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Fast communication

Weiss–Weinstein bound for MIMO radar with colocated linear arrays for SNR threshold prediction ☆

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ABSTRACT

Several works have suggested that a multi-input multi-output (MIMO) radar system offers improvement in terms of performance in comparison with classical phased-array radar. However, under the widely spread assumption of a uniform *a priori* distribution for one parameter of interest, there is no result concerning lower bounds on the mean-square error in the case of a Gaussian observation model with parameterized mean. This Fast Communication fills this lack by using the Weiss–Weinstein bound (WWB) which can be calculated under this difficult scenario. As we will show, the proposed bound for MIMO Radar with colocated linear arrays has no closed-form expression. To solve this problem, we propose a closed-form approximation that, as we will show by simulations, is close to the actual bound. This approximated bound is then analyzed for a design purpose in terms of array geometry. Simulations confirm the good ability of the proposed bound to predict the mean square error (MSE) of the maximum *a posteriori* (MAP) in all ranges of SNR. Particularly, the tightness of the bound to predict the SNR threshold effect is shown.

1. Introduction

Since it was first introduced [1], the MIMO radar has gained interest in investigation to improve multisensor systems performance. A MIMO radar utilizes multiple sensors in both the transmitter and the receiver to send out and collect a set of probing waveforms which can be fully uncorrelated or partially correlated. If all the transmitted waveforms are correlated, it refers to the conventional phased-array radar. Thanks to these additional degrees of freedom, a MIMO radar has many advantages in comparison with a phased-array radar. Depending on the distance between elements of antennas and the positions of the transmitting and receiving arrays (colocated or widely separated), each configuration of MIMO radar has its own interests such as improving parameter identifiability [2,3]; offering higher flexibility in transmit beam pattern designs: improving the angular resolution, lowering sidelobes, or decreasing the spatial power density of transmitted signals [2]; increasing detection performance, and allowing direct applicability of adaptive techniques for parameter estimation [4].

If we focus on target localization, the superiority of MIMO radar's performance w.r.t. the phased-array radar has been shown in terms of lower bound on the mean square error (MSE). To the best of our knowledge, all the existing works available in the literature concern only the case of deterministic parameter for the location of the target. In [1], preliminary calculations of the Cramér–Rao bound (CRB) have been introduced. The considered model was the colocated uniform linear array (ULA) for both transmitting and receiving arrays. In [5,6], a MIMO radar with widely separated antennas was studied and the CRB has been calculated for target position and velocity. Besides, Li et al. have investigated the waveform optimization of MIMO radar based on the CRB using colocated ULA transmitting and receiving antennas [7,8]. While all

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the above works have exploited the asymptotic region of the MSE over signal-to-noise ratio (SNR) performance, the Barankin bound has been shown to give a SNR threshold approximation in the scenario of colocated circular array configuration [9,2, Chapter 4].

All these aforementioned investigations handle only the cases where the unknown parameters are assumed to be deterministic. Since the support of the parameters was not taken into account, these bounds cannot describe the overall MSE performance. As an alternative to the deterministic case, we propose to handle the problem in the Bayesian framework which will provide a tight lower bound over all the ranges of SNR and a good prediction of the SNR threshold. The Bayesian bounds deal with the random parameter and presume an *a priori* probability density function (pdf). Hence, they will allow us to characterize and to predict more completely MIMO radar performance. Note that not all the Bayesian bounds proposed in the literature are able to take into account the case when the parameters of interest are supposed to be uniformly distributed. Therefore, among various types of Bayesian bounds, such as the Ziv–Zakai bound [10], the Bell–Steinberg–Ephraim–Van Trees bound [11], the Bobrovsky–Zakai bound [12], and the Bayesian Abel bound [13], we concentrate, in this work, on the Weiss–Weinstein bound (WWB) (see [14] and the recent work of Rapoport and Oshman [15,16]), which can deal with the uniformly distributed prior assumption and is one of the tightest bound of the Weiss and Weinstein family [17,18]. The scenario under investigation is a MIMO radar estimating the direction-of-arrival (DOA) of a target and the complex radar-cross-section (RCS) in the Swerling 0 case. Our results are derived in the case of linear (possibly non-uniform) colocated arrays at emission and reception. We first propose a strict bound for which it is not possible to obtain a closed-form expression. While a numerical integration is still possible, we also propose a closed-form expression based on an approximation.

The rest of this paper is organized as follows: Section 2 presents the general problem setup with colocated linear arrays. In Section 3, we calculate the corresponding WWB matrix (and its approximation) for the DOA and the complex RCS and we prove its diagonal structure. In Section 4, simulations are presented to confirm the good ability of WWB to predict the mean square error (MSE) of the maximum *a posteriori* (MAP) in all ranges of SNR (asymptotic and threshold regions). Numerical procedure is introduced to analyze the MIMO radar system in terms of antenna geometry. Finally, Section 5 draws the conclusions.

