VI. Supplementary materials

A. Proof of Theorem 1

For $\mathbf{\Omega} \in T_{\mathbf{\Sigma}}\mathcal{H}_M^{++}$, the Fisher Information Metric is obtained according to Theorem 1 of [1] as

$$g_{\mathbf{\Sigma}}^{fim}(\mathbf{\Omega},\mathbf{\Omega}) = -\mathbb{E}\left[\left. \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}\left(\{\mathbf{z}_k\}|\mathbf{\Sigma}+t\partial\mathbf{\Omega},g\right)\right|_{t=0}\right]. \tag{18}$$

First, recall that the log-likelihood of the sample set is

$$\mathcal{L}\left(\{\mathbf{z}_k\}|\mathbf{\Sigma},g\right) = \sum_{k=1}^{K}\log\left(g\left(\mathrm{Tr}\left\{\mathbf{\Sigma}^{-1}\mathbf{Z}_k\right\}\right)\right) - K\log|\mathbf{\Sigma}|, \tag{19}$$

where $\mathbf{Z}_k = \mathbf{z}_k\mathbf{z}_k^H$. Through the second-order Taylor expansions around $\mathbf{\Sigma}$, one can obtain

$$\begin{aligned}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}\left(\{\mathbf{z}_k\}|\mathbf{\Sigma}+t\mathbf{\Omega}\right)\right|_{t=0} &= K\mathrm{Tr}\left\{\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\right)^2\right\}\\ &+ 2\sum_{k=1}^{K}\mathrm{Tr}\left\{\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\right)^2\mathbf{Z}_k\mathbf{\Sigma}^{-1}\right\}\phi\left(Tr\left\{\mathbf{\Sigma}^{-1}\mathbf{Z}_k\right\}\right)\\ &+ \sum_{k=1}^{K}\mathrm{Tr}^2\left\{\mathbf{\Sigma}^{-1}\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{Z}_k\right\}\phi'\left(Tr\left\{\mathbf{\Sigma}^{-1}\mathbf{Z}_k\right\}\right).\end{aligned} \tag{20}$$

In order to compute the expectations, we recall that $\mathbf{z}_k$ has the stochastic representation $\mathbf{z}_k \overset{d}{=} \sqrt{\mathcal{Q}_k}\ \mathbf{\Sigma}^{1/2}\ \mathbf{u}_k$. This allows us some simplifications since $\mathrm{Tr}\left\{\mathbf{\Sigma}^{-1}\mathbf{Z}_k\right\} = \mathcal{Q}_k$, $\mathbf{u}_k^H\mathbf{u}_k = 1$, and since that $\mathbf{u}_k$ and $\mathcal{Q}_k$ are independent (allowing to split the expectations). Hence we have for the first term:

$$\begin{aligned}&\mathbb{E}\left[\mathrm{Tr}\left\{\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\right)^2\mathbf{Z}_k\mathbf{\Sigma}^{-1}\right\}\phi\left(\mathrm{Tr}\left\{\mathbf{\Sigma}^{-1}\mathbf{Z}_k\right\}\right)\right]\\ &\mathbb{E}\left[\mathrm{Tr}\left\{\mathbf{\Sigma}^{H/2}\mathbf{\Sigma}^{-1}\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\right)^2\mathbf{\Sigma}^{1/2}\mathbf{u}_k\mathbf{u}_k^H\right\}\right]\mathbb{E}\left[\mathcal{Q}_k\phi\left(\mathcal{Q}_k\right)\right]\\ &= -\mathrm{Tr}\left\{\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\right)^2\right\},\end{aligned} \tag{21}$$

where we used $\mathbb{E}\left[\mathbf{u}_k\mathbf{u}_k^H\right] = \mathbf{I}_M/M$ (since $\mathbf{u}_k \sim \mathcal{U}(\mathbb{C}\mathcal{S}^M)$), and (2) to obtain the result $\mathbb{E}\left[\mathcal{Q}_k\phi\left(\mathcal{Q}_k\right)\right] = -M$. The second expectation is obtained by the same method as

$$\begin{aligned}&\mathbb{E}\left[\mathrm{Tr}^2\left\{\mathbf{\Sigma}^{-1}\mathbf{\Omega}\mathbf{\Sigma}^{-1}\mathbf{Z}_k\right\}\phi'\left(\mathrm{Tr}\left\{\mathbf{\Sigma}^{-1}\mathbf{Z}_k\right\}\right)\right]\\ &= \frac{\mathbb{E}\left[\mathcal{Q}_k^2\phi'\left(\mathcal{Q}_k\right)\right]}{M(M+1)}\left(\mathrm{Tr}^2\left\{\mathbf{\Omega}\mathbf{\Sigma}^{-1}\right\} + \mathrm{Tr}\left\{\left(\mathbf{\Omega}\mathbf{\Sigma}^{-1}\right)^2\right\}\right),\end{aligned} \tag{22}$$

