

Bayesian Time-Series Econometrics

Book 2 - algebraic derivations

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Second edition

Bayesian Time-Series Econometrics

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Cover illustration: Thomas Bayes (d. 1761) in Terence O'Donnell, *History of Life Insurance in Its Formative Years* (Chicago: American Conservation Co., 1936), p. 335.

To my wife, Mélanie.

To my sons, Tristan and Arnaud.

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PART I

Bayesian statistics

Three applied examples

derivations for equation (1.3.3)

$$f(y|p) = \prod_{i=1}^n p^{y_i} (1-p)^{1-y_i} = p^{\sum_{i=1}^n y_i} (1-p)^{\sum_{i=1}^n 1-y_i} = p^m (1-p)^{n-m} \quad (\text{a.1.3.1})$$

derivations for equation (1.3.5)

The derivative is given by:

$$\frac{d \log(f(y|p))}{dp} = \frac{m}{p} - \frac{n-m}{1-p} \quad (\text{a.1.3.2})$$

Set the value to 0 and solve for p :

$$\begin{aligned} \frac{m}{p} - \frac{n-m}{1-p} &= 0 \\ \Leftrightarrow \frac{m}{p} &= \frac{n-m}{1-p} \\ \Leftrightarrow m(1-p) &= p(n-m) \\ \Leftrightarrow m - mp &= np - mp \\ \Leftrightarrow m &= np \\ \Leftrightarrow p &= \frac{m}{n} \end{aligned} \quad (\text{a.1.3.3})$$

derivations for equation (1.3.11)

$$f(y|p) = \prod_{i=1}^n \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} = \frac{\prod_{i=1}^n \lambda^{y_i} \prod_{i=1}^n e^{-\lambda}}{\prod_{i=1}^n y_i!} = \frac{\lambda^{\sum_{i=1}^n y_i} e^{-n\lambda}}{\prod_{i=1}^n y_i!} \quad (\text{a.1.3.4})$$

derivations for equation (1.3.13)

The derivative is given by:

$$\frac{d \log(f(y|\lambda))}{d\lambda} = \frac{\sum_{i=1}^n y_i}{\lambda} - n \quad (\text{a.1.3.5})$$

Set the value to 0 and solve for λ :

$$\begin{aligned} \frac{\sum_{i=1}^n y_i}{\lambda} - n &= 0 \\ \Leftrightarrow \frac{\sum_{i=1}^n y_i}{\lambda} &= n \\ \Leftrightarrow \lambda &= \frac{1}{n} \sum_{i=1}^n y_i \end{aligned} \quad (\text{a.1.3.6})$$

derivations for equation (1.3.16)

$$\begin{aligned}
 & \lambda^{\sum_{i=1}^n y_i} e^{-n\lambda} \times \lambda^{a-1} e^{-\lambda/b} \\
 = & \lambda^{a+\sum_{i=1}^n y_i-1} e^{-\lambda(n+1/b)} \\
 = & \lambda^{a+\sum_{i=1}^n y_i-1} e^{-\lambda/(n+1/b)^{-1}}
 \end{aligned} \tag{a.1.3.7}$$

Now:

$$(n+1/b)^{-1} = \frac{1}{n+1/b} = \frac{b}{bn+1} \tag{a.1.3.8}$$

Hence:

$$\begin{aligned}
 & \lambda^{\sum_{i=1}^n y_i} e^{-n\lambda} \times \lambda^{a-1} e^{-\lambda/b} \\
 = & \lambda^{a+\sum_{i=1}^n y_i-1} e^{-\lambda/\frac{b}{bn+1}}
 \end{aligned} \tag{a.1.3.9}$$

derivations for equation (1.3.19)

$$\begin{aligned}
 & f(y|\mu) \\
 = & \prod_{i=1}^n (2\pi\sigma)^{-1/2} \exp\left(-\frac{1}{2} \frac{(y_i - \mu)^2}{\sigma}\right) \\
 = & \prod_{i=1}^n (2\pi\sigma)^{-1/2} \prod_{i=1}^n \exp\left(-\frac{1}{2} \frac{(y_i - \mu)^2}{\sigma}\right) \\
 = & (2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma}\right)
 \end{aligned} \tag{a.1.3.10}$$

derivations for equation (1.3.21)

The derivative is given by:

$$\frac{d \log(f(y|\mu))}{d\mu} = \sum_{i=1}^n \frac{(y_i - \mu)}{\sigma} \tag{a.1.3.11}$$

Set the value to 0 and solve for μ :

$$\begin{aligned}
 & \sum_{i=1}^n \frac{(y_i - \mu)}{\sigma} = 0 \\
 \Leftrightarrow & \sum_{i=1}^n (y_i - \mu) = 0 \\
 \Leftrightarrow & \sum_{i=1}^n y_i - n\mu = 0 \\
 \Leftrightarrow & \sum_{i=1}^n y_i = n\mu \\
 \Leftrightarrow & \mu = \frac{1}{n} \sum_{i=1}^n y_i
 \end{aligned} \tag{a.1.3.12}$$

derivations for equation (1.3.24)

First group the exponential terms:

$$\pi(\mu|y) \propto \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma}\right) \times \exp\left(-\frac{1}{2} \frac{(\mu - m)^2}{v}\right) = \exp\left(-\frac{1}{2} \left[\sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma} + \frac{(\mu - m)^2}{v} \right]\right) \quad (\text{a.1.3.13})$$

Develop the term within the square bracket:

$$\begin{aligned} & \sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma} + \frac{(\mu - m)^2}{v} \\ &= \frac{1}{\sigma} \sum_{i=1}^n (y_i^2 + \mu^2 - 2\mu y_i) + \frac{1}{v} (\mu^2 + m^2 - 2\mu m) \\ &= \frac{1}{\sigma} \left(\sum_{i=1}^n y_i^2 + n\mu^2 - 2\mu \sum_{i=1}^n y_i \right) + \frac{1}{v} (\mu^2 + m^2 - 2\mu m) \end{aligned} \quad (\text{a.1.3.14})$$

Group the terms:

$$= \mu^2 \left(\frac{n}{\sigma} + \frac{1}{v} \right) - 2\mu \left(\frac{1}{\sigma} \sum_{i=1}^n y_i + \frac{m}{v} \right) + \frac{1}{\sigma} \sum_{i=1}^n y_i^2 + \frac{m^2}{v} \quad (\text{a.1.3.15})$$

Set back in (a.1.3.13):

$$\pi(\mu|y) \propto \exp\left(-\frac{1}{2} \left[\mu^2 \left(\frac{n}{\sigma} + \frac{1}{v} \right) - 2\mu \left(\frac{1}{\sigma} \sum_{i=1}^n y_i + \frac{m}{v} \right) + \frac{1}{\sigma} \sum_{i=1}^n y_i^2 + \frac{m^2}{v} \right]\right) \quad (\text{a.1.3.16})$$

Further aspects of Bayesian priors and posteriors

derivations for equation (1.4.5)

First group the terms:

$$\begin{aligned}
 & \pi(\mu, \sigma|y) \\
 & \propto \sigma^{-n/2} \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma}\right) \times \sigma^{-1/2} \exp\left(-\frac{1}{2} \frac{(\mu - m)^2}{v\sigma}\right) \times \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right) \\
 & = \sigma^{-(n+\alpha)/2-1} \times \sigma^{-1/2} \times \exp\left(-\frac{1}{2\sigma} \left[\sum_{i=1}^n (y_i - \mu)^2 + \frac{(\mu - m)^2}{v} + \delta \right]\right)
 \end{aligned} \tag{a.1.4.1}$$

Develop the term in the square bracket:

$$\begin{aligned}
 & \sum_{i=1}^n (y_i - \mu)^2 + \frac{(\mu - m)^2}{v} + \delta \\
 & = \sum_{i=1}^n (y_i^2 + \mu^2 - 2\mu y_i) + \frac{\mu^2}{v} + \frac{m^2}{v} - 2\mu \frac{m}{v} + \delta \\
 & = \sum_{i=1}^n y_i^2 + n\mu^2 - 2\mu \sum_{i=1}^n y_i + \frac{\mu^2}{v} + \frac{m^2}{v} - 2\mu \frac{m}{v} + \delta \\
 & = \mu^2 \left(n + \frac{1}{v}\right) - 2\mu \left(\sum_{i=1}^n y_i + \frac{m}{v}\right) + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} + \delta
 \end{aligned} \tag{a.1.4.2}$$

Complete the squares:

$$= \mu^2 \left(n + \frac{1}{v}\right) - 2\mu \frac{\bar{v}}{\bar{v}} \left(\sum_{i=1}^n y_i + \frac{m}{v}\right) + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} + \delta + \frac{\bar{m}^2}{\bar{v}} - \frac{\bar{m}^2}{\bar{v}} \tag{a.1.4.3}$$

Define:

$$\bar{v} = \left(n + \frac{1}{v}\right)^{-1} \quad \bar{m} = \bar{v} \left(\sum_{i=1}^n y_i + \frac{m}{v}\right) \tag{a.1.4.4}$$

Then (a.1.4.3) rewrites:

$$\begin{aligned}
 & = \frac{\mu^2}{\bar{v}} + \frac{\bar{m}^2}{\bar{v}} - 2\mu \frac{\bar{m}}{\bar{v}} + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}} \\
 & = \frac{(\mu - \bar{m})^2}{\bar{v}} + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}}
 \end{aligned} \tag{a.1.4.5}$$