2. Problem setup

We consider a single-target localization scenario with a MIMO radar system formed with colocated linear (possibly non-uniform) transmitting and receiving arrays with M and N antennas, respectively. Each transmitting antenna sends out a different waveform, which is known. The $N\times 1$ complex received signal is then given by [7]

$$\mathbf{y}(t)=\beta\mathbf{b}(\theta)\mathbf{a}^T(\theta)\mathbf{x}(t)+\mathbf{n}(t)=\beta\mathbf{C}(\theta)\mathbf{x}(t)+\mathbf{n}(t),\quad t=1\ldots T, \tag{1}$$

where T is the number of snapshots. Since both transmitting and receiving arrays are linear, the steering vectors have the following forms $\mathbf{a}(\theta)=[\exp(-j(2\pi/\lambda)a_1\sin\theta),\ \ldots,\exp(-j(2\pi/\lambda)a_M\sin\theta)]^T$, and $\mathbf{b}(\theta)=[\exp(-j(2\pi/\lambda)b_1\sin\theta),\ldots,\exp(-j(2\pi/\lambda)b_N\sin\theta)]^T$, where λ denotes the wavelength, where $a_i, i=1\ldots M$, and $b_j, j=1\ldots N$, are the positions of the elements (w.r.t. a reference point) of the transmitting and receiving arrays, respectively. Note that for the colocated scenario, the DOA is the same as the direction-of-departure (DOD), hence θ is the only one location parameter of the target. Moreover, θ is assumed to have an *a priori* uniform distribution over the support $[0,\pi]$. For simplicity in the derivation, we define $u=\sin(\theta)$ as the parameter of interest instead of working directly with θ. Note that, the pdf for u is $p(u)=1/\pi\sqrt{1-u^2}$. We denote $\mathbf{C}(\theta)=\mathbf{b}(\theta)\mathbf{a}^T(\theta)\in\mathbb{C}^{N\times M}$. The elements of $\mathbf{C}(\theta)$ are given by $[\mathbf{C}(\theta)]_{k,l}=[\exp(-j(2\pi/\lambda)(a_l+b_k)\sin\theta)]$. $\mathbf{x}(t)$ is the vector of each transmitted waveform. We suppose that these waveforms are independent and have the following empirical covariance matrix: $\mathbf{R}_x=(1/T)\sum_{t=1}^{T}\mathbf{x}(t)\mathbf{x}^H(t)\ =Diag(\boldsymbol{\sigma}_s^2)$, where $\boldsymbol{\sigma}_s^2=[\sigma_1^2,\ldots,\sigma_M^2]^T$. We also define $\{\mathbf{n}(t)\}_{t=1}^{T}$ as the noise vectors, which are assumed to be independent and identically distributed circularly complex Gaussian with zero-mean and covariance matrix $\mathbf{R}=\sigma_n^2\mathbf{I}$. β is the target complex amplitude related to the RCS of the target in the Swerling 0 case, and has an *a priori* circular complex Gaussian distribution with zero mean and variance σ_β^2, namely $\beta\sim\mathcal{CN}(0,\sigma_\beta^2)$. The elements of the unknown parameter vector $\boldsymbol{\Theta}=[u,\beta_R,\beta_I]^T$ with $\beta_R=\mathrm{Re}\{\beta\}$, $\beta_I=\mathrm{Im}\{\beta\}$ are considered to be statistically independent such that the joint pdf $p(\boldsymbol{\Theta})=p(u)p(\beta_R)p(\beta_I)$.

Under the assumption of independent observations, the likelihood of the full set of observations $\mathbf{Y}=[\mathbf{y}(1),\mathbf{y}(2),\ldots,\mathbf{y}(T)]$ is given by

$$p(\mathbf{Y};\boldsymbol{\Theta})=\frac{1}{\pi^{NT}\sigma_n^{NT}}\exp\left(-\frac{1}{\sigma_n^2}\sum_{t=1}^{T}(\mathbf{y}(t)-\beta\mathbf{C}(\theta)\mathbf{x}(t))^H(\mathbf{y}(t)-\beta\mathbf{C}(\theta)\mathbf{x}(t))\right). \tag{2}$$

