Absorbing Boundary Layers in Time Domain Elastodynamics :

Two-Dimensional Perfectly Matched Layer

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Student : Alexandre Poulain

Project supervisor : Michael Brun

Laboratory director : Jean-François Molinari

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Abstract

Absorbing boundary layers is a common method to solve numerically wave propagation phenomena in infinite or unbounded domains. Several techniques exist to construct these boundaries such as the introduction of Rayleigh damping in the absorbing layer or as presented in this report, perfectly matched layers (PML). They find their application in many fields such as electromagnetism and elastodynamics. Due to the complexity of the formulation of these PMLs, their construction and use remain a challenge. A large number of formulations and implementations have been proposed through the years to tackle some problems such as instability and performance. We attend in this report to present the formulation of unsplit field two-dimensional perfectly matched layer based on the weak form of the equations of elastodynamics.

First of all, the presentation of the construction of the PML will be detailed, including an extensive description of the governing equations. The second part will be dedicated to the numerical results for a simple test case and a special attention will be dedicated to the reflection of the waves. We will shortly see that the PML is efficient to attenuate incident wave with less than 1 percent of the incident wave reflected in some extreme cases where the length of the PML and the coefficients of the attenuation function are reduced to their minimum. The last part will describe the proof of the numerical stability of the integration method associated with the PML. It will be proved that the scheme imployed is stable and the theoretical stability will be reviewed after that since stability of one doesn't lead to stability of the other.

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Introduction

The study of wave propagation in unbounded media is an growing topic of research and takes its applications in many engineering fields. The solution of the elastodynamic problem using infinite domains is interesting for the simulation of earthquake ground motion or for soil-structure problems where the reflection of the wave at the boundaries needs to be eliminated. To address this problem, a common practice is to add a layer surrounding the domain of interest in order to absorb the outgoing wave.
Different methods have been developed to simulate wave propagation in unbounded media : the infinite elements methods, originally developed by Ungless [26] and Bettess [23], which is close to the concept of finite elements but adds a new formulation including an infinite extent of the element region and shape functions. This method allows the approximation of the decaying laws governing the waves radiation process at infinity. The technique used here is to use finite element with their end nodes placed at infinity. The issues encountered with such technique is the same as the following method. The appropriate absorbing boundary conditions is a method involving specific conditions at the model boundaries to approximate the radiation condition for the elastic waves [4]. The definition of these boundaries using this method leads to non stable scheme and spurious reflections cannot be avoided. These conditions are also not useful for practical calculations since they involve complex system of equations. The two first methods presented here presents the same drawbacks in terms of computation and analysis. However in order to solve these problems other methods have been developed leading to more efficient schemes.
As presented in this report, another method is to define a new layer to the simulation : An absorbing layer. The Rayleigh Damping layers is based on a Rayleigh/Caughey damping formulation to express the damping matrix $[C]$ in the classic formulation of elastodynamic problems. This damping matrix in the Rayleigh formulation can be expressed as a combination of stiffness and mass matrix. This damping matrix for finite elements is often already available in existing Finite Element software which makes this method really practical to use. In [28], the efficiency of the method is given and shows a satisfactory behaviour of the method in terms of efficiency for the one and the two dimensional case.
Perfectly matched layers is another absorbing boundary layers method, that absorbs almost perfectly incident waves without any reflection from the truncation interface for all angles of incidence and frequencies. The wave entering into the PML decays with distance according to a user-defined decay function. The property of the non-reflection at the truncation interface is true in theory for the continuum case. Once a spatial discretization is used, numerical reflections are present but they can be attenuated using the parameters of the PML. These user-defined parameters can also increase the accuracy of the scheme used and even reduce the computational cost. Perfectly matched layers is a concept first introduced by Bérenger for the simulation of electromagnetic waves [19]. He used a split-field formulation and it arises from the use of complex-valued coordinate stretching in the electromagnetic wave equations [10]. The field-splitting formulation permits to avoid convolutional operations in the time domain when the resulting forms are inverted back into the frequency domain and it is based on the partition of the variables into two components : parallel and perpendicular to the truncation boundary. The drawback of this technique is that it alters the structure of the underlying differential equations and thus increases the number of unknowns. Another problem with Bérenger's split-field PML is that the problem is only weakly well-posed and thus prones to instability [2]. This led to the development of strongly well-posed unsplit formulation [1] but it turns out that these formulations also suffer from instability and need further manipulation of the equations to ensure it [3]. However, the PML have been adapted for other linear wave equations such as the Helmholtz equation (scalar wave equation) [25, 30, 17], linearised Euler equations [16] or for the wave propagation in poroelastic materials [33]. An extensive discussion of these different methods is beyond the scope of this report since we focus on elastodynamic problems.
Indeed, the concept of PML was first adapted to elastodynamic wave propagation problems by Hastings et al [18]. This formulation was obtained by taking the split-field formulation of Bérenger and directly

