1.4 ANALYSIS OF ALGORITHMS

AS PEOPLE GAIN EXPERIENCE USING COMPUTERS, they use them to solve difficult problems or to process large amounts of data and are invariably led to questions like these:

How long will my program take?

Why does my program run out of memory?

You certainly have asked yourself these questions, perhaps when rebuilding a music or photo library, installing a new application, working with a large document, or working with a large amount of experimental data. The questions are much too vague to be answered precisely—the answers depend on many factors such as properties of the particular computer being used, the particular data being processed, and the particular program that is doing the job (which implements some algorithm). All of these factors leave us with a daunting amount of information to analyze.

Despite these challenges, the path to developing useful answers to these basic questions is often remarkably straightforward, as you will see in this section. This process is based on the *scientific method*, the commonly accepted body of techniques used by scientists to develop knowledge about the natural world. We apply *mathematical analysis* to develop concise models of costs and do *experimental studies* to validate these models.

Scientific method The very same approach that scientists use to understand the natural world is effective for studying the running time of programs:

- *Observe* some feature of the natural world, generally with precise measurements.
- *Hypothesize* a model that is consistent with the observations.
- *Predict* events using the hypothesis.
- *Verify* the predictions by making further observations.
- *Validate* by repeating until the hypothesis and observations agree.

One of the key tenets of the scientific method is that the experiments we design must be *reproducible*, so that others can convince themselves of the validity of the hypothesis. Hypotheses must also be *falsifiable*, so that we can know for sure when a given hypothesis is wrong (and thus needs revision). As Einstein famously is reported to have said ("*No amount of experimentation can ever prove me right; a single experiment can prove me wrong*"), we can never know for sure that any hypothesis is absolutely correct; we can only validate that it is consistent with our observations.

Observations Our first challenge is to determine how to make quantitative measurements of the running time of our programs. This task is far easier than in the natural sciences. We do not have to send a rocket to Mars or kill laboratory animals or split an atom—we can simply run the program. Indeed, *every* time you run a program, you are performing a scientific experiment that relates the program to the natural world and answers one of our core questions: *How long will my program take?*

Our first qualitative observation about most programs is that there is a *problem size* that characterizes the difficulty of the computational task. Normally, the problem size is either the size of the input or the value of a command-line argument. Intuitively, the running time should increase with problem size, but the question of *by how much* it increases naturally comes up every time we develop and run a program.

Another qualitative observation for many programs is that the running time is relatively insensitive to the input itself; it depends primarily on the problem size. If this relationship does not hold, we need to take steps to better understand and perhaps better control the running time's sensitivity to the input. But it does often hold, so we now focus on the goal of better quantifying the relationship between problem size and running time.

Example. As a running example, we will work with the program `ThreeSum` shown here, which counts the number of triples in a file of `N` integers that sum to 0 (assuming that overflow plays no role). This computation may seem contrived to you, but it is deeply related to numerous fundamental computational tasks (for example, see EXERCISE 1.4.26). As a test input, consider the file `1Mints.txt` from the booksite, which contains 1 million randomly generated `int` values. The second, eighth, and tenth entries in `1Mints.txt` sum to 0. How many more such triples are there in the file? `ThreeSum` can tell us, but can it do so in a reasonable amount of time? What is the relationship between the problem size `N` and running time for `ThreeSum`? As a first experiment, try running `ThreeSum` on your computer for the files `1Kints.txt`, `2Kints.txt`, `4Kints.txt`, and `8Kints.txt` on the

```
public class ThreeSum
{
   public static int count(int[] a)
   {  // Count triples that sum to 0.
      int N = a.length;
      int cnt = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0)
                  cnt++;
      return cnt;
   }

   public static void main(String[] args)
   {
      int[] a = In.readInts(args[0]);
      StdOut.println(count(a));
   }
}
```

Given N, how long will this program take?

```
% more 1Mints.txt
 324110
-442472
 626686
-157678
 508681
 123414
 -77867
 155091
 129801
 287381
 604242
 686904
-247109
  77867
 982455
-210707
-922943
-738817
  85168
 855430
 ...
```

booksite that contain the first 1,000, 2,000, 4,000, and 8,000 integers from `1Mints.txt`, respectively. You can quickly determine that there are 70 triples that sum to 0 in `1Kints.txt` and that there are 528 triples that sum to 0 in `2Kints.txt`. The program takes substantially more time to determine that there are 4,039 triples that sum to 0 in `4Kints.txt`, and as you wait for the program to finish for `8Kints.txt`, you will find yourself asking the question *How long will my program take*? As you will see, answering this question for this program turns out to be easy. Indeed, you can often come up with a fairly accurate prediction while the program is running.

Stopwatch. Reliably measuring the exact running time of a given program can be difficult. Fortunately, we are usually happy with estimates. We want to be able to distinguish programs that will finish in a few seconds or a few minutes from those that might require a few days or a few months or more, and we want to know when one program is twice as fast as another for the same task. Still, we need accurate measurements to generate experimental data that we can use to formulate and to check the validity of hypotheses about the relationship between running time and problem size. For this purpose, we use the `Stopwatch` data type shown on the facing page. Its `elapsedTime()` method returns the elapsed time since it was created, in seconds. The implementation is based on using the Java system's `currentTimeMillis()` method, which gives the current time in milliseconds, to save the time when the constructor is invoked, then uses it again to compute the elapsed time when `elapsedTime()` is invoked.

```
% java ThreeSum 1000 1Kints.txt
```

tick tick tick

```
70
% java ThreeSum 2000 2Kints.txt
```

tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick

```
528
% java ThreeSum 4000 4Kints.txt
```

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```
4039
```

Observing the running time of a program

API

public class Stopwatch	
Stopwatch()	*create a stopwatch*
double elapsedTime()	*return elapsed time since creation*

typical client

```
public static void main(String[] args)
{
   int N = Integer.parseInt(args[0]);
   int[] a = new int[N];
   for (int i = 0; i < N; i++)
      a[i] = StdRandom.uniform(-1000000, 1000000);
   Stopwatch timer = new Stopwatch();
   int cnt = ThreeSum.count(a);
   double time = timer.elapsedTime();
   StdOut.println(cnt + " triples " + time);
}
```

application

```
% java Stopwatch 1000
51 triples 0.488 seconds

% java Stopwatch 2000
516 triples 3.855 seconds
```

implementation

```
public class Stopwatch
{
   private final long start;

   public Stopwatch()
   {  start = System.currentTimeMillis();  }

   public double elapsedTime()
   {
      long now = System.currentTimeMillis();
      return (now - start) / 1000.0;
   }

}
```

An abstract data type for a stopwatch

Analysis of experimental data. The program `DoublingTest` on the facing page is a more sophisticated `Stopwatch` client that produces experimental data for `ThreeSum`. It generates a sequence of random input arrays, doubling the array size at each step, and prints the running times of `ThreeSum.count()` for each input size. These experiments are certainly reproducible—you can also run them on your own computer, as many times as you like. When you run `DoublingTest`, you will find yourself in a prediction-verification cycle: it prints several lines very quickly, but then slows down considerably. Each time it prints a line, you find yourself wondering how long it will be until it prints the next line. Of course, since you have a different computer from ours, the actual running times that you get are likely to be different from those shown for our computer. Indeed, if your computer is twice as fast as ours, your running times will be about half ours, which leads immediately to the well-founded hypothesis that running times on different computers are likely to differ by a constant factor. Still, you will find yourself asking the more detailed question *How long will my program take, as a function of the input size?* To help answer this question, we plot the data. The diagrams at the bottom of the facing page show the result of plotting the data, both on a normal and on a log-log scale, with the problem size N on the x-axis and the running time $T(N)$ on the y-axis. The log-log plot immediately leads to a hypothesis about the running time—the data fits a straight line of slope 3 on the log-log plot. The equation of such a line is

$$\lg(T(N)) = 3 \lg N + \lg a$$

(where a is a constant) which is equivalent to

$$T(N) = a N^3$$

the running time, as a function of the input size, as desired. We can use one of our data points to solve for a—for example, $T(8000) = 51.1 = a\,8000^3$, so $a = 9.98 \times 10^{-11}$—and then use the equation

$$T(N) = 9.98 \times 10^{-11} N^3$$

to predict running times for large N. Informally, we are checking the hypothesis that the data points on the log-log plot fall close to this line. Statistical methods are available for doing a more careful analysis to find estimates of a and the exponent b, but our quick calculations suffice to estimate running time for most purposes. For example, we can estimate the running time on our computer for $N = 16{,}000$ to be about $9.98 \times 10^{-11}\,16000^3 = 408.8$ seconds, or about 6.8 minutes (the actual time was 409.3 seconds). While waiting for your computer to print the line for $N = 16{,}000$ in `DoublingTest`, you might use this method to predict when it will finish, then check the result by waiting to see if your prediction is true.

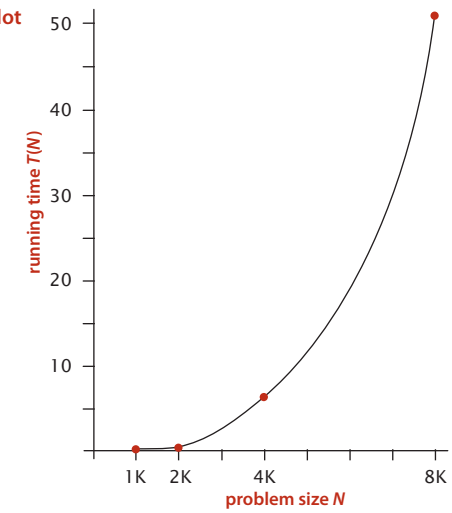
program to perform experiments

```
public class DoublingTest
{
   public static double timeTrial(int N)
   {  // Time ThreeSum.count() for N random 6-digit ints.
      int MAX = 1000000;
      int[] a = new int[N];
      for (int i = 0; i < N; i++)
         a[i] = StdRandom.uniform(-MAX, MAX);
      Stopwatch timer = new Stopwatch();
      int cnt = ThreeSum.count(a);
      return timer.elapsedTime();
   }

   public static void main(String[] args)
   {  // Print table of running times.
      for (int N = 250; true; N += N)
      {  // Print time for problem size N.
         double time = timeTrial(N);
         StdOut.printf("%7d %5.1f\n", N, time);
      }
   }
}
```

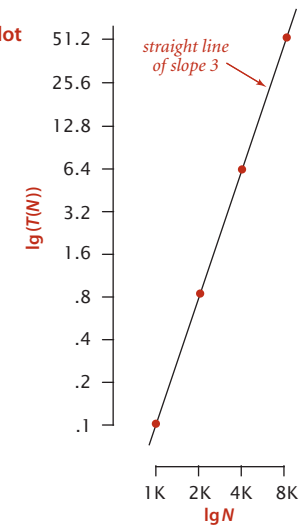
results of experiments