3. Weiss–Weinstein Bound for MIMO radar's parameter estimation

3.1. Numerical WWB

The Weiss–Weinstein bound [14], denoted by **WWB**, of the unknown target parameters $\boldsymbol{\Theta}=[u,\beta_R,\beta_I]^T$ is a 3×3 matrix such that the following relation holds:

$$E\{(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta})(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta})^T\}\geq\mathbf{WWB}=\sup_{\substack{\mathbf{h}_i,s_i\\ i=1,2,3}}\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T, \tag{3}$$

where $E\{(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta})(\hat{\boldsymbol{\Theta}}-\boldsymbol{\Theta})^T\}$ is the MSE of any Bayesian estimator $\hat{\boldsymbol{\Theta}}$ and $\mathbf{H}=[\mathbf{h}_1,\mathbf{h}_2,\mathbf{h}_3]$ is the 3×3 matrix of test-points. The elements of the 3×3 matrix $\mathbf{G}$ are given by

$$\mathbf{G}_{kl}=\frac{E\{[L^{s_k}(\mathbf{Y};\boldsymbol{\Theta}+\mathbf{h}_k,\boldsymbol{\Theta})-L^{1-s_k}(\mathbf{Y};\boldsymbol{\Theta}-\mathbf{h}_k,\boldsymbol{\Theta})][L^{s_l}(\mathbf{Y};\boldsymbol{\Theta}+\mathbf{h}_l,\boldsymbol{\Theta})-L^{1-s_l}(\mathbf{Y};\boldsymbol{\Theta}-\mathbf{h}_l,\boldsymbol{\Theta})]\}}{E\{L^{s_k}(\mathbf{Y};\boldsymbol{\Theta}+\mathbf{h}_k,\boldsymbol{\Theta})\}E\{L^{s_l}(\mathbf{Y};\boldsymbol{\Theta}+\mathbf{h}_l,\boldsymbol{\Theta})\}}, \tag{4}$$

where $L(\mathbf{Y};\mathbf{\Theta}+\mathbf{h}_i,\mathbf{\Theta})=p(\mathbf{Y},\mathbf{\Theta}+\mathbf{h}_i)/p(\mathbf{Y},\mathbf{\Theta})$, and $s_i\in[0,1]$, $i=1,2,3$. The expectations are taken over the joint pdf $p(\mathbf{Y},\mathbf{\Theta})$.

Rigorously, the quantity $\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T$ must be maximized w.r.t. $\mathbf{h}_i$ and s_i leading to a high computational complexity. However, it has been shown [19,20] that choosing $s_i=\frac{1}{2},\forall i=1,2,3$, and a diagonal matrix $\mathbf{H}$ leads to a bound still tight. Therefore, we assume here that $s_i=\frac{1}{2}$, $\forall i$, and that $\mathbf{H}=Diag([h_1,h_2,h_3]^T)$. In this case, the elements of the matrix $\mathbf{G}$ can be written as

$$\mathbf{G}_{kl}=\frac{\eta(\mathbf{h}_k,\mathbf{h}_l)+\eta(-\mathbf{h}_k,-\mathbf{h}_l)-\eta(\mathbf{h}_k,-\mathbf{h}_l)-\eta(-\mathbf{h}_k,\mathbf{h}_l)}{\eta(\mathbf{h}_k,\mathbf{0})\eta(\mathbf{h}_l,\mathbf{0})},\tag{5}$$

where we define $\eta(\boldsymbol{\alpha},\boldsymbol{\gamma})=\int_{\mathbf{\Upsilon}}\int_{\mathbf{\Omega}}p(\mathbf{Y},\mathbf{\Theta}+\boldsymbol{\alpha})^{1/2}p(\mathbf{Y},\mathbf{\Theta}+\boldsymbol{\gamma})^{1/2}\,d\mathbf{Y}\,d\mathbf{\Theta}$, where $\mathbf{\Omega}$ and $\mathbf{\Upsilon}$ are the observation and parameter spaces, respectively, and where $\boldsymbol{\alpha}=[\alpha_1,\alpha_2,\alpha_3]^T$, and $\boldsymbol{\gamma}=[\gamma_1,\gamma_2\gamma_3]^T$.

By denoting

$$\eta'(\mathbf{\Theta},\boldsymbol{\alpha},\boldsymbol{\gamma})=\int_{\mathbf{\Omega}}p(\mathbf{Y};\mathbf{\Theta}+\boldsymbol{\alpha})^{1/2}p(\mathbf{Y};\mathbf{\Theta}+\boldsymbol{\gamma})^{1/2}\,d\mathbf{Y},\tag{6}$$

$\eta(\boldsymbol{\alpha},\boldsymbol{\gamma})$ can be rewritten as

$$\eta(\boldsymbol{\alpha},\boldsymbol{\gamma})=\int_{\mathbf{\Upsilon}}\eta'(\mathbf{\Theta},\boldsymbol{\alpha},\boldsymbol{\gamma})p(\mathbf{\Theta}+\boldsymbol{\alpha})^{1/2}p(\mathbf{\Theta}+\boldsymbol{\gamma})^{1/2}\,d\mathbf{\Theta}.\tag{7}$$