applying it to the P- and S-wave potentials. This formulation was obtained in term of displacement potentials and yields to a velocity-stress finite-difference method. The proof of the absorptive property of the PML was developed by Chew and Liu [9] : they developed in the same time a new split-field formulation for isotropic media using complex-valued coordinate stretching to obtain the equations governing the PML. Following the same idea Liu [24] introduced a split-field PMLs for time-dependent elastic waves in cylindrical and spherical coordinates. Other split-field, time-domain PMLs for the velocity–stress formulation have been obtained and we refer to [34, 12, 8] for the details of these methods and the presentation of a finite-differences-time-domain (FDTD) implementation of them. Another split-field formulation was introduced by Komatitsch and Tromp [20] where the stress term is eliminated and the displacement is split into four components. This results in a third-order in time semi-discrete forms for the four displacement fields or can be expressed by a second-order system coupled with one first-order equation for one of the displacement field. The main drawback of their method is its complexity but it is the first displacement only formulation for elastodynamics.

As we have seen before split-field PML suffer from instability since they are weakly well-posed. Wang [32] introduced an unsplit formulation for finite-difference modeling of elastic wave propagation using convolution features (CPML). In contrast of the original formulation of CPML from the electromagnetic where they used complex-frequency-shifted stretching functions [29, 27], Wang used standard stretching functions for its PML implementation. In this report, we will develop an unsplit formulation using the finite element framework in the same spirit as Basu and Chopra. In [5], they introduced an unsplit-field PML for time-harmonic elastodynamics in 2D media. In [6], they developed the time-domain implementation of their PML and in [31], Basu extended its 2D formulation to 3D media using an explicit scheme.

Based on a decomposition of the elastodynamics equations as a first-order system, Cohen and Fauqueux [11] derived a split-field formulation where the strain tensor is split and they had to introduce independent stress variables to account for the split strain tensor components. This method was implemented using a mixed finite element approach and spectral elements. A different formulation was obtained by Festa and Vilotte [15] where they followed classic lines for reducing the second-order displacement-only elastodynamic problem to a first-order in time system. Instead of slitting the strain tensor, they used split-fields for both the velocity and stress components. In the framework of unsplit PML, Drossaert and Giannopoulos [14] described an alternative implementation based on recursive integration (RIPML). But this implementation presents less performance than the CPML for elastodynamics using the complex-frequency-shifted stretching functions[13]. Meza-Fajardo and Papageorgiou [22] discussed a novel PML approach. In the standard approach of PML the coordinate-stretching and associated decay functions are used along the direction normal to the PML interface. Meza-Fajardo and Papageorgiou introduced them along all coordinate directions resulting in a split-field, non-convolutional M-PML which shows superior performance compared to standard PML.

All this literature survey of the different formulation of the PML can be summarised in the following table borrowed from [21] :

Implementation	split-field	unsplit-field
FD	Chew and Liu [9] Hastings et al. [18] Liu[24] Collino and Tsogka [12]	Wang and Tang [32] Drossaert and Giannopoulos [14, 13] Komatitsch and Martin [20]
FE/SE	Bécache et al. [8] Komatitsch and Tromp [20] Cohen and Fauqueux [11] Festa and Vilotte [15] Meza-Fajardo and Papageorgiou [22]	Basu and Chopra [5] Basu [6]

Table 1 – PML implementations in time-domain elastodynamics

The stability of the PML have been studied for mostly isotropic cases but Collino and Tsogka [12] showed that the split-field standard PML is adequate in the case of anisotropic conditions. Also Bécache [7] studied the stability of the PML and the effect of anisotropy : She showed that the standard PML is stable for isotropic cases and conditionally unstable for anisotropic applications. Bécache also proposed necessary conditions for stability in the form of inequalities choice of the stretching function.

In the light of these previous works we will attend in this report to describe the formulations of a one and a two dimensional unsplit-field displacement-based PML. The implementations will be realised in the framework of finite elements. First of all we will describe in details the construction of the equations of the PML and also its formulation in the context of finite elements. After this theoretical description, the numerical results obtained on test cases involving a Ricker wave will be presented. In the last part, we will discuss the stability of the PML : Theoretical and numerical stability will be presented as well as the proof of the well-posedness of the problem.

1 Description

1.1 One dimensional perfectly matched layer

In this part we will describe the construction of the one dimensional PML from elastodynamics wave equations to the finite elements formulation.