```
% java DoublingTest
    250   0.0
    500   0.0
   1000   0.1
   2000   0.8
   4000   6.4
   8000  51.1
...
```

standard plot



log-log plot



Analysis of experimental data (the running time of `ThreeSum.count()`)

So far, this process mirrors the process scientists use when trying to understand properties of the real world. A straight line in a log-log plot is equivalent to the hypothesis that the data fits the equation $T(N) = aN^b$. Such a fit is known as a *power law*. A great many natural and synthetic phenomena are described by power laws, and it is reasonable to hypothesize that the running time of a program does, as well. Indeed, for the analysis of algorithms, we have mathematical models that strongly support this and similar hypotheses, to which we now turn.

Mathematical models In the early days of computer science, D. E. Knuth postulated that, despite all of the complicating factors in understanding the running times of our programs, it is possible, in principle, to build a mathematical model to describe the running time of any program. Knuth's basic insight is simple: the total running time of a program is determined by two primary factors:

- The cost of executing each statement
- The frequency of execution of each statement

The former is a property of the computer, the Java compiler and the operating system; the latter is a property of the program and the input. If we know both for all instructions in the program, we can multiply them together and sum for all instructions in the program to get the running time.

The primary challenge is to determine the frequency of execution of the statements. Some statements are easy to analyze: for example, the statement that sets `cnt` to 0 in `ThreeSum.count()` is executed exactly once. Others require higher-level reasoning: for example, the `if` statement in `ThreeSum.count()` is executed precisely

$$N(N-1)(N-2)/6$$

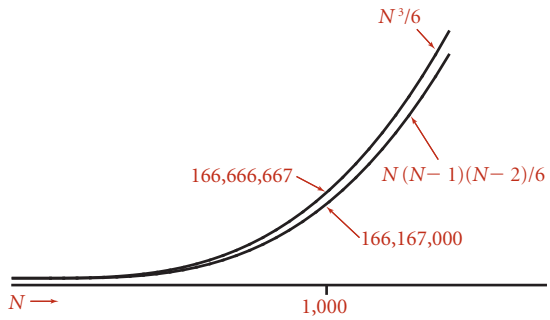
times (the number of ways to pick three different numbers from the input array—see EXERCISE 1.4.1). Others depend on the input data: for example the number of times the instruction `cnt++` in `ThreeSum.count()` is executed is precisely the number of triples that sum to 0 in the input, which could range from 0 of them to all of them. In the case of `DoublingTest`, where we generate the numbers randomly, it is possible to do a probabilistic analysis to determine the expected value of this quantity (see EXERCISE 1.4.40).

Tilde approximations. Frequency analyses of this sort can lead to complicated and lengthy mathematical expressions. For example, consider the count just considered of the number of times the `if` statement in `ThreeSum` is executed:

$$N(N-1)(N-2)/6 = N^3/6 - N^2/2 + N/3$$

As is typical in such expressions, the terms after the leading term are relatively small (for example, when $N = 1{,}000$ the value of $-N^2/2 + N/3$ 499,667 is certainly insignificant by comparison with $N^3/6 \approx 166{,}666{,}667$). To allow us to ignore insignificant terms and therefore substantially simplify the mathematical formulas that we work with, we often use a mathematical device known as the *tilde notation* ($\sim$). This notation allows us to work with *tilde approximations*, where we throw away low-order terms that complicate formulas and represent a negligible contribution to values of interest:

Definition. We write $\sim f(N)$ to represent any function that, when divided by $f(N)$, approaches 1 as N grows, and we write $g(N) \sim f(N)$ to indicate that $g(N)/f(N)$ approaches 1 as N grows.



Leading-term approximation

function	tilde approximation	order of growth
$N^3/6 - N^2/2 + N/3$	$\sim N^3/6$	N^3
$N^2/2 - N/2$	$\sim N^2/2$	N^2
$\lg N + 1$	$\sim \lg N$	$\lg N$
3	~ 3	1

Typical tilde approximations

order of growth	
description	**function**
constant	1
logarithmic	$\log N$
linear	N
linearithmic	$N \log N$
quadratic	N^2
cubic	N^3
exponential	2^N

Commonly encountered order-of-growth functions

For example, we use the approximation $\sim N^3/6$ to describe the number of times the `if` statement in `ThreeSum` is executed, since $N^3/6 - N^2/2 + N/3$ divided by $N^3/6$ approaches 1 as N grows. Most often, we work with tilde approximations of the form $g(N) \sim af(N)$ where $f(N) = N^b(\log N)^c$ with a, b, and c constants and refer to $f(N)$ as the *order of growth* of $g(N)$. When using the logarithm in the order of growth, we generally do not specify the base, since the constant a can absorb that detail. This usage covers the relatively few functions that are commonly encountered in studying the order of growth of a program's running time shown in the table at left (with the exception of the exponential, which we defer to CONTEXT). We will describe these functions in more detail and briefly discuss why they appear in the analysis of algorithms after we complete our treatment of `ThreeSum`.

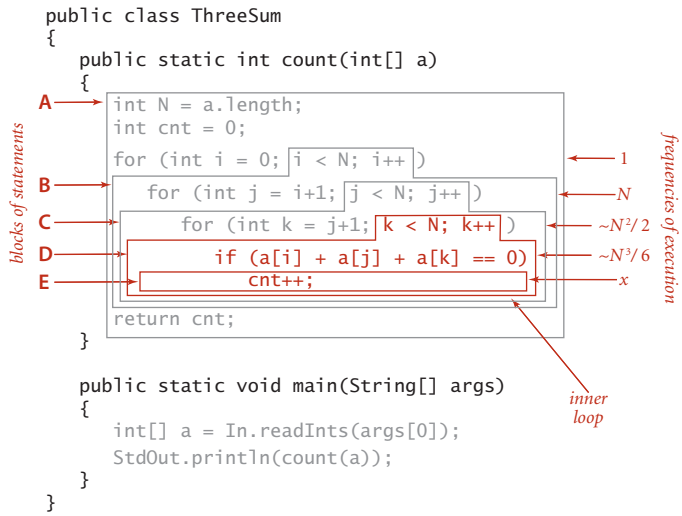
Approximate running time. To follow through on Knuth's approach to develop a mathematical expression for the total running time of a Java program, we can (in principle) study our Java compiler to find the number of machine instructions corresponding to each Java instruction and study our machine specifications to find the time of execution of each of the machine instructions, to produce a grand total. This process, for `ThreeSum`, is briefly summarized on the facing page. We classify blocks of Java statements by their frequency of execution, develop leading-term approximations for the frequencies, determine the cost of each statement, and then compute a total. Note that some frequencies may depend on the input. In this case, the number of times `cnt++` is executed certainly depends on the input—it is the number of triples that sum to 0, and could range from 0 to $\sim N^3/6$. We stop short of exhibiting the details (values of the constants) for any particular system, except to highlight that by using constant values t_0, t_1, t_2, ... for the time taken by the blocks of statements, we are assuming that each block of Java statements corresponds to machine instructions that require a specified fixed amount of time. A key observation from this exercise is to note that only the instructions that are executed the most frequently play a role in the final total—we refer to these instructions as the *inner loop* of the program. For `ThreeSum`, the inner loop is the statements that increment `k` and test that it is less than `N` and the statements that test whether the sum of three given numbers is 0 (and possibly the statement that implements the count, depending on the input). This behavior is typical: the running times of a great many programs depend only on a small subset of their instructions.

Order-of-growth hypothesis. In summary, the experiments on page 177 and the mathematical model on page 181 both support the following hypothesis:

Property A. The order of growth of the running time of `ThreeSum` (to compute the number of triples that sum to 0 among N numbers) is N^3.

Evidence: Let $T(N)$ be the running time of `ThreeSum` for N numbers. The mathematical model just described suggests that $T(N) \sim aN^3$ for some machine-dependent constant a; experiments on many computers (including yours and ours) validate that approximation.

Throughout this book, we use the term *property* to refer to a hypothesis that needs to be validated through experimentation. The end result of our mathematical analysis is precisely the same as the end result of our experimental analysis—the running time of `ThreeSum` is $\sim aN^3$ for a machine-dependent constant a. This match validates both the experiments and the mathematical model and also exhibits more insight about the



Anatomy of a program's statement execution frequencies

statement block	time in seconds	frequency	total time
E	t_0	x *(depends on input)*	$t_0 x$
D	t_1	$N^3/6 - N^2/2 + N/3$	$t_1 (N^3/6 - N^2/2 + N/3)$
C	t_2	$N^2/2 - N/2$	$t_2 (N^2/2 - N/2)$
B	t_3	N	$t_3 N$
A	t_4	1	t_4

grand total	$(t_1/6)\, N^3 + (t_2/2 - t_1/2)\, N^2 + (t_1/3 - t_2/2 + t_3)\, N + t_4 + t_0 x$
tilde approximation	$\sim (t_1 / 6)\, N^3$ *(assuming x is small)*
order of growth	N^3

Analyzing the running time of a program (example)

program because it does not require experimentation to determine the exponent. With some effort, we could validate the value of a on a particular system as well, though that activity is generally reserved for experts in situations where performance is critical.

Analysis of algorithms. Hypotheses such as PROPERTY A are significant because they relate the abstract world of a Java program to the real world of a computer running it. Working with the order of growth allows us to take one further step: to separate a program from the algorithm it implements. The idea that the order of growth of the running time of `ThreeSum` is N^3 does not depend on the fact that it is implemented in Java or that it is running on your laptop or someone else's cellphone or a supercomputer; it depends primarily on the fact that it examines all the different triples of numbers in the input. The *algorithm* that you are using (and sometimes the input model) determines the order of growth. Separating the algorithm from the implementation on a particular computer is a powerful concept because it allows us to develop knowledge about the performance of algorithms and then apply that knowledge to any computer. For example, we might say that `ThreeSum` is an implementation of the brute-force algorithm "*compute the sum of all different triples, counting those that sum to* 0"—we expect that an implementation of this algorithm in any programming language on any computer will lead to a running time that is proportional to N^3. In fact, much of the knowledge about the performance of classic algorithms was developed decades ago, but that knowledge is still relevant to today's computers.

3-sum cost model. When studying algorithms to solve the 3-sum problem, we count *array accesses* (the number of times an array entry is accessed, for read or write).

Cost model. We focus attention on properties of algorithms by articulating a *cost model* that defines the basic operations used by the algorithms we are studying to solve the problem at hand. For example, an appropriate cost model for the 3-sum problem, shown at right, is the number of times we access an array entry. With this cost model, we can make precise mathematical statements about properties of an algorithm, not just a particular implementation, as follows:

Proposition B. The brute-force 3-sum algorithm uses $\sim N^3/2$ array accesses to compute the number of triples that sum to 0 among N numbers.

Proof: The algorithm accesses each of the 3 numbers for each of the $\sim N^3/6$ triples.

We use the term *proposition* to refer to mathematical truths about algorithms in terms of a cost model. Throughout this book, we study the algorithms that we consider within

the framework of a specific cost model. Our intent is to articulate cost models such that the order of growth of the running time for a given implementation is the same as the order of growth of the cost of the underlying algorithm (in other words, the cost model should include operations that fall within the inner loop). We seek precise mathematical results about algorithms (propositions) and also hypotheses about performance of implementations (properties) that you can check through experimentation. In this case, PROPOSITION B is a mathematical truth that supports the hypothesis stated in PROPERTY A, which we have validated with experiments, in accordance with the scientific method.

Summary. For many programs, developing a mathematical model of running time reduces to the following steps:

- Develop an *input model*, including a definition of the problem size.
- Identify the *inner loop*.
- Define a *cost model* that includes operations in the inner loop.
- Determine the frequency of execution of those operations for the given input. Doing so might require mathematical *analysis*—we will consider some examples in the context of specific fundamental algorithms later in the book.

If a program is defined in terms of multiple methods, we normally consider the methods separately. As an example, consider our example program of SECTION 1.1, `BinarySearch`.

> ***Binary search.*** The *input model* is the array `a[]` of size N; the *inner loop* is the statements in the single `while` loop; the *cost model* is the compare operation (compare the values of two array entries); and the *analysis*, discussed in SECTION 1.1 and given in full detail in PROPOSITION B in SECTION 3.1, shows that the number of compares is at most $\lg N + 1$.

> ***Whitelist.*** The *input model* is the N numbers in the whitelist and the M numbers on standard input where we assume $M >> N$; the *inner loop* is the statements in the single `while` loop; the *cost model* is the compare operation (inherited from binary search); and the *analysis* is immediate given the analysis of binary search—the number of compares is at most M $(\lg N + 1)$.

Thus, we draw the conclusion that the order of growth of the running time of the whitelist computation is at most $M \lg N$, subject to the following considerations:

- If N is small, the input-output cost might dominate.
- The number of compares depends on the input—it lies between $\sim M$ and $\sim M \lg N$, depending on how many of the numbers on standard input are in the whitelist and on how long the binary search takes to find the ones that are (typically it is $\sim M \lg N$).
- We are assuming that the cost of `Arrays.sort()` is small compared to $M \lg N$. `Arrays.sort()` implements the *mergesort* algorithm, and in SECTION 2.2, we will see that the order of growth of the running time of mergesort is $N \log N$ (see PROPOSITION G in CHAPTER 2), so this assumption is justified.

Thus, the model supports our hypothesis from SECTION 1.1 that the *binary search algorithm* makes the computation feasible when M and N are large. If we double the length of the standard input stream, then we can expect the running time to double; if we double the size of the whitelist, then we can expect the running time to increase only slightly.

DEVELOPING MATHEMATICAL MODELS for the analysis of algorithms is a fruitful area of research that is somewhat beyond the scope of this book. Still, as you will see with binary search, mergesort, and many other algorithms, understanding certain mathematical models is critical to understanding the efficiency of fundamental algorithms, so we often present details and/or quote the results of classic studies. When doing so, we encounter various functions and approximations that are widely used in mathematical analysis. For reference, we summarize some of this information in the tables below.

description	notation	definition
floor	$\lfloor x \rfloor$	largest integer not greater than x
ceiling	$\lceil x \rceil$	smallest integer not smaller than x
natural logarithm	$\ln N$	$\log_e N$ (x such that $e^x = N$)
binary logarithm	$\lg N$	$\log_2 N$ (x such that $2^x = N$)
integer binary logarithm	$\lfloor \lg N \rfloor$	largest integer not greater than $\lg N$ (# bits in binary representation of N) – 1
harmonic numbers	H_N	$1 + 1/2 + 1/3 + 1/4 + \ldots + 1/N$
factorial	$N!$	$1 \times 2 \times 3 \times 4 \times \ldots \times N$

Commonly encountered functions in the analysis of algorithms

description	approximation
harmonic sum	$H_N = 1 + 1/2 + 1/3 + 1/4 + \ldots + 1/N \sim \ln N$
triangular sum	$1 + 2 + 3 + 4 + \ldots + N \sim N^2/2$
geometric sum	$1 + 2 + 4 + 8 + \ldots + N = 2N - 1 \sim 2N$ when $N = 2^n$
Stirling's approximation	$\lg N! = \lg 1 + \lg 2 + \lg 3 + \lg 4 + \ldots + \lg N \sim N \lg N$
binomial coefficients	$\binom{N}{k} \sim N^k/k!$ when k is a small constant
exponential	$(1 - 1/x)^x \sim 1/e$

Useful approximations for the analysis of algorithms

Order-of-growth classifications We use just a few structural primitives (statements, conditionals, loops, nesting, and method calls) to implement algorithms, so very often the order of growth of the cost is one of just a few functions of the problem size N. These functions are summarized in the table on the facing page, along with the names that we use to refer to them, typical code that leads to each function, and examples.

Constant. A program whose running time's order of growth is *constant* executes a fixed number of operations to finish its job; consequently its running time does not depend on N. Most Java operations take constant time.

Logarithmic. A program whose running time's order of growth is *logarithmic* is barely slower than a constant-time program. The classic example of a program whose running time is logarithmic in the problem size is *binary search* (see `BinarySearch` on page 47). The base of the logarithm is not relevant with respect to the order of growth (since all logarithms with a constant base are related by a constant factor), so we use log N when referring to order of growth.

Linear. Programs that spend a constant amount of time processing each piece of input data, or that are based on a single `for` loop, are quite common. The order of growth of such a program is said to be *linear*—its running time is proportional to N.

Linearithmic. We use the term *linearithmic* to describe programs whose running time for a problem of size N has order of growth N log N. Again, the base of the logarithm is not relevant with respect to the order of growth. The prototypical examples of linearithmic algorithms are `Merge.sort()` (see ALGORITHM 2.4) and `Quick.sort()` (see ALGORITHM 2.5).

Quadratic. A typical program whose running time has order of growth N^2 has two nested `for` loops, used for some calculation involving all pairs of N elements. The elementary sorting algorithms `Selection.sort()` (see ALGORITHM 2.1) and `Insertion.sort()` (see ALGORITHM 2.2) are prototypes of the programs in this classification.

Cubic. A typical program whose running time has order of growth N^3 has three nested `for` loops, used for some calculation involving all triples of N elements. Our example for this section, `ThreeSum`, is a prototype.

Exponential. In CHAPTER 6 (but not until then!) we will consider programs whose running times are proportional to 2^N or higher. Generally, we use the term *exponential* to refer to algorithms whose order of growth is b^N for any constant $b > 1$, even though different values of b lead to vastly different running times. Exponential algorithms are extremely slow—you will never run one of them to completion for a large problem. Still, exponential algorithms play a critical role in the theory of algorithms because

description	order of growth	typical code framework	description	example
constant	1	`a = b + c;`	*statement*	*add two numbers*
logarithmic	$\log N$	[***see page 47***]	*divide in half*	*binary search*

linear	N	`double max = a[0];`
		`for (int i = 1; i < N; i++)`
		`  if (a[i] > max) max = a[i];`
			loop	*find the maximum*

linearithmic	$N \log N$	[***see*** ALGORITHM 2.4]	*divide and conquer*	*mergesort*

quadratic	N^2	`for (int i = 0; i < N; i++)`
		`  for (int j = i+1; j < N; j++)`
		`  if (a[i] + a[j] == 0)`
		`  cnt++;`
			double loop	*check all pairs*

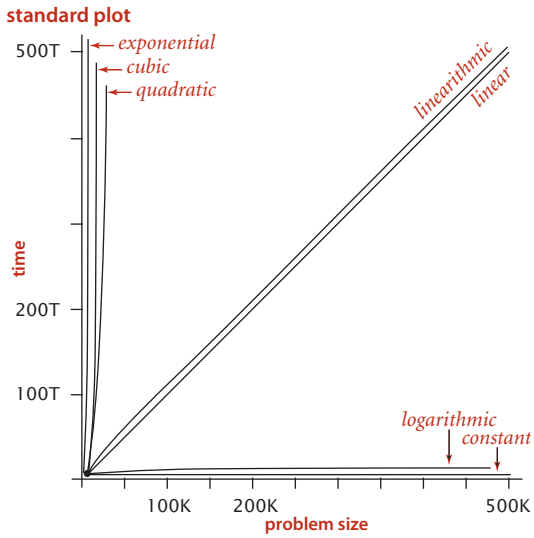
cubic	N^3	`for (int i = 0; i < N; i++)`
		`  for (int j = i+1; j < N; j++)`
		`  for (int k = j+1; k < N; k++)`
		`  if (a[i] + a[j] + a[k] == 0)`
		`  cnt++;`
			triple loop	*check all triples*

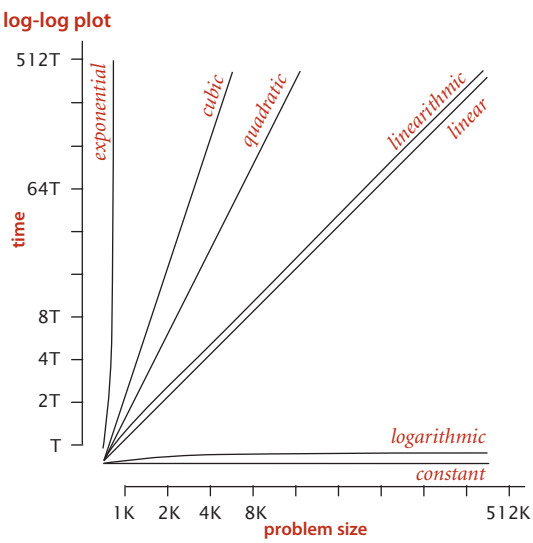
exponential	2^N	[***see*** CHAPTER 6]	*exhasutive search*	*check all subsets*

Summary of common order-of-growth hypotheses

there exists a large class of problems for which it seems that an exponential algorithm is the best possible choice.

THESE CLASSIFICATIONS ARE THE MOST COMMON, but certainly not a complete set. The order of growth of an algorithm's cost might be $N^2 \log N$ or $N^{3/2}$ or some similar function. Indeed, the detailed analysis of algorithms can require the full gamut of mathematical tools that have been developed over the centuries.





Typical orders of growth

A great many of the algorithms that we consider have straightforward performance characteristics that can be accurately described by one of the orders of growth that we have considered. Accordingly, we can usually work with specific propositions with a cost model, such as *mergesort uses between ½ N lg N and N lg N compares* that immediately imply hypotheses (properties) such as *the order of growth of mergesort's running time is linearithmic*. For economy, we abbreviate such a statement to just say *mergesort is linearithmic*.

The plots at left indicate the importance of the order of growth in practice. The *x*-axis is the problem size; the *y*-axis is the running time. These charts make plain that quadratic and cubic algorithms are not feasible for use on large problems. As it turns out, several important problems have natural solutions that are quadratic but clever algorithms that are linearithmic. Such algorithms (including mergesort) are critically important in practice because they enable us to address problem sizes far larger than could be addressed with quadratic solutions. Naturally, we therefore focus in this book on developing logarithmic, linear, and linearithmic algorithms for fundamental problems.

Designing faster algorithms One of the primary reasons to study the order of growth of a program is to help design a faster algorithm to solve the same problem. To illustrate this point, we consider next a faster algorithm for the 3-sum problem. How can we devise a faster algorithm, before even embarking on the study of algorithms? The answer to this question is that we *have* discussed and used two classic algorithms, *mergesort* and *binary search*, have introduced the facts that the mergesort is linearithmic and binary search is logarithmic. How can we take advantage of these algorithms to solve the 3-sum problem?

Warmup: 2-sum. Consider the easier problem of determining the number of *pairs* of integers in an input file that sum to 0. To simplify the discussion, assume also that the integers are distinct. This problem is easily solved in quadratic time by deleting the `k` loop and `a[k]` from `ThreeSum.count()`, leaving a double loop that examines all pairs, as shown in the *quadratic* entry in the table on page 187 (we refer to such an implementation as `TwoSum`). The implementation below shows how mergesort and binary search (see page 47) can serve as a basis for a *linearithmic* solution to the 2-sum problem. The improved algorithm is based on the fact that an entry `a[i]` is one of a pair that sums to 0 if and only if the value `-a[i]` is in the array (and `a[i]` is not zero). To solve the problem, we sort the array (to enable binary search) and then, for every entry `a[i]` in the array, do a binary search for `-a[i]` with `rank()` in `BinarySearch`. If the result is an index `j` with `j > i`, we increment the count. This succinct test covers three cases:

- An unsuccessful binary search returns `-1`, so we do not increment the count.
- If the binary search returns `j > i`, we have `a[i] + a[j] = 0`, so we increment the count.
- If the binary search returns `j` between 0 and `i`, we also have `a[i] + a[j] = 0` but do not increment the count, to avoid double counting.

The result of the computation is precisely the same as the result of the quadratic algorithm, but it takes much less time. The running time of the mergesort is