Plugging (2) into (6), we obtain a closed-form expression for η' by using Eqs. (16) and (17) of [21]

$$\eta'(\mathbf{\Theta},\boldsymbol{\alpha},\boldsymbol{\gamma})=\exp\left(-\frac{1}{4\sigma_n^2}\sum_{t=1}^{T}[\mathbf{f}(\mathbf{\Theta}+\boldsymbol{\alpha},t)-\mathbf{f}(\mathbf{\Theta}+\boldsymbol{\gamma},t)]^H[\mathbf{f}(\mathbf{\Theta}+\boldsymbol{\alpha},t)-\mathbf{f}(\mathbf{\Theta}+\boldsymbol{\gamma},t)]\right),\tag{8}$$

where $\mathbf{f}(\mathbf{\Theta},t)=\beta\mathbf{C}(\theta)\mathbf{x}(t)$. Consequently,

$$\eta(\boldsymbol{\alpha},\boldsymbol{\gamma})=\int_{\mathbf{\Upsilon}}\eta'(\mathbf{\Theta},\boldsymbol{\alpha},\boldsymbol{\gamma})\frac{1}{\pi((1-(u+\alpha_1)^2))^{1/4}((1-(u+\gamma_1)^2))^{1/4}}\frac{1}{\pi\sigma_\beta^2}\exp\left(-\frac{(\beta_R-\alpha_2)^2+(\beta_R-\gamma_2)^2}{\sigma_\beta^2}\right)\exp\left(-\frac{(\beta_I-\alpha_3)^2+(\beta_I-\gamma_3)^2}{\sigma_\beta^2}\right)du\,d\beta_R\,d\beta_I.\tag{9}$$

Unfortunately, in this case, no closed-form expression of the WWB can be obtained (see also [22] where the same thing appears in the context of classical bearing estimation). This is why we propose in the following an approximation for which we will have a closed-form expression of the bound. A closed-form expression is always interesting because it allows an easy interpretation of the bound, i.e. we can study the performance of the MIMO radar system w.r.t. some design parameters.

3.2. Analytical approximation of the bound

In this section, with the intention of obtaining a closed-form expression of the WWB, we propose a different approach where we assume a uniform *a priori* pdf on u. Note that this approach has already been successfully used in [23–26]. We also show by simulation (in Section 4.2) that the two approaches give very close bounds and the same prediction of the SNR threshold.

Assuming a uniform *a priori* pdf on u, the structure of the matrix **WWB** is given as follows (see Appendix A.1 for more details):

$$\mathbf{WWB}=Diag\left(\sup_{h_1}\frac{h_1^2}{\mathbf{G}_{11}},\frac{1}{2\left(TASNR+\frac{1}{\sigma_\beta^2}\right)},\frac{1}{2\left(TASNR+\frac{1}{\sigma_\beta^2}\right)}\right),\tag{10}$$

where the array signal to noise ($ASNR$) is defined as PN/σ_n^2 where $P=\sum_{k=1}^{M}\sigma_k^2$ is the total transmitted power, and where $\mathbf{G_{11}}$ is given by

$$\mathbf{G}_{11}=4\frac{[(2-h_1)f(2,h_1)-2(1-h_1)]f(1,h_1)^2}{(2-h_1)^2f(2,h_1)},\tag{11}$$

where we define $f(\varphi,h_1)=\sigma_\beta^2\frac{1}{2}T\sum_{k=1}^{M}\sum_{r=1}^{N}\ (\sigma_k^2/\sigma_n^2)[1-\cos((\ \varphi 2\pi/\lambda)h_1(a_k+b_r))]+1$. In the following, we analyse the proposed bound in terms of SNR threshold prediction and array geometry design.

4. Analysis

4.1. Properties of the bound

There are three properties of the WWB matrix (10): (1) Due to the diagonal structure of the WWB matrix, the lower bounds on estimation errors are decoupled. (2) Thanks to this derivation, the initial multidimensional optimization problem w.r.t. h_1, h_2, and h_3 has been reduced to a monodimensional optimization problem w.r.t. h_1 only. (3) The WWB for β_R and β_I are the same as their Bayesian CRB in the case where u is assumed to be known (the proof is straightforward).

4.2. WWB performance in predicting the global MSE and the SNR threshold

The aim of this part is to examine the usefulness of using the WWB to predict the ultimate global MSE and the SNR threshold. For that reason, we consider the empirical global MSE of the MAP estimator, which is evaluated over 1000 Monte Carlo trials. The simulation is performed for three scenarios with configurations as follows: the transmit array is sparse with $M=8$ (scenario A and B) and $M=5$ (scenario C) sensors and with inter-element spacing (in unit of wavelengths) equals to $0.5M$ and the receive array is an ULA with $N=8$ (scenario A and B) and $N=5$ (scenario C) sensors and with inter-element spacing (in unit of wavelengths) equals to 0.5. Such array configuration is well-known in MIMO radar [2, p. 76] to create a Nyquist virtual array with good performance. The uncorrelated MIMO radar waveforms are generated using Hadamard codes with $T=64$ (scenario A) and $T=32$ (scenario B and C) snapshots. The transmitted power is assumed to be uniformly distributed on all transmit antennas. h_1 is chosen on the support $[-1,1]$. Fig. 1 shows the approximated WWB with $s_i=\frac{1}{2}$ and the MAP of the parameter of interest θ versus ASNR for three scenarios. We also draw the numerical optimization of approximated WWB over s_i.