where we used the relation from [24], giving

$$\mathbb{E}\left[\left(\mathbf{u}_k^H\mathbf{B}\mathbf{u}_k\right)^2\right] = \left(\mathrm{Tr}\left\{\mathbf{B}^2\right\} + \mathrm{Tr}^2\left\{\mathbf{B}\right\}\right)/\left(M(M+1)\right), \tag{23}$$

for an arbitrary constant matrix $\mathbf{B}$ and $\mathbf{u}_k \sim \mathcal{U}\left(\mathbb{C}\mathcal{S}^M\right)$. Eventually, by plugging (21) and (22) into (18) and (20), the Fisher Information Metric is given as:

$$g_{\mathbf{\Sigma}}^{fim}(\mathbf{\Omega},\mathbf{\Omega}) = K\alpha\mathrm{Tr}\left\{\left(\mathbf{\Sigma}^{-1}\mathbf{\Omega}\right)^2\right\} + K\beta\mathrm{Tr}^2\left\{\mathbf{\Omega}\mathbf{R}^{-1}\right\}, \tag{24}$$

with coefficients α and β defined in (7). Notice that the dependency on k in $\mathcal{Q}$ is omitted since these parameters are assumed to be i.i.d.. Also, some manipulations with $\phi'(t) = g''(t)/g(t) - \phi^2(t)$ and (2) allow to show that

$$M(M+1) - \mathbb{E}\left[\mathcal{Q}^2\phi'\left(\mathcal{Q}\right)\right] = \mathbb{E}\left[\mathcal{Q}^2\phi^2\left(\mathcal{Q}\right)\right], \tag{25}$$

which is consistent with the coefficients obtained in the parametric case of [24]. To obtain the metric $g_{\mathbf{\Sigma}}^{fim}(\mathbf{\Omega}_1,\mathbf{\Omega}_2)$ use a standard polarization formula which, after some expansions and simplifications leads to (6) and (7).

To derive the metric (8) we first remark that

$$g_{\mathbf{\Sigma}}^{ces}(\mathbf{\Omega}_1,\mathbf{\Omega}_2) = \tilde{\alpha}\mathrm{Tr}\left\{\mathbf{V}^{-1}\mathbf{\Omega}_1\mathbf{V}^{-1}\mathbf{\Omega}_2\right\} + \tilde{\beta}\mathrm{Tr}\left\{\mathbf{V}^{-1}\mathbf{\Omega}_1\right\}\mathrm{Tr}\left\{\mathbf{V}^{-1}\mathbf{\Omega}_2\right\}.$$

With such expression, $g_{\mathbf{V}}^{fim}$ can be identified to $g_{\mathbf{\Sigma}}^{fim}$ on the restricted set $\mathcal{SH}_M^{++}$, as they describe the same underlying distribution. Note that this is also consistent with the change of variable $\mathbf{\Sigma} = \sigma^2\mathbf{V}$ and Theorem 3 of [1]. Second, we also notice that the terms $\mathrm{Tr}\left\{\mathbf{V}^{-1}\mathbf{\Omega}\right\}$ are equal to 0 when $\mathbf{\Omega} \in T_{\mathbf{V}}\mathcal{SH}_M^{++}$ which cancels all the terms in β in (8).

B. Proof of Theorem 2

It is readily checked that for all $\mathbf{\Sigma} \in \mathcal{H}_M^{++}$ the function (6) is symmetric and bilinear. It remains to determine whether it is positive-definite. Let $\mathbf{\Sigma} \in \mathcal{H}_M^{++}$ and $\mathbf{\Omega} \in \mathcal{H}_M$, and let $\mathbf{U}\,\mathbf{\Lambda}\,\mathbf{U}^T$ be the eigenvalue decomposition of $\mathbf{\Sigma}^{-1/2}\,\mathbf{\Omega}\,\mathbf{\Sigma}^{-1/2}$. One can first check that we need $\alpha > 0$ because the term on the right can be canceled for $\mathbf{\Omega}$ different from $\mathbf{0}$. We have

$$\begin{aligned}g_{\mathbf{\Sigma}}^{ces}(\mathbf{\Omega},\mathbf{\Omega}) &= \alpha\,\mathrm{Tr}(\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}\,\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}) + \beta\left(\mathrm{Tr}(\mathbf{\Sigma}^{-1}\,\mathbf{\Omega})\right)^2\\ &= \alpha\,\mathrm{Tr}(\mathbf{U}\,\mathbf{\Lambda}^2\,\mathbf{U}^T) + \beta\left(\mathrm{Tr}(\mathbf{U}\,\mathbf{\Lambda}\,\mathbf{U}^T)\right)^2\\ &= \alpha\,\mathrm{Tr}(\mathbf{\Lambda}^2) + \beta\,(\mathrm{Tr}(\mathbf{\Lambda}))^2.\end{aligned} \tag{26}$$