```
import java.util.Arrays;

public class TwoSumFast
{
   public static int count(int[] a)
   {  // Count pairs that sum to 0.
      Arrays.sort(a);
      int N = a.length;
      int cnt = 0;
      for (int i = 0; i < N; i++)
         if (BinarySearch.rank(-a[i], a) > i)
            cnt++;
      return cnt;
   }

   public static void main(String[] args)
   {
      int[] a = In.readInts(args[0]);
      StdOut.println(count(a));
   }
}
```

Linearithmic solution to the 2-sum problem

proportional to N log N, and the N binary searches each take time proportional to log N, so the running time of the whole algorithm is proportional to N log N. Developing a faster algorithm like this is not merely an academic exercise—the faster algorithm enables us to address much larger problems. For example, you are likely to be able to solve the 2-sum problem for 1 million integers (`1Mints.txt`) in a reasonable amount of time on your computer, but you would have to wait quite a long time to do it with the quadratic algorithm (see EXERCISE 1.4.41).

Fast algorithm for 3-sum. The very same idea is effective for the 3-sum problem. Again, assume also that the integers are distinct. A pair `a[i]` and `a[j]` is part of a triple that sums to 0 if and only if the value `-(a[i] + a[j])` is in the array (and not `a[i]` or `a[j]`). The code below sorts the array, then does $N(N-1)/2$ binary searches that each take time proportional to log N, for a total running time proportional to $N^2 \log N$. Note that in this case the cost of the sort is insignificant. Again, this solution enables us to address much larger problems (see EXERCISE 1.4.42). The plots in the figure at the bottom of the next page show the disparity in costs among these four algorithms for problem sizes in the range we have considered. Such differences certainly motivate the search for faster algorithms.

Lower bounds. The table on page 191 summarizes the discussion of this section. An interesting question immediately arises: Can we find algorithms for the 2-sum and 3-sum problems that are substantially faster than `TwoSumFast` and `ThreeSumFast`? Is there a linear algorithm for 2-sum or a linearithmic algorithm for 3-sum? The answer to this question is *no* for 2-sum (under a model that counts and allows only comparisons of linear or quadratic functions of the numbers) and *no one knows* for 3-sum, though experts believe that the best possible algorithm for 3-sum is quadratic. The idea of a lower bound on the order of growth of the worst-case running time for all possible algorithms to solve a problem is a very powerful one, which we will

```
import java.util.Arrays;

public class ThreeSumFast
{
   public static int count(int[] a)
   {  // Count triples that sum to 0.
      Arrays.sort(a);
      int N = a.length;
      int cnt = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            if (BinarySearch.rank(-a[i]-a[j], a) > j)
               cnt++;
      return cnt;
   }

   public static void main(String[] args)
   {
      int[] a = In.readInts(args[0]);
      StdOut.println(count(a));
   }
}
```

N^2 lg N solution to the 3-sum problem

revisit in detail in SECTION 2.2 in the context of sorting. Nontrivial lower bounds are difficult to establish, but very helpful in guiding our search for efficient algorithms.

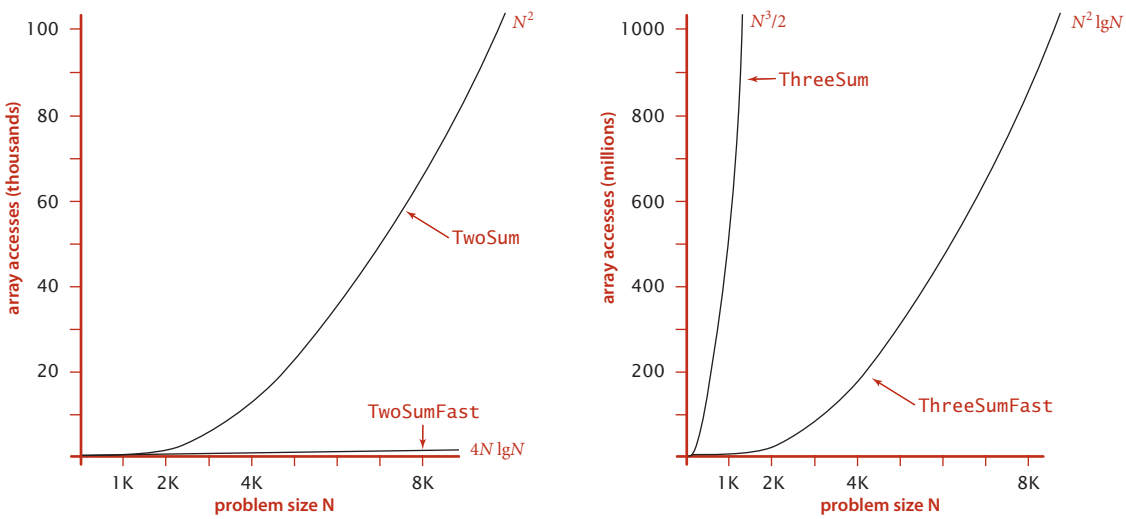
algorithm	order of growth of running time
`TwoSum`	N^2
`TwoSumFast`	$N \log N$
`ThreeSum`	N^3
`ThreeSumFast`	$N^2 \log N$

Summary of running times

THE EXAMPLES IN THIS SECTION SET THE STAGE for our treatment of algorithms in this book. Throughout the book, our strategy for addressing new problems is the following:

- Implement and analyze a straighforward solution to the problem. We usually refer to such solutions, like `ThreeSum` and `TwoSum`, as the *brute-force* solution.
- Examine algorithmic improvements, usually designed to reduce the order of growth of the running time, such as `TwoSumFast` and `ThreeSumFast`.
- Run experiments to validate the hypotheses that the new algorithms are faster.

In many cases, we examine *several* algorithms for the same problem, because running time is only one consideration when choosing an algorithm for a practical problem. We will develop this idea in detail in the context of fundamental problems throughout the book.



Costs of algorithms to solve the 2-sum and 3-sum problems

Doubling ratio experiments The following is a simple and effective shortcut for predicting performance and for determining the approximate order of growth of the running time of any program:

- Develop an input generator that produces inputs that model the inputs expected in practice (such as the random integers in `timeTrial()` in `DoublingTest`.
- Run the program `DoublingRatio` given below, a modification of `DoublingTest` that calculates the ratio of each running time with the previous.
- Run until the ratios approach a limit 2^b.

This test is not effective if the ratios do not approach a limiting value, but they do for many, many programs, implying the following conclusions:

- The order of growth of the running time is approximately N^b.
- To predict running times, multiply the last observed running time by 2^b and double N, continuing as long as desired. If you want to predict for an input size that is not a power of 2 times N, you can adjust ratios accordingly (see EXERCISE 1.4.9).

As illustrated below, the ratio for `ThreeSum` is about 8 and we can predict the running times for N = 16,000, 32,000, 64,000 to be 408.8, 3270.4, 26163.2 seconds, respectively, just by successively multiplying the last time for 8,000 (51.1) by 8.

program to perform experiments

```
public class DoublingRatio
{
   public static double timeTrial(int N)
   // same as for DoublingTest (page 177)

   public static void main(String[] args)
   {
      double prev = timeTrial(125);
      for (int N = 250; true; N += N)
      {
         double time = timeTrial(N);
         StdOut.printf("%6d %7.1f ", N, time);
         StdOut.printf("%5.1f\n", time/prev);
         prev = time;
      }
   }
}
```

results of experiments

```
% java DoublingRatio
   250     0.0   2.7
   500     0.0   4.8
  1000     0.1   6.9
  2000     0.8   7.7
  4000     6.4   8.0
  8000    51.1   8.0
```

predictions