1.1.1 Elastic medium

In the following of this report we will consider a one dimensional homogeneous isotropic elastic continuum. In such medium the displacement $u(x,t)$ is governed by the following equations :

$$\begin{aligned} \frac{\partial \sigma}{\partial x} &= \rho \ddot{u} \\ \sigma &= E\epsilon \\ \epsilon &= \frac{\partial u}{\partial x} \end{aligned} \tag{1}$$

In the equations of 1 and in the following of this report we will omit the dependence on x and t since, without any indications, is obvious. In fact the displacement u, the stress σ, the strain ϵ and the acceleration $\ddot{u}$ are scalar functions. ρ is the density and E is the young modulus of the medium.

1.1.2 Strong form of the PML in the frequency domain

As in the work of Basu and Chopra [5] we first begin by introducing the complex-valued coordinate stretching functions λ which is a non-zeros function everywhere. The idea is to replace the real coordinates x by the complex ones $x \to \tilde{x} : \mathbb{R} \to \mathbb{C}$. This function represents a mapping of the real spatial coordinates onto the complex space. Let us also introduce the attenuation functions f^p and f^e : an explicit formulation of this functions will be described later in this report. f^p is used to attenuate propagating waves and f^e in other hand serves to attenuate evanescent waves. Both of these real-valued positive functions vanish at the interface between the physical medium and the PML so that the PML matches perfectly the physical domain. The complex coordinates are defined by :

$$\begin{aligned} \tilde{x} &= \int_{s=0}^{x} \lambda(s,\omega) ds \\ &= \int_{s=0}^{x} \left[1 + f^e(s) + \frac{c_s}{i\omega L_p} f^p(s) \right] ds \end{aligned} \tag{2}$$

And the derivative with respect to x by :

$$\frac{\partial \tilde{x}}{\partial x} = 1 + f^e(x) + \frac{c_s}{i\omega L_p} f^p(x) = \lambda(x) = 1 + f^e(x) - \frac{ic_s}{\omega L_p} f^p(x) \tag{3}$$

In the equations 2 and 3 c_s stands for the celerity of the S-waves and L_p is the length of the PML. The functions of attenuation have the following the expressions :

$$f^p(x) = \alpha_p \left(\frac{x - x_0}{L_p} \right)^m \tag{4}$$

$$f^e(x) = \alpha_e \left(\frac{x - x_0}{L_p} \right)^m \tag{5}$$

In the equations 4 and 5, $\alpha_p > 0$ and $\alpha_e > 0$ respectively are the coefficients of attenuation for propagating and evanescent waves. Using these complex coordinates we can express the governing equations of motion in the frequency domain :

$$\frac{1}{\lambda(x)}\frac{\partial \overline{\sigma}}{\partial x} = -\rho\omega^2\overline{u} \tag{6}$$

In equation 6, ρ is the density of the PML, σ is the stress tensor and $u(x,t)$ is the position. The line over the elements define that they are defined in the frequency domain. In the following of this report we will omit the dependence on x. We can express the derivative of the stress tensor :

$$\begin{aligned}\frac{\partial \overline{\sigma}}{\partial x} &= \rho(i\omega)^2\overline{u}\left[(1+f^e(x)) + \frac{c_s}{i\omega L_p}f^p(x)\right] \\ &= \rho(i\omega)^2(1+f^e(x))\overline{u} + \frac{c_s}{L_p}(i\omega)f^p(x)\overline{u}\end{aligned} \tag{7}$$

Using the inverse Fourier transform, we obtain the equation of motion in the time domain :

$$\frac{\partial \sigma}{\partial x} = (1+f^e(x))\rho\ddot{u} + \frac{\rho c_s}{L_p}f^p(x)\dot{u} \tag{8}$$

Since we place ourselves in the context of linear elastic we have the constitutive relationship :

$$\sigma = E\epsilon \tag{9}$$

with E the Young modulus of the PML. Let us now introduce the strain-displacement relationship in the frequency domain :

$$\overline{\epsilon} = \frac{1}{2}\left(\frac{\partial \overline{u}}{\partial x}\frac{\partial x}{\partial \tilde{x}} + \frac{\partial \overline{u}}{\partial x}\frac{\partial x}{\partial \tilde{x}}\right) = \frac{\partial \overline{u}}{\partial x}\frac{\partial x}{\partial \tilde{x}} = \frac{\partial \overline{u}}{\partial x}\frac{1}{\lambda} \tag{10}$$

