N-body Initial Conditions – Hernquist Profile

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This is a documentation for my N-body project done in course. A Hernquist profile is taken and a N-body profile is constructed using properties derivable using kinetic theory. I discuss the theoretical approach to the method realizing the profile and constructing the condition generators using python.

- The file `hernquist_profile.py` contains the equations and the initial condition generator used to compile all the analytical values of the Hernquist profile.
- To compile the velocities and positions of the $N = 10^6$ particles, compile `test.py` with python (make sure it is in the same directory as `hernquist_profile.py`).
  - Each i-th particle can be index from the main generator function `generate_nbody()[i]`
  - For the i-th particle position in the volume, index `generate_nbody()[i][0]`
  - For the i-th particle velocity in the volume, index `generate_nbody()[i][1]`
- To generate positional vectors alone, compile `compile_pos.py` with python.
- To compile the plots in this document, you must re-define the base path to the directory that has `hernquist_profile.py`.

I. The Hernquist Profile

The Hernquist density profile (Hernquist 1990) is a double power law profile that has been shown to be a good match to galaxy bulges and dark matter halos. Furthermore, it is also convenient in that the total mass is finite, and the density profile can be expressed directly as a function of the gravitational potential.

$$\rho(r) = \frac{M}{2\pi} \frac{a}{r\,(r+a)^3} \tag{1}$$

where M is the mass of the system, a is the scale radius, and r is the radial distribution. Results derived from this characteristic density is taken from (Hernquist 1990) unless otherwise specified.

For our halo of interest, we take the parameters:

$$M = 10^{12} M_\odot, \quad a = 35\,\mathrm{kpc}, \quad N = 10^6 \tag{2}$$

II. Numerical Process

A. Radii Sampling

To compute the distribution of radii for the particle following this system, we know that the mass profile can be constructed to assign probabilistic values of the particle radii. The mass element dM of the volume element $dV = 4\pi r^2 dr$ for the Hernquist profile can be solved to compute the enclosed mass $M(<r)$

$$M(<r) = 4\pi r^2 dr \rho(r) = M\frac{r^2}{(r+a)^2} \tag{3}$$

Normalizing the mass distribution by the total mass M gives us the probabilistic cumulative distribution for the radii:

$$\mathcal{P}(r) = \frac{r^2}{(a+r)^2} \tag{4}$$

We can then invert this distribution to solve for the allowed values of r that is dependent on this cumulative distribution:

$$r = \frac{a\sqrt{x}}{1-\sqrt{x}} \tag{5}$$

where $x = \mathcal{P}(<r)$.

Since x is just the cumulative distribution found between 0 and 1, the value x for each i-th particle can be uniformly sampled in $x_i \in \mathrm{Uni}(0,1)$ where $i \in \{1,\ldots,N\}$. To ensure that particles are uniformly distributed, the sampled radial values for 10^4 particles are plotted against Equation (1) in Figure 1. The values of r were logarithmically binned, where within each bin the mass is divided by the volume of the spherical shell corresponding to that bin. Furthermore, all N particles will have the same mass, such that $\sum_i^N m_i = M$. This simply implies that $m_i = M/N$.

Acquiring the set of radial quantities r_i from $\mathcal{P}(r)$, we calculate their positions in spherical coordinates. The radial values are randomly positioned by a uniform distribution of random points in the polar and azimuthal bounds $\theta_i \in [0,\pi]$ and $\varphi_i \in [0, 2\pi)$, respectively. Once uniformly distributing over the sphere, these spherical coordinates are converted to Cartesian coordinates:

$$\begin{aligned} x_i &= r_i \sin\theta_i \cos\varphi_i \\ y_i &= r_i \sin\theta_i \sin\varphi_i \\ z_i &= r_i \cos\theta_i \end{aligned} \tag{6}$$

The right plot in Figure 1 shows a projection of the the same number of randomly sampled particles in Cartesian coordinates.

B. Phase Space Distribution

The Hernquist profile has a defined spherical potential

$$\phi(r) = -\frac{GM}{r+a} \tag{7}$$

If this system is confined to this energy, then it is possible to have a system uniquely ergodic that depends on the phase-space coordinates only through the Hamiltonian:

$$f = f(\vec{x}, \vec{v}) d^3\vec{x} d^3\vec{v} \tag{8}$$

A system that can be characterized by the binding energy $\mathcal{E}$ is ergodic and isotropic ($\beta = 0$) everywhere. Therefore the DF is a function of the just the relative binding energy $\mathcal{E}$

$$f(\mathcal{E}) = f\left(\phi(r) + \frac{v^2}{2}\right) \tag{9}$$

Obtaining $f(\mathcal{E})$ in the Hernquist model owes itself between the relation of the density and potential given previously. The functional form is the Eddington formula (Binney and Tremaine 2011)