```
 16000   408.8   8.0
 32000  3270.4   8.0
 64000 26163.2   8.0
```

This test is roughly equivalent to the process described on page 176 (run experiments, plot values on a log-log plot to develop the hypothesis that the running time is aN^b, determine the value of b from the slope of the line, then solve for a), but it is simpler to apply. Indeed, you can accurately predict preformance by hand when you run `DoublingRatio`. As the ratio approaches a limit, just multiply by that ratio to fill in later values in the table. Your approximate model of the order of growth is a power law with the binary logarithm of that ratio as the power.

Why does the ratio approach a constant? A simple mathematical calculation shows that to be the case for all of the common orders of growth just discussed (except exponential):

Proposition C. (Doubling ratio) If $T(N) \sim aN^b \lg N$ then $T(2N)/T(N) \sim 2^b$.

Proof: Immediate from the following calculation:

$$\begin{aligned} T(2N)/T(N) &= a(2N)^b \lg(2N) / aN^b \lg N \\ &= 2^b (1 + \lg 2 / \lg N) \\ &\sim 2^b \end{aligned}$$

Generally, the logarithmic factor cannot be ignored when developing a mathematical model, but it plays a less important role in predicting performance with a doubling hypothesis.

YOU SHOULD CONSIDER running doubling ratio experiments for every program that you write where performance matters—doing so is a very simple way to estimate the order of growth of the running time, perhaps revealing a performance bug where a program may turn out to be not as efficient as you might think. More generally, we can use hypotheses about the order of growth of the running time of programs to predict performance in one of the following ways:

Estimating the feasibility of solving large problems. You need to be able to answer this basic question for every program that you write: *Will the program be able to process this given input data in a reasonable amount of time?* To address such questions for a large amount of data, we extrapolate by a much larger factor than for doubling, say 10, as shown in the fourth column in the table at the bottom of the next page. Whether it is an investment banker running daily financial models or a scientist running a program to analyze experimental data or an engineer running simulations to test a design, it is not unusual for people to regularly run programs that take several hours to complete,

so the table focuses on that situation. Knowing the order of growth of the running time of an algorithm provides precisely the information that you need to understand limitations on the size of the problems that you can solve. *Developing such understanding is the most important reason to study performance.* Without it, you are likely have no idea how much time a program will consume; with it, you can make a back-of-the-envelope calculation to estimate costs and proceed accordingly.

Estimating the value of using a faster computer. You also may be faced with this basic question, periodically: *How much faster can I solve the problem if I get a faster computer?* Generally, if the new computer is x times faster than the old one, you can improve your running time by a factor of x. But it is usually the case that you can address larger problems with your new computer. How will that change affect the running time? Again, the order of growth is precisely the information needed to answer that question.

A FAMOUS RULE OF THUMB known as *Moore's Law* implies that you can expect to have a computer with about double the speed and double the memory 18 months from now, or a computer with about 10 times the speed and 10 times the memory in about 5 years. The table below demonstrates that you cannot keep pace with Moore's Law if you are using a quadratic or a cubic algorithm, and you can quickly determine whether that is the case by doing a doubling ratio test and checking that the ratio of running times as the input size doubles approaches 2, not 4 or 8.

order of growth of time				for a program that takes a few hours for input of size N	
description	function	2x factor	10x factor	predicted time for 10*N*	predicted time for10*N* on a 10x faster computer
linear	N	2	10	a day	a few hours
linearithmic	$N \log N$	2	10	a day	a few hours
quadratic	N^2	4	100	a few weeks	a day
cubic	N^3	8	1,000	several months	a few weeks
exponential	2^N	2^N	2^{9N}	never	never

Predictions on the basis of order-of-growth function

Caveats There are many reasons that you might get inconsistent or misleading results when trying to analyze program performance in detail. All of them have to do with the idea that one or more of the basic assumptions underlying our hypotheses might be not quite correct. We can develop new hypotheses based on new assumptions, but the more details that we need to take into account, the more care is required in the analysis.

Large constants. With leading-term approximations, we ignore constant coefficients in lower-order terms, which may not be justifed. For example, when we approximate the function $2N^2 + cN$ by $\sim 2N^2$, we are assuming that c is small. If that is not the case (suppose that c is 10^3 or 10^6) the approximation is misleading. Thus, we have to be sensitive to the possibility of large constants.

Nondominant inner loop. The assumption that the inner loop dominates may not always be correct. The cost model might miss the true inner loop, or the problem size N might not be sufficiently large to make the leading term in the mathematical description of the frequency of execution of instructions in the inner loop so much larger than lower-order terms that we can ignore them. Some programs have a significant amount of code outside the inner loop that needs to be taken into consideration. In other words, the cost model may need to be refined.

Instruction time. The assumption that each instruction always takes the same amount of time is not always correct. For example, most modern computer systems use a technique known as *caching* to organize memory, in which case accessing elements in huge arrays can take much longer if they are not close together in the array. You might observe the effect of caching for `ThreeSum` by letting `DoublingTest` run for a while. After seeming to converge to 8, the ratio of running times may jump to a larger value for large arrays because of caching.

System considerations. Typically, there are many, many things going on in your computer. Java is one application of many competing for resources, and Java itself has many options and controls that significantly affect performance. A garbage collector or a just-in-time compiler or a download from the internet might drastically affect the results of experiments. Such considerations can interfere with the bedrock principle of the scientific method that experiments should be reproducible, since what is happening at this moment in your computer will never be reproduced again. Whatever else is going on in your system should *in principle* be negligible or possible to control.

Too close to call. Often, when we compare two different programs for the same task, one might be faster in some situations, and slower in others. One or more of the considerations just mentioned could make the difference. There is a natural tendency among

some programmers (and some students) to devote an extreme amount of energy running races to find the "best" implementation, but such work is best left for experts.

Strong dependence on inputs. One of the first assumptions that we made in order to determine the order of growth of the program's running time of a program was that the running time should be relatively insensitive to the inputs. When that is not the case, we may get inconsistent results or be unable to validate our hypotheses. For example, suppose that we modify `ThreeSum` to answer the question *Does the input have a triple that sums to 0?* by changing it to return a `boolean` value, replacing `cnt++` by `return true` and adding `return false` as the last statement. The order of growth of the running time of this program is *constant* if the first three integers sum to 0 and *cubic* if there are no such triples in the input.

Multiple problem parameters. We have been focusing on measuring performance as a function of a *single* parameter, generally the value of a command-line argument or the size of the input. However, it is not unusual to have several parameters. A typical example arises when an algorithm involves building a data structure and then performing a sequence of operations that use that data structure. Both the size of the data structure and the number of operations are parameters for such applications. We have already seen an example of this in our analysis of the problem of whitelisting using binary search, where we have N numbers in the whitelist and M numbers on standard input and a typical running time proportional to $M \log N$.

Despite all these caveats, understanding the order of growth of the running time of each program is valuable knowledge for any programmer, and the methods that we have described are powerful and broadly applicable. Knuth's insight was that we can carry these methods through to the last detail *in principle* to make detailed, accurate predictions. Typical computer systems are extremely complex and close analysis is best left for experts, but the same methods are effective for developing approximate estimates of the running time of any program. A rocket scientist needs to have some idea of whether a test flight will land in the ocean or in a city; a medical researcher needs to know whether a drug trial will kill or cure all the subjects; and any scientist or engineer using a computer program needs to have some idea of whether it will run for a second or for a year.

Coping with dependence on inputs For many problems, one of the most significant of the caveats just mentioned is the dependence on inputs, because running times can vary widely. The running time of the modification of `ThreeSum` mentioned on the facing page ranges from constant to cubic, depending on the input, so a closer analysis is required if we want to predict performance. We briefly consider here some of the approaches that are effective and that we will consider for specific algorithms later in the book.

Input models. One approach is to more carefully model the kind of input to be processed in the problems that we need to solve. For example, we might assume that the numbers in the input to `ThreeSum` are random `int` values. This approach is challenging for two reasons:

- The model may be unrealistic.
- The analysis may be extremely difficult, requiring mathematical skills quite beyond those of the typical student or programmer.

The first of these is the more significant, often because the goal of a computation is to *discover* characteristics of the input. For example, if we are writing a program to process a genome, how can we estimate its performance on a different genome? A good model describing the genomes found in nature is precisely what scientists seek, so estimating the running time of our programs on data found in nature actually amounts to contributing to that model! The second challenge leads to a focus on mathematical results only for our most important algorithms. We will see several examples where a simple and tractable input model, in conjunction with classical mathematical analysis, helps us predict performance.

Worst-case performance guarantees. Some applications demand that the running time of a program be less than a certain bound, no matter what the input. To provide such performance *guarantees*, theoreticians take an extremely pessimistic view of the performance of algorithms: what would the running time be in the *worst case*? For example, such a conservative approach might be appropriate for the software that runs a nuclear reactor or a pacemaker or the brakes in your car. We want to guarantee that such software completes its job within the bounds that we set because the result could be catastrophic if it does not. Scientists normally do not contemplate the worst case when studying the natural world: in biology, the worst case might be the extinction of the human race; in physics, the worst case might be the end of the universe. But the worst case can be a very real concern in computer systems, where the input may be generated by another (potentially malicious) user, rather than by nature. For example, websites that do not use algorithms with performance guarantees are subject to *denial-of-service* attacks, where hackers flood them with pathological requests that make them

run much more slowly than planned. Accordingly, many of our algorithms are designed to provide performance guarantees, such as the following:

Proposition D. In the linked-list implementations of `Bag` (ALGORITHM 1.4), `Stack` (ALGORITHM 1.2), and `Queue` (ALGORITHM 1.3), all operations take constant time in the worst case.

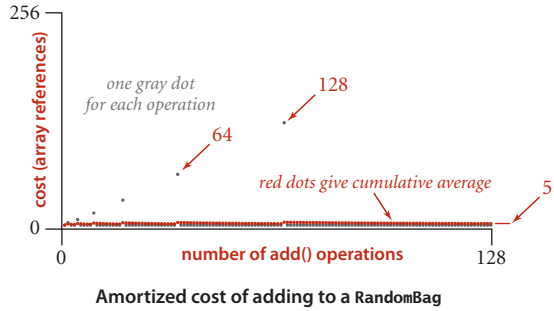
Proof: Immediate from the code. The number of instructions executed for each operation is bounded by a small constant. *Caveat*: This argument depends upon the (reasonable) assumption that the Java system creates a new `Node` in constant time.

Randomized algorithms. One important way to provide a performance guarantee is to introduce randomness. For example, the quicksort algorithm for sorting that we study in SECTION 2.3 (perhaps the most widely used sorting algorithm) is quadratic in the worst case, but randomly ordering the input gives a probabilistic guarantee that its running time is linearithmic. Every time you run the algorithm, it will take a different amount of time, but the chance that the time will not be linearithmic is so small as to be negligible. Similarly, the hashing algorithms for symbol tables that we study in SECTION 3.4 (again, perhaps the most widely used approach) are linear-time in the worst case, but constant-time under a probabilistic guarantee. These guarantees are not absolute, but the chance that they are invalid is less than the chance your computer will be struck by lightning. Thus, such guarantees are as useful in practice as worst-case guarantees.

Sequences of operations. For many applications, the algorithm "input" might be not just data, but the sequence of operations performed by the client. For example, a pushdown stack where the client pushes N values, then pops them all, may have quite different performance characteristics from one where the client issues an alternating sequence N of push and pop operations. Our analysis has to take both situations into account (or to include a reasonable model of the sequence of operations).

Amortized analysis. Accordingly, another way to provide a performance guarantee is to *amortize* the cost, by keeping track of the total cost of all operations, divided by the number of operations. In this setting, we can allow some expensive operations, while keeping the average cost of operations low. The prototypical example of this type of analysis is the study of the resizing array data structure for `Stack` that we considered in SECTION 1.3 (ALGORITHM 1.1 on page 141). For simplicity, suppose that N is a power of 2. Starting with an empty structure, how many array entries are accessed for N consecutive calls to `push()`? This quantity is easy to calculate: the number of array accesses is

$$N + 4 + 8 + 16 + \ldots + 2N = 5N - 4$$



Amortized cost of adding to a `RandomBag`

The first term accounts for the array access within each of the N calls to `push()`; the subsequent terms account for the array accesses to initialize the data structure each time it doubles in size. Thus the *average number of array accesses per operation* is constant, even though the last operation takes linear time. This is known as an "amortized" analysis because we spread the cost of the few expensive operations, by assigning a portion of it to each of a large number of inexpensive operations. `VisualAccumulator` provides an easy way to illustrate the process, shown above.

Proposition E. In the resizing array implementation of `Stack` (ALGORITHM 1.1), the average number of array accesses for any sequence of operations starting from an empty data structure is constant in the worst case.

Proof sketch: For each `push()` that causes the array to grow (say from size N to size $2N$), consider the $N/2 - 1$ `push()` operations that most recently caused the stack size to grow to k, for k from $N/2 + 2$ to N. Averaging the $4N$ array accesses to grow the array with $N/2$ array accesses (one for each push), we get an average cost of 9 array accesses per operation. Proving that the number of array accesses used by any sequence of M operations is proportional to M is more intricate (see EXERCISE 1.4.32)

This kind of analysis is widely applicable. In particular, we use resizing arrays as the underlying data structure for several algorithms that we consider later in this book.

IT IS THE TASK OF THE ALGORITHM ANALYST to discover as much relevant information about an algorithm as possible, and it is the task of the applications programmer to apply that knowledge to develop programs that effectively solve the problems at hand. Ideally, we want algorithms that lead to clear and compact code that provides both a good guarantee and good performance on input values of interest. Many of the classic algorithms that we consider in this chapter are important for a broad variety of applications precisely because they have these properties. Using them as models, you can develop good solutions yourself for typical problems that you face while programming.

Memory As with running time, a program's memory usage connects directly to the physical world: a substantial amount of your computer's circuitry enables your program to store values and later retrieve them. The more values you need to have stored at any given instant, the more circuitry you need. You probably are aware of limits on memory usage on your computer (even more so than for time) because you probably have paid extra money to get more memory.

Memory usage is well-defined for Java on your computer (every value requires precisely the same amount of memory each time that you run your program), but Java is implemented on a very wide range of computational devices, and memory consumption is implementation-dependent. For economy, we use the word *typical* to signal that values are subject to machine dependencies.

type	bytes
`boolean`	1
`byte`	1
`char`	2
`int`	4
`float`	4
`long`	8
`double`	8

Typical memory requirements for primitive types

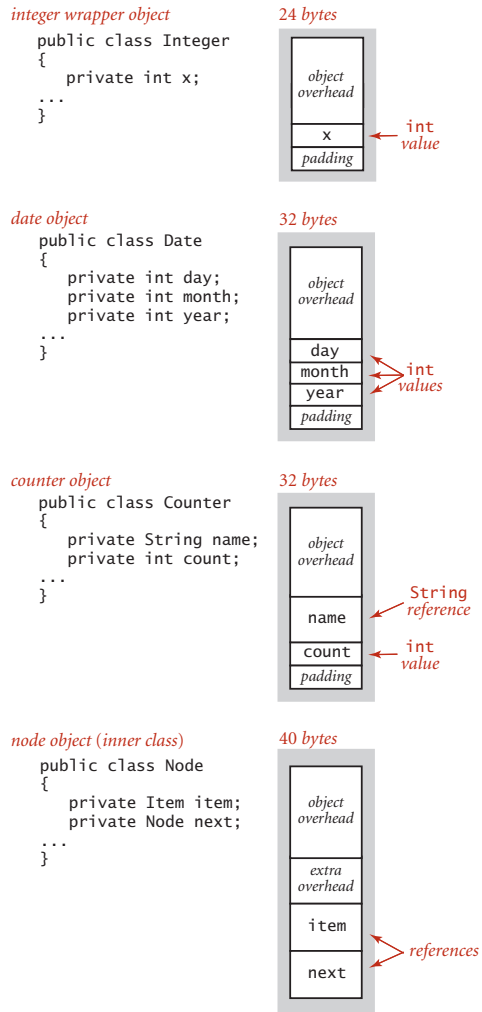
One of Java's most significant features is its memory allocation system, which is supposed to relieve you from having to worry about memory. Certainly, you are well-advised to take advantage of this feature when appropriate. Still, it is your responsibility to know, at least approximately, when a program's memory requirements will prevent you from solving a given problem.

Analyzing memory usage is much easier than analyzing running time, primarily because not as many program statements are involved (just declarations) and because the analysis reduces complex objects to the primitive types, whose memory usage is well-defined and simple to understand: we can count up the number of variables and weight them by the number of bytes according to their type. For example, since the Java `int` data type is the set of integer values between −2,147,483,648 and 2,147,483,647, a grand total of 2^{32} different values, typical Java implementations use 32 bits to represent `int` values. Similar considerations hold for other primitive types: typical Java implementations use 8-bit bytes, representing each `char` value with 2 bytes (16 bits), each `int` value with 4 bytes (32 bits), each `double` and each `long` value with 8 bytes (64 bits), and each `boolean` value with 1 byte (since computers typically access memory one byte at a time). Combined with knowledge of the amount of memory available, you can calculate limitations from these values. For example, if you have 1GB of memory on your computer (1 billion bytes), you cannot fit more than about 32 million `int` values or 16 million `double` values in memory at any one time.

On the other hand, analyzing memory usage is subject to various differences in machine hardware and in Java implementations, so you should consider the specific examples that we give as indicative of how you might go about determining memory usage when warranted, not the final word for your computer. For example, many data structures involve representation of machine addresses, and the amount of memory

needed for a machine address varies from machine to machine. For consistency, we assume that 8 bytes are needed to represent addresses, as is typical for 64-bit architectures that are now widely used, recognizing that many older machines use a 32-bit architecture that would involve just 4 bytes per machine address.

Objects. To determine the memory usage of an object, we add the amount of memory used by each instance variable to the overhead associated with each object, typically 16 bytes. The overhead includes a reference to the object's class, garbage collection information, and synchronization information. Moreover, the memory usage is typically padded to be a multiple of 8 bytes (machine words, on a 64-bit machine). For example, an `Integer` object uses 24 bytes (16 bytes of overhead, 4 bytes for its `int` instance variable, and 4 bytes of padding). Similarly, a `Date` (page 91) object also uses 32 bytes: 16 bytes of overhead, 4 bytes for each of its three `int` instance variables, and 4 bytes of padding. A reference to an object typically is a memory address and thus uses 8 bytes of memory. For example, a `Counter` (page 89) object uses 32 bytes: 16 bytes of overhead, 8 bytes for its `String` instance variable (a reference), 4 bytes for its `int` instance variable, and 4 bytes of padding. When we account for the memory for a reference, we account separately for the memory for the object itself, so this total does not count the memory for the `String` value.



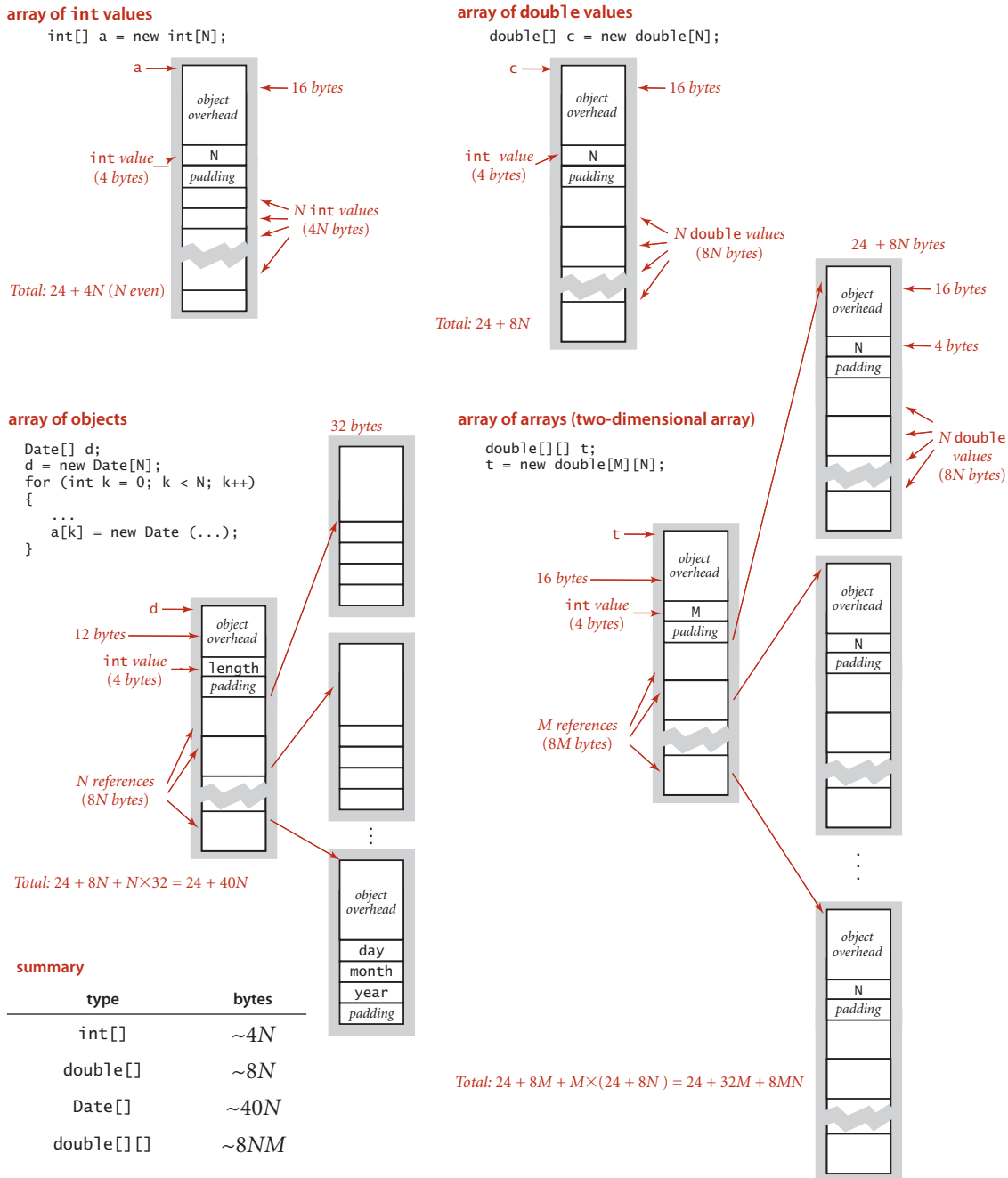
Typical object memory requirements

Linked lists. A nested non-static (inner) class such as our `Node` class (page 142) requires an extra 8 bytes of overhead (for a reference to the enclosing instance). Thus, a `Node` object uses 40 bytes (16 bytes of object overhead, 8 bytes each for the references to the `Item` and `Node` objects, and 8 bytes for the extra overhead). Thus, since an `Integer` object uses 24 bytes, a stack with N integers built with a linked-list representation (ALGORITHM 1.2) uses $32 + 64N$ bytes, the usual 16 for object overhead for `Stack`, 8 for its reference instance variable, 4 for its `int` instance variable, 4 for padding, and 64 for each entry, 40 for a `Node` and 24 for an `Integer`.

Arrays. Typical memory requirements for various types of arrays in Java are summarized in the diagrams on the facing page. Arrays in Java are implemented as objects, typically with extra overhead for the length. An *array of primitive-type values* typically requires 24 bytes of header information (16 bytes of object overhead, 4 bytes for the length, and 4 bytes of padding) plus the memory needed to store the values. For example, an array of N `int` values uses $24 + 4N$ bytes (rounded up to be a multiple of 8), and an array of N `double` values uses $24 + 8N$ bytes. An *array of objects* is an array of references to the objects, so we need to add the space for the references to the space required for the objects. For example, an array of N `Date` objects (page 91) uses 24 bytes (array overhead) plus $8N$ bytes (references) plus 32 bytes for each object and 4 bytes of padding, for a grand total of $24 + 40N$ bytes. A *two-dimensional array* is an array of arrays (each array is an object). For example, a two-dimensional M-by-N array of `double` values uses 24 bytes (overhead for the array of arrays) plus $8M$ bytes (references to the row arrays) plus M times 16 bytes (overhead from the row arrays) plus M times N times 8 bytes (for the N `double` values in each of the M rows) for a grand total of $8NM + 32M + 24 \sim 8NM$ bytes. When array entries are objects, a similar accounting leads to a total of $8NM + 32M + 24 \sim 8NM$ bytes for the array of arrays filled with references to objects, plus the memory for the objects themselves.