Substituting back (a.1.4.5) in (a.1.4.1) eventually yields:

$$\begin{aligned}
& \pi(\mu, \sigma | y) \\
& \propto \sigma^{-(n+\alpha)/2-1} \times \sigma^{-1/2} \times \exp \left(-\frac{1}{2\sigma} \left[\frac{(\mu - \bar{m})^2}{\bar{v}} + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}} \right] \right) \\
& = \sigma^{-(n+\alpha)/2-1} \times \sigma^{-1/2} \times \exp \left(-\frac{1}{2} \frac{(\mu - \bar{m})^2}{\sigma \bar{v}} \right) \times \exp \left(-\frac{1}{2\sigma} \left[\sum_{i=1}^n y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}} \right] \right) \\
& = \sigma^{-\bar{\alpha}/2-1} \times \sigma^{-1/2} \times \exp \left(-\frac{1}{2} \frac{(\mu - \bar{m})^2}{\sigma \bar{v}} \right) \times \exp \left(-\frac{\bar{\delta}}{2\sigma} \right) \tag{a.1.4.6}
\end{aligned}$$

with:

$$\bar{\alpha} = n + \alpha \quad \bar{\delta} = \sum_{i=1}^n y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}} \tag{a.1.4.7}$$

derivations for equation (1.4.10)

Rearrange the terms:

$$\begin{aligned}
& \pi(\mu | y) \\
& \propto \Gamma \left(\frac{\bar{\alpha} + 1}{2} \right) \left(\frac{\bar{\delta} + (\mu - \bar{m})^2 / \bar{v}}{2} \right)^{-\frac{\bar{\alpha}+1}{2}} \\
& \propto \left(\frac{\bar{\delta} + (\mu - \bar{m})^2 / \bar{v}}{2} \right)^{-\frac{\bar{\alpha}+1}{2}} \\
& \propto \left(\bar{\delta} + \frac{(\mu - \bar{m})^2}{\bar{v}} \right)^{-\frac{\bar{\alpha}+1}{2}} \\
& = \bar{\delta} \left(1 + \frac{(\mu - \bar{m})^2}{\bar{\delta} \bar{v}} \right)^{-\frac{\bar{\alpha}+1}{2}} \\
& \propto \left(1 + \frac{(\mu - \bar{m})^2}{\bar{\delta} \bar{v}} \right)^{-\frac{\bar{\alpha}+1}{2}} \\
& = \left(1 + \frac{1}{\bar{\alpha}} \frac{(\mu - \bar{m})^2}{\bar{\delta} \bar{v} / \bar{\alpha}} \right)^{-\frac{\bar{\alpha}+1}{2}} \tag{a.1.4.8}
\end{aligned}$$

derivations for equation (1.4.13)

Solve for the derivative:

$$\begin{aligned}
& 2 \int (\hat{\theta} - \theta) \pi(\theta | y) d\theta = 0 \\
& \Leftrightarrow \int (\hat{\theta} - \theta) \pi(\theta | y) d\theta = 0 \\
& \Leftrightarrow \int \hat{\theta} \pi(\theta | y) d\theta - \int \theta \pi(\theta | y) d\theta = 0 \\
& \Leftrightarrow \hat{\theta} \int \pi(\theta | y) d\theta = \int \theta \pi(\theta | y) d\theta \\
& \Leftrightarrow \hat{\theta} = \int \theta \pi(\theta | y) d\theta \tag{a.1.4.9}
\end{aligned}$$

derivations for equation (1.4.16)

Rearrange the expression:

$$\begin{aligned}
 f(y) &= \int \int (2\pi)^{-n/2} (2\pi)^{-1/2} v^{-1/2} \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\
 &\quad \times \sigma^{-n/2} \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma}\right) \times \sigma^{-1/2} \exp\left(-\frac{1}{2} \frac{(\mu - m)^2}{v\sigma}\right) \times \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right) d\mu d\sigma
 \end{aligned} \tag{a.1.4.10}$$

The second row can be recognised as equation (a.1.4.1). Using the same manipulations, one obtains equation (a.1.4.6), and thus the previous expression rewrites as:

$$\begin{aligned}
 f(y) &= \int \int (2\pi)^{-n/2} (2\pi)^{-1/2} v^{-1/2} \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\
 &\quad \times \sigma^{-1/2} \exp\left(-\frac{1}{2} \frac{(\mu - \bar{m})^2}{\sigma \bar{v}}\right) \times \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\mu d\sigma
 \end{aligned} \tag{a.1.4.11}$$

with \bar{m} , \bar{v} , $\bar{\alpha}$ and $\bar{\delta}$ defined as in (a.1.4.4) and (a.1.4.7). Now add multiplicative terms to obtain normal and inverse Gamma probability density functions, and take constants out of the integral:

$$\begin{aligned}
 f(y) &= (2\pi)^{-n/2} v^{-1/2} \bar{v}^{1/2} \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \frac{\Gamma(\bar{\alpha}/2)}{\bar{\delta}/2^{\bar{\alpha}/2}} \\
 &\quad \times \int \int (2\pi \bar{v} \sigma)^{-1/2} \exp\left(-\frac{1}{2} \frac{(\mu - \bar{m})^2}{\sigma \bar{v}}\right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\mu d\sigma
 \end{aligned} \tag{a.1.4.12}$$

The expression can simplify further. Consider only the constant on the first line:

$$\begin{aligned}
 &(2\pi)^{-n/2} v^{-1/2} \bar{v}^{1/2} \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \frac{\Gamma(\bar{\alpha}/2)}{\bar{\delta}/2^{\bar{\alpha}/2}} \\
 &= 2^{-n/2} \pi^{-n/2} v^{-1/2} ((n+1/v)^{-1})^{1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{2^{\bar{\alpha}/2}}{2^{\alpha/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
 &= 2^{-n/2} \pi^{-n/2} v^{-1/2} (n+1/v)^{-1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{2^{(\alpha+n)/2}}{2^{\alpha/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
 &= \pi^{-n/2} (1+vn)^{-1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}
 \end{aligned} \tag{a.1.4.13}$$

Substitute back in (a.1.4.12):

$$\begin{aligned}
 f(y) &= \pi^{-n/2} (1+vn)^{-1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
 &\quad \times \int \int (2\pi \bar{v} \sigma)^{-1/2} \exp\left(-\frac{1}{2} \frac{(\mu - \bar{m})^2}{\sigma \bar{v}}\right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\mu d\sigma
 \end{aligned} \tag{a.1.4.14}$$

derivations for equation (1.4.19)

Rearrange the expression:

$$\begin{aligned}
\mathbb{P}(M_i|y) &= \frac{f(y|M_i) \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow \mathbb{P}(M_i|y) &= \frac{f(y, M_i) / \pi(M_i) \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow \mathbb{P}(M_i|y) &= \frac{\int f(y, M_i, \theta_i) / \pi(M_i) d\theta_i \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow \mathbb{P}(M_i|y) &= \frac{\int \frac{f(y, M_i, \theta_i)}{\pi(M_i, \theta)} \frac{\pi(M_i, \theta)}{\pi(M_i)} d\theta_i \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow \mathbb{P}(M_i|y) &= \frac{\int f(y|M_i, \theta_i) \pi(\theta|M_i) d\theta_i \mathbb{P}(M_i)}{f(y)} \tag{a.1.4.15}
\end{aligned}$$

derivations for equation (1.4.24)

Rearrange the expression to obtain:

$$\begin{aligned}
&f(\hat{y}|y) \\
&= \int \int \sigma^{-1/2} \exp\left(-\frac{1}{2} \frac{(\hat{y} - \mu)^2}{\sigma}\right) \times \sigma^{-n/2} \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma}\right) \\
&\quad \times \sigma^{-1/2} \exp\left(-\frac{1}{2} \frac{(\mu - m)^2}{\nu \sigma}\right) \times \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right) d\mu d\sigma \\
&= \int \int \sigma^{-1/2} \exp\left(-\frac{1}{2\sigma} \left[(\hat{y} - \mu)^2 + \sum_{i=1}^n (y_i - \mu)^2 + \frac{(\mu - m)^2}{\nu} + \delta \right]\right) \sigma^{-(\alpha+n+1)/2-1} d\mu d\sigma \\
&= \int \int \sigma^{-1/2} \exp\left(-\frac{1}{2\sigma} \left[(\hat{y} - \mu)^2 + \sum_{i=1}^n (y_i - \mu)^2 + \frac{(\mu - m)^2}{\nu} + \delta \right]\right) \sigma^{-\hat{\alpha}/2-1} d\mu d\sigma \tag{a.1.4.16}
\end{aligned}$$

with:

$$\hat{\alpha} = \alpha + n + 1 \tag{a.1.4.17}$$

Consider the term in square brackets:

$$\begin{aligned}
& (\hat{y} - \mu)^2 + \sum_{i=1}^n (y_i - \mu)^2 + \frac{(\mu - m)^2}{v} + \delta \\
&= \hat{y}^2 + \mu^2 - 2\hat{y}\mu + \sum_{i=1}^n (y_i^2 + \mu^2 - 2y_i\mu) + \frac{\mu^2}{v} + \frac{m^2}{v} - 2\mu\frac{m}{v} + \delta \\
&= \hat{y}^2 + \mu^2 - 2\hat{y}\mu + \sum_{i=1}^n y_i^2 + n\mu^2 - 2\mu \sum_{i=1}^n y_i + \frac{\mu^2}{v} + \frac{m^2}{v} - 2\mu\frac{m}{v} + \delta \\
&= \left(\delta + \hat{y}^2 + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} \right) + \mu^2 \left(1 + n + \frac{1}{v} \right) - 2\mu \left(\hat{y} + \sum_{i=1}^n y_i + \frac{m}{v} \right) \\
&= \left(\delta + \hat{y}^2 + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} \right) + \mu^2 \left(1 + n + \frac{1}{v} \right) - 2\mu \frac{\hat{v}}{\hat{v}} \left(\hat{y} + \sum_{i=1}^n y_i + \frac{m}{v} \right) + \frac{\hat{m}^2}{\hat{v}} - \frac{\hat{m}^2}{\hat{v}} \\
&= \left(\delta + \hat{y}^2 + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \frac{\hat{m}^2}{\hat{v}} \right) + \mu^2 \left(1 + n + \frac{1}{v} \right) - 2\mu \frac{\hat{v}}{\hat{v}} \left(\hat{y} + \sum_{i=1}^n y_i + \frac{m}{v} \right) + \frac{\hat{m}^2}{\hat{v}} \tag{a.1.4.18}
\end{aligned}$$