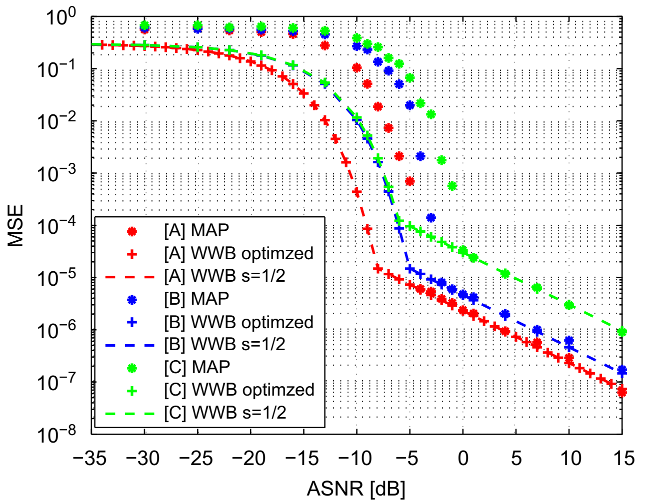

Fig. 1. MAP estimator empirical global MSE and approximated WWB of θ versus SNR.

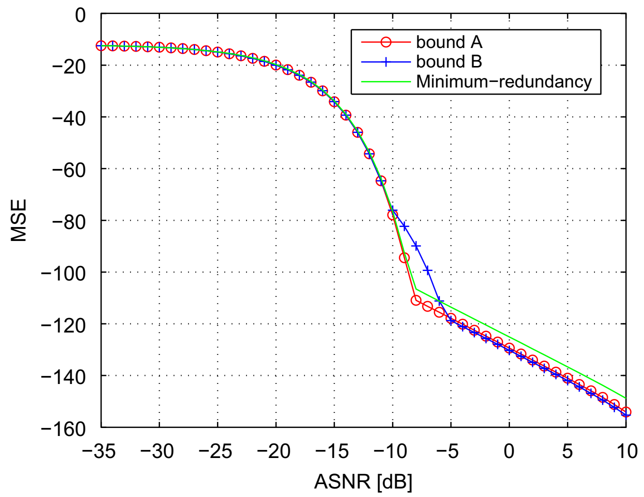

Fig. 3. Receiver geometry investigation: WWB for the lowest-threshold-point, the lowest-asymptotic-region, and the minimum-redundancy arrays.

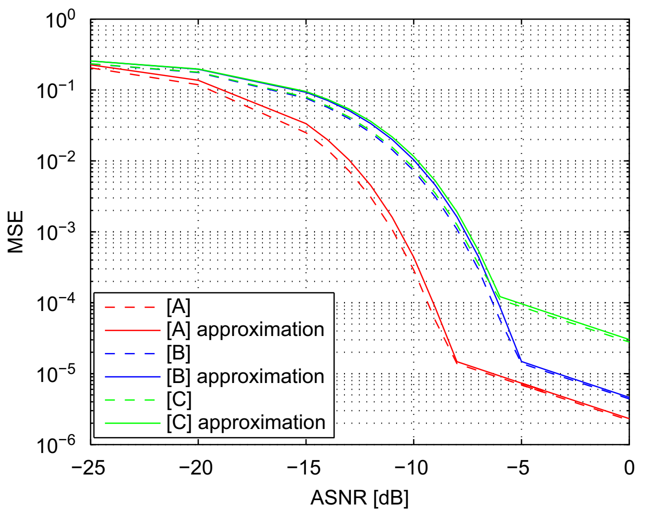

Fig. 2. WWB of θ and its approximation versus SNR.

It can be seen that the WWB provides a good prediction of the MSE in all regions. It predicts the threshold SNR location to be 4 dB below the threshold SNR indicated by the MAP.

In Fig. 2, we compare, under the three aforementioned scenarios, the true WWB (computed with numerical integration) and our approximation of the WWB. Both bounds are very close and lead to the same prediction of the SNR threshold. Note that, in our simulation, time needed to calculate the bound with numerical integration is over 100 times longer than that in the other case. Consequently, to have a (still tight) bound with an efficiently computational cost and an easy interpretation, our WWB approximation should be preferred especially for designing a MIMO radar system.

