (12) \end{aligned}$$

- Perturbed solution :

$$X'_{n+1} = H^{n+1}X'_0 + \sum_{j=0}^{n+1} H^{n-j+1}g_j \tag{13}$$

- Effect of initial disturbance at time t_{n+1} :

$$\delta X_{n+1} = H^{n+1}\delta X_0 \tag{14}$$

- Associated eigenvalue problem :

$$det(H - \lambda I) = 0 \tag{15}$$

- $\lambda_r, x_{(r)}$ associated eigenvalues and eigevectors.

- Modal expension :

$$\delta X_0 = \sum_{s=1}^{2N} a_s x_{(s)} \tag{16}$$

- Transform the recurrence relation to :

$$\delta X_{n+1} = H^{n+1} \sum_{s=1}^{2N} a_s x_{(s)} \tag{17}$$

$$= \sum_{s=1}^{2N} a_s \lambda_s^{n+1} x_{(s)} \tag{18}$$

- Disturbance will be amplified if eigenvalues are higher than unity.

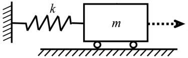
- Equation of motion at time t_n and t_{n+1} :

$$\begin{cases} M\ddot{U}^n = -C\dot{U}^n - KU^n + P_{int}^n \\ M\ddot{U}^{n+1} = -C\dot{U}^{n+1} - KU^{n+1} + P_{int}^{n+1} \end{cases} \tag{19}$$

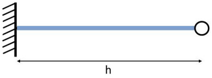
- And the recurrence relationships by Newmark method (it could be another method) :

$$\begin{cases} \dot{U}_{n+1} = \dot{U}_n + (1-\gamma)dt\ddot{U}_n + \gamma dt\ddot{U}_{n+1} \\ U_{n+1} = U_n + dt\dot{U}_n + dt^2\left(\frac{1}{2} - \beta\right)\ddot{U}_n + dt^2\beta\ddot{U}_{n+1} \end{cases} \tag{20}$$

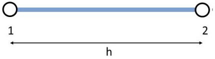
1D : four cases



Single degree of freedom spring



Fixed extremity 1D bar element



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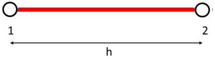

1D PML bar element

SDOF spring

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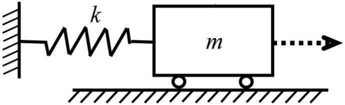
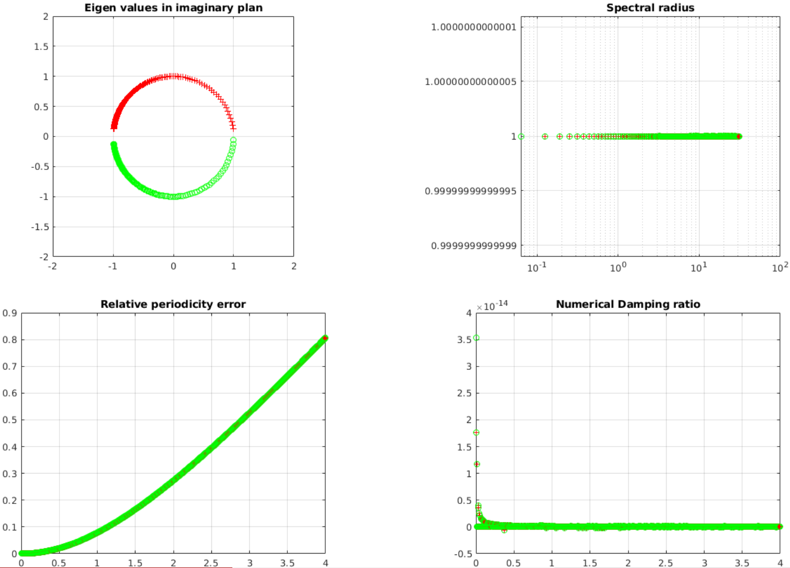