Using this equation and the expression of the stretching function 3 we can define the following equivalence :

$$\lambda\overline{\epsilon} = \frac{\partial \overline{u}}{\partial x} \iff \left(1 + f^e(x) + \frac{c_s}{i\omega L_p}f^p(x)\right)\overline{\epsilon} = \frac{\partial \overline{u}}{\partial x} \tag{11}$$

Again using the inverse Fourier transform, we can recast the previous equation 11 in the time domain :

$$(1+f^e(x))\epsilon + \frac{c_s}{L_p}f^p(x)H = \frac{\partial u}{\partial x} \tag{12}$$

with $H = \int_0^t \epsilon(x,s)ds$.
The strong form of the PML in the time domain is obtained using the equations 12, 8 and 9. To summarize the problem obtained so far we can recall these equation into the following problem :

$$\begin{aligned}\frac{\partial \sigma}{\partial x} &= (1+f^e(x))\rho\ddot{u} + \frac{\rho c_s}{L_p}f^p(x)\dot{u} \\ \sigma &= E\epsilon \\ \frac{\partial u}{\partial x} &= (1+f^e(x))\epsilon + \frac{c_s}{L_p}f^p(x)H\end{aligned} \tag{13}$$

1.1.3 Weak form of the PML and finite elements formulation

Following the work of Basu and Chopra [5], we will use a displacement-based space discretization in the framework of standard finite elements. Thus let v be the compactly supported test function vanishing at the boundaries and belonging to an appropriate space V. The weak formulation of the PML is obtained by multiplying the left and right sides of 8 by v and then integrating the all equation over the computational domain Ω.

$$\int_\Omega \frac{\partial \sigma}{\partial x} v \mathrm{d}\Omega = \int_\Omega (1 + f^e(x))\rho \ddot{u} v \mathrm{d}\Omega + \int_\Omega \frac{\rho c_s}{L_p} f^p(x) \dot{u} v \mathrm{d}\Omega \tag{14}$$

Let us focus on the left hand side of the equation 14, integration by parts yields to the following expression :

$$LHS = -\int_\Omega \sigma \frac{\partial v}{\partial x} \mathrm{d}\Omega + \int_{\partial\Omega} v \cdot \sigma n \mathrm{d}S \tag{15}$$

The test have been chosen such that it vanishes at the boundary of the computational domain. Therefore the rightmost term of the equation 15 is equal to $\int_{\partial\Omega_p} v \cdot \sigma n \mathrm{d}S = 0$. Using the finite elements method, we can define the basis functions as :

$$N_i(x_j) \begin{cases} 1, & \text{if } x_j = x_i \\ 0, & \text{Otherwise} \end{cases} \tag{16}$$

Therefore for one bar element composed of two nodes $[-1, 1]$ we can define the following basis functions :

$$\begin{cases} N_1(x) = \frac{1}{2}(1 - x) \\ N_2(x) = \frac{1}{2}(1 + x) \end{cases} \tag{17}$$

Using this basis functions for each element the components of the problem can be decomposed as a linear combination of this two functions.

$$u_e(x) = \sum_{i=1}^{2} u_i(t) N_i(x) \tag{18}$$

with $x \in [-1, 1]$, $u_i(t)$ a time dependent coefficient. As we can see we get rid of the spatial dependence for the coefficient u_i and we can move it out of the integrals. Thus for each element Ω_e the weak form of the equation of motion in the PML can be rewrite as :

$$\{v_e\}^T \int_{\Omega_e} (1 + f^e(x))\, \rho\, [N]^T [N] \,\mathrm{d}\Omega_e \{\ddot{u}_e\} + \{v_e\}^T \int_{\Omega_e} \frac{\rho c_s}{L_p} [N]^T [N] \,\mathrm{d}\Omega_e \{\dot{u}_e\} + \int_{\Omega_e} \left(\frac{\partial v}{\partial x}\sigma\right) \mathrm{d}\Omega_e = 0 \tag{19}$$

With $\{v_e\} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ with v_1 and v_2 being respectively the value of the coefficient at the node 1 and 2. To solve the problem of the partial derivative with respect to x, we need to introduce the simple first order approximation operator $[B] = \frac{1}{L_e}\begin{bmatrix} 1 & -1 \end{bmatrix}$ with L_e the length of an element.

$$\frac{\partial v_e}{\partial x} = [B]\{v_e\} \tag{20}$$

Thus the right most term can be rewrite as :

$$\int_{\Omega_e} \left(\frac{\partial v}{\partial x}\sigma\right) \mathrm{d}\Omega_e = \{v_e\}^T \int_{\Omega_e} [B]^T \sigma \mathrm{d}\Omega_e \tag{21}$$