4.3. MIMO array geometry investigation

This part investigates the behavior of the derived WWB w.r.t. the array geometry of both the receiver and transmitter. Note that the proposed bound has been derived in the context of linear arrays but possibly non-uniform. Consequently, it remains a degree of freedom to enhance the performance. We first analyze the impact of the receiver array geometry while the transmitter is made from a classical ULA. Both transmitter and receiver are made with $M=N=8$ sensors. The transmitter is a fixed ULA (half-wavelength inter-element spacing) array. The receiver has different linear array geometries keeping the array aperture $D=23$ (in unit of $\lambda/2$) fixed (i.e. the positions of the two extreme sensors of the linear array are fixed). In other words, their is 22 positions available for the six other sensors (74 613 possibilities). The number of observations is $T=64$. We have generated all the array configurations and computed the associated WWB. In Fig. 3, among these 74 613 bounds we have plotted the bound which leads to the lowest SNR threshold (denoted bound A) and the bound which has the lowest MSE in the asymptotic area (denoted bound B). We have noticed that both geometries leading to bounds A and B are not with minimum redundancy or minimum gap. Indeed, for this configuration, it can be seen that it exists four minimum-redundancy arrays. We have computed the bounds associated to these four possibilities and plotted the best one in terms of MSE for comparison purpose. The bound A (i.e. the bound with the lowest SNR threshold) shows that better asymptotic performance can be obtained. There exists a geometry configuration leading to slightly better asymptotic performance (bound B) but with a worst SNR threshold. Note that, we have obtained the same results, i.e. same bounds associated to the same optimal geometries, in the opposite case where the receiver is a fixed ULA and where the transmitter array geometry is optimized, following the same aforementioned way.

5. Conclusions

In this correspondence, we have derived a closed-form expression of the Weiss–Weinstein bound for MIMO radar with colocated linear arrays. It has been seen that the Weiss–Weinstein bound for the parameter of interest provides a good prediction of the MSE in all regions. It also predicts the threshold SNR location near the threshold SNR indicated by the MAP. We have also investigated the influence of array geometry on the behavior of the Weiss–Weinstein bound.

Appendix A

*A.1. Derivation of matrix **WWB***

From (8), the set of functions involved in $\mathbf{G}_{11}$ are given by

$$\begin{cases} \eta'(\boldsymbol{\Theta},\mathbf{h}_1,\mathbf{h}_1)=1, \quad \eta'(\boldsymbol{\Theta},-\mathbf{h}_1,-\mathbf{h}_1)=1, \\ \eta'(\boldsymbol{\Theta},\mathbf{h}_1,-\mathbf{h}_1)=\exp\left\{-\dfrac{1}{2}(\beta_R^2+\beta_I^2)T\displaystyle\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left[1-\cos\left(\frac{4\pi}{\lambda}h_1(a_k+b_r)\right)\right]\right\}, \\ \eta'(\boldsymbol{\Theta},\mathbf{h}_1,0)=\exp\left\{-\dfrac{1}{2}(\beta_R^2+\beta_I^2)2T\displaystyle\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left[1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right]\right\}. \end{cases} \tag{12}$$

Then, since all the functions η' do not depend on u, by integrating w.r.t. $\boldsymbol{\Theta}$, we obtain

$$\begin{cases} \eta(\mathbf{h}_1,\mathbf{h}_1)=\dfrac{2-h_1}{2}, \quad \eta(-\mathbf{h}_1,-\mathbf{h}_1)=\dfrac{2-h_1}{2}, \\ \eta(\mathbf{h}_1,-\mathbf{h}_1)=\dfrac{1-h_1}{\sigma_\beta^2\frac{1}{2}T\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left[1-\cos\left(\frac{4\pi}{\lambda}h_1(a_k+b_r)\right)\right]+1}, \\ \eta(\mathbf{h}_1,0)=\dfrac{2-h_1}{2}\dfrac{1}{\sigma_\beta^2\frac{1}{2}T\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left[1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right]+1}. \end{cases} \tag{13}$$