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^{\mathcal{E}} \frac{d^2\rho}{d\Psi^2}\frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{d\rho}{d\Psi}\right)_{\Psi=0}\right] \tag{10}$$

The analytical form for this model is already solved for

$$\begin{aligned} f(\mathcal{E}) = &\frac{M/a^3}{4\pi^3(2GM/a)^{3/2}} \\ &\times\left[\frac{3\sin^{-1}q + q\sqrt{1-q^2}(1-2q^2)(8q^4-8q^2-3)}{(1-q^2)^{3/2}}\right] \end{aligned} \tag{11}$$

where

$$q = \sqrt{-\frac{a}{GM}\mathcal{E}} \tag{12}$$

C. Local Velocity Distribution

The values of the particle velocities are calculated through the following steps:

- For each i-th particle, the potential energy can be calculated at its associated radius

$$\phi(r_i) = -\frac{GM}{r_i+a} \tag{13}$$

- The maximum velocity of the particle, one where to have the particle stable in orbit and stay in the system, is its escape velocity:

$$v_{\rm esc}(r_i) = \sqrt{-2\phi(r_i)} = \sqrt{\frac{2GM}{r_i+a}} \tag{14}$$

  The minimum velocity is with a dominating potential, implying $v = 0$. Thus, the velocity of the i-th particle should be found in the range $v \in [0, v_{\rm esc}(r_i)]$.

- To start the rejection method (Press 2007), notice that for an isotropic system that

$$\mathcal{P}(v|r_i)dv \propto f\left[\phi(r_i) + \frac{v^2}{2}\right] v^2 dv \tag{15}$$

- A value of the velocity v is randomly sample from a variable found in

$$v \in \mathrm{Uni}(0,1) \times v_{\rm esc}(r_i). \tag{16}$$

  and with a fixed value of the potential for the i-th particle, the binding energy for the particle with this velocity is calculated.

- With every v randomly sampled, the maximum value for the probability is also sampled as the variable $\widetilde{w}$

$$\widetilde{w} \in \mathrm{Uni}(0,1) \times f\left(\phi(r_i) + \frac{1}{2}v_{\rm esc}(r_i)^2\right) v_{\rm esc}(r_i)^2 \tag{17}$$

- The value of v is then checked through the criterion

$$\widetilde{w} < f\left(\phi(r_i) + \frac{v^2}{2}\right) v^2 \tag{18}$$

  If the condition is met, v will be the accepted velocity magnitude for the i-th particle. Else, a new value of $\widetilde{w}$ and $\widetilde{v}$ will be sampled until the condition is met.

Finally, components of the velocity are computed in a velocity sphere and then converted to Cartesian coordinates. This is similar to the transformation applied previously but with the calculated value of v acting as the magnitude in this volume:

$$\begin{aligned} v_{i,x} &= v_i \sin\theta_i \cos\varphi_i \\ v_{i,y} &= v_i \sin\theta_i \sin\varphi_i \\ v_{i,z} &= v_i \cos\theta_i \end{aligned} \tag{19}$$

To check if these generated values are correctly computed, a comparison between the radial velocity dispersion of the particles and the analytical form from the isotropic profile can lead to a successful indication. For the isotropic Hernquist profile, the 1D radial velocity dispersion is derived from Jeans equation Binney and Tremaine (2011):

$$\sigma^2(r) = \frac{1}{\rho(r)}\int_r^\infty dr' \rho(r') \frac{d\phi(r')}{dr} \tag{20}$$

$$= \frac{GM^2 a}{2\pi\rho(r)}\int_r^\infty \frac{dr'}{r'(r'+a)^5} \tag{21}$$

The radial velocity from the generated particles can be computed with the sampled vector components:

$$v_{\rm rad} = \frac{\vec{v} \cdot \vec{r}}{|r|} \tag{22}$$

To compute the radial velocity dispersion profile, the population of particles are logarithmically binned by values of r. In each bin, the radial velocity dispersion is computed as $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\langle v_{{\rm rad},i}^2\rangle - \langle v_{\rm rad}\rangle}$, where n is the number of particles in each bin and $\langle v_{\rm rad}\rangle$ is the average velocity in that same bin (which can just be approximated as zero since the system is isotropic and in a steady state to begin with). Figure 2 shows the comparison between the generate values (white dots) and the analytical profile (black line).

James Binney and Scott Tremaine. Galactic dynamics. Princeton university press, 2011.

Lars Hernquist. An analytical model for spherical galaxies and bulges. The Astrophysical Journal, 356:359–364, 1990.