Since we consider an one dimensional bar element, the integral over the element can be re-expressed as :

$$\mathrm{d}\Omega_e = S \frac{L_e}{2} \mathrm{d}\xi \tag{22}$$

With S the section of the bar element. This leads us to the following formulation :

$$M_e \{\ddot{u}_e\} + C_e \{\dot{u}_e\} + P_{e,int} = 0 \tag{23}$$

With

$$M_e = \int_{\xi=-1}^{1} (1 + f^e(\xi))\rho [N]^T [N] S \frac{L_e}{2} \mathrm{d}\xi \tag{24}$$

$$C_e = \int_{\xi=-1}^{1} \frac{\rho c_s}{L_p} f^p(\xi) [N]^T [N] S \frac{L_e}{2} \mathrm{d}\xi \tag{25}$$

And

$$P_{e,int} = \int_{\xi=-1}^{1} [B]^T \sigma S \frac{L_e}{2} \mathrm{d}\xi \tag{26}$$

We can clearly recognise here the same formulation as the standard elastodynamic problem. In fact if we think about the formulation over the all computational domain, we retrieve the following formulation :

$$M \{\ddot{u}\} + C \{\dot{u}\} + P_{int} = 0 \tag{27}$$

Where M, the mass matrix, is defined per blocks of shape 2×2 on the diagonal using the equation 24 and we can obtain the same for C, the damping matrix, using the formula 25.
The only problem remaining in this formulation is the stress in the expression of the internal forces P_{int}. In fact we will give an explicit formulation of this latter concurrently with the description of the temporal discretization and the presentation of the temporal integration method used.

1.1.4 Temporal discretization

We will consider a simple time discretization $t = nh = t^n$ with h the step of the temporal discretization and n the index. If we consider the time step $n+1$ and go back to the equation 12 we can write :

$$\begin{aligned} \frac{\partial u^{n+1}}{\partial x} &= (1 + f^e(x))\epsilon^{n+1} + \frac{c_s}{L_p} H^{n+1} \\ &= (1 + f^e(x))\epsilon^{n+1} + \frac{c_s}{L_p} \left(H^n + h\epsilon^{n+1}\right) \end{aligned} \tag{28}$$

This expression is obtained by the temporal discretization and its effect on $H = \int_0^t \epsilon(x,s)ds$ turning this expression into $H^{n+1} = k \sum_{i=0}^{n+1} \epsilon(x, t^i)$. We can change the equation 28 by factorising by ϵ^{n+1} :

$$\left[(1 + f^e(x)) + \frac{hc_s}{L_p} f^p(x)\right] \epsilon^{n+1} = \frac{\partial u^{n+1}}{\partial x} - \frac{c_s f^p(x)}{L_p} H^n \tag{29}$$

Using this equation and the first order approximation of the spatial derivative we can express the stress using Hooke's law 9 :

$$\begin{aligned} \sigma^{n+1} &= E\epsilon^{n+1} \\ &= \frac{E}{\alpha(x)} [B] \{u_e^{n+1}\} - \frac{E c_s f^p(x)}{\alpha(x) L_p} H^n \end{aligned} \tag{30}$$

With $\alpha(x) = \left[(1 + f^e(x)) + \frac{hc_s}{L_p} f^p(x)\right]$. The expression of the internal forces for an element is given by the equation 26, expressing this equation at time $n+1$ and using the expression of the stress 30 we obtain :

$$\begin{aligned} P_{e,int}^{n+1} &= \int_{\xi=-1}^{1} [B]^T \sigma^{n+1} S \frac{L_e}{2} \mathrm{d}\xi \\ &= \left(\int_{\xi=-1}^{1} \frac{E}{\alpha(\xi)} [B]^T [B] S \frac{L_e}{2} \mathrm{d}\xi\right) \{u_e^{n+1}\} - \left(\int_{\xi=-1}^{1} [B]^T \frac{c_s}{L_p} \frac{f^p(\xi)}{\alpha(\xi)} E H^n S \frac{L_e}{2} \mathrm{d}\xi\right) \end{aligned} \tag{31}$$

Thus the stiffness matrix K can be defined per bloc as the mass and damping matrices. The blocs have the following form :

$$K_e = \int_{\xi=-1}^{1} \frac{E}{\alpha(\xi)} [B]^T [B] S \frac{L_e}{2} \mathrm{d}\xi \tag{32}$$

Therefore for each element we obtain the following equation :

$$[M_e]\{\ddot{u}_e^n\} + [C_e]\{\dot{u}_e^n\} + [K_e]\{u_e^n\} = \int_{\xi=-1}^{1} [B]^T \frac{c_s}{L_p} \frac{f^p(\xi)}{\alpha(\xi)} EH^n S \frac{L_e}{2} \mathrm{d}\xi \tag{33}$$