String objects. We account for memory in Java's `String` objects in the same way as for any other object, except that aliasing is common for strings. The standard `String` implementation has four instance variables: a reference to a character array (8 bytes) and three `int` values (4 bytes each). The first `int` value is an offset into the character array; the second is a count (the string length). In terms of the instance variable names in the drawing on the facing page, the string that is represented consists of the characters `value[offset]` through `value[offset + count - 1]`. The third `int` value in `String` objects is a hash code that saves recomputation in certain circumstances that need not concern us now. Therefore, each `String` object uses a total of 40 bytes (16 bytes for object overhead plus 4 bytes for each of the three `int` instance variables plus 8 bytes for the array reference plus 4 bytes of padding). This space requirement is in addition to the space needed for the characters themselves, which are in the array. The space needed for the characters is accounted for separately because the `char` array is often shared among strings. Since `String` objects are immutable, this arrangement allows the implementation to save memory when `String` objects have the same underlying `value[]`.

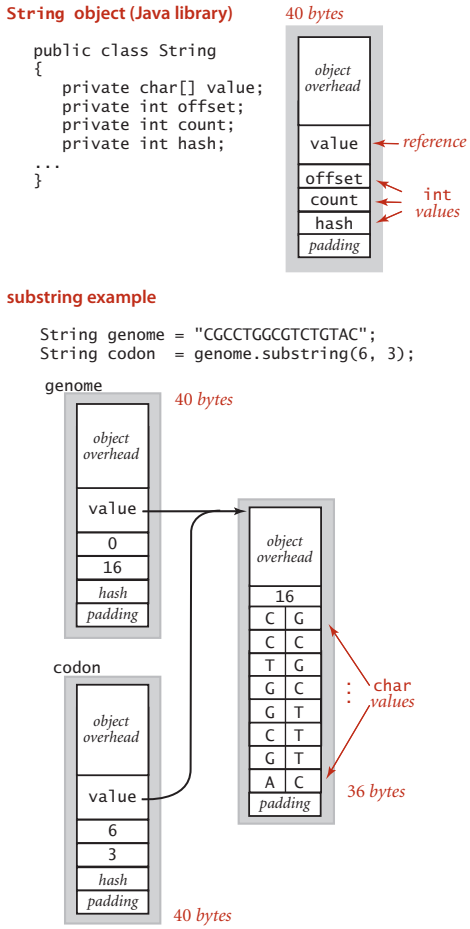
String values and substrings. A `String` of length N typically uses 40 bytes (for the `String` object) plus $24 + 2N$ bytes (for the array that contains the characters) for a total of $64 + 2N$ bytes. But it is typical in string processing to work with substrings, and Java's representation is meant to allow us to do so without having to make copies of



summary

type	bytes
`int[]`	~4*N*
`double[]`	~8*N*
`Date[]`	~40*N*
`double[][]`	~8*NM*

Typical memory requirements for arrays of `int` values, `double` values, objects, and arrays



A `String` and a substring

the string's characters. When you use the `substring()` method, you create a new `String` object (40 bytes) but reuse the same `value[]` array, so a substring of an existing string takes just 40 bytes. The character array containing the original string is aliased in the object for the substring; the offset and length fields identify the substring. In other words, *a substring takes constant extra memory and forming a substring takes constant time*, even when the lengths of the string and the substring are huge. A naive representation that requires copying characters to make substrings would take linear time and space. The ability to create a substring using space (and time) independent of its length is the key to efficiency in many basic string-processing algorithms.

THESE BASIC MECHANISMS ARE EFFECTIVE for estimating the memory usage of a great many programs, but there are numerous complicating factors that can make the task significantly more difficult. We have already noted the potential effect of aliasing. Moreover, memory consumption is a complicated dynamic process when function calls are involved because the system memory allocation mechanism plays a more important role, with more system dependencies. For example, when your program calls a method, the system allocates the memory needed for the method (for its local variables) from a special area of memory called the *stack* (a system pushdown stack), and when the method returns to the caller, the memory is returned to the stack. For this reason, creating arrays or other large objects in recursive programs is dangerous, since each recursive call implies significant memory usage. When you create an object with `new`, the system allocates the memory needed for the object from another special area of memory known as the *heap* (not the same as the binary heap data structure we consider in SECTION 2.4), and you must remember that every object lives until no references to it remain, at which point a system process known as *garbage collection* reclaims its memory for the heap. Such dynamics can make the task of precisely estimating memory usage of a program challenging.

Perspective Good performance is important. An impossibly slow program is almost as useless as an incorrect one, so it is certainly worthwhile to pay attention to the cost at the outset, to have some idea of which kinds of problems you might feasibly address. In particular, it is always wise to have some idea of which code constitutes the inner loop of your programs.

Perhaps the most common mistake made in programming is to pay too much attention to performance characteristics. Your first priority is to make your code clear and correct. Modifying a program for the sole purpose of speeding it up is best left for experts. Indeed, doing so is often counterproductive, as it tends to create code that is complicated and difficult to understand. C. A. R. Hoare (the inventor of quicksort and a leading proponent of writing clear and correct code) once summarized this idea by saying that "*premature optimization is the root of all evil*," to which Knuth added the qualifier "(*or at least most of it*) *in programming*." Beyond that, improving the running time is not worthwhile if the available cost benefits are insignificant. For example, improving the running time of a program by a factor of 10 is inconsequential if the running time is only an instant. Even when a program takes a few minutes to run, the total time required to implement and debug an improved algorithm might be substantially more than the time required simply to run a slightly slower one—you may as well let the computer do the work. Worse, you might spend a considerable amount of time and effort implementing ideas that should in theory improve a program but do not do so in practice.

Perhaps the second most common mistake made in programming is to ignore performance characteristics. Faster algorithms are often more complicated than brute-force ones, so you might be tempted to accept a slower algorithm to avoid having to deal with more complicated code. However, you can sometimes reap huge savings with just a few lines of good code. Users of a surprising number of computer systems lose substantial time unknowingly waiting for brute-force quadratic algorithms to finish solving a problem, when linear or linearithmic algorithms are available that could solve the problem in a fraction of the time. When we are dealing with huge problem sizes, we often have no choice but to seek better algorithms.

We generally take as implicit the methodology described in this section to estimate memory usage and to develop an order-of-growth hypothesis of the running time from a tilde approximation resulting from a mathematical analysis within a cost model, and to check those hypotheses with experiments. Improving a program to make it more clear, efficient, and elegant should be your goal every time that you work on it. If you pay attention to the cost all the way through the development of a program, you will reap the benefits every time you use it.

Q&A

Q. Why not use `StdRandom` to generate random values instead of maintaining the file `1Mints.txt`?

A. It is easier to debug code in development and to reproduce experiments. `StdRandom` produces different values each time it is called, so running a program after fixing a bug may not test the fix! You could use the `initialize()` method in `StdRandom` to address this problem, but a reference file such as `1Mints.txt` makes it easier to add test cases while debugging. Also, different programmers can compare performance on different computers, without worrying about the input model. Once you have debugged a program and have a good idea of how it performs, it is certainly worthwhile to test it on random data. For example, `DoublingTest` and `DoublingRatio` take this approach.

Q. I ran `DoublingRatio` on my computer, but the results were not as consistent as in the book. Some of the ratios were not close to 8. Why?

A. That is why we discussed "caveats" on page 195. Most likely, your computer's operating system decided to do something else during the experiment. One way to mitigate such problems is to invest more time in more experiments. For example, you could change `DoublingTest` to run the experiments 1,000 times for each N, giving a much more accurate estimate for the running time for each size (see EXERCISE 1.4.39).

Q. What, exactly, does "as N grows" mean in the definition of the tilde notation?

A. The formal definition of $f(N) \sim g(N)$ is $\lim_{N \to \infty} f(N)/g(N) = 1$.

Q. I've seen other notations for describing order of growth. What's the story?

A. The "big-Oh" notation is widely used: we say that $f(N)$ is $O(g(N))$ if there exist constants c and N_0 such that $|f(N)| < c\, g(N)$ for all $N > N_0$. This notation is very useful in providing asymptotic upper bounds on the performance of algorithms, which is important in the theory of algorithms. But it is not useful for predicting performance or for comparing algorithms.

Q. Why not?

A. The primary reason is that it describes only an *upper bound* on the running time. Actual performance might be much better. The running time of an algorithm might be both $O(N^2)$ and $\sim a\, N \log N$. As a result, it cannot be used to justify tests like our doubling ratio test (see PROPOSITION C on page 193).

Q. So why is the big-Oh notation so widely used?

A. It facilitates development of bounds on the order of growth, even for complicated algorithms for which more precise analysis might not be feasible. Moreover, it is compatible with the "big-Omega" and "big-Theta" notations that theoretical computer scientists use to classify algorithms by bounding their worst-case performance. We say that $f(N)$ is $\Omega(g(N))$ if there exist constants c and N_0 such that $|f(N)| > c\,g(N)$ for $N > N_0$; and if $f(N)$ is $O(g(N))$ and $\Omega(g(N))$, we say that $f(N)$ is $\Theta(g(N))$. The "big-Omega" notation is typically used to describe a *lower bound* on the worst case, and the "big-Theta" notation is typically used to describe the performance of algorithms that are *optimal* in the sense that no algorithm can have better asymptotic worst-case order of growth. Optimal algorithms are certainly worth considering in practical applications, but there are many other considerations, as you will see.

Q. Aren't upper bounds on asymptotic performance important?

A. Yes, but we prefer to discuss precise results in terms of frequency of statement execution with respect to cost models, because they provide more information about algorithm performance and because deriving such results is feasible for the algorithms that we discuss. For example, we say "`ThreeSum` uses $\sim N^3/2$ array accesses" and "the number of times `cnt++` is executed in `ThreeSum` is $\sim N^3/6$ in the worst case," which is a bit more verbose but much more informative than the statement "the running time of `ThreeSum` is $O(N^3)$."

Q. When the order of growth of the running time of an algorithm is $N \log N$, the doubling test will lead to the hypothesis that the running time is $\sim a\,N$ for a constant a. Isn't that a problem?

A. We have to be careful not to try to infer that the experimental data implies a particular mathematical model, but when we are just predicting performance, this is not really a problem. For example, when N is between 16,000 and 32,000, the plots of $14N$ and $N \lg N$ are very close to one another. The data fits both curves. As N increases, the curves become closer together. It actually requires some care to experimentally check the hypothesis that an algorithm's running time is linearithmic but not linear.

Q. Does `int[] a = new int[N]` count as N array accesses (to initialize entries to 0)?

A. Most likely yes, so we make that assumption in this book, though a sophisticated compiler implementation might try to avoid this cost for huge sparse arrays.

EXERCISES

1.4.1 Show that the number of different triples that can be chosen from N items is precisely $N(N-1)(N-2)/6$. *Hint*: Use mathematical induction.

1.4.2 Modify `ThreeSum` to work properly even when the `int` values are so large that adding two of them might cause overflow.

1.4.3 Modify `DoublingTest` to use `StdDraw` to produce plots like the standard and log-log plots in the text, rescaling as necessary so that the plot always fills a substantial portion of the window.

1.4.4 Develop a table like the one on page 181 for `TwoSum`.

1.4.5 Give tilde approximations for the following quantities:

a. $N + 1$

b. $1 + 1/N$

c. $(1 + 1/N)(1 + 2/N)$

d. $2N^3 - 15N^2 + N$

e. $\lg(2N)/\lg N$

f. $\lg(N^2 + 1) / \lg N$

g. $N^{100} / 2^N$

1.4.6 Give the order of growth (as a function of N) of the running times of each of the following code fragments:

a.
```
int sum = 0;
for (int n = N; n > 0; n /= 2)
  for(int i = 0; i < n; i++)
     sum++;
```

b.
```
int sum = 0;
for (int i = 1 i < N; i *= 2)
    for (int j = 0; j < i; j++)
        sum++;
```

c.
```
int sum = 0;
for (int i = 1 i < N; i *= 2)
   for (int j = 0; j < N; j++)
       sum++;
```

1.4.7 Analyze `ThreeSum` under a cost model that counts arithmetic operations (and comparisons) involving the input numbers.

1.4.8 Write a program to determine the number pairs of values in an input file that are equal. If your first try is quadratic, think again and use `Arrays.sort()` to develop a linearithmic solution.

1.4.9 Give a formula to predict the running time of a program for a problem of size N when doubling experiments have shown that the doubling factor is 2^b and the running time for problems of size N_0 is T.

1.4.10 Modify binary search so that it always returns the element with the smallest index that matches the search element (and still guarantees logarithmic running time).

1.4.11 Add an instance method `howMany()` to `StaticSETofInts` (page 99) that finds the number of occurrences of a given key in time proportional to log N in the worst case.

1.4.12 Write a program that, given two sorted arrays of N `int` values, prints all elements that appear in both arrays, in sorted order. The running time of your program should be proportional to N in the worst case.

1.4.13 Using the assumptions developed in the text, give the amount of memory needed to represent an object of each of the following types:

a. `Accumulator`

b. `Transaction`

c. `FixedCapacityStackOfStrings` with capacity C and N entries

d. `Point2D`

e. `Interval1D`

f. `Interval2D`

g. `Double`

CREATIVE PROBLEMS

1.4.14 *4-sum.* Develop an algorithm for the *4-sum* problem.

1.4.15 *Faster 3-sum.* As a warmup, develop an implementation `TwoSumFaster` that uses a *linear* algorithm to count the pairs that sum to zero after the array is sorted (instead of the binary-search-based linearithmic algorithm). Then apply a similar idea to develop a quadratic algorithm for the 3-sum problem.

1.4.16 *Closest pair (in one dimension).* Write a program that, given an array `a[]` of N `double` values, finds a *closest pair*: two values whose difference is no greater than the the difference of any other pair (in absolute value). The running time of your program should be linearithmic in the worst case.

1.4.17 *Farthest pair (in one dimension).* Write a program that, given an array `a[]` of N `double` values, finds a *farthest pair*: two values whose difference is no smaller than the the difference of any other pair (in absolute value). The running time of your program should be linear in the worst case.

1.4.18 *Local minimum of an array.* Write a program that, given an array `a[]` of N distinct integers, finds a *local minimum*: an index `i` such that `a[i-1] < a[i] < a[i+1]`. Your program should use ~2lg N compares in the worst case..

Answer: Examine the middle value `a[N/2]` and its two neighbors `a[N/2 - 1]` and `a[N/2 + 1]`. If `a[N/2]` is a local minimum, stop; otherwise search in the half with the smaller neighbor.

1.4.19 *Local minimum of a matrix.* Given an N-by-N array `a[]` of N^2 distinct integers, design an algorithm that runs in time proportional to N to find a *local minimum*: a pair of indices `i` and `j` such that `a[i][j] < a[i+1][j]`, `a[i][j] < a[i][j+1]`, `a[i][j] < a[i-1][j]`, and `a[i][j] < a[i][j-1]`. The running time of your program should be proportional to N in the worst case.

1.4.20 *Bitonic search.* An array is *bitonic* if it is comprised of an increasing sequence of integers followed immediately by a decreasing sequence of integers. Write a program that, given a bitonic array of N distinct `int` values, determines whether a given integer is in the array. Your program should use ~3lg N compares in the worst case.

1.4.21 *Binary search on distinct values.* Develop an implementation of binary search for `StaticSETofInts` (see page 98) where the running time of `contains()` is guaranteed

to be ~lg R, where R is the number of different integers in the array given as argument to the constructor.

1.4.22 *Binary search with only addition and subtraction.* [Mihai Patrascu] Write a program that, given an array of N distinct `int` values in ascending order, determines whether a given integer is in the array. You may use only additions and subtractions and a constant amount of extra memory. The running time of your program should be proportional to log N in the worst case.

Answer: Instead of searching based on powers of two (binary search), use Fibonacci numbers (which also grow exponentially). Maintain the current search range to be the interval $[i, i + F_k]$ and keep F_k and F_{k-1} in two variables. At each step compute F_{k-2} via subtraction, check element $i + F_{k-2}$, and update the current range to either $[i, i + F_{k-2}]$ or $[i + F_{k-2}, i + F_{k-2} + F_{k-1}]$.

1.4.23 *Binary search for a fraction.* Devise a method that uses a logarithmic number of queries of the form *Is the number less than x?* to find a rational number p/q such that $0 < p < q < N$. *Hint*: Two fractions with denominators less than N cannot differ by more than $1/N^2$.

1.4.24 *Throwing eggs from a building.* Suppose that you have an N-story building and plenty of eggs. Suppose also that an egg is broken if it is thrown off floor F or higher, and unhurt otherwise. First, devise a strategy to determine the value of F such that the number of broken eggs is ~lg N when using ~lg N throws, then find a way to reduce the cost to ~2lg F.

1.4.25 *Throwing two eggs from a building.* Consider the previous question, but now suppose you only have two eggs, and your cost model is the number of throws. Devise a strategy to determine F such that the number of throws is at most $2\sqrt{N}$, then find a way to reduce the cost to $\sim c\sqrt{F}$. This is analogous to a situation where search hits (egg intact) are much cheaper than misses (egg broken).

1.4.26 *3-collinearity.* Suppose that you have an algorithm that takes as input N distinct points in the plane and can return the number of triples that fall on the same line. Show that you can use this algorithm to solve the 3-sum problem. *Strong hint*: Use algebra to show that (a, a^3), (b, b^3), and (c, c^3) are collinear if and only if $a + b + c = 0$.

1.4.27 *Queue with two stacks.* Implement a queue with two stacks so that each queue

CREATIVE PROBLEMS *(continued)*

operation takes a constant amortized number of stack operations. *Hint*: If you push elements onto a stack and then pop them all, they appear in reverse order. If you repeat this process, they're now back in order.

1.4.28 *Stack with a queue.* Implement a stack with a single queue so that each stack operations takes a linear number of queue operations. *Hint*: To delete an item, get all of the elements on the queue one at a time, and put them at the end, except for the last one which you should delete and return. (This solution is admittedly very inefficient.)

1.4.29 *Steque with two stacks.* Implement a steque with two stacks so that each steque operation (see EXERCISE 1.3.32) takes a constant amortized number of stack operations.

1.4.30 *Deque with a stack and a steque.* Implement a deque with a stack and a steque (see EXERCISE 1.3.32) so that each deque operation takes a constant amortized number of stack and steque operations.

1.4.31 *Deque with three stacks.* Implement a deque with three stacks so that each deque operation takes a constant amortized number of stack operations.

1.4.32 *Amortized analysis.* Prove that, starting from an empty stack, the number of array accesses used by any sequence of M operations in the resizing array implementation of `Stack` is proportional to M.

1.4.33 *Memory requirements on a 32-bit machine.* Give the memory requirements for `Integer`, `Date`, `Counter`, `int[]`, `double[]`, `double[][]`, `String`, `Node`, and `Stack` (linked-list representation) for a 32-bit machine. Assume that references are 4 bytes, object overhead is 8 bytes, and padding is to a multiple of 4 bytes.

1.4.34 *Hot or cold.* Your goal is to guess a secret integer between 1 and N. You repeatedly guess integers between 1 and N. After each guess you learn if your guess equals the secret integer (and the game stops). Otherwise, you learn if the guess is hotter (closer to) or colder (farther from) the secret number than your previous guess. Design an algorithm that finds the secret number in at most $\sim 2 \lg N$ guesses. Then design an algorithm that finds the secret number in at most $\sim 1 \lg N$ guesses.

1.4.35 *Time costs for pushdown stacks.* Justify the entries in the table below, which shows typical time costs for various pushdown stack implementations, using a cost model that counts both *data references* (references to data pushed onto the stack, either an array reference or a reference to an object's instance variable) and *objects created*.

data structure	item type	cost to push N `int` values	
		data references	objects created
linked list	`int`	$2N$	N
	`Integer`	$3N$	$2N$
resizing array	`int`	$\sim 5N$	$\lg N$
	`Integer`	$\sim 5N$	$\sim N$

Time costs for pushdown stacks (various implementations)

1.4.36 *Space usage for pushdown stacks.* Justify the entries in the table below, which shows typical space usage for various pushdown stack implementations. Use a static nested class for linked-list nodes to avoid the non-static nested class overhead.

data structure	item type	space usage for N `int` values (bytes)
linked list	`int`	$\sim 32N$
	`Integer`	$\sim 64N$
resizing array	`int`	between $\sim 4N$ and $\sim 16N$
	`Integer`	between $\sim 32N$ and $\sim 56N$

Space usage in pushdown stacks (various implementations)

EXPERIMENTS

1.4.37 *Autoboxing performance penalty.* Run experiments to determine the performance penalty on your machine for using autoboxing and auto-unboxing. Develop an implementation `FixedCapacityStackOfInts` and use a client such as `DoublingRatio` to compare its performance with the generic `FixedCapacityStack<Integer>`, for a large number of `push()` and `pop()` operations.

1.4.38 *Naive 3-sum implementation.* Run experiments to evaluate the following implementation of the inner loop of `ThreeSum`:

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      for (int k = 0; k < N; k++)
         if (i < j && j < k)
            if (a[i] + a[j] + a[k] == 0)
               cnt++;
```