The set of functions involved in $\mathbf{G}_{kk}, k=2,3$, are given by

$$\begin{cases} \eta'(\boldsymbol{\Theta},\mathbf{h}_k,\mathbf{h}_k)=1, \quad \eta'(\boldsymbol{\Theta},-\mathbf{h}_k,-\mathbf{h}_k)=1 \\ \eta'(\boldsymbol{\Theta},\mathbf{h}_k,-\mathbf{h}_k)=\exp\left(-h_k^2NT\dfrac{\sum_{k=1}^{M}\sigma_k^2}{\sigma_n^2}\right) \\ \eta'(\boldsymbol{\Theta},\mathbf{h}_k,0)=\exp\left(-\dfrac{1}{4}h_k^2NT\dfrac{\sum_{k=1}^{M}\sigma_k^2}{\sigma_n^2}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \eta(\mathbf{h}_k,\mathbf{h}_k)=1, \quad \eta(-\mathbf{h}_k,-\mathbf{h}_k)=1, \\ \eta(\mathbf{h}_k,-\mathbf{h}_k)=\exp\left(-h_k^2NT\dfrac{\sum_{k=1}^{M}\sigma_k^2}{\sigma_n^2}\right)\exp\left(-\dfrac{h_k^2}{\sigma_\beta^2}\right), \\ \eta(\mathbf{h}_k,0)=\exp\left(-\dfrac{1}{4}h_k^2NT\dfrac{\sum_{k=1}^{M}\sigma_k^2}{\sigma_n^2}\right)\exp\left(-\dfrac{h_k^2}{4\sigma_\beta^2}\right). \end{cases} \tag{14}$$

The set of functions involved in $\mathbf{G}_{12}$ are given by

$$\begin{cases} \eta'(\boldsymbol{\Theta},\mathbf{h}_1,\mathbf{h}_2) = \exp\left\{2T\displaystyle\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left[\beta_R^2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right)\right.\right. \\ \quad +\beta_R h_2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right) \\ \quad +\beta_I^2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right) \\ \quad \left.\left.-\beta_I h_2\sin\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)+\frac{1}{2}h_2^2\right]\right\}, \\ \eta'(\boldsymbol{\Theta},-\mathbf{h}_1,-\mathbf{h}_2) = \exp\left\{2T\displaystyle\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left[\beta_R^2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right)\right.\right. \\ \quad -\beta_R h_2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right) \\ \quad +\beta_I^2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right) \\ \quad +\beta_I h_2\sin\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right) \\ \quad \left.\left.+\frac{1}{2}h_2^2\right]\exp\right\}, \\ \eta'(\boldsymbol{\Theta},-\mathbf{h}_1,\mathbf{h}_2) = \exp\left\{2T\displaystyle\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left[\beta_R^2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right)\right.\right. \\ \quad +\beta_R h_2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right) \\ \quad +\beta_I^2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right) \\ \quad +\beta_I h_2\sin\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right) \\ \quad \left.\left.+\frac{1}{2}h_2^2\right]\exp\right\}, \\ \eta'(\boldsymbol{\Theta},-\mathbf{h}_1,-\mathbf{h}_2) = \exp\left\{2T\displaystyle\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left[\beta_R^2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right)\right.\right. \\ \quad -\beta_R h_2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right) \\ \quad +\beta_I^2\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right) \\ \quad -\beta_I h_2\sin\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right) \\ \quad \left.\left.+\frac{1}{2}h_2^2\right]\exp\right\}. \end{cases} \tag{15}$$

Consequently,

$$\eta(\mathbf{h}_1,\mathbf{h}_2)=\eta(-\mathbf{h}_1,\mathbf{h}_2)=\frac{2-h_1}{2}t_1t_2, \tag{16}$$

where

$$t_1=\exp\left(-\frac{h_2^2}{2\sigma_\beta^2}\right)\exp\left\{\frac{h_2^2}{4}\left[\frac{T}{2}\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right)+\frac{1}{\sigma_\beta^2}\right]\right\}$$

$$\times\frac{\sqrt{\pi}\sigma_\beta}{4\left\{\frac{\sigma_\beta^2}{2}\left[\frac{T}{2}\sum_{k=1}^{M}\sum_{r=1}^{N}\frac{\sigma_k^2}{\sigma_n^2}\left(1-\cos\left(\frac{2\pi}{\lambda}h_1(a_k+b_r)\right)\right)+\frac{1}{\sigma_\beta^2}\right]\right\}^{1/2}}, \tag{17}$$

$$t_2 = \exp\left\{ \frac{\left[h_2 \frac{T}{2} \sum_{k=1}^{M} \sum_{r=1}^{N} \frac{\sigma_k^2}{\sigma_n^2} \sin\left(\frac{2\pi}{\lambda} h_1 (a_k + b_r) \right) \right]^2}{4\left[\frac{T}{2} \sum_{k=1}^{M} \sum_{r=1}^{N} \frac{\sigma_k^2}{\sigma_n^2} \left(1 - \cos\left(\frac{2\pi}{\lambda} h_1 (a_k + b_r) \right) \right) + \frac{1}{\sigma_\beta^2} \right]} \right\} \times \frac{\sqrt{\pi}\sigma_\beta}{4\left\{ \frac{\sigma_\beta^2}{2} \left[\frac{T}{2} \sum_{k=1}^{M} \sum_{r=1}^{N} \frac{\sigma_k^2}{\sigma_n^2} \left(1 - \cos\left(\frac{2\pi}{\lambda} h_1 (a_k + b_r) \right) \right) + \frac{1}{\sigma_\beta^2} \right] \right\}^{1/2}}. \tag{18}$$