Define:

$$\hat{\delta} = \left(\delta + \hat{y}^2 + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \frac{\hat{m}^2}{\hat{v}} \right) \quad \hat{v} = \left(1 + n + \frac{1}{v} \right)^{-1} \quad \hat{m} = \hat{v} \left(\hat{y} + \sum_{i=1}^n y_i + \frac{m}{v} \right) \tag{a.1.4.19}$$

Then (a.1.4.17) becomes:

$$\begin{aligned}
&= \hat{\delta} + \frac{\mu^2}{\hat{v}} - 2\mu \frac{\hat{m}}{\hat{v}} + \frac{\hat{m}^2}{\hat{v}} \\
&= \hat{\delta} + \frac{(\mu - \hat{m})^2}{\hat{v}} \tag{a.1.4.20}
\end{aligned}$$

Substitute back in (a.1.4.16):

$$\begin{aligned}
& f(\hat{y}|y) \\
&= \int \int \sigma^{-1/2} \exp \left(-\frac{1}{2\sigma} \left[\hat{\delta} + \frac{(\mu - \hat{m})^2}{\hat{v}} \right] \right) \sigma^{-\hat{\alpha}/2-1} d\mu d\sigma \\
&= \int \int \sigma^{-1/2} \exp \left(-\frac{1}{2} \frac{(\mu - \hat{m})^2}{\hat{v}\sigma} \right) \times \sigma^{-\hat{\alpha}/2-1} \exp \left(-\frac{\hat{\delta}}{2\sigma} \right) d\mu d\sigma \\
&= \int \sigma^{-\hat{\alpha}/2-1} \exp \left(-\frac{\hat{\delta}}{2\sigma} \right) \int \sigma^{-1/2} \exp \left(-\frac{1}{2} \frac{(\mu - \hat{m})^2}{\hat{v}\sigma} \right) d\mu d\sigma \tag{a.1.4.21}
\end{aligned}$$

The second integral contains the kernel of a normal distribution with mean \hat{m} and variance $\hat{v}\sigma$. It thus integrates to a constant (not involving \hat{y}) and can be relegated to the normalization constant, yielding:

$$\propto \int \sigma^{-(\hat{\alpha}+1)/2-1} \exp \left(-\frac{\hat{\delta}}{2\sigma} \right) d\sigma \tag{a.1.4.22}$$

The remaining integral contains the kernel of an inverse Gamma distribution with shape $\hat{\alpha}$ and scale $\hat{\delta}$. It integrates to the reciprocal of the normalization constant of the inverse Gamma distribution (see book1, section 4.3), which does involve \hat{y} . The term must thus be retained, yielding:

$$\begin{aligned}
& f(\hat{y}|y) \\
& \propto \Gamma(\hat{\alpha})(\hat{\delta}/2)^{-\hat{\alpha}/2} \\
& \propto (\hat{\delta}/2)^{-\hat{\alpha}/2} \\
& \propto (\hat{\delta})^{-\hat{\alpha}/2} \\
& = \left(\delta + \hat{y}^2 + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \frac{\hat{m}^2}{\hat{v}} \right)^{-\hat{\alpha}/2} \\
& = \left(\delta + \hat{y}^2 + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \hat{v} \left[\hat{y} + \sum_{i=1}^n y_i + \frac{m}{v} \right]^2 \right)^{-\hat{\alpha}/2}
\end{aligned} \tag{a.1.4.23}$$

Define:

$$\tilde{m} = \sum_{i=1}^n y_i + \frac{m}{v} \tag{a.1.4.24}$$

Then (a.1.4.23) becomes:

$$\begin{aligned}
& = \left(\delta + \hat{y}^2 + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \hat{v}[\hat{y} + \tilde{m}]^2 \right)^{-\hat{\alpha}/2} \\
& = \left(\delta + \hat{y}^2 + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \hat{v}\hat{y}^2 - \hat{v}\tilde{m}^2 - 2\hat{v}\tilde{m}\hat{y} \right)^{-\hat{\alpha}/2} \\
& = \left(\delta + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \hat{v}\tilde{m}^2 + \hat{y}^2(1 - \hat{v}) - 2\hat{v}\tilde{m}\hat{y} \right)^{-\hat{\alpha}/2}
\end{aligned} \tag{a.1.4.25}$$

Complete the squares:

$$\begin{aligned}
& = \left(\delta + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \hat{v}\tilde{m}^2 + \hat{y}^2(1 - \hat{v}) - 2\hat{v}\frac{\ddot{v}}{\ddot{v}}\tilde{m}\hat{y} + \frac{\dot{m}^2}{\ddot{v}} - \frac{\ddot{m}^2}{\ddot{v}} \right)^{-\hat{\alpha}/2} \\
& = \left(\left[\delta + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \hat{v}\tilde{m}^2 - \frac{\ddot{m}^2}{\ddot{v}} \right] + \hat{y}^2(1 - \hat{v}) - 2\hat{v}\frac{\ddot{v}}{\ddot{v}}\tilde{m}\hat{y} + \frac{\dot{m}^2}{\ddot{v}} \right)^{-\hat{\alpha}/2}
\end{aligned} \tag{a.1.4.26}$$

Define:

$$\bar{\alpha} = \alpha + n \quad \bar{\delta} = \delta + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \hat{v}\tilde{m}^2 - \frac{\ddot{m}^2}{\ddot{v}} \quad \ddot{v} = (1 - \hat{v})^{-1} \quad \ddot{m} = \hat{v}\ddot{v}\tilde{m} \tag{a.1.4.27}$$

Then (a.1.4.26) becomes:

$$\begin{aligned}
&= \left(\ddot{\delta} + \frac{\hat{y}^2}{\hat{v}} - 2\hat{y}\frac{\dot{m}}{\hat{v}} + \frac{\dot{m}^2}{\hat{v}} \right)^{-(\bar{\alpha}+1)/2} \\
&= \left(\ddot{\delta} + \frac{(\hat{y} - \dot{m})^2}{\hat{v}} \right)^{-(\bar{\alpha}+1)/2} \\
&= \ddot{\delta}^{-(\bar{\alpha}+1)/2} \left(1 + \frac{(\hat{y} - \dot{m})^2}{\ddot{\delta}\hat{v}} \right)^{-(\bar{\alpha}+1)/2} \\
&\propto \left(1 + \frac{(\hat{y} - \dot{m})^2}{\ddot{\delta}\hat{v}} \right)^{-(\bar{\alpha}+1)/2} \\
&= \left(1 + \frac{1}{\bar{\alpha}} \frac{(\hat{y} - \dot{m})^2}{\ddot{\delta}\hat{v}/\bar{\alpha}} \right)^{-(\bar{\alpha}+1)/2} \tag{a.1.4.28}
\end{aligned}$$

Finally, reformulate all the messy terms:

$$\begin{aligned}
\hat{v} &= (1 - \hat{v})^{-1} = \frac{1}{1 - \hat{v}} = \frac{1}{1 - \frac{1}{1+n+\frac{1}{v}}} = \frac{1}{\frac{1+n+\frac{1}{v}-1}{1+n+\frac{1}{v}}} = \frac{1}{\frac{n+\frac{1}{v}}{1+n+\frac{1}{v}}} = \frac{1+n+\frac{1}{v}}{n+\frac{1}{v}} = \frac{v+vn+1}{vn+1} \\
&= 1 + \frac{v}{vn+1} = 1 + \frac{1}{n+1/v} = 1 + \left(n + \frac{1}{v} \right)^{-1} = 1 + \bar{v} \tag{a.1.4.29}
\end{aligned}$$

with \bar{v} defined as in (a.1.4.4).

Also:

$$\frac{\hat{v}}{1 - \hat{v}} = \frac{\frac{1}{1+n+\frac{1}{v}}}{1 - \frac{1}{1+n+\frac{1}{v}}} = \frac{\frac{1}{1+n+\frac{1}{v}}}{\frac{1+n+\frac{1}{v}-1}{1+n+\frac{1}{v}}} = \frac{1}{\frac{n+\frac{1}{v}}{1+n+\frac{1}{v}}} = \frac{1}{n+\frac{1}{v}} = \left(n + \frac{1}{v} \right)^{-1} = \bar{v} \tag{a.1.4.30}$$

Then:

$$\ddot{m} = \hat{v}\ddot{m} = \frac{\hat{v}}{1 - \hat{v}}\ddot{m} = \bar{v}\ddot{m} = \bar{v} \left(\sum_{i=1}^n y_i + \frac{m}{v} \right) = \bar{m} \tag{a.1.4.31}$$

with \bar{m} defined as in (a.1.4.4).