Do so by developing a version of `DoublingTest` that computes the ratio of the running times of this program and `ThreeSum`.

1.4.39 *Improved accuracy for doubling test.* Modify `DoublingRatio` to take a second command-line argument that specifies the number of calls to make to `timeTrial()` for each value of `N`. Run your program for 10, 100, and 1,000 trials and comment on the precision of the results.

1.4.40 *3-sum for random values.* Formulate and validate a hypothesis describing the number of triples of N random `int` values that sum to 0. If you are skilled in mathematical analysis, develop an appropriate mathematical model for this problem, where the values are uniformly distributed between $-M$ and M, where M is not small.

1.4.41 *Running times.* Estimate the amount of time it would take to run `TwoSumFast`, `TwoSum`, `ThreeSumFast` and `ThreeSum` on your computer to solve the problems for a file of 1 million numbers. Use `DoublingRatio` to do so.

1.4.42 *Problem sizes.* Estimate the size of the largest value of P for which you can run `TwoSumFast`, `TwoSum`, `ThreeSumFast`, and `ThreeSum` on your computer to solve the problems for a file of 2^P thousand numbers. Use `DoublingRatio` to do so.

1.4.43 *Resizing arrays versus linked lists.* Run experiments to validate the hypothesis that resizing arrays are faster than linked lists for stacks (see EXERCISE 1.4.35 and EXERCISE 1.4.36). Do so by developing a version of `DoublingRatio` that computes the ratio

of the running times of the two programs.

1.4.44 *Birthday problem.* Write a program that takes an integer N from the command line and uses `StdRandom.uniform()` to generate a random sequence of integers between 0 and $N - 1$. Run experiments to validate the hypothesis that the number of integers generated before the first repeated value is found is $\sim\sqrt{\pi N/2}$.

1.4.45 *Coupon collector problem.* Generating random integers as in the previous exercise, run experiments to validate the hypothesis that the number of integers generated before all possible values are generated is $\sim N H_N$.

1.5 CASE STUDY: UNION-FIND

TO ILLUSTRATE our basic approach to developing and analyzing algorithms, we now consider a detailed example. Our purpose is to emphasize the following themes.

- Good algorithms can make the difference between being able to solve a practical problem and not being able to address it at all.
- An efficient algorithm can be as simple to code as an inefficient one.
- Understanding the performance characteristics of an implementation can be an interesting and satisfying intellectual challenge.
- The scientific method is an important tool in helping us choose among different methods for solving the same problem.
- An iterative refinement process can lead to increasingly efficient algorithms.

These themes are reinforced throughout the book. This prototypical example sets the stage for our use of the same general methodology for many other problems.

The problem that we consider is not a toy problem; it is a fundamental computational task, and the solution that we develop is of use in a variety of applications, from percolation in physical chemistry to connectivity in communications networks. We start with a simple solution, then seek to understand that solution's performance characteristics, which help us to see how to improve the algorithm.

Dynamic connectivity We start with the following problem specification: The input is a sequence of pairs of integers, where each integer represents an object of some type and we are to interpret the pair `p q` as meaning "`p` is connected to `q`." We assume that "is connected to" is an *equivalence* relation, which means that it is

- *Reflexive*: `p` is connected to `p`.
- *Symmetric*: If `p` is connected to `q`, then `q` is connected to `p`.
- *Transitive*: If `p` is connected to `q` and `q` is connected to `r`, then `p` is connected to `r`.

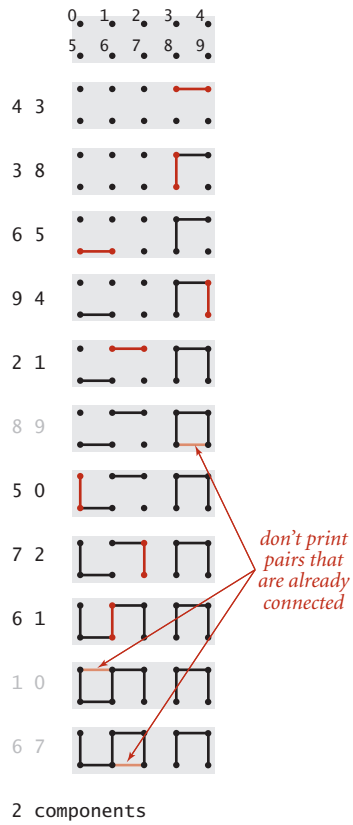
An equivalence relation partitions the objects into *equivalence classes*. In this case, two objects are in the same equivalence class if and only if they are connected. Our goal is to write a program to filter out extraneous pairs (pairs where both objects are in the same equivalence class) from the sequence. In other words, when the program reads a pair `p q` from the input, it should write the pair to the output only if the pairs it has seen to that point *do not* imply that `p` is connected to `q`. If the previous pairs *do* imply that `p` is connected to `q`, then the program should ignore the pair `p q` and proceed to read in the next pair. The figure on the facing page gives an example of this process. To achieve the desired goal, we need to devise a data structure that can remember sufficient

information about the pairs it has seen to be able to decide whether or not a new pair of objects is connected. Informally, we refer to the task of designing such a method as the *dynamic connectivity* problem. This problem arises applications such as the following:

Networks. The integers might represent computers in a large network, and the pairs might represent connections in the network. Then, our program determines whether we need to establish a new direct connection for `p` and `q` to be able to communicate or whether we can use existing connections to set up a communications path. Or, the integers might represent contact sites in an electrical circuit, and the pairs might represent wires connecting the sites. Or, the integers might represent people in a social network, and the pairs might represent friendships. In such applications, we might need to process millions of objects and billions of connections.

Variable-name equivalence. In certain programming environments, it is possible to declare two variable names as being equivalent (references to the same object). After a sequence of such declarations, the system needs to be able to determine whether two given names are equivalent. This application is an early one (for the FORTRAN programming language) that motivated the development of the algorithms that we are about to consider.

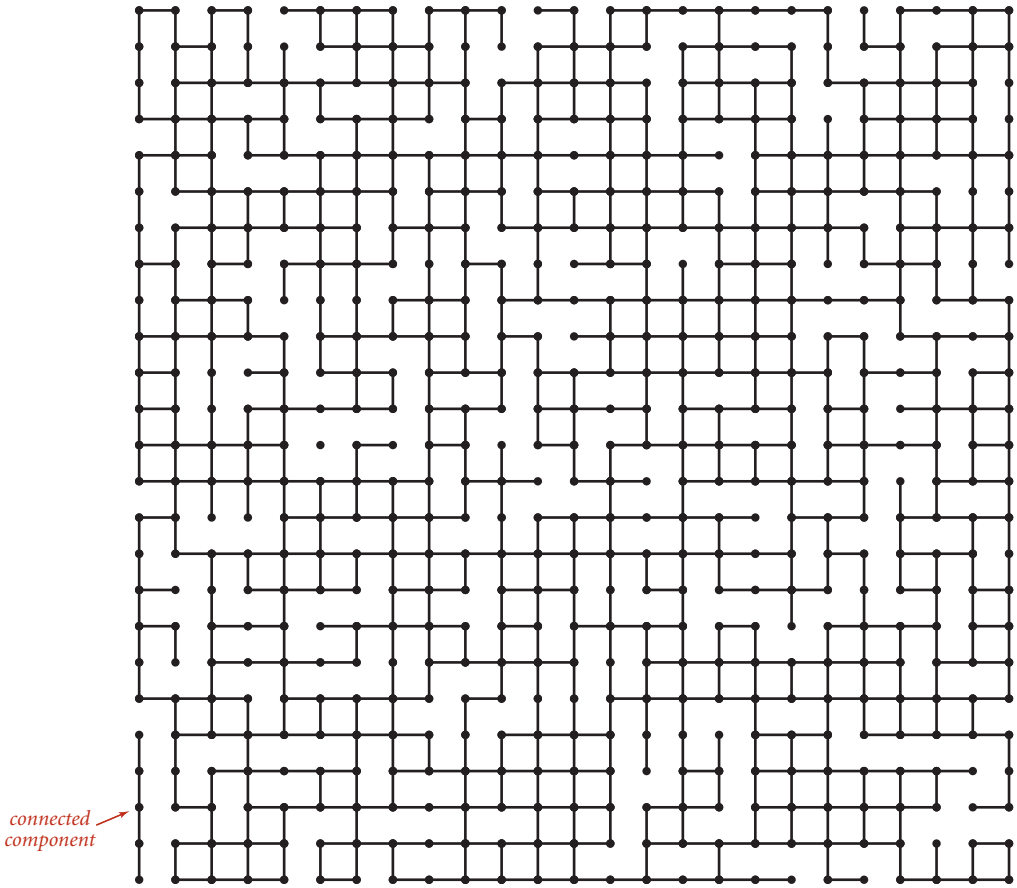
Mathematical sets. On a more abstract level, you can think of the integers as belonging to mathematical sets. When we process a pair `p q`, we are asking whether they belong to the same set. If not, we unite `p`'s set and `q`'s set, putting them in the same set.



Dynamic connectivity example

TO FIX IDEAS, we will use networking terminology for the rest of this section and refer to the objects as *sites*, the pairs as *connections*, and the equivalence classes as *connected components*, or just *components* for short. For simplicity, we assume that we have `N` sites with integer names, from `0` to `N-1`. We do so without loss of generality because we shall be considering a host of algorithms in CHAPTER 3 that can associate arbitrary names with such integer identifiers in an efficient manner.

A larger example that gives some indication of the difficulty of the connectivity problem is depicted in the figure at the top of the next page. You can quickly identify the component consisting of a single site in the left middle of the diagram and the



Medium connectivity example (625 sites, 900 edges, 3 connected components)

component consisting of five sites at the bottom left, but you might have difficulty verifying that all of the other sites are connected to one another. For a program, the task is even more difficult, because it has to work just with site names and connections and has no access to the geometric placement of sites in the diagram. How can we tell quickly whether or not any given two sites in such a network are connected?

The first task that we face in developing an algorithm is to specify the problem in a precise manner. The more we require of an algorithm, the more time and space we may expect it to need to finish the job. It is impossible to quantify this relationship *a priori*, and we often modify a problem specification on finding that it is difficult or expensive to solve or, in happy circumstances, on finding that an algorithm can provide information more useful than what was called for in the original specification. For example, our

connectivity problem specification requires only that our program be able to determine whether or not any given pair `p q` is connected, and not that it be able to demonstrate a set of connections that connect that pair. Such a requirement makes the problem more difficult and leads us to a different family of algorithms, which we consider in SECTION 4.1.

To specify the problem, we develop an API that encapsulates the basic operations that we need: initialize, add a connection between two sites, identify the component containing a site, determine whether two sites are in the same component, and count the number of components. Thus, we articulate the following API:

`public class UF`

	`UF(int N)`	*initialize N sites with integer names (*`0` *to* `N-1`*)*
`void`	`union(int p, int q)`	*add connection between* `p` *and* `q`
`int`	`find(int p)`	*component identifier for* `p` *(*`0` *to* `N-1`*)*
`boolean`	`connected(int p, int q)`	*return* `true` *if* `p` *and* `q` *are in the same component*
`int`	`count()`	*number of components*

Union-find API

The `union()` operation merges two components if the two sites are in different components, the `find()` operation returns an integer component identifier for a given site, the `connected()` operation determines whether two sites are in the same component, and the `count()` method returns the number of components. We start with `N` components, and each `union()` that merges two different components decrements the number of components by 1.

As we shall soon see, the development of an algorithmic solution for dynamic connectivity thus reduces to the task of developing an implementation of this API. Every implementation has to

- Define a data structure to represent the known connections
- Develop efficient `union()`, `find()`, `connected()`, and `count()` implementations that are based on that data structure

As usual, the nature of the data structure has a direct impact on the efficiency of the algorithms, so data structure and algorithm design go hand in hand. The API already specifies the convention that both sites and components will be identified by `int` values between `0` and `N-1`, so it makes sense to use a *site-indexed array* `id[]` as our basic

data structure to represent the components. We always use the name of one of the sites in a component as the component identifier, so you can think of each component as being represented by one of its sites. Initially, we start with `N` components, each site in its own component, so we initialize `id[i]` to `i` for all `i` from `0` to `N-1`. For each site `i`, we keep the information needed by `find()` to determine the component containing `i` in `id[i]`, using various algorithm-dependent strategies. All of our implementations use a one-line implementation of `connected()` that returns the `boolean` value `find(p) == find(q)`.

```
% more tinyUF.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7

% more mediumUF.txt
625
528 503
548 523
...
[900 connections]

% more largeUF.txt
1000000
786321 134521
696834 98245
...
[2000000 connections]
```

IN SUMMARY, our starting point is ALGORITHM 1.5 on the facing page. We maintain two instance variables, the count of components and the array `id[]`. Implementations of `find()` and `union()` are the topic of the remainder of this section.

To test the utility of the API and to provide a basis for development, we include a client in `main()` that uses it to solve the dynamic connectivity problem. It reads the value of `N` followed by a sequence of pairs of integers (each in the range `0` to `N-1`), calling `find()` for each pair: If the two sites in the pair are already connected, it moves on to the next pair; if they are not, it calls `union()` and prints the pair. Before considering implementations, we also prepare test data: the file `tinyUF.txt` contains the 11 connections among 10 sites used in the small example illustrated on page 217, the file `mediumUF.txt` contains the 900 connections among 625 sites illustrated on page 218, and the file `largeUF.txt` is an example with 2 million connections among 1 millions sites. Our goal is to be able to handle inputs such as `largeUF.txt` in a reasonable amount of time.

To analyze the algorithms, we focus on the number of times each algorithm accesses an array entry. By doing so, we are implicitly formulating the hypothesis that the running times of the algorithms on a particular machine are within a constant factor of this quantity. This hypothesis is immediate from the code, is not difficult to validate through experimentation, and provides a useful starting point for comparing algorithms, as we will see.

Union-find cost model. When studying algorithms to implement the union-find API, we count *array accesses* (the number of times an array entry is accessed, for read or write).

ALGORITHM 1.5 Union-find implementation

```
public class UF
{
   private int[] id;     // access to component id (site indexed)
   private int count;    // number of components

   public UF(int N)
   {  // Initialize component id array.
      count = N;
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }

   public int count()
   {  return count;  }

   public boolean connected(int p, int q)
   {  return find(p) == find(q);  }

   public int  find(int p)
   public void union(int p, int q)
  // See page 222 (quick-find),page 224 (quick-union) andpage 228 (weighted).

   public static void main(String[] args)
   {  // Solve dynamic connectivity problem on StdIn.
      int N = StdIn.readInt();               // Read number of sites.
      UF uf = new UF(N);                     // Initialize N components.
      while (!StdIn.isEmpty())
      {
         int p = StdIn.readInt();
         int q = StdIn.readInt();            // Read pair to connect.
         if (uf.connected(p, q)) continue;   // Ignore if connected.
         uf.union(p, q);                     // Combine components
         StdOut.println(p + " " + q);        //   and print connection.
      }
      StdOut.println(uf.count() + " components");
   }

}
```

```
% java UF < tinyUF.txt
4 3
3 8
6 5
9 4
2 1
5 0
7 2
6 1
2 components
```

Our `UF` implementations are based on this code, which maintains an array of integers `id[]` such that the `find()` method returns the same integer for every site in each connected component. The `union()` method must maintain this invariant.

Implementations We shall consider three different implementations, all based on using the site-indexed `id[]` array, to determine whether two sites are in the same connected component.

Quick-find. One approach is to maintain the invariant that `p` and `q` are connected if and only if `id[p]` is equal to `id[q]`. In other words, all sites in a component must have the same value in `id[]`. This method is called *quick-find* because `find(p)` just returns `id[p]`, which immediately implies that `connected(p, q)` reduces to just the test `id[p] == id[q]` and returns `true` if and only if `p` and `q` are in the same component. To maintain the invariant for the call `union(p, q)`, we first check whether they are already in the same component, in which case there is nothing to do. Otherwise, we are faced with the situation that all of the `id[]` entries corresponding to sites in the same component as `p` have one value and all of the `id[]` entries corresponding to sites in the same component as `q` have another value. To combine the two components into one, we have to make all of the `id[]` entries corresponding to both sets of sites the same value, as shown in the example at right. To do so, we go through the array, changing all the entries with values equal to `id[p]` to the value `id[q]`. We could have decided to change all the entries equal to `id[q]` to the value `id[p]`—the choice between these two alternatives is arbitrary. The code for `find()` and `union()` based on these descriptions, given at left, is straightforward. A full trace for our development client with our sample test data `tinyUF.txt` is shown on the next page.

find examines `id[5]` *and* `id[9]`

p	q	0	1	2	3	4	5	6	7	8	9
5	9	1	1	1	8	8	1	1	1	8	8

union has to change all 1*s to* 8*s*

p	q	0	1	2	3	4	5	6	7	8	9
5	9	1	1	1	8	8	1	1	1	8	8
		8	8	8	8	8	8	8	8	8	8

Quick-find overview

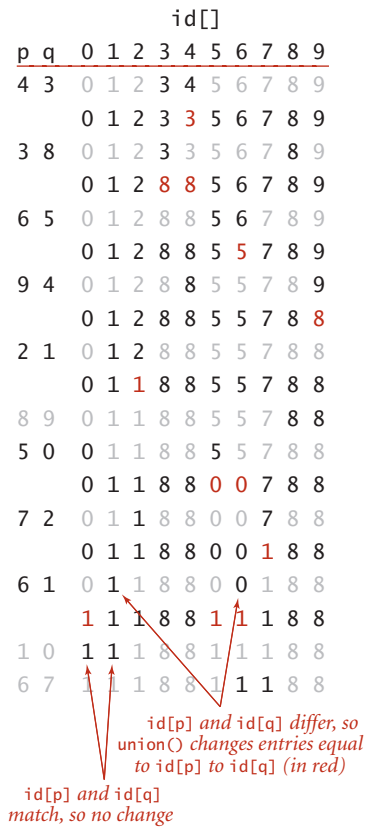
```
public int find(int p)
{  return id[p];  }

public void union(int p, int q)
{  // Put p and q into the same component.
   int pID = find(p);
   int qID = find(q);

   // Nothing to do if p and q are already
        in the same component.
   if (pID == qID) return;

   // Rename p's component to q's name.
   for (int i = 0; i < id.length; i++)
       if (id[i] == pID) id[i] = qID;
   count--;
}
```

Quick-find

Quick-find analysis. The `find()` operation is certainly quick, as it only accesses the `id[]` array once in order to complete the operation. But quick-find is typically not useful for large problems because `union()` needs to scan through the whole `id[]` array for each input pair.