Similarly, it is straightforward to see that $\eta(\mathbf{h}_1, -\mathbf{h}_2) = \eta(-\mathbf{h}_1, -\mathbf{h}_2)$ which, together with (16), lead to $\mathbf{G}_{12} = 0$. We also have $\mathbf{G}_{21} = 0$, $\mathbf{G}_{13} = 0$, $\mathbf{G}_{31} = 0$, $\mathbf{G}_{23} = 0$, and $\mathbf{G}_{32} = 0$. Therefore, the matrix $\mathbf{G}$ is diagonal with elements given by

$$\mathbf{G}_{11} = 2 \frac{\frac{2-h_1}{2} - \frac{1-h_1}{\frac{\sigma_\beta^2}{\sigma_n^2} \frac{1}{2} T \sum_{k=1}^{M} \sum_{r=1}^{N} \sigma_k^2 \left[1 - \cos\left(\frac{4\pi}{\lambda} h_1 (a_k + b_r) \right) \right] + 1}}{\left[\frac{2-h_1}{2} \frac{1}{\frac{\sigma_\beta^2}{\sigma_n^2} \frac{1}{2} T \sum_{k=1}^{M} \sum_{r=1}^{N} \sigma_k^2 \left[1 - \cos\left(\frac{2\pi}{\lambda} h_1 (a_k + b_r) \right) \right] + 1} \right]^2}, \tag{19}$$

$$\mathbf{G}_{22} = 2 \frac{1 - \exp\left(-h_2^2 NT \frac{\sum_{k=1}^{M} \sigma_k^2}{\sigma_n^2} \right) \exp\left(-\frac{h_2^2}{\sigma_\beta^2} \right)}{\exp\left(-\frac{1}{2} h_2^2 NT \frac{\sum_{k=1}^{M} \sigma_k^2}{\sigma_n^2} \right) \exp\left(-\frac{h_2^2}{2\sigma_\beta^2} \right)}, \tag{20}$$

$$\mathbf{G}_{33} = 2 \frac{1 - \exp\left(-h_3^2 NT \frac{\sum_{k=1}^{M} \sigma_k^2}{\sigma_n^2} \right) \exp\left(-\frac{h_3^2}{\sigma_\beta^2} \right)}{\exp\left(-\frac{1}{2} h_3^2 NT \frac{\sum_{k=1}^{M} \sigma_k^2}{\sigma_n^2} \right) \exp\left(-\frac{h_3^2}{2\sigma_\beta^2} \right)}. \tag{21}$$

Using all the assumptions and results above, the closed-form expression of the matrix **WWB** can be written as $\mathbf{WWB} = Diag(\sup_{h_1} h_1^2/\mathbf{G}_{11}, \sup_{h_2} h_2^2/\mathbf{G}_{22}, \sup_{h_3} h_3^2/\mathbf{G}_{33})$, with $\mathbf{G}_{11}, \mathbf{G}_{22}, \mathbf{G}_{33}$ given in (11), (20) and (21), respectively.

Note that in the original form of the Weiss–Weinstein bound (3), we have to optimize the matrix **WWB** w.r.t. h_1, h_2, and h_3. Since $\mathbf{G}_{ii}$ depends only on h_i, the optimization task is reduced to $\sup_{h_k} h_k^2/\mathbf{G}_{kk}, \forall k$ which is more tractable. Next, we give a closed-form expression for $\sup_{h_k} h_k^2/\mathbf{G}_{kk}$, $k = 2,3$,

$$\mathbf{WWB}_{kk} = \sup_{x \geq 0} \frac{x \exp(-ax)}{2[1 - \exp(-2ax)]}, \quad a > 0,$$

where $x = h_k^2$ and $a = NT \sum_{k=1}^{M} \sigma_k^2 / 2\sigma_n^2 + 1/2\sigma_\beta^2$. If we denote $g(x) = x \exp(-ax)/2[1 - \exp(-2ax)]$, then

$$\frac{dg(x)}{dx} = \frac{\exp(-ax)[1 - ax - \exp(-2ax) - ax \exp(-2ax)]}{4[1 - \exp(-2ax)]^2}, \tag{22}$$

which is always negative $\forall x > 0$. Consequently, $g(x)$ is a monotonically decreasing function and has its maximum at $x = 0$. Finally, we obtain:

$$\sup_{h_k} \frac{h_k^2}{\mathbf{G}_{kk}} = \frac{1}{2\left(NT \frac{\sum_{k=1}^{M} \sigma_k^2}{\sigma_n^2} + \frac{1}{\sigma_\beta^2} \right)}, \quad k = 2, 3,$$

which concludes the proof.

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