SDOF spring : Implicit



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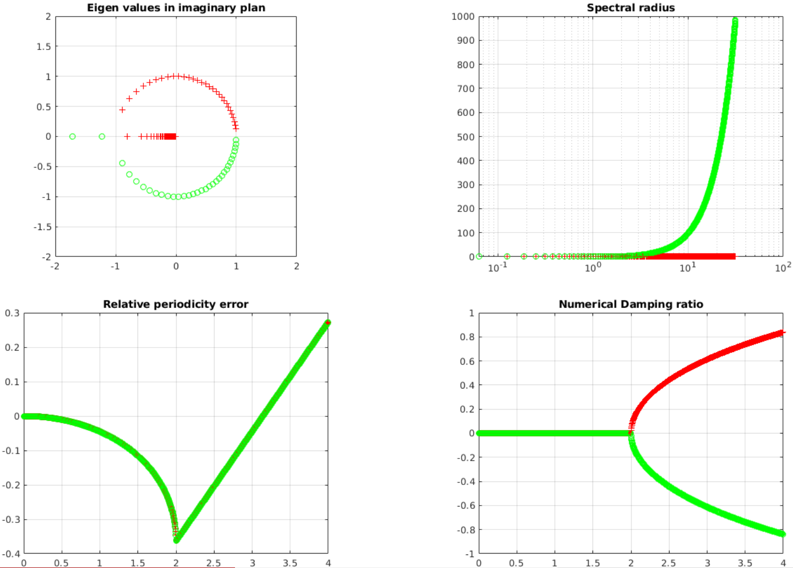

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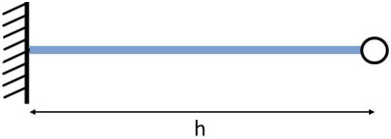
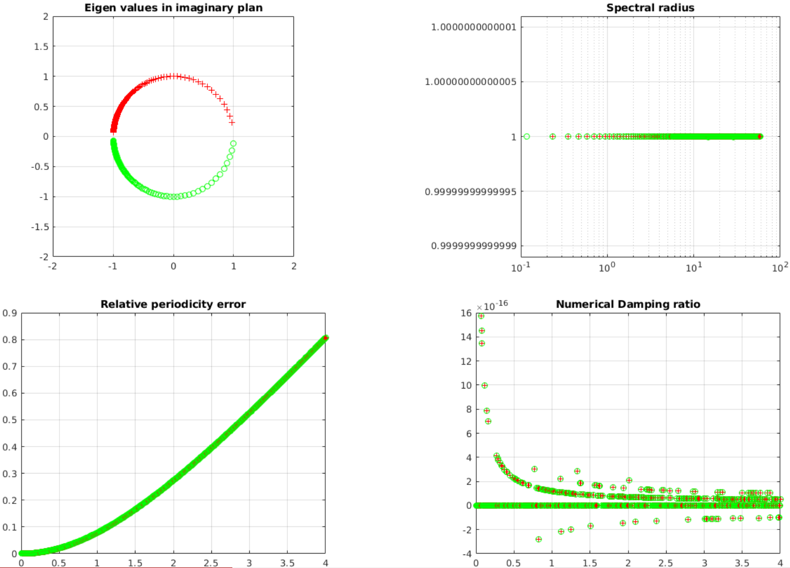