Finally:

$$\begin{aligned}
\hat{v}\ddot{m}^2 + \frac{\dot{m}^2}{\hat{v}} &= \hat{v}\ddot{m}^2 + (\hat{v}\ddot{m})^2/\hat{v} = \hat{v}\ddot{m}^2 + \hat{v}^2\ddot{m}^2 = \hat{v}\ddot{m}^2(1 + \hat{v}) = \hat{v}\ddot{m}^2 \left(1 + \frac{\hat{v}}{1 - \hat{v}} \right) \\
&= \hat{v}\ddot{m}^2 \left(\frac{1 - \hat{v} + \hat{v}}{1 - \hat{v}} \right) = \hat{v}\ddot{m}^2 \left(\frac{1}{1 - \hat{v}} \right) = \ddot{m}^2 \left(\frac{\hat{v}}{1 - \hat{v}} \right) = \ddot{m}^2 \bar{v} = \bar{m}\bar{m} = \frac{\bar{m}}{\bar{v}}\bar{m} = \frac{\bar{m}^2}{\bar{v}} \tag{a.1.4.32}
\end{aligned}$$

Substitute in (a.1.4.27) to obtain:

$$\ddot{\delta} = \delta + \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \frac{\bar{m}^2}{\bar{v}} = \bar{\delta} \tag{a.1.4.33}$$

with $\bar{\delta}$ defined as in (a.1.4.7).

Substitute (a.1.4.29), (a.1.4.31) and (a.1.4.33) in (a.1.4.28) to eventually obtain:

$$f(\hat{y}|y) \propto \left(1 + \frac{1}{\bar{\alpha}} \frac{(\hat{y} - \bar{m})^2}{\bar{\delta}(1 + \bar{v})/\bar{\alpha}} \right)^{-(\bar{\alpha}+1)/2} \tag{a.1.4.34}$$

Properties of Bayesian estimates

derivations for equation (1.5.1)

The mean of a Beta distribution with shapes a and b is given by $\frac{a}{a+b}$. Given the posterior hyperparameters $\bar{\alpha} = \alpha + m$ and $\bar{\beta} = \beta + n - m$, the posterior mean writes as:

$$\begin{aligned}
 & \mathbb{E}(p|y) \\
 &= \frac{\bar{\alpha}}{\bar{\alpha} + \bar{\beta}} \\
 &= \frac{\alpha + m}{\alpha + m + \beta + n - m} \\
 &= \frac{\alpha + m}{\alpha + \beta + n} \\
 &= \frac{\alpha}{\alpha + \beta + n} + \frac{m}{\alpha + \beta + n} \\
 &= \frac{\alpha}{\alpha + \beta} \frac{\alpha + \beta}{\alpha + \beta + n} + \frac{m}{n} \frac{n}{\alpha + \beta + n} \\
 &= \gamma \mathbb{E}(p) + (1 - \gamma) \hat{p}
 \end{aligned} \tag{a.1.5.1}$$

with:

$$\mathbb{E}(p) = \frac{\alpha}{\alpha + \beta} \quad \hat{p} = \frac{m}{n} \quad \gamma = \frac{\alpha + \beta}{\alpha + \beta + n} \tag{a.1.5.2}$$

derivations for equation (1.5.2)

The mean of a Gamma distribution with shape a and scale b is given by ab . Given the posterior hyperparameters $\bar{a} = a + \sum_{i=1}^n y_i$ and $\bar{b} = \frac{b}{bn+1}$, the posterior mean writes as:

$$\begin{aligned}
 & \mathbb{E}(\lambda|y) \\
 &= \frac{(a + \sum_{i=1}^n y_i)b}{bn + 1} \\
 &= \frac{ab}{bn + 1} + \frac{b \sum_{i=1}^n y_i}{bn + 1} \\
 &= ab \left(\frac{1}{bn + 1} \right) + \frac{\sum_{i=1}^n y_i}{n} \left(\frac{bn}{bn + 1} \right) \\
 &= \gamma \mathbb{E}(\lambda) + (1 - \gamma) \hat{\lambda}
 \end{aligned} \tag{a.1.5.3}$$

with:

$$\mathbb{E}(\lambda) = ab \quad \hat{\lambda} = \frac{\sum_{i=1}^n y_i}{n} \quad \gamma = \frac{1}{bn + 1} \tag{a.1.5.4}$$

derivations for equation (1.5.3)

The mean of a normal distribution with mean μ and variance σ is given by μ . Given the posterior hyperparameters $\bar{v} = \left(\frac{n}{\sigma} + \frac{1}{v}\right)^{-1}$ and $\bar{m} = \bar{v} \left(\frac{1}{\sigma} \sum_{i=1}^n y_i + \frac{m}{v}\right)$, the posterior variance writes as:

$$\bar{v} = \left(\frac{n}{\sigma} + \frac{1}{v}\right)^{-1} = \frac{1}{n/\sigma + 1/v} = \frac{\sigma}{n + \sigma/v} \quad (\text{a.1.5.5})$$

Then the posterior mean can be expressed as:

$$\begin{aligned} & \mathbb{E}(\mu|y) \\ &= \frac{\sigma}{n + \sigma/v} \left(\frac{1}{\sigma} \sum_{i=1}^n y_i + \frac{m}{v} \right) \\ &= \frac{1}{n + \sigma/v} \left(\sum_{i=1}^n y_i \right) + \frac{\sigma}{n + \sigma/v} \left(\frac{m}{v} \right) \\ &= \frac{n}{n + \sigma/v} \left(\frac{\sum_{i=1}^n y_i}{n} \right) + \frac{\sigma/v}{n + \sigma/v} m \\ &= \frac{vn}{vn + \sigma} \left(\frac{\sum_{i=1}^n y_i}{n} \right) + \frac{\sigma}{vn + \sigma} m \\ &= \gamma \mathbb{E}(\mu) + (1 - \gamma) \hat{\mu} \end{aligned} \quad (\text{a.1.5.6})$$

with:

$$\mathbb{E}(\mu) = m \quad \hat{\mu} = \frac{\sum_{i=1}^n y_i}{n} \quad \gamma = \frac{\sigma}{vn + \sigma} \quad (\text{a.1.5.7})$$

PART II

Simulation methods

The Gibbs sampling algorithm

derivations for equation (2.6.17)

Combine all the terms to obtain:

$$\begin{aligned}
 f(y) &\approx (2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma}\right) \frac{(2\pi v)^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^J (2\pi\bar{v})^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right)} \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
 &= (2\pi)^{-n/2} \sigma^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(y_i - \mu)^2}{\sigma}\right) \frac{v^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^J \bar{v}^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right)} \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-(\alpha+n)/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
 &= (2\pi)^{-n/2} \exp \frac{v^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^J \bar{v}^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right)} \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \exp\left(-\frac{1}{2\sigma} [\delta + \sum_{i=1}^n (y_i - \mu)^2]\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
 &= (2\pi)^{-n/2} \exp \frac{v^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^J \bar{v}^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right)} \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}} \\
 &= 2^{-n/2} \pi^{-n/2} \exp \frac{v^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^J \bar{v}^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right)} \frac{\frac{\delta^{\alpha/2}}{\Gamma(\alpha/2)} 2^{\bar{\alpha}/2}}{\frac{\bar{\delta}^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} 2^{\alpha/2}} \\
 &= \pi^{-n/2} \exp \frac{v^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^J \bar{v}^{-1/2} \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right)} \frac{\frac{\delta^{\alpha/2}}{\Gamma(\alpha/2)}}{\frac{\bar{\delta}^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}} \\
 &= \pi^{-n/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \frac{\exp\left(-\frac{1}{2}\frac{(\mu - m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^J (v/\bar{v})^{1/2} \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right)} \tag{a.2.6.1}
 \end{aligned}$$

Now:

$$v/\bar{v} = v(n/\sigma + 1/v) = vn/\sigma + 1 \tag{a.2.6.2}$$

Hence:

$$f(y) \approx \pi^{-n/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \frac{\exp\left(-\frac{1}{2}\frac{(\mu - m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^J (1 + vn/\sigma)^{1/2} \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right)} \tag{a.2.6.3}$$

The Metropolis-Hastings algorithm

derivations for equation (2.7.7)

Rearrange:

$$\begin{aligned}
 & \pi(\mu|y, \lambda) \\
 & \propto \exp\left(-\frac{1}{2} \sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda)}\right) \times \exp\left(-\frac{1}{2} \frac{(\mu - m)^2}{v}\right) \\
 & = \exp\left(-\frac{1}{2} \left[\sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda)} + \frac{(\mu - m)^2}{v} \right]\right)
 \end{aligned} \tag{a.2.7.1}$$

Develop the term within the square bracket:

$$\begin{aligned}
 & \sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda)} + \frac{(\mu - m)^2}{v} \\
 & = \frac{1}{\exp(\lambda)} \sum_{i=1}^n (y_i^2 + \mu^2 - 2\mu y_i) + \frac{1}{v} (\mu^2 + m^2 - 2\mu m) \\
 & = \frac{1}{\exp(\lambda)} \left(\sum_{i=1}^n y_i^2 + n\mu^2 - 2\mu \sum_{i=1}^n y_i \right) + \frac{1}{v} (\mu^2 + m^2 - 2\mu m)
 \end{aligned} \tag{a.2.7.2}$$