The right hand side term is the internal forces at time n : $P_{e,int}^n$. Therefore we obtain for the all computational domain the following relation :

$$[M]\{\ddot{u}^n\} + [C]\{\dot{u}^n\} + [K]\{u^n\} = P_{int}^n \tag{34}$$

We can observe that our unknowns are the displacement u, the velocity $\dot{u}$, the acceleration $\ddot{u}$ and the internal forces P_{int}. An important remark is the number of unknowns of our formulation of the unsplit-field PML is lesser than for split-field standard PML.
In order to resolve the temporal integration, we will employ a standard Newmark-β scheme and the integrals over the each element will be evaluated using Gauss quadrature. In our case we used two points of quadrature in order to obtain an approximate value of the integral. For each element we have the two quadrature points $\xi_1 = -\frac{1}{\sqrt{3}}$ and $\xi_2 = \frac{1}{\sqrt{3}}$ with their associated weights $w_1 = w_2 = 1$. Thus the integrals over the elements can be expressed as a sum of the interior of the integral evaluated at the quadrature points.

1.2 Two-dimensional perfectly matched layer

Before describing the governing equations of the two-dimensional PML let us introduce the following notations : Ω_{PML} will denote the PML domain which is bounded by $\Gamma PML = \Gamma_{PML}^D \cup \Gamma_{PML}^N$. Γ_{PML}^D corresponds to the boundary where the Dirichlet condition will be applied (imposed displacement). Γ_{PML}^N is the boundary where the tractions will be applied and represents the boundary of the Neumann conditions. The intersection of these two boundary defined by their imposed conditions is null : $\Gamma_{PML}^D \cap \Gamma_{PML}^N = \emptyset$. The temporal domain will be denoted by $J = [0, T]$ with T the end time.

1.2.1 Strong form of PML

As in the context of the one-dimensional PML, the classical formulation begins with the introduction of the complex-valued coordinates stretching functions λ_i. They are used to replace the real coordinates by the complex ones : $x_i \to \tilde{x}_i : \mathbb{R} \to \mathbb{C}$.

$$\lambda_i(x_i) = \frac{\partial \tilde{x}_i}{\partial x_i} = 1 + f_i^e(x_i) - \frac{i}{bk_s} f_i^p(x_i) \tag{35}$$

where b is the characteristic length of the problem. $k_s = \frac{\omega}{c_s}$ denotes the wavenumber (ω is the pulsation and c_s is the celerity of shear waves). f_i^p and f_i^e are the attenuation functions for respectively propagating and evanescent waves. They are written as polynomial of order n.

$$f_i^\alpha = a_\alpha \left(\frac{x_i - x_0}{L_p}\right), x_i \in [x0, x0 + d] \tag{36}$$

The tunable property of the attenuation function relies mainly on the formulation of the attenuation functions. The value of the coefficients of attenuation a_p, a_e, the order of the polynomial n, and the size of the PML L_p can be defined by the user. As we will see in the section concerning the numercal results these parameters need to be chosen carefully depending on the problem to obtain the "best" result possible. The concept of "best result" depends of course of the expectations of the user. Accuracy versus performance is the recurent dilemma for all numerical simulations. The perfectly matched layer does not get out of this rule. Moreover an increase of the attenuation coefficients leads to a higher reflection of the wave from the interface between the physical medium and the PML. In order to choose the value these coefficients, we can use the reflection coefficient for an incident pressure wave given by [6] :

$$R_{pp} = \frac{cos(\theta + \theta_s)}{cos(\theta - \theta_s)} exp\left[-2\frac{c_s}{c_p} F_1(L_p) cos(\theta)\right] \tag{37}$$

Where c_p stands for the velocity of P-waves. The incident P-wave is characterised by θ its angle of incidence and θ_s its reflective angle after being reflected at the end of the PML. F_1 corresponds to the integral over the PML of the attenuation function for propagating waves : $F_1(L_p) = \int_{s=0}^{L_p} f^p(s)ds = \frac{\beta_0 L_p}{n+1}$. Thus, the attenuation coefficients can be expressed in function of the reflection coefficient :

$$a_\alpha = \ln\left(\frac{cos(\theta + \theta_s)}{R_{pp} cos(\theta - \theta_s)}\right) \frac{c_p}{c_s} \frac{n+1}{L_p cos(\theta)} \tag{38}$$

If we consider that the incident wave has an angle of $\theta = \theta_s = 0$ therefore the value of the coeffecient has the form :

$$a_\alpha = \ln\left(\frac{1}{R_{pp}}\right) \frac{c_p}{c_s} \frac{n+1}{L_p cos(\theta)} \tag{39}$$