1D encastred element : Implicit



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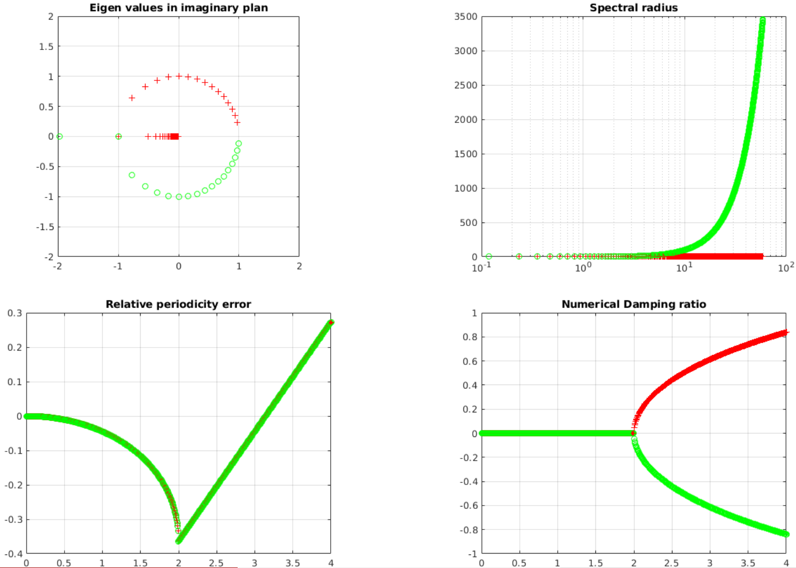

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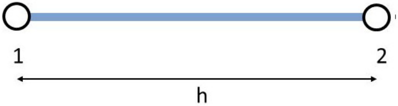

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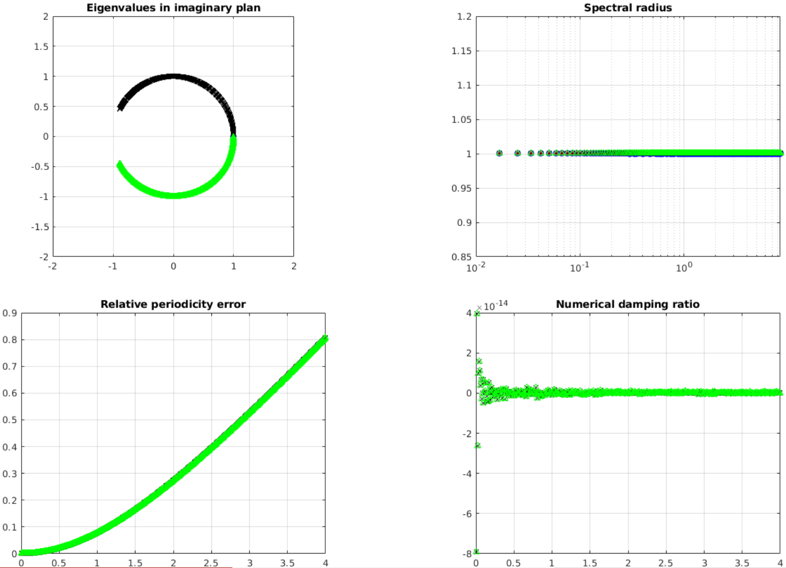

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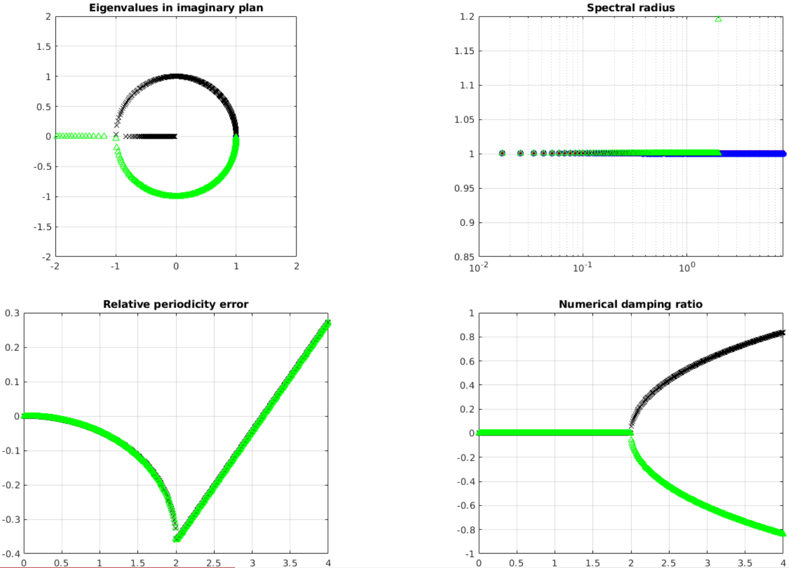