Group the terms and complete the squares:

$$\begin{aligned}
 & = \mu^2 \left(\frac{n}{\exp(\lambda)} + \frac{1}{v} \right) - 2\mu \left(\frac{1}{\exp(\lambda)} \sum_{i=1}^n y_i + \frac{m}{v} \right) + \frac{1}{\exp(\lambda)} \sum_{i=1}^n y_i^2 + \frac{m^2}{v} \\
 & = \mu^2 \left(\frac{n}{\exp(\lambda)} + \frac{1}{v} \right) - 2\mu \frac{\bar{v}}{\bar{v}} \left(\frac{1}{\exp(\lambda)} \sum_{i=1}^n y_i + \frac{m}{v} \right) + \frac{1}{\exp(\lambda)} \sum_{i=1}^n y_i^2 + \frac{m^2}{v} + \frac{\bar{m}^2}{\bar{v}} - \frac{\bar{m}^2}{\bar{v}}
 \end{aligned} \tag{a.2.7.3}$$

Define:

$$\bar{v} = \left(\frac{n}{\exp(\lambda)} + \frac{1}{v} \right)^{-1} \quad \bar{m} = \bar{v} \left(\frac{1}{\exp(\lambda)} \sum_{i=1}^n y_i + \frac{m}{v} \right) \tag{a.2.7.4}$$

Then (a.2.7.3) rewrites:

$$\begin{aligned}
 & = \frac{\mu^2}{\bar{v}} + \frac{\bar{m}^2}{\bar{v}} - 2\mu \frac{\bar{m}}{\bar{v}} + \frac{1}{\exp(\lambda)} \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \frac{\bar{m}^2}{\bar{v}} \\
 & = \frac{(\mu - \bar{m})^2}{\bar{v}} + \frac{1}{\exp(\lambda)} \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \frac{\bar{m}^2}{\bar{v}}
 \end{aligned} \tag{a.2.7.5}$$

Substitute back in (a.2.7.1):

$$\begin{aligned}
& \pi(\mu|y, \lambda) \\
& \propto \exp\left(-\frac{1}{2}\left[\frac{(\mu - \bar{m})^2}{\bar{v}} + \frac{1}{\exp(\lambda)} \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \frac{\bar{m}^2}{\bar{v}}\right]\right) \\
& = \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right) \exp\left(-\frac{1}{2}\left[\frac{1}{\exp(\lambda)} \sum_{i=1}^n y_i^2 + \frac{m^2}{v} - \frac{\bar{m}^2}{\bar{v}}\right]\right) \\
& \propto \exp\left(-\frac{1}{2}\frac{(\mu - \bar{m})^2}{\bar{v}}\right)
\end{aligned} \tag{a.2.7.6}$$

derivations for equation (2.7.14)

$$\begin{aligned}
& \alpha(\lambda^{(j-1)}, \lambda^{(j)}) \\
& = \frac{\exp(\lambda^{(j)})^{-n/2}}{\exp(\lambda^{(j-1)})^{-n/2}} \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda^{(j)})}\right)}{\exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda^{(j-1)})}\right)} \frac{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j)} - g)^2}{z}\right)}{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j-1)} - g)^2}{z}\right)}
\end{aligned} \tag{a.2.7.7}$$

Consider the first term:

$$\begin{aligned}
& \frac{\exp(\lambda^{(j)})^{-n/2}}{\exp(\lambda^{(j-1)})^{-n/2}} \\
& = \exp(\lambda^{(j)} - \lambda^{(j-1)})^{-n/2} \\
& = \exp\left(\frac{n}{2}(\lambda^{(j-1)} - \lambda^{(j)})\right)
\end{aligned} \tag{a.2.7.8}$$

Consider the second term:

$$\begin{aligned}
& \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda^{(j)})}\right)}{\exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda^{(j-1)})}\right)} \\
& = \frac{\exp\left(-\frac{1}{2}\exp(-\lambda^{(j)})\sum_{i=1}^n (y_i - \mu)^2\right)}{\exp\left(-\frac{1}{2}\exp(-\lambda^{(j-1)})\sum_{i=1}^n (y_i - \mu)^2\right)} \\
& = \exp\left(-\frac{1}{2}\left[\exp(-\lambda^{(j)}) - \exp(-\lambda^{(j-1)})\right]\sum_{i=1}^n (y_i - \mu)^2\right) \\
& = \exp\left(\frac{1}{2}\left[\exp(-\lambda^{(j-1)}) - \exp(-\lambda^{(j)})\right]\sum_{i=1}^n (y_i - \mu)^2\right)
\end{aligned} \tag{a.2.7.9}$$

Consider the third term:

$$\begin{aligned}
& \frac{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j)} - g)^2}{z}\right)}{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j-1)} - g)^2}{z}\right)} \\
& = \exp\left(-\frac{1}{2}\left[\frac{(\lambda^{(j)} - g)^2 - (\lambda^{(j-1)} - g)^2}{z}\right]\right) \\
& = \exp\left(\frac{1}{2}\left[\frac{(\lambda^{(j-1)} - g)^2 - (\lambda^{(j)} - g)^2}{z}\right]\right)
\end{aligned} \tag{a.2.7.10}$$

Substitute back in (a.2.7.7):

$$\begin{aligned} & \alpha(\lambda^{(j-1)}, \lambda^{(j)}) \\ = & \exp \left(\frac{1}{2} \left[\frac{n(\lambda^{(j-1)} - \lambda^{(j)}) + [\exp(-\lambda^{(j-1)}) - \exp(-\lambda^{(j)})] \sum_{i=1}^n (y_i - \mu)^2}{(\lambda^{(j-1)} - g)^2 - (\lambda^{(j)} - g)^2} \right] \right) \end{aligned} \quad (\text{a.2.7.11})$$

derivations for equation (2.7.21)

Rearrange the expression:

$$\begin{aligned} & \frac{1}{f(y)} \\ \approx & \frac{1}{J} \sum_{j=1}^J \frac{g(\theta^{(j)})}{f(y|\mu^{(j)}, \lambda^{(j)}) \pi(\mu^{(j)}) \pi(\lambda^{(j)})} \\ = & \frac{1}{J} \sum_{j=1}^J \frac{\mathbb{1}(\theta \in \hat{\Theta}) \times \omega^{-1} (2\pi)^{-k/2} |\hat{\Sigma}|^{-1/2} \exp \left(-\frac{1}{2} (\theta - \hat{\theta})' \hat{\Sigma}^{-1} (\theta - \hat{\theta}) \right) \mathbb{1}(\theta \in \hat{\Theta})}{(2\pi \exp(\lambda))^{-n/2} \exp \left(-\frac{1}{2} \sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda)} \right) (2\pi v)^{-1/2} \exp \left(-\frac{1}{2} \frac{(\mu - m)^2}{v} \right) (2\pi z)^{-1/2} \exp \left(-\frac{1}{2} \frac{(\lambda - g)^2}{z} \right)} \\ = & \mathbb{1}(\theta \in \hat{\Theta}) \times \omega^{-1} (2\pi)^{(n+2-k)/2} |\hat{\Sigma}|^{-1/2} (vz)^{1/2} \\ & \times \frac{1}{J} \sum_{j=1}^J \exp \left(\frac{1}{2} \left[n\lambda + \sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda)} + \frac{(\mu - m)^2}{v} + \frac{(\lambda - g)^2}{z} - (\theta - \hat{\theta})' \hat{\Sigma}^{-1} (\theta - \hat{\theta}) \right] \right) \\ = & \mathbb{1}(\theta \in \hat{\Theta}) \times (\omega J)^{-1} (2\pi)^{n/2} |\hat{\Sigma}|^{-1/2} (vz)^{1/2} \\ & \times \sum_{j=1}^J \exp \left(\frac{1}{2} \left[n\lambda + \sum_{i=1}^n \frac{(y_i - \mu)^2}{\exp(\lambda)} + \frac{(\mu - m)^2}{v} + \frac{(\lambda - g)^2}{z} - (\theta - \hat{\theta})' \hat{\Sigma}^{-1} (\theta - \hat{\theta}) \right] \right) \end{aligned} \quad (\text{a.2.7.12})$$

Mathematical theory

derivations for equation (2.8.11)

The definition of an invariant distribution implies that:

$$(\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4 \quad \cdots) \begin{pmatrix} p+q & r & 0 & 0 & 0 & \cdots \\ p & q & r & 0 & 0 & \cdots \\ 0 & p & q & r & 0 & \cdots \\ 0 & 0 & p & q & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4 \quad \cdots) \quad (\text{a.2.8.1})$$

The product with the first column of P yields:

$$\begin{aligned} \pi_1(p+q) + \pi_2p &= \pi_1 \\ \Leftrightarrow \pi_2p &= \pi_1(1-p-q) \\ \Leftrightarrow \pi_2p &= \pi_1r \\ \Leftrightarrow \pi_2 &= (r/p)\pi_1 \end{aligned} \quad (\text{a.2.8.2})$$