William H Press. Numerical recipes 3rd edition: The art of scientific computing. Cambridge university press, 2007.

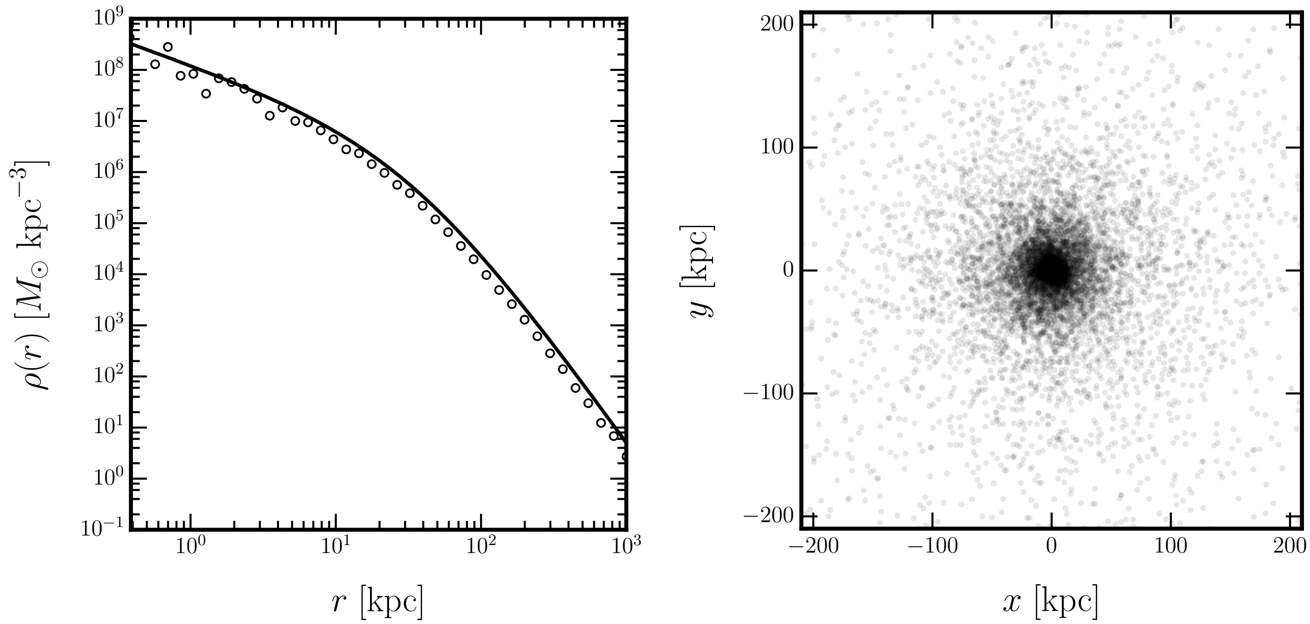

FIG. 1: $N \sim 10^4$ randomly sampled particles for the Hernquist halo using this method for $M = 10^{12} M_\odot$ and $a = 35$ kpc. **Left**: The uniformly generated variables of the radii compared to the analytical density profile for the Hernquist system. The curve is the analytical value from Equation (1) and the circles depict the values generate from the realization. **Right**: A projection of the distributed particles. This can be generated by compiling the `halo_pos.py` file.

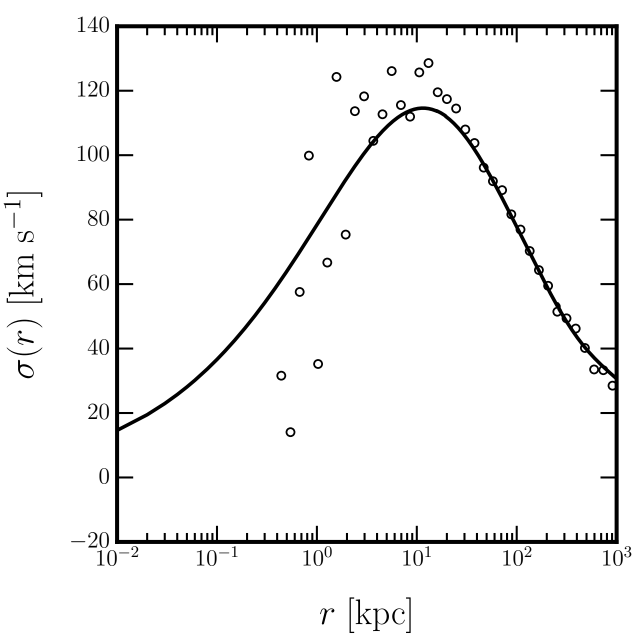

FIG. 2: The radial velocity dispersion profile for 10^4 particles.