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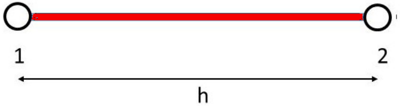

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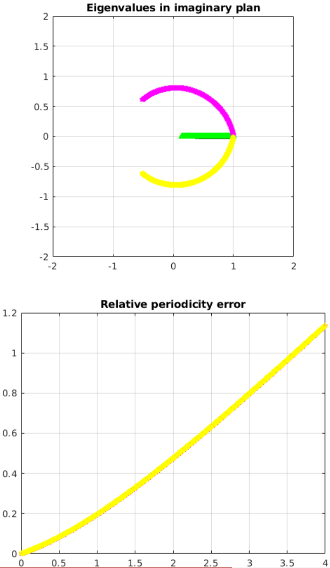

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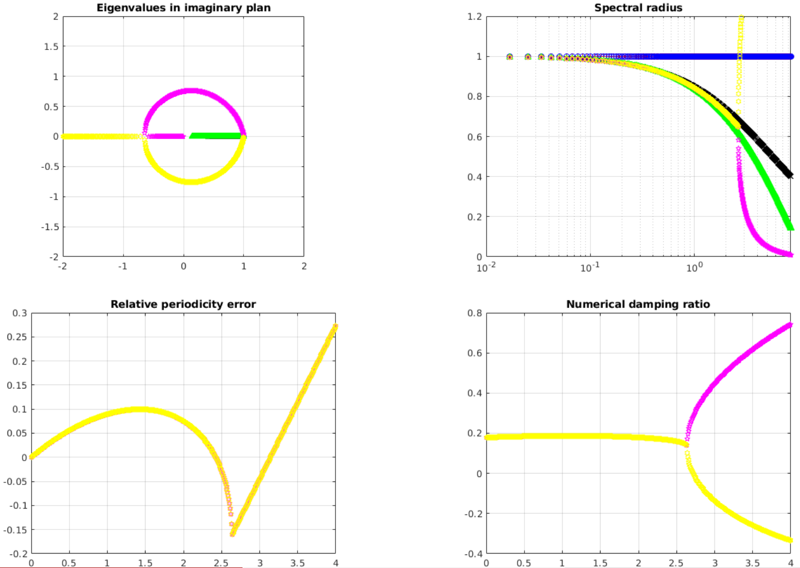
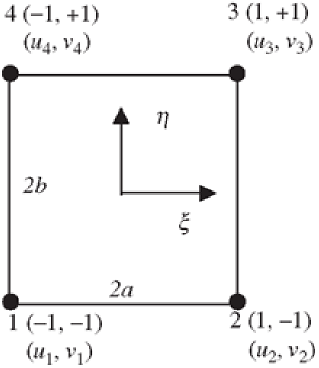


2D element stability

- 2D linear 4-noded element :



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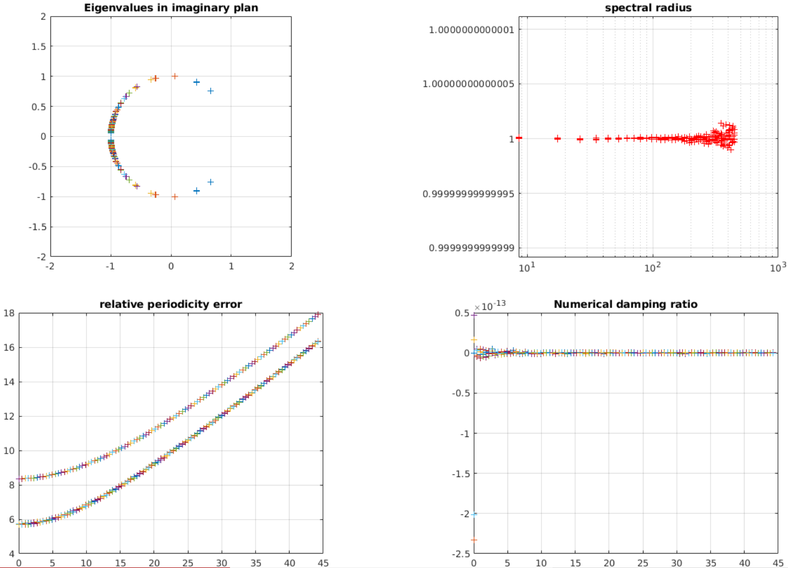