The second column yields:

$$\begin{aligned} \pi_1r + \pi_2q + \pi_3p &= \pi_2 \\ \Leftrightarrow \pi_1r + \pi_3p &= \pi_2(1-q) \\ \Leftrightarrow \pi_1r + \pi_3p &= \pi_1(r/p)(1-q) \\ \Leftrightarrow \pi_1 + \pi_3(p/r) &= \pi_1(1-q)/p \\ \Leftrightarrow \pi_3(p/r) &= \pi_1(1-q-p)/p \\ \Leftrightarrow \pi_3(p/r) &= \pi_1(r/p) \\ \Leftrightarrow \pi_3(p/r) &= \pi_2 \\ \Leftrightarrow \pi_3 &= (r/p)\pi_2 \\ \Leftrightarrow \pi_3 &= (r/p)^2\pi_1 \end{aligned} \quad (\text{a.2.8.3})$$

The third column yields:

$$\begin{aligned}
& \pi_2 r + \pi_3 q + \pi_4 p = \pi_3 \\
\Leftrightarrow & \pi_2 r + \pi_4 p = \pi_3(1 - q) \\
\Leftrightarrow & \pi_2 r + \pi_4 p = \pi_2(r/p)(1 - q) \\
\Leftrightarrow & \pi_2 + \pi_4(p/r) = \pi_2(1 - q)/p \\
\Leftrightarrow & \pi_4(p/r) = \pi_2(1 - q - p)/p \\
\Leftrightarrow & \pi_4(p/r) = \pi_2(r/p) \\
\Leftrightarrow & \pi_4(p/r) = \pi_3 \\
\Leftrightarrow & \pi_4 = (r/p)\pi_3 \\
\Leftrightarrow & \pi_4 = (r/p)^2\pi_2 \\
\Leftrightarrow & \pi_4 = (r/p)^3\pi_1
\end{aligned} \tag{a.2.8.4}$$

Continuing this way, one obtains in general that $\pi_j = (r/p)^{j-1}\pi_1$. If an invariant distribution exists, we must have $\pi_1 + \pi_2 + \pi_3 + \dots = 1$. Hence:

$$\begin{aligned}
& \pi_1 + \pi_2 + \pi_3 + \dots = 1 \\
\Leftrightarrow & \pi_1 + (r/p)\pi_1 + (r/p)^2\pi_1 + \dots = 1 \\
\Leftrightarrow & \pi_1(1 + (r/p) + (r/p)^2 + \dots) = 1 \\
\Leftrightarrow & \pi_1 \frac{1}{1 - r/p} = 1 \\
\Leftrightarrow & \pi_1 = 1 - r/p
\end{aligned} \tag{a.2.8.5}$$

derivations for equation (2.8.16)

Start from the definition and rearrange:

$$\begin{aligned}
\pi(y_t) &= \int \pi(y_{t-1}) p(y_{t-1}, y_t) dy_{t-1} \\
&\propto \int \exp\left(-\frac{1}{2} \frac{(y_{t-1} - \mu)^2}{\sigma}\right) \exp\left(-\frac{1}{2} \frac{(y_t - c - \gamma y_{t-1})^2}{s}\right) dy_{t-1} \\
&= \int \exp\left(-\frac{1}{2} \left[\frac{(y_{t-1} - \mu)^2}{\sigma} + \frac{(y_t - c - \gamma y_{t-1})^2}{s} \right]\right) dy_{t-1} \\
&= \int \exp\left(-\frac{1}{2} \left[\frac{(y_{t-1} - \mu)^2(1 - \gamma^2)}{s} + \frac{(y_t - c - \gamma y_{t-1})^2}{s} \right]\right) dy_{t-1} \\
&= \int \exp\left(-\frac{1}{2s} [(y_{t-1} - \mu)^2(1 - \gamma^2) + (y_t - \mu(1 - \gamma) - \gamma y_{t-1})^2]\right) dy_{t-1}
\end{aligned} \tag{a.2.8.6}$$

Consider the term within the square brackets:

$$\begin{aligned}
& (y_{t-1} - \mu)^2(1 - \gamma^2) + (y_t - \mu(1 - \gamma) - \gamma y_{t-1})^2 \\
&= (1 - \gamma^2)y_{t-1}^2 + (1 - \gamma^2)\mu^2 - 2(1 - \gamma^2)\mu y_{t-1} + y_t^2 + \mu^2(1 - \gamma)^2 + \gamma^2 y_{t-1}^2 \\
&\quad - 2\mu(1 - \gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma(1 - \gamma)y_{t-1} \\
&= y_{t-1}^2 + (1 - \gamma^2)\mu^2 - 2(1 - \gamma^2)\mu y_{t-1} + y_t^2 + \mu^2(1 - \gamma)^2 \\
&\quad - 2\mu(1 - \gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma(1 - \gamma)y_{t-1} \\
&= y_{t-1}^2 + (1 - \gamma^2)\mu^2 - 2\mu y_{t-1} + 2\gamma^2 \mu y_{t-1} + y_t^2 + \mu^2(1 - \gamma)^2 \\
&\quad - 2\mu(1 - \gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma y_{t-1} - 2\mu\gamma^2 y_{t-1}
\end{aligned} \tag{a.2.8.7}$$

$$\begin{aligned}
&= y_{t-1}^2 + (1 - \gamma^2)\mu^2 - 2\mu y_{t-1} + y_t^2 + \mu^2(1 - \gamma)^2 - 2\mu(1 - \gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma y_{t-1} \\
&= y_{t-1}^2 + (1 - \gamma^2)\mu^2 - 2\mu y_{t-1}(1 - \gamma) + y_t^2 + \mu^2(1 - \gamma)^2 - 2\mu(1 - \gamma)y_t - 2\gamma y_t y_{t-1} \\
&= y_{t-1}^2 + (1 - \gamma^2)\mu^2 - 2\mu y_{t-1}(1 - \gamma) + y_t^2 + \mu^2(1 - \gamma)^2 - 2\mu(1 - \gamma^2)y_t + 2\mu(1 - \gamma)\gamma y_t - 2\gamma y_t y_{t-1} \\
&= y_{t-1}^2 + (1 - \gamma^2)\mu^2 - 2\mu y_{t-1}(1 - \gamma) + (1 - \gamma^2)y_t^2 + \gamma^2 y_t^2 + \mu^2(1 - \gamma)^2 \\
&\quad - 2\mu(1 - \gamma^2)y_t + 2\mu(1 - \gamma)\gamma y_t - 2\gamma y_t y_{t-1} \\
&= (1 - \gamma^2)y_t^2 + (1 - \gamma^2)\mu^2 - 2\mu(1 - \gamma^2)y_t \\
&\quad + y_{t-1}^2 + \mu^2(1 - \gamma)^2 + \gamma^2 y_t^2 - 2\gamma y_t y_{t-1} - 2\mu y_{t-1}(1 - \gamma) + 2\mu(1 - \gamma)\gamma y_t \\
&= (1 - \gamma^2)y_t^2 + (1 - \gamma^2)\mu^2 - 2\mu(1 - \gamma^2)y_t \\
&\quad + y_{t-1}^2 + c^2 + \gamma^2 y_t^2 - 2\gamma y_t y_{t-1} - 2c y_{t-1} + 2c\gamma y_t \\
&= (1 - \gamma^2)(y_t - \mu)^2 + (y_{t-1} - c - \gamma y_t)^2 \tag{a.2.8.8}
\end{aligned}$$

Substitute back in (a.2.8.6):

$$\pi(y_t) = \int \exp\left(-\frac{1}{2s} [(1 - \gamma^2)(y_t - \mu)^2 + (y_{t-1} - c - \gamma y_t)^2]\right) dy_{t-1} \tag{a.2.8.9}$$

and this eventually reformulates as:

$$\pi(y_t) = \exp\left(-\frac{1}{2} \frac{(y_t - \mu)^2}{\sigma}\right) \int \exp\left(-\frac{1}{2} \frac{(y_{t-1} - c - \gamma y_t)^2}{s}\right) dy_{t-1} \tag{a.2.8.10}$$

PART III

Econometrics

The linear regression model

derivations for equation (3.9.7)

Consider first β . To do so, rewrite the likelihood function as:

$$\log(f(y|\beta, \sigma)) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma) - \frac{1}{2\sigma} (y'y + \beta'X'X\beta - 2\beta'X'y) \quad (\text{a.3.9.1})$$

Then solve for the partial derivative:

$$\begin{aligned} \frac{\partial \log(f(y|\beta, \sigma))}{\partial \beta} &= 0 \\ \Leftrightarrow -\frac{1}{2\sigma} (2\beta'X'X - 2y'X) &= 0 \\ \Leftrightarrow \beta'X'X - y'X &= 0 \\ \Leftrightarrow \beta'X'X &= y'X \\ \Leftrightarrow X'X\beta &= X'y \\ \Leftrightarrow \beta &= (X'X)^{-1}X'y \end{aligned} \quad (\text{a.3.9.2})$$

Hence the estimate is $\hat{\beta} = (X'X)^{-1}X'y$. Consider now σ . Solve for the partial derivative:

$$\begin{aligned} \frac{\partial \log(f(y|\beta, \sigma))}{\partial \sigma} &= 0 \\ \Leftrightarrow -\frac{n}{2} \frac{1}{\sigma} + \frac{1}{2} \frac{(y - X\beta)'(y - X\beta)}{\sigma^2} &= 0 \\ \Leftrightarrow -n + \frac{(y - X\beta)'(y - X\beta)}{\sigma} &= 0 \\ \Leftrightarrow \frac{(y - X\beta)'(y - X\beta)}{\sigma} &= n \\ \Leftrightarrow \sigma &= \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{n} \end{aligned} \quad (\text{a.3.9.3})$$