Proposition F. The quick-find algorithm uses one array access for each call to `find()` and between $N + 3$ and $2N + 1$ array accesses for each call to `union()` that combines two components.

Proof: Immediate from the code. Each call to `connected()` tests two entries in the `id[]` array, one for each of the two calls to `find()`. Each call to `union()` that combines two components does so by making two calls to `find()`, testing each of the N entries in the `id[]` array, and changing between 1 and $N - 1$ of them.



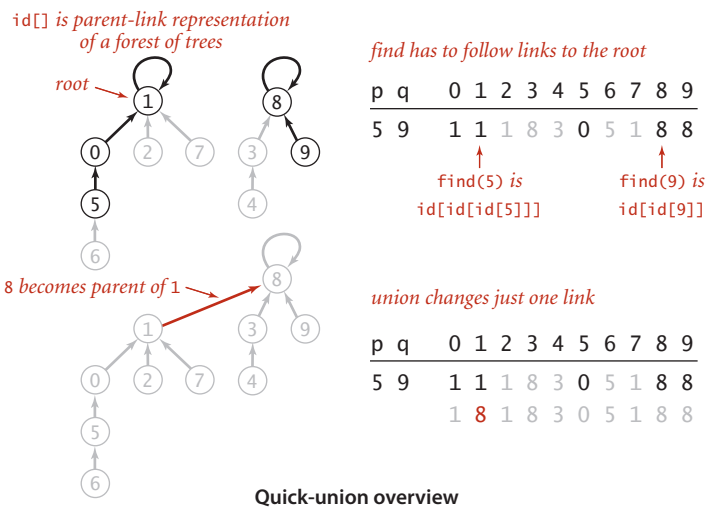
Quick-find trace

In particular, suppose that we use quick-find for the dynamic connectivity problem and wind up with a single component. This requires at least $N-1$ calls to `union()`, and, consequently, at least $(N+3)(N-1) \sim N^2$ array accesses—we are led immediately to the hypothesis that dynamic connectivity with quick-find can be a *quadratic*-time process. This analysis generalizes to say that quick-find is quadratic for typical applications where we end up with a small number of components. You can easily validate this hypothesis on your computer with a doubling test (see EXERCISE 1.5.23 for an instructive example). Modern computers can execute hundreds of millions or billions of instructions per second, so this cost is not noticeable if N is small, but we also might find ourselves with millions or billions of sites and connections to process in a modern application, as represented by our test file `largeUF.txt`. If you are still not convinced and feel that you have a particularly fast computer, try using quick-find to determine the number of components implied by the pairs in `largeUF.txt`. The inescapable conclusion is that we cannot feasibly solve such a problem using the quick-find algorithm, so we seek better algorithms.

Quick-union. The next algorithm that we consider is a complementary method that concentrates on speeding up the `union()` operation. It is based on the same data structure—the site-indexed `id[]` array—but we interpret the values differently, to define more complicated structures. Specifically, the `id[]` entry for each site is the name of another site in the same component (possibly itself)—we refer to this connection as a *link*. To implement `find()`, we start at the given site, follow its link to another site, follow that site's link to yet another site, and so forth, following links until reaching a *root*, a site that has a link to itself (which is guaranteed to happen, as you will see). Two sites are in the same component if and only if this process leads them to the same root. To validate this process, we need `union(p, q)` to maintain this invariant, which is easily arranged: we follow links to find the roots associated with `p` and `q`, then rename one of the components by linking one of these roots to the other; hence the name *quick-union*. Again, we have an arbitrary choice of whether to rename the component containing `p` or the component containing `q`; the implementation above renames the one containing `p`. The figure on the next page shows a trace of the quick-union algorithm for `tinyUF.txt`. This trace is best understood in terms of the graphical representation depicted at left, which we consider next.

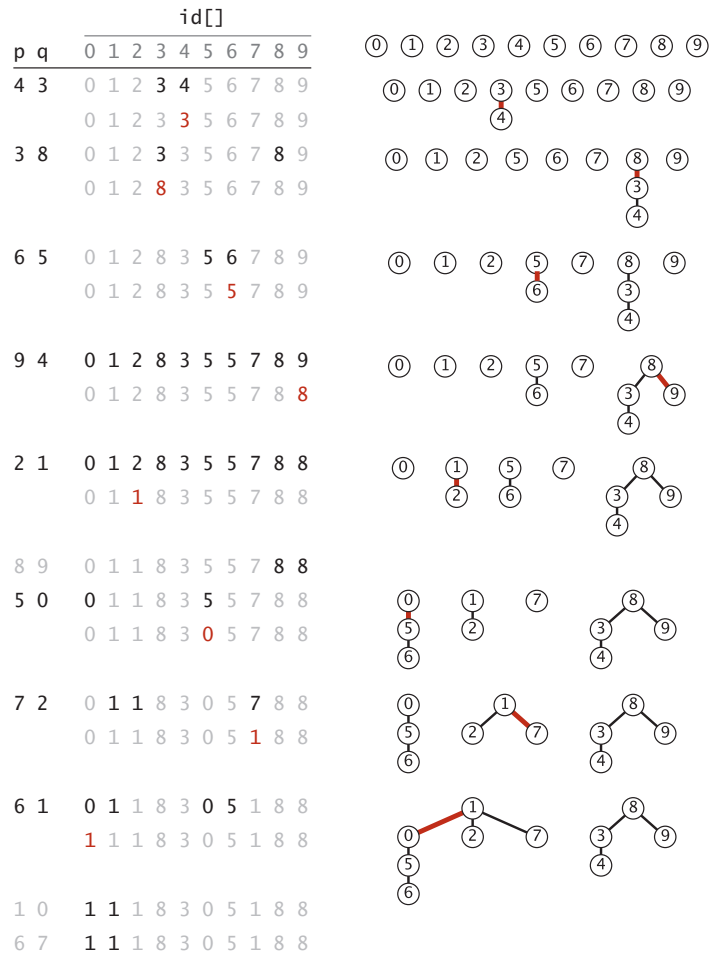
```
private int find(int p)
{  // Find component name.
   while (p != id[p]) p = id[p];
   return p;
}

public void union(int p, int q)
{  // Give p and q the same root.
   int pRoot = find(p);
   int qRoot = find(q);
   if (pRoot == qRoot) return;

   id[pRoot] = qRoot;

   count--;
}
```

Quick-union



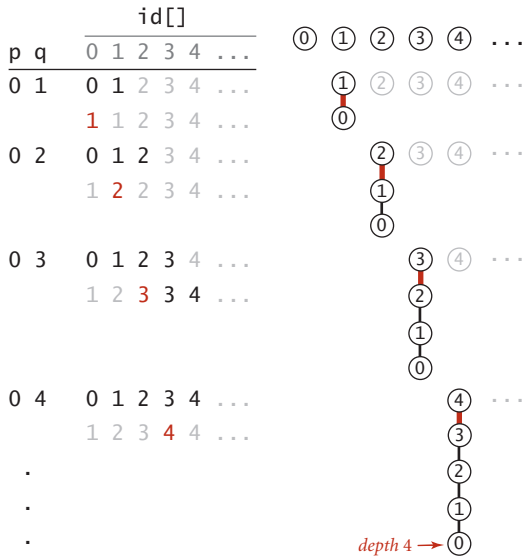
Quick-union overview

Forest-of-trees representation. The code for quick-union is compact, but a bit opaque. Representing sites as *nodes* (labeled circles) and links as arrows from one node to another gives a graphical representation of the data structure that makes it relatively easy to understand the operation of the algorithm. The resulting structures are *trees*—in technical terms, our `id[]` array is a parent-link representation of a forest (set) of trees. To simplify the diagrams, we often omit both the arrowheads in the links (because they all point upwards) and the self-links in the roots of the trees. The forests corresponding to the `id[]` array for `tinyUF.txt` are shown at right. When we start at the node corresponding to any site and follow links, we eventually end up at the root of the tree containing that node. We can prove this property to be true by induction: It is true after the array is initialized to have every node link to itself, and if it is true before a given `union()` operation, it is certainly true afterward. Thus, the `find()` method on page 224 returns the name of the site at the root (so that `connected()` checks whether two sites are in the same tree). This representation is useful for this problem because the nodes corresponding to two sites are in the same tree if and only if the sites are in the same component. Moreover, the trees are not difficult to build: the `union()` implementation on page 224 combines two trees into one in a single statement, by making the root of one the parent of the other.



Quick-union trace (with corresponding forests of trees)

Quick-union analysis. The quick-union algorithm would seem to be faster than the quick-find algorithm, because it does not have to go through the entire array for each input pair; but how much faster is it? Analyzing the cost of quick-union is more difficult than it was for quick-find, because the cost is more dependent on the nature of the input. In the best case, `find()` just needs one array access to find the identifier associated with a site, as in quick-find; in the worst case, it needs $2N + 1$ array accesses, as for 0 in the example at left (this count is conservative since compiled code will typically *not* do an array access for the second reference to `id[p]` in the `while` loop). Accordingly, it is not difficult to construct a best-case input for which the running time of our dynamic connectivity client is linear; on the other hand it is also not difficult to construct a worst-case input for which the running time is quadratic (see the diagram at left and PROPOSITION G below). Fortunately, we do not need to face the problem of analyzing quick union and we will not dwell on comparative performance of quick-find and quick-union because we will next examine another variant that is far more efficient than either. For the moment, you can regard quick-union as an improvement over quick-find because it removes quick-find's main liability (that `union()` always takes linear time). This difference certainly represents an improvement for typical data, but quick-union still has the liability that we cannot *guarantee* it to be substantially faster than quick-find in every case (for certain input data, quick-union is no faster than quick-find).



Quick-union worst case

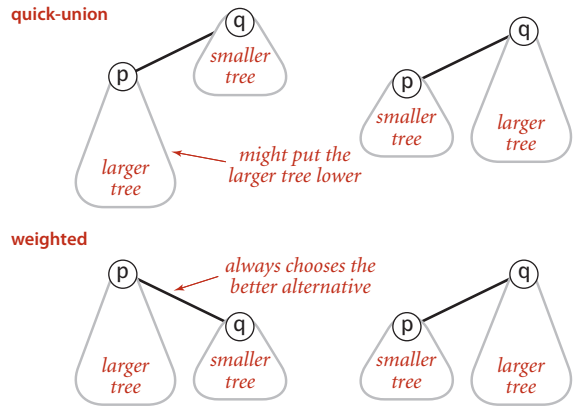
Definition. The *size* of a tree is its number of nodes. The *depth* of a node in a tree is the number of links on the path from it to the root. The *height* of a tree is the maximum depth among its nodes.

Proposition G. The number of array accesses used by `find()` in quick-union is 1 plus the twice the depth of the node corresponding to the given site. The number of array accesses used by `union()` and `connected()` is the cost of the two `find()` operations (plus 1 for `union()` if the given sites are in different trees).

Proof: Immediate from the code.

Again, suppose that we use quick-union for the dynamic connectivity problem and wind up with a single component. An immediate implication of PROPOSITION G is that the running time is quadratic, in the worst case. Suppose that the input pairs come in the order `0-1`, then `0-2`, then `0-3`, and so forth. After $N-1$ such pairs, we have N sites all in the same set, and the tree that is formed by the quick-union algorithm has height $N-1$, with `0` linking to `1`, which links to `2`, which links to `3`, and so forth (see the diagram on the facing page). By PROPOSITION G, the number of array accesses for the `union()` operation for the pair `0 i` is exactly $2i+2$ (site `0` is at depth i and site `i` at depth 0). Thus, the total number of array accesses for the `find()` operations for these N pairs is $2\,(1+2+\ldots+N) \sim N^2$.

Weighted quick-union. Fortunately, there is an easy modification to quick-union that allows us to guarantee that bad cases such as this one do not occur. Rather than arbitrarily connecting the second tree to the first for `union()`, we keep track of the *size* of each tree and always connect the smaller tree to the larger. This change requires slightly more code and another array to hold the node counts, as shown on page 228, but it leads to substantial improvements in efficiency. We refer to this algorithm as the *weighted quick-union* algorithm. The forest of trees constructed by this algorithm for `tinyUF.txt` is shown in the figure at left on the top of page 229. Even for this small example, the tree height is substantially smaller than the height for the unweighted version.



Weighted quick-union

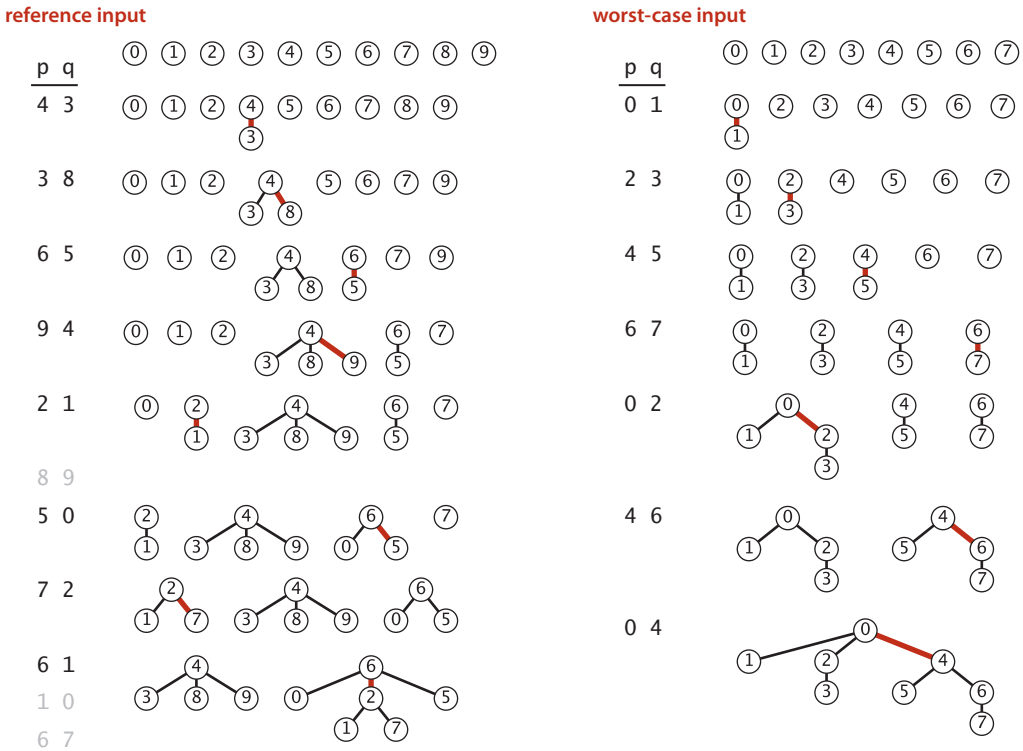
Weighted quick-union analysis. The figure at right on the top of page 229 illustrates the worst case for weighted quick union, when the sizes of the trees to be merged by `union()` are always equal (and a power of 2). These tree structures look complex, but they have the simple property that the height of a tree of 2^n nodes is n. Furthermore, when we merge two trees of 2^n nodes, we get a tree of 2^{n+1} nodes, and we increase the height of the tree to $n+1$. This observation generalizes to provide a proof that the weighted algorithm can guarantee *logarithmic* performance.

```
% java WeightedQuickUnionUF < mediumUF.txt
528 503
548 523
...
3 components

% java WeightedQuickUnionUF < largeUF.txt
786321 134521
696834 98245
...
6 components
```

ALGORITHM 1.5 (continued) Union-find implementation (weighted quick-union)

```
public class WeightedQuickUnionUF
{
   private int[] id;     // parent link (site indexed)
   private int[] sz;     // size of component for roots (site indexed)
   private int count;    // number of components

   public WeightedQuickUnionUF(int N)
   {
      count = N;
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
      sz = new int[N];
      for (int i = 0; i < N; i++) sz[i] = 1;

   }

   public int count()
   {  return count;  }

   public boolean connected(int p, int q)
   {  return find(p) == find(q);  }

   private int find(int p)
   {  // Follow links to find a root.
      while (p != id[p]) p = id[p];
      return p;
   }

   public void union(int p, int q)
   {
      int i = find(p);
      int j = find(q);
      if (i == j) return;

      // Make smaller root point to larger one.
      if   (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
      else                 { id[j] = i; sz[i] += sz[j]; }
      count--;
   }
}
```

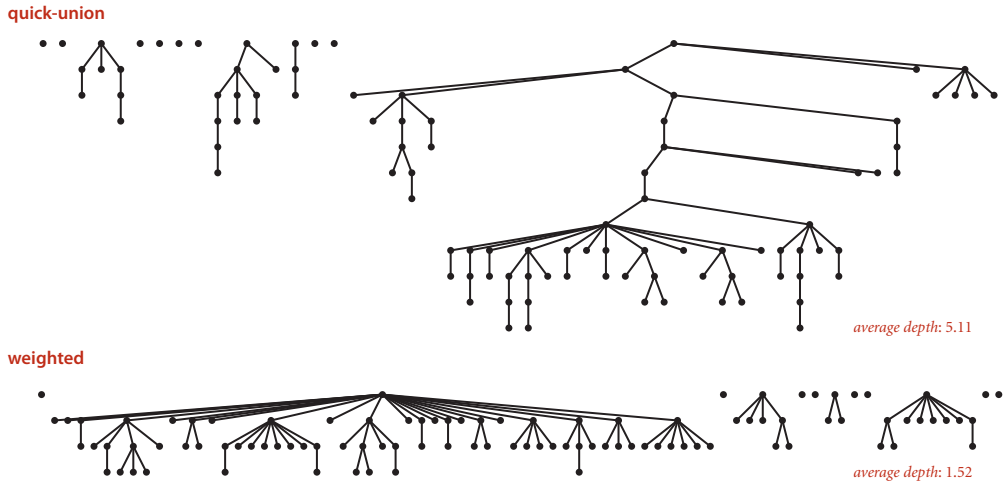
This code is best understood in terms of the forest-of-trees representation described in the text. We add a site-indexed array `sz[]` as an instance variable so that `union()` can link the root of the smaller tree to the root of the larger tree. This addition makes it feasible to address large problems.