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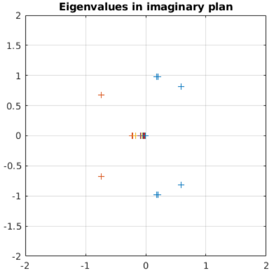
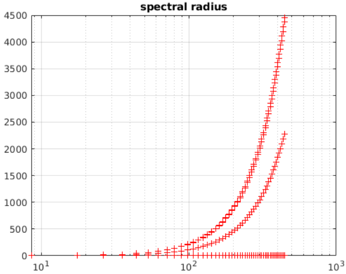
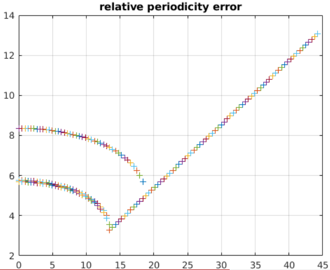
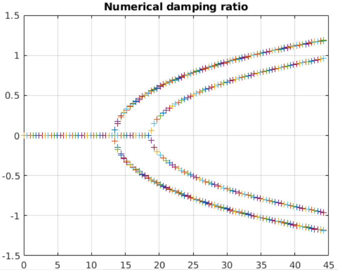
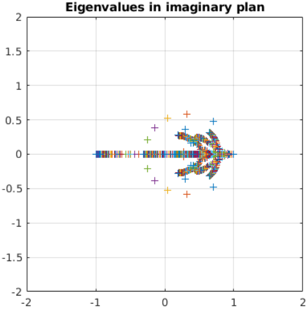


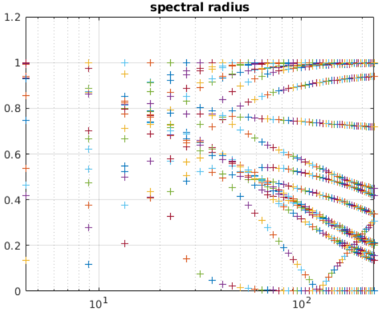






2D element : Implicit





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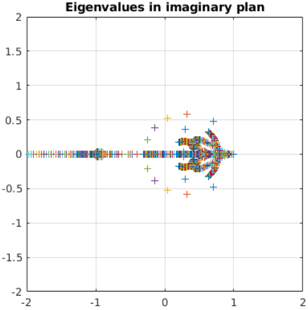

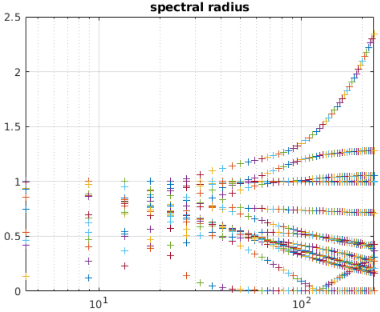

Conclusion

- Overview of the analysis of stability for integration method.
- Stability of 1D and 2D perfectly matched layer (for implicit time integration method).
- Properties of attenuation and delay of unstability for 1D PML.
- Further work :
  - Same analysis of relative periodicity error and numerical damping for 2D PML.
  - Implementation of 2D Perfectly matched layer within Akantu.