Using the complex-valued coordinates stretching functions 35, the strong form of the equation of motion for the PML in the frequency domain is defined by :

$$\begin{cases} \sum_j \frac{1}{\lambda_j(x_j)} \frac{\partial \sigma_{ij}}{\partial x_j} & = -\omega^2 \rho u_j \\ \sigma_{ij} & = \sum_{k,l} C_{ijkl} \epsilon_{ij} \\ \epsilon_{ij} & = \frac{1}{2}\left(\frac{1}{\lambda_j(x_j)} \frac{\partial u_i}{\partial x_j} + \frac{1}{\lambda_i(x_i)} \frac{\partial u_j}{\partial x_i}\right) \end{cases} \tag{40}$$

Where C_{ijkl} are the components of the elastic constitutive tensor.

The strong form of the PML in the temporal domain can be obtained by inverse Fourier transform. Indeed the introduction of the complex-valued coordinates stretching functions makes the application of this inverse easier. In the following equation the number of lines below a tensor will precise its order.

$$\begin{cases} div(\underline{\underline{\sigma}} \tilde{F}^e + \underline{\underline{\Sigma}} \tilde{F}^p) = \rho f_m \underline{\ddot{u}} + \rho \frac{c_s}{b} f_c \underline{\dot{u}} + \frac{\mu}{b^2} f_k \underline{u}, & \text{In } \Omega_{PML} \times J \\ \underline{\underline{\sigma}} = C : \underline{\underline{\epsilon}}, & \text{In } \Omega_{PML} \times J \\ F^{eT} \underline{\underline{\dot{\epsilon}}} F^e + F^{pT} \underline{\underline{\epsilon}} F^e + F^{eT} \underline{\underline{\epsilon}} F^p + F^{pT} \underline{\underline{E}} F^p = ... & \\ \frac{1}{2}(\nabla \underline{\dot{u}}^T F^e + F^{eT} \nabla \underline{\dot{u}}) + \frac{1}{2}(\nabla \underline{u}^T F^p + F^{pT} \nabla \underline{u}), & \text{In } \Omega_{PML} \times J \end{cases} \tag{41}$$

submitted to the homogeneous boundary conditions :

$$\begin{cases} \underline{u} = 0, & \text{on } \Gamma^D_{PML} \\ (\underline{\underline{\sigma}} \tilde{F}^e + \underline{\underline{\Sigma}} \tilde{F}^p).n, & \text{on } \Gamma^N_{PML} \end{cases} \tag{42}$$

Let us now summarize the form of the different matrices $F^e, F^p, \tilde{F}^e$ and $\tilde{F}^p$ in the above equations.

$$F^e = \begin{bmatrix} 1 + f_1^e(x1) & 0 \\ 0 & 1 + f_2^e(x2) \end{bmatrix}, F^p = \begin{bmatrix} \frac{c_s}{b} f_1^p(x1) & 0 \\ 0 & \frac{c_s}{b} f_2^p(x2) \end{bmatrix} \tag{43}$$

$$\tilde{F}^e = \begin{bmatrix} 1+f_2^e(x2) & 0 \\ 0 & 1+f_1^e(x1) \end{bmatrix}, \tilde{F}^p = \begin{bmatrix} \frac{c_s}{b} f_2^p(x2) & 0 \\ 0 & \frac{c_s}{b} f_1^p(x1) \end{bmatrix} \tag{44}$$

Let is now focus on the first equation of 41, the functions f_m, f_c and f_k depend on the attenuation function and take the form :

$$\begin{cases} f_m = (1+f_1^e(x1))(1+f_2^e(x2)) \\ f_c = (1+f_1^e(x1))f_2^p(x2) + (1+f_2^e(x2))f_1^p(x1) \\ f_k = f_1^p(x_1)f_2^p(x_2) \end{cases} \tag{45}$$

The next elements to define, which appear in the first and the third equations of 41 is the integral of the stress and the strain.

$$\underline{\underline{\Sigma}} = \int_0^t \underline{\underline{\sigma}} dt, \underline{\underline{E}} = \int_0^t \underline{\underline{\epsilon}} dt \tag{46}$$