Weighted quick-union traces (forests of trees)

Proposition H. The depth of any node in a forest built by weighted quick-union for N sites is at most $\lg N$.

Proof: We prove a stronger fact by (strong) induction: The height of every tree of size k in the forest is at most $\lg k$. The base case follows from the fact that the tree height is 0 when k is 1. By the inductive hypothesis, assume that the tree height of a tree of size i is at most $\lg i$ for all $i < k$. When we combine a tree of size i with a tree of size j with $i \leq j$ and $i + j = k$, we increase the depth of each node in the smaller set by 1, but they are now in a tree of size $i + j = k$, so the property is preserved because $1 + \lg i = \lg(i + i) \leq \lg(i + j) = \lg k$.



Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Corollary. For weighted quick-union with N sites, the worst-case order of growth of the cost of `find()`, `connected()`, and `union()` is log N.

Proof. Each operation does at most a constant number of array accesses for each node on the path from a node to a root in the forest.

For dynamic connectivity, the practical implication of PROPOSITION H and its corollary is that weighted quick-union is the only one of the three algorithms that can feasibly be used for huge practical problems. The weighted quick-union algorithm uses *at most* $c\ M \lg N$ array accesses to process M connections among N sites for a small constant c. This result is in stark contrast to our finding that quick-find always (and quick-union sometimes) uses *at least* MN array accesses. Thus, with weighted quick-union, we can guarantee that we can solve huge practical dynamic connectivity problems in a reasonable amount of time. For the price of a few extra lines of code, we get a program that can be millions of times faster than the simpler algorithms for the huge dynamic connectivity problems that we might encounter in practical applications.

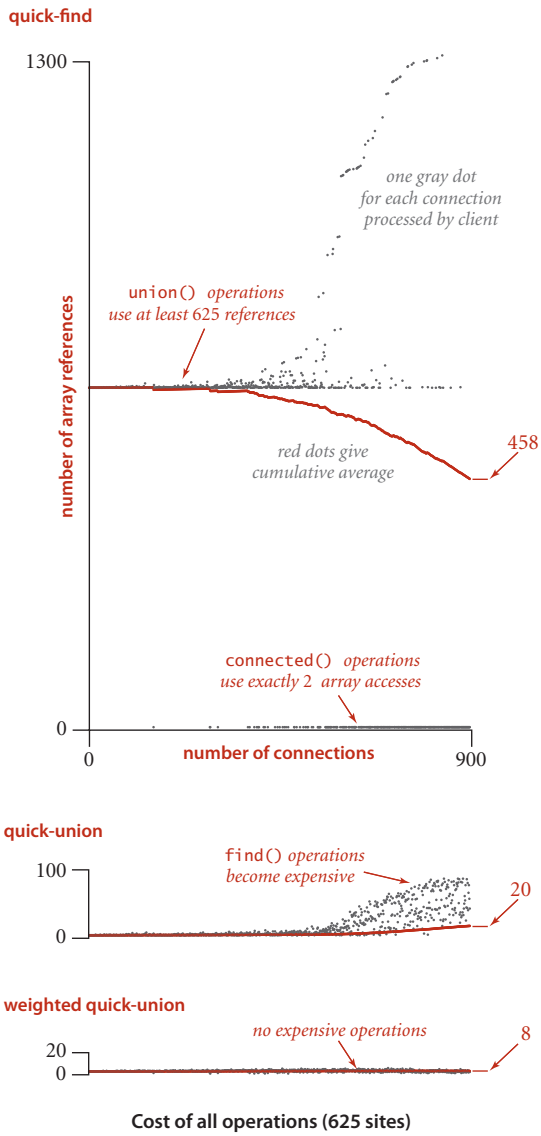
A 100-site example is shown on the top of this page. It is evident from this diagram that relatively few nodes fall far from the root with weighted quick-union. Indeed it is frequently the case that a 1-node tree is merged with a larger tree, which puts the node just one link from the root. Empirical studies on huge problems tell us that weighted quick-union typically solves practical problems in *constant* time per operation. We could hardly expect to find a more efficient algorithm.

algorithm	*order of growth for N sites (worst case)*		
	constructor	union	find
quick-find	N	N	1
quick-union	N	*tree height*	*tree height*
weighted quick-union	N	$\lg N$	$\lg N$
weighted quick-union with path compresson	N	*very, very nearly, but not quite* 1 (*amortized*) (see EXERCISE 1.5.13)	
impossible	N	1	1

Performance characteristics of union-find algorithms

Optimal algorithms. Can we find an algorithm that has *guaranteed* constant-time-per-operation performance? This question is an extremely difficult one that plagued researchers for many years. In pursuit of an answer, a number of variations of quick-union and weighted quick-union have been studied. For example, the following method, known as *path compression*, is easy to implement. Ideally, we would like every node to link directly to the root of its tree, but we do not want to pay the price of changing a large number of links, as we did in the quick-find algorithm. We can approach the ideal simply by making all the nodes that we *do* examine directly link to the root. This step seems drastic at first blush, but it is easy to implement, and there is nothing sacrosanct about the structure of these trees: if we can modify them to make the algorithm more efficient, we should do so. To implement path compression, we just add another loop to `find()` that sets the `id[]` entry corresponding to each node encountered along the way to link directly to the root. The net result is to flatten the trees almost completely, approximating the ideal achieved by the quick-find algorithm. The method is simple and effective, but you are not likely to be able to discern any improvement over weighted quick-union in a practical situation (see EXERCISE 1.5.24). Theoretical results about the situation are extremely complicated and quite remarkable. *Weighted quick union with path compression is optimal but not quite constant-time per operation.* That is, not only is weighted quick-find with path compression not constant-time per operation in the worst case (amortized), but also there exists *no* algorithm that can guarantee to perform each union-find operation in amortized constant time (under the very general "cell probe" model of computation). Weighted quick-union with path compression is very close to the best that we can do for this problem.

Amortized cost plots. As with any data type implementation, it is worthwhile to run experiments to test the validity of our performance hypotheses for typical clients, as discussion in SECTION 1.4. The figure at left shows details of the performance of the algorithms for our dynamic connectivity development client when solving our 625-site connectivity example (`mediumUF.txt`). Such diagrams are easy to produce (see EXERCISE 1.5.16): For the *i*th connection processed, we maintain a variable `cost` that counts the number of array accesses (to `id[]` or `sz[]`) and a variable `total` that is the sum of the total number of array accesses so far. Then we plot a gray dot at `(i, cost)` and a red dot at `(i, total/i)`. The red dots are the average cost per operation, or amortized cost. These plots provide good insights into algorithm behavior. For *quick-find*, every `union()` operation uses at least 625 accesses (plus 1 for each component merged, up to another 625) and every `connected()` operation uses 2 accesses. Initially, most of the connections lead to a call on `union()`, so the cumulative average hovers around 625; later, most connections are calls to `connected()` that cause the call to `union()` to be skipped, so the cumulative average decreases, but still remains relatively high. (Inputs that lead to a large number of `connected()` calls that cause `union()` to be skipped will exhibit significantly better performance—see EXERCISE 1.5.23 for an example). For *quick-union*, all operations initially require only a few array accesses; eventually, the height of the trees becomes a significant factor and the amortized cost grows noticably. For *weighted quick-union*, the tree height stays small, none of the operations are expensive, and the amortized cost is low. These experiments validate our conclusion that weighted quick-union is certainly worth implementing and that there is not much further room for improvement for practical problems.



Cost of all operations (625 sites)

Perspective Each of the `UF` implementations that we considered is an improvement over the previous in some intuitive sense, but the process is artificially smooth because we have the benefit of hindsight in looking over the development of the algorithms as they were studied by researchers over the years. The implementations are simple and the problem is well specified, so we can evaluate the various algorithms directly by running empirical studies. Furthermore, we can use these studies to validate mathematical results that quantify the performance of these algorithms. When possible, we follow the same basic steps for fundamental problems throughout the book that we have taken for union–find algorithms in this section, some of which are highlighted in this list:

- Decide on a complete and specific problem statement, including identifying fundamental abstract operations that are intrinsic to the problem and an API.
- Carefully develop a succinct implementation for a straightforward algorithm, using a well-thought-out development client and realistic input data.
- Know when an implementation could not possibly be used to solve problems on the scale contemplated and must be improved or abandoned.
- Develop improved implementations through a process of stepwise refinement, validating the efficacy of ideas for improvement through empirical analysis, mathematical analysis, or both.
- Find high-level abstract representations of data structures or algorithms in operation that enable effective high-level design of improved versions.
- Strive for worst-case performance guarantees when possible, but accept good performance on typical data when available.
- Know when to leave further improvements for detailed in-depth study to skilled researchers and move on to the next problem.

The potential for spectacular performance improvements for practical problems such as those that we saw for union–find makes algorithm design a compelling field of study. What other design activities hold the potential to reap savings factors of millions or billions, or more?

Developing an efficient algorithm is an intellectually satisfying activity that can have direct practical payoff. As the dynamic connectivity problem indicates, a simply stated problem can lead us to study numerous algorithms that are not only both useful and interesting, but also intricate and challenging to understand. We shall encounter many ingenious algorithms that have been developed over the years for a host of practical problems. As the scope of applicability of computational solutions to scientific and commercial problems widens, so also grows the importance of being able to use efficient algorithms to solve known problems and of being able to develop efficient solutions to new problems.

Q&A

Q. I'd like to add a `delete()` method to the API that allows clients to delete connections. Any advice on how to proceed?

A. No one has devised an algorithm as simple and efficient as the ones in this section that can handle deletions. This theme recurs throughout this book. Several of the data structures that we consider have the property that deleting something is much more difficult than adding something.

Q. What is the cell-probe model?

A. A model of computation where we only count accesses to a random-access memory large enough to hold the input and consider all other operations to be free.

EXERCISES

1.5.1 Show the contents of the `id[]` array and the number of times the array is accessed for each input pair when you use quick-find for the sequence `9-0 3-4 5-8 7-2 2-1 5-7 0-3 4-2`.

1.5.2 Do EXERCISE 1.5.1, but use quick-union (page 224). In addition, draw the forest of trees represented by the `id[]` array after each input pair is processed.

1.5.3 Do EXERCISE 1.5.1, but use weighted quick-union (page 228).

1.5.4 Show the contents of the `sz[]` and `id[]` arrays and the number of array accesses for each input pair corresponding to the weighted quick-union examples in the text (both the reference input and the worst-case input).

1.5.5 Estimate the minimum amount of time (in days) that would be required for quick-find to solve a dynamic connectivity problem with 10^9 sites and 10^6 input pairs, on a computer capable of executing 10^9 instructions per second. Assume that each iteration of the inner `for` loop requires 10 machine instructions.

1.5.6 Repeat EXERCISE 1.5.5 for weighted quick-union.

1.5.7 Develop classes `QuickUnionUF` and `QuickFindUF` that implement quick-union and quick-find, respectively.

1.5.8 Give a counterexample that shows why this intuitive implementation of `union()` for quick-find is not correct:

```
public void union(int p, int q)
{
   if (connected(p, q)) return;

   // Rename p's component to q's name.
   for (int i = 0; i < id.length; i++)
       if (id[i] == id[p]) id[i] = id[q];
   count--;
}
```

1.5.9 Draw the tree corresponding to the `id[]` array depicted at right. Can this be the result of running weighted quick-union? Explain why this is impossible or give a sequence of operations that results in this array.

i	0	1	2	3	4	5	6	7	8	9
id[i]	1	1	3	1	5	6	1	3	4	5

EXERCISES *(continued)*

1.5.10 In the weighted quick-union algorithm, suppose that we set `id[find(p)]` to `q` instead of to `id[find(q)]`. Would the resulting algorithm be correct?

Answer: Yes, but it would increase the tree height, so the performance guarantee would be invalid.

1.5.11 Implement *weighted quick-find*, where you always change the `id[]` entries of the smaller component to the identifier of the larger component. How does this change affect performance?

CREATIVE PROBLEMS

1.5.12 *Quick-union with path compression.* Modify quick-union (page 224) to include *path compression*, by adding a loop to `union()` that links every site on the paths from `p` and `q` to the roots of their trees to the root of the new tree. Give a sequence of input pairs that causes this method to produce a path of length 4. *Note*: The amortized cost per operation for this algorithm is known to be logarithmic.

1.5.13 *Weighted quick-union with path compression.* Modify weighted quick-union (ALGORITHM 1.5) to implement path compression, as described in EXERCISE 1.5.12. Give a sequence of input pairs that causes this method to produce a tree of height 4. *Note*: The amortized cost per operation for this algorithm is known to be bounded by a function known as the *inverse Ackermann function* and is less than 5 for any conceivable practical value of N.

1.5.14 *Weighted quick-union by height.* Develop a `UF` implementation that uses the same basic strategy as weighted quick-union but keeps track of tree height and always links the shorter tree to the taller one. Prove a logarithmic upper bound on the height of the trees for N sites with your algorithm.

1.5.15 *Binomial trees.* Show that the number of nodes at each level in the worst-case trees for weighted quick-union are *binomial coefficients*. Compute the average depth of a node in a worst-case tree with $N = 2^n$ nodes.

1.5.16 *Amortized costs plots.* Instrument your implementations from EXERCISE 1.5.7 to make amortized costs plots like those in the text.

1.5.17 *Random connections.* Develop a `UF` client `ErdosRenyi` that takes an integer value `N` from the command line, generates random pairs of integers between `0` and `N-1`, calling `connected()` to determine if they are connected and then `union()` if not (as in our development client), looping until all sites are connected, and printing the number of connections generated. Package your program as a static method `count()` that takes `N` as argument and returns the number of connections and a `main()` that takes `N` from the command line, calls `count()`, and prints the returned value.

1.5.18 *Random grid generator.* Write a program `RandomGrid` that takes an `int` value `N` from the command line, generates all the connections in an `N`-by-`N` grid, puts them in random order, randomly orients them (so that `p q` and `q p` are equally likely to occur), and prints the result to standard output. To randomly order the connections, use a `RandomBag` (see EXERCISE 1.3.34 on page 167). To encapsulate `p` and `q` in a single object,

CREATIVE PROBLEMS *(continued)*

use the `Connection` nested class shown below. Package your program as two static methods: `generate()`, which takes `N` as argument and returns an array of connections, and `main()`, which takes `N` from the command line, calls `generate()`, and iterates through the returned array to print the connections.

1.5.19 *Animation.* Write a `RandomGrid` client (see EXERCISE 1.5.18) that uses `UnionFind` as in our development client to check connectivity and uses `StdDraw` to draw the connections as they are processed.

1.5.20 *Dynamic growth.* Using linked lists or a resizing array, develop a weighted quick-union implementation that removes the restriction on needing the number of objects ahead of time. Add a method `newSite()` to the API, which returns an `int` identifier.

```
private class Connection
{
   int p;
   int q;

   public Connection(int p, int q)
   {  this.p = p; this.q = q;  }
}
```

Record to encapsulate connections

EXPERIMENTS

1.5.21 *Erdös-Renyi model.* Use your client from EXERCISE 1.5.17 to test the hypothesis that the number of pairs generated to get one component is ~ ½N ln N.

1.5.22 *Doubling test for Erdös-Renyi model.* Develop a performance-testing client that takes an `int` value `T` from the command line and performs `T` trials of the following experiment: Use your client from EXERCISE 1.5.17 to generate random connections, using `UnionFind` to determine connectivity as in our development client, looping until all sites are connected. For each `N`, print the value of `N`, the average number of connections processed, and the ratio of the running time to the previous. Use your program to validate the hypotheses in the text that the running times for quick-find and quick-union are quadratic and weighted quick-union is near-linear.

1.5.23 *Compare quick-find with quick-union for Erdös-Renyi model.* Develop a performance-testing client that takes an `int` value `T` from the command line and performs `T` trials of the following experiment: Use your client from EXERCISE 1.5.17 to generate random connections. Save the connections, so that you can use both quick-union and quick-find to determine connectivity as in our development client, looping until all sites are connected. For each `N`, print the value of `N` and the ratio of the two running times.

1.5.24 *Fast algorithms for Erdös-Renyi model.* Add weighted quick-union and weighted quick-union with path compression to your tests from EXERCISE 1.5.23 . Can you discern a difference between these two algorithms?

1.5.25 *Doubling test for random grids.* Develop a performance-testing client that takes an `int` value `T` from the command line and performs `T` trials of the following experiement: Use your client from EXERCISE 1.5.18 to generate the connections in an `N`-by-`N` square grid, randomly oriented and in random order, then use `UnionFind` to determine connectivity as in our development client, looping until all sites are connected. For each `N`, print the value of `N`, the average number of connections processed, and the ratio of the running time to the previous. Use your program to validate the hypotheses in the text that the running times for quick-find and quick-union are quadratic and weighted quick-union is near-linear. *Note*: As `N` doubles, the number of sites in the grid increases by a factor of 4, so expect a doubling factor of 16 for quadratic and 4 for linear.

EXPERIMENTS *(continued)*

1.5.26 *Amortized plot for Erdös-Renyi.* Develop a client that takes an `int` value `N` from the command line and does an amortized plot of the cost of all operations in the style of the plots in the text for the process of generating random pairs of integers between `0` and `N-1`, calling `connected()` to determine if they are connected and then `union()` if not (as in our development client), looping until all sites are connected.