One can notice that $\mathrm{Tr}(\mathbf{\Lambda}^2) = \|\mathrm{diag}(\mathbf{\Lambda})\|_2^2$ and $(\mathrm{Tr}(\mathbf{\Lambda}))^2 \leq \|\mathrm{diag}(\mathbf{\Lambda})\|_1^2$, where $\mathrm{diag}(\cdot)$ returns the vector of diagonal elements of its argument, and $\|\cdot\|_2$ and $\|\cdot\|_1$ denote the L2 and L1 norms, respectively. From the Cauchy-Schwarz inequality, we have $\|\mathrm{diag}(\mathbf{\Lambda})\|_1^2 \leq M\|\mathrm{diag}(\mathbf{\Lambda})\|_2^2$. It follows that $g_{\mathbf{\Sigma}}^{ces}(\mathbf{\Omega},\mathbf{\Omega}) > 0$ if $\alpha + M\beta > 0$. Now, the directional derivative of $g_{\mathbf{\Sigma}}^{ces}(\mathbf{\Omega}_1,\mathbf{\Omega}_2)$ in the direction $\mathbf{\Omega}_3$, where $\mathbf{\Sigma} \in \mathcal{H}_M^{++}$ and $\mathbf{\Omega}_1,\mathbf{\Omega}_2,\mathbf{\Omega}_3 \in T_{\mathbf{\Sigma}}\mathcal{H}_M^{++}$ is

$$\begin{aligned}\mathrm{D}\,g_{\mathbf{\Sigma}}^{ces}(\mathbf{\Omega}_1,\mathbf{\Omega}_2)[\mathbf{\Omega}_3] &= g_{\mathbf{\Sigma}}^{ces}(\mathrm{D}\,\mathbf{\Omega}_1[\mathbf{\Omega}_3],\mathbf{\Omega}_2) + g_{\mathbf{\Sigma}}^{ces}(\mathbf{\Omega}_1,\mathrm{D}\,\mathbf{\Omega}_2[\mathbf{\Omega}_3])\\ &- \beta\,\mathrm{Tr}(\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_3\,\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_1)\,\mathrm{Tr}(\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_2)\\ &- \beta\,\mathrm{Tr}(\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_1)\,\mathrm{Tr}(\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_3\,\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_2)\\ &- \alpha\,\mathrm{Tr}(\mathbf{\Sigma}^{-1}(\mathbf{\Omega}_3\,\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_1 + \mathbf{\Omega}_1\,\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_3)\,\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_2)\end{aligned}$$

It then follows from the Koszul formula (equation (5.11) in [34]) that the Levi-Civita connection ∇ of $\mathbf{\Omega}_2$ in the direction $\mathbf{\Omega}_1$ on $\mathcal{H}_M^{++}$ endowed with metric (6) which is defined for all $\mathbf{\Sigma} \in \mathcal{H}_M^{++}$

$$\nabla_{\mathbf{\Omega}_1}\mathbf{\Omega}_2 = \mathrm{D}\,\mathbf{\Omega}_2[\mathbf{\Omega}_1] - \mathrm{sym}(\mathbf{\Omega}_2\,\mathbf{\Sigma}^{-1}\,\mathbf{\Omega}_1), \tag{27}$$

where $\mathrm{sym}(\cdot)$ is the operator that returns the symmetrical part of its argument. The Levi-Civita connection is the same as for the classical Riemannian metric in our case and we therefore have the same geodesics, i.e. γ on $\mathcal{H}_M^{++}$, defined for all $\mathbf{\Sigma} \in \mathcal{H}_M^{++}$ and $\mathbf{\Omega} \in T_{\mathbf{\Sigma}}\mathcal{H}_M^{++}$ as

$$\gamma(t) = \mathbf{\Sigma}^{1/2}\exp(t\,\mathbf{\Sigma}^{-1/2}\,\mathbf{\Omega}\,\mathbf{\Sigma}^{-1/2})\,\mathbf{\Sigma}^{1/2}, \tag{28}$$

where $\exp(\cdot)$ denotes the matrix exponential. Furthermore, one can check that the metric (6) is invariant by congruence, *i.e.*

$$g_{\mathbf{U}\,\mathbf{\Sigma}\,\mathbf{U}^T}^{ces}(\mathbf{U}\,\mathbf{\Omega}_1\,\mathbf{U}^T,\mathbf{U}\,\mathbf{\Omega}_2\,\mathbf{U}^T) = g_{\mathbf{\Sigma}}^{ces}(\mathbf{\Omega}_1,\mathbf{\Omega}_2), \tag{29}$$

for all $\mathbf{\Sigma} \in \mathcal{H}_M^{++}$, $\mathbf{\Omega}_1,\mathbf{\Omega}_2 \in T_{\mathbf{\Sigma}}\mathcal{H}_M^{++}$ and invertible matrix $\mathbf{U}$. Since we have the same geodesic and the congruence invariance property, the proof is completed by using the same steps given in [32] for the proof of the Riemannian distance on $\mathcal{H}_M^{++}$ equipped with the classical Riemannian metric (where $\alpha = 1$ and $\beta = 0$). Therefore, we can show that the distance on $\mathcal{H}_M^{++}$ is equal to (9). The distance (10) on $\mathcal{SH}_M^{++}$ can be obtained by following the same steps.