This expression gives the optimum for any value of β . To obtain a global maximum we must choose the value of β that maximizes the likelihood, namely $\hat{\beta}$. Therefore, the estimate for σ is given by $\hat{\sigma} = (y - X\hat{\beta})'(y - X\hat{\beta})/n$.

derivations for equation (3.9.12)

Develop and group:

$$\begin{aligned} &\exp\left(-\frac{1}{2} \frac{(y - X\beta)'(y - X\beta)}{\sigma}\right) \times \exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right) \\ &= \exp\left(-\frac{1}{2} [(y - X\beta)' \sigma^{-1} (y - X\beta) + (\beta - b)'V^{-1}(\beta - b)]\right) \end{aligned} \quad (\text{a.3.9.4})$$

Consider the term in square brackets:

$$\begin{aligned}
& (y - X\beta)' \sigma^{-1} (y - X\beta) + (\beta - b)' V^{-1} (\beta - b) \\
&= y' \sigma^{-1} y + \beta' X' \sigma^{-1} X \beta - 2\beta' X' \sigma^{-1} y + \beta' V^{-1} \beta + b' V^{-1} b - 2\beta' V^{-1} b \\
&= \beta' (V^{-1} + \sigma^{-1} X' X) \beta - 2\beta' (V^{-1} b + \sigma^{-1} X' y) + b' V^{-1} b + y' \sigma^{-1} y
\end{aligned} \tag{a.3.9.5}$$

Substitute back in (a.3.9.4):

$$= \exp \left(-\frac{1}{2} [\beta' (V^{-1} + \sigma^{-1} X' X) \beta - 2\beta' (V^{-1} b + \sigma^{-1} X' y) + b' V^{-1} b + y' \sigma^{-1} y] \right) \tag{a.3.9.6}$$

derivations for equation (3.9.23)

Group and rearrange:

$$\begin{aligned}
& \pi(\beta, \sigma | y) \\
&\propto \sigma^{-n/2} \exp \left(-\frac{1}{2} \frac{(y - X\beta)' (y - X\beta)}{\sigma} \right) \times |\sigma V|^{-1/2} \exp \left(-\frac{1}{2} (\beta - b)' (\sigma V)^{-1} (\beta - b) \right) \\
&\times \sigma^{-\alpha/2-1} \exp \left(-\frac{\delta}{2\sigma} \right) \\
&= \sigma^{-k/2} \exp \left(-\frac{1}{2\sigma} [(y - X\beta)' (y - X\beta) + (\beta - b)' V^{-1} (\beta - b)] \right) \sigma^{-(\alpha+n)/2-1} \exp \left(-\frac{\delta}{2\sigma} \right)
\end{aligned} \tag{a.3.9.7}$$

Define:

$$\bar{\alpha} = \alpha + n \tag{a.3.9.8}$$

Then:

$$= \sigma^{-k/2} \exp \left(-\frac{1}{2\sigma} [(y - X\beta)' (y - X\beta) + (\beta - b)' V^{-1} (\beta - b)] \right) \sigma^{-\bar{\alpha}/2-1} \exp \left(-\frac{\delta}{2\sigma} \right) \tag{a.3.9.9}$$

Consider the term between the square brackets:

$$\begin{aligned}
& (y - X\beta)' (y - X\beta) + (\beta - b)' V^{-1} (\beta - b) \\
&= y' y + \beta' X' X \beta - 2\beta' X' y + \beta' V^{-1} \beta + b' V^{-1} b - 2\beta' V^{-1} b \\
&= \beta' (V^{-1} + X' X) \beta - 2\beta' (V^{-1} b + X' y) + y' y + b' V^{-1} b \\
&= \beta' (V^{-1} + X' X) \beta - 2\beta' \bar{V}^{-1} \bar{V} (V^{-1} b + X' y) + y' y + b' V^{-1} b + \bar{b}' \bar{V}^{-1} \bar{b} - \bar{b}' \bar{V}^{-1} \bar{b}
\end{aligned} \tag{a.3.9.10}$$

Define:

$$\bar{V} = (V^{-1} + X' X)^{-1} \quad \bar{b} = \bar{V} (V^{-1} b + X' y) \tag{a.3.9.11}$$

Then (a.3.9.10) becomes:

$$\begin{aligned}
&= \beta' \bar{V}^{-1} \beta - 2\beta' \bar{V}^{-1} \bar{b} + y' y + b' V^{-1} b + \bar{b}' \bar{V}^{-1} \bar{b} - \bar{b}' \bar{V}^{-1} \bar{b} \\
&= (\beta' \bar{V}^{-1} \beta - 2\beta' \bar{V}^{-1} \bar{b} + \bar{b}' \bar{V}^{-1} \bar{b}) + y' y + b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} \\
&= (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + y' y + b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b}
\end{aligned} \tag{a.3.9.12}$$

Substitute back in (a.3.9.9):

$$= \sigma^{-k/2} \exp \left(-\frac{1}{2} (\beta - \bar{b})' (\sigma \bar{V})^{-1} (\beta - \bar{b}) \right) \sigma^{-\bar{\alpha}/2-1} \exp \left(-\frac{\delta + y' y + b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b}}{2\sigma} \right) \tag{a.3.9.13}$$

Define:

$$\bar{\delta} = \delta + y'y + b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} \quad (\text{a.3.9.14})$$

Then (a.3.9.13) eventually rewrites:

$$\pi(\beta, \sigma|y) \propto \sigma^{-k/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'(\sigma\bar{V})^{-1}(\beta - \bar{b})\right) \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right) \quad (\text{a.3.9.15})$$

derivations for equation (3.9.28)

Rearrange:

$$\begin{aligned} & \Gamma\left(\frac{\bar{\alpha}+k}{2}\right) \left(\frac{\bar{\delta} + (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})}{2}\right)^{-\frac{\bar{\alpha}+k}{2}} \\ & \propto \left(\frac{\bar{\delta} + (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})}{2}\right)^{-\frac{\bar{\alpha}+k}{2}} \\ & \propto (\bar{\delta} + (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b}))^{-\frac{\bar{\alpha}+k}{2}} \\ & \propto (1 + (\beta - \bar{b})'(\bar{\delta}\bar{V})^{-1}(\beta - \bar{b}))^{-\frac{\bar{\alpha}+k}{2}} \\ & \propto \left(1 + \frac{1}{\bar{\alpha}}(\beta - \bar{b})'(\bar{\delta}\bar{V}/\bar{\alpha})^{-1}(\beta - \bar{b})\right)^{-\frac{\bar{\alpha}+k}{2}} \end{aligned} \quad (\text{a.3.9.16})$$

derivations for equation (3.9.38)

Rearrange the likelihood function:

$$\begin{aligned} & f(y|\beta, \sigma) \\ &= (2\pi)^{-n/2} |\sigma W|^{-1/2} \exp\left(-\frac{1}{2}(y - X\beta)'(\sigma W)^{-1}(y - X\beta)\right) \\ &= (2\pi\sigma)^{-n/2} |W|^{-1/2} \exp\left(-\frac{1}{2}\frac{(y - X\beta)'W^{-1}(y - X\beta)}{\sigma}\right) \end{aligned} \quad (\text{a.3.9.17})$$

This reformulates further as:

$$\begin{aligned} &= (2\pi\sigma)^{-n/2} \left(\prod_{i=1}^n w_i^{-1/2}\right) \exp\left(-\frac{1}{2}\frac{(y - X\beta)' \text{diag}(\exp(-Z\gamma)) (y - X\beta)}{\sigma}\right) \\ &= (2\pi\sigma)^{-n/2} \left(\prod_{i=1}^n \exp(z_i'\gamma)\right)^{-1/2} \exp\left(-\frac{1}{2}\frac{(y - X\beta)' \text{diag}(\exp(-Z\gamma)) (y - X\beta)}{\sigma}\right) \\ &= (2\pi\sigma)^{-n/2} \left(\exp\left(\sum_{i=1}^n z_i'\gamma\right)\right)^{-1/2} \exp\left(-\frac{1}{2}\frac{(y - X\beta)' \text{diag}(\exp(-Z\gamma)) (y - X\beta)}{\sigma}\right) \\ &= (2\pi\sigma)^{-n/2} (\exp(1_n'Z\gamma))^{-1/2} \exp\left(-\frac{1}{2}\frac{(y - X\beta)' \text{diag}(\exp(-Z\gamma)) (y - X\beta)}{\sigma}\right) \\ &= (2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2}1_n'Z\gamma\right) \exp\left(-\frac{1}{2}\frac{(y - X\beta)' \text{diag}(\exp(-Z\gamma)) (y - X\beta)}{\sigma}\right) \\ &= (2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2}\left[1_n'Z\gamma + (y - X\beta)' \text{diag}(\exp(-Z\gamma)) (y - X\beta)/\sigma\right]\right) \end{aligned} \quad (\text{a.3.9.18})$$

derivations for equation (3.9.44)

Rearrange the terms:

$$\begin{aligned}
& \pi(\beta|y, \sigma, w) \\
&= \exp\left(-\frac{1}{2} \frac{(y - X\beta)'W^{-1}(y - X\beta)}{\sigma}\right) \times \exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right) \\
&= \exp\left(-\frac{1}{2}[(y - X\beta)'(\sigma W)^{-1}(y - X\beta) + (\beta - b)'V^{-1}(\beta - b)]\right) \tag{a.3.9.19}
\end{aligned}$$