1.3 Weak form of 2D PML

Let us introduce the test function $\underline{v}$ belonging to an admisible space of solution V. Premultiplying the first equation of the strong form of the equation of motion within the PML 41 by $\underline{v}$ and integrating over the PML domain gives :

$$\int_{\Omega_{PML}} \underline{v}.div(\underline{\underline{\sigma}}\tilde{F}^e + \underline{\underline{\Sigma}}\tilde{F}^p) d\Omega_{PML} = \int_{\Omega_{PML}} \rho f_m \underline{v}.\underline{\ddot{u}} d\Omega_{PML} + ...$$
$$\int_{\Omega_{PML}} \rho \frac{c_s}{b} f_c \underline{v}.\underline{\dot{u}} d\Omega_{PML} + \int_{\Omega_{PML}} \frac{\mu}{b^2} f_k \underline{v}.\underline{u} d\Omega_{PML}, \text{In } \Omega_{PML} \times J \tag{47}$$

Using Gauss divergence theorem and integration per parts give :

$$\int_{\Gamma_{PML}} \underline{v}.\left(\underline{\underline{\sigma}}\tilde{F}^e + \underline{\underline{\Sigma}}\tilde{F}^p\right).\underline{n} d\Gamma_{PML} = \int_{\Omega_{PML}} \rho f_m \underline{\ddot{u}}.\underline{v} d\Omega_{PML} + ...$$
$$\int_{\Omega_{PML}} \rho \frac{c_s}{b} f_c \underline{\dot{u}}.\underline{v} d\Omega_{PML} + \int_{\Omega_{PML}} \frac{\mu}{b^2} f_k \underline{u}.\underline{v} d\Omega_{PML} + \int_{\Omega_{PML}} \underline{\underline{\tilde{\epsilon}}}^e : \underline{\underline{\sigma}} d\Omega_{PML} + \int_{\Omega_{PML}} \underline{\underline{\tilde{\epsilon}}}^p : \underline{\underline{\Sigma}} d\Omega_{PML}, \text{In } \Omega_{PML} \times J$$

$\underline{n}$ is the normal vector to the boundary of the PML Γ_{PML}. The tensors $\underline{\underline{\tilde{\epsilon}}}^p$ and $\underline{\underline{\tilde{\epsilon}}}^e$ depend on the attenuation functions as :

$$\begin{cases} \underline{\underline{\tilde{\epsilon}}}^e = \frac{1}{2}\left(\nabla \underline{v}\tilde{F}^e + \tilde{F}^{eT}\nabla\underline{v}^T\right) \\ \underline{\underline{\tilde{\epsilon}}}^p = \frac{1}{2}\left(\nabla \underline{v}\tilde{F}^p + \tilde{F}^{pT}\nabla\underline{v}^T\right) \end{cases} \tag{48}$$

Using the weak form of the equation of motion within the PML in the temporal domain **??** we can define the mass, damping and stiffness matrices as :

$$m^e = \int_{\Omega_e} \rho f_m N_I N_J d\Omega_e I_d \tag{49}$$

$$c^e = \int_{\Omega_e} \rho f_c \frac{c_s}{b} N_I N_J d\Omega_e I_d \tag{50}$$

$$k^e = \int_{\Omega_e} \frac{\mu}{b^2} f_k N_I N_J d\Omega_e I_d \tag{51}$$

Conclusion

The development and the analysis of the PML show two of the main aspects of the perfectly matched layers method : its power to attenuate propagating waves and but also its complexity. Indeed we have seen the full development of the equations required to construct the one dimensional PML. Starting from the simple governing equations of a linear elastodynamic problem such as the equation of motion, the constitutive equation and the strain-displacement relationship, we derived a new problem by introducing the complex coordinates and the stretching function λ. This development allows us to obtain the strong form of the PML. Going from this formulation, we were able to construct the weak form of the PML by introducing virtual displacement in order to implement these equations using the finite elements method. Using a Newmark temporal integration method, we were capable of construct an implicit scheme on the PML for wave propagation and therefore its attenuation. As we have seen the scheme presents a very accurate behavior : it attenuates completely the wave and without any reflection from the truncation interface (using a appropriate set of parameters that we highlighted in the analysis of the effects of the parameters).

The analysis of the numerical stability proves that this implicit scheme is also unconditionally stable and shows the critical value of time step h for the explicit scheme used for the propagation of the wave in the physical medium.

However certain points remains unclear in this analysis and we need to focus in detail in how to interpret correctly the results of the numerical damping and the relative periodicity error. They will give us more information about the behavior of the scheme and its stability for the PML but also for the physical medium.

In future works we will also include a complete analysis of the computational cost of this method : computational resources and time of execution. This will be a prerequisite before implementing the PML into a finite elements code (Akantu).

We will also extend this method to the two and three dimensional cases and for anisotropic medium. The same analysis of complexity, stability and computational cost will also be done in order to provide the perfectly matched layers method for the simulation of wave propagation on unbounded domains with the maximum of information and analysis.

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