Consider the term in square brackets and complete the squares:

$$\begin{aligned}
& (y - X\beta)'(\sigma W)^{-1}(y - X\beta) + (\beta - b)'V^{-1}(\beta - b) \\
&= y'(\sigma W)^{-1}y + \beta'X'(\sigma W)^{-1}X\beta - 2\beta'X'(\sigma W)^{-1}y + \beta'V^{-1}\beta + b'V^{-1}b - 2\beta'V^{-1}b \\
&= \beta'(V^{-1} + \sigma^{-1}X'W^{-1}X)\beta - 2\beta'(V^{-1}b + \sigma^{-1}X'W^{-1}y) + y'(\sigma W)^{-1}y + b'V^{-1}b \\
&= \beta'(V^{-1} + \sigma^{-1}X'W^{-1}X)\beta - 2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b + \sigma^{-1}X'W^{-1}y) \\
&\quad + y'(\sigma W)^{-1}y + b'V^{-1}b + \bar{b}'\bar{V}^{-1}\bar{b} - \bar{b}'\bar{V}^{-1}\bar{b} \tag{a.3.9.20}
\end{aligned}$$

Define:

$$\bar{V} = (V^{-1} + \sigma^{-1}X'W^{-1}X)^{-1} \quad \bar{b} = \bar{V}(V^{-1}b + \sigma^{-1}X'W^{-1}y) \tag{a.3.9.21}$$

Then (a.3.9.20) rewrites:

$$\begin{aligned}
&= \beta'\bar{V}^{-1}\beta - 2\beta'\bar{V}^{-1}\bar{b} + \bar{b}'\bar{V}^{-1}\bar{b} + y'(\sigma W)^{-1}y + b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} \\
&= (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b}) + y'(\sigma W)^{-1}y + b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} \tag{a.3.9.22}
\end{aligned}$$

Substitute back in (a.3.9.19):

$$\begin{aligned}
& \pi(\beta|y, \sigma, w) \\
&= \exp\left(-\frac{1}{2}[(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b}) + y'(\sigma W)^{-1}y + b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b}]\right) \\
&= \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right) \exp\left(-\frac{1}{2}[y'(\sigma W)^{-1}y + b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b}]\right) \\
&\propto \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right) \tag{a.3.9.23}
\end{aligned}$$

derivations for equation (3.9.69)

Rearrange the terms:

$$\begin{aligned}
& \pi(\phi|y, \beta, \sigma) \\
&\propto \exp\left(-\frac{1}{2} \frac{(\varepsilon - E\phi)'(\varepsilon - E\phi)}{\sigma}\right) \times \exp\left(-\frac{1}{2}(\phi - p)'H^{-1}(\phi - p)\right) \\
&= \exp\left(-\frac{1}{2}[(\varepsilon - E\phi)'\sigma^{-1}(\varepsilon - E\phi) + (\phi - p)'H^{-1}(\phi - p)]\right) \tag{a.3.9.24}
\end{aligned}$$

Consider the term in square brackets and complete the squares:

$$\begin{aligned}
& (\varepsilon - E\phi)' \sigma^{-1} (\varepsilon - E\phi) + (\phi - p)' H^{-1} (\phi - p) \\
&= \varepsilon' \sigma^{-1} \varepsilon + \phi' E' \sigma^{-1} E \phi - 2\phi' E' \sigma^{-1} \varepsilon + \phi' H^{-1} \phi + p' H^{-1} p - 2\phi' H^{-1} p \\
&= \phi' (H^{-1} + \sigma^{-1} E' E) \phi - 2\phi' (H^{-1} p + \sigma^{-1} E' \varepsilon) + \varepsilon' \sigma^{-1} \varepsilon + p' H^{-1} p \\
&= \phi' (H^{-1} + \sigma^{-1} E' E) \phi - 2\phi' \bar{H}^{-1} \bar{H} (H^{-1} p + \sigma^{-1} E' \varepsilon) + \varepsilon' \sigma^{-1} \varepsilon + p' H^{-1} p + \bar{p}' \bar{H}^{-1} \bar{p} - \bar{p}' \bar{H}^{-1} \bar{p}
\end{aligned} \tag{a.3.9.25}$$

Define:

$$\bar{H} = (H^{-1} + \sigma^{-1} E' E)^{-1} \quad \bar{p} = \bar{H} (H^{-1} p + \sigma^{-1} E' \varepsilon) \tag{a.3.9.26}$$

Then (a.3.9.25) becomes:

$$\begin{aligned}
&= \phi' \bar{H}^{-1} \phi - 2\phi' \bar{H}^{-1} \bar{p} + \bar{p}' \bar{H}^{-1} \bar{p} + \varepsilon' \sigma^{-1} \varepsilon + p' H^{-1} p - \bar{p}' \bar{H}^{-1} \bar{p} \\
&= (\phi - \bar{p})' \bar{H}^{-1} (\phi - \bar{p}) + \varepsilon' \sigma^{-1} \varepsilon + p' H^{-1} p - \bar{p}' \bar{H}^{-1} \bar{p}
\end{aligned} \tag{a.3.9.27}$$

Substitute back in (a.3.9.24) to obtain:

$$\begin{aligned}
& \pi(\phi|y, \beta, \sigma) \\
&= \exp \left(-\frac{1}{2} [(\phi - \bar{p})' \bar{H}^{-1} (\phi - \bar{p}) + \varepsilon' \sigma^{-1} \varepsilon + p' H^{-1} p - \bar{p}' \bar{H}^{-1} \bar{p}] \right) \\
&= \exp \left(-\frac{1}{2} (\phi - \bar{p})' \bar{H}^{-1} (\phi - \bar{p}) \right) \exp \left(-\frac{1}{2} [\varepsilon' \sigma^{-1} \varepsilon + p' H^{-1} p - \bar{p}' \bar{H}^{-1} \bar{p}] \right) \\
&\propto \exp \left(-\frac{1}{2} (\phi - \bar{p})' \bar{H}^{-1} (\phi - \bar{p}) \right)
\end{aligned} \tag{a.3.9.28}$$

Applications with the linear regression model

derivations for equation (3.10.4)

Rearrange the expression:

$$\begin{aligned}
 & f(\hat{y}|y) \\
 & \propto \int \exp\left(-\frac{1}{2} \frac{(\hat{y} - \hat{X}\beta)'(\hat{y} - \hat{X}\beta)}{\sigma}\right) \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right) d\beta \\
 & = \int \exp\left(-\frac{1}{2} [\sigma^{-1}(\hat{y} - \hat{X}\beta)'(\hat{y} - \hat{X}\beta) + (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})]\right) d\beta \tag{a.3.10.1}
 \end{aligned}$$

Consider the term in square brackets:

$$\begin{aligned}
 & \sigma^{-1}(\hat{y} - \hat{X}\beta)'(\hat{y} - \hat{X}\beta) + (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b}) \\
 & = \sigma^{-1}\hat{y}'\hat{y} + \sigma^{-1}\beta'\hat{X}'\hat{X}\beta - 2\sigma^{-1}\beta'\hat{X}'\hat{y} + \beta'\bar{V}^{-1}\beta + \bar{b}'\bar{V}^{-1}\bar{b} - 2\beta'\bar{V}^{-1}\bar{b} \\
 & = \beta'(\bar{V}^{-1} + \sigma^{-1}\hat{X}'\hat{X})\beta - 2\beta'(\bar{V}^{-1}\bar{b} + \sigma^{-1}\hat{X}'\hat{y}) + \sigma^{-1}\hat{y}'\hat{y} + \bar{b}'\bar{V}^{-1}\bar{b} \\
 & = \beta'(\bar{V}^{-1} + \sigma^{-1}\hat{X}'\hat{X})\beta - 2\beta'\hat{V}^{-1}\hat{V}(\bar{V}^{-1}\bar{b} + \sigma^{-1}\hat{X}'\hat{y}) + \sigma^{-1}\hat{y}'\hat{y} + \bar{b}'\bar{V}^{-1}\bar{b} + \hat{b}'\hat{V}^{-1}\hat{b} - \hat{b}'\hat{V}^{-1}\hat{b} \tag{a.3.10.2}
 \end{aligned}$$

Define:

$$\hat{V} = (\bar{V}^{-1} + \sigma^{-1}\hat{X}'\hat{X})^{-1} \quad \hat{b} = \hat{V}(\bar{V}^{-1}\bar{b} + \sigma^{-1}\hat{X}'\hat{y}) \tag{a.3.10.3}$$

Then (a.3.10.2) becomes:

$$\begin{aligned}
 & = \beta'\hat{V}^{-1}\beta - 2\beta'\hat{V}^{-1}\hat{b} + \hat{b}'\hat{V}^{-1}\hat{b} + \sigma^{-1}\hat{y}'\hat{y} + \bar{b}'\bar{V}^{-1}\bar{b} - \hat{b}'\hat{V}^{-1}\hat{b} \\
 & = (\beta - \hat{b})'\hat{V}^{-1}(\beta - \hat{b}) + \sigma^{-1}\hat{y}'\hat{y} + \bar{b}'\bar{V}^{-1}\bar{b} - \hat{b}'\hat{V}^{-1}\hat{b} \tag{a.3.10.4}
 \end{aligned}$$

Substituting back in (a.3.10.1):

$$\begin{aligned}
 & f(\hat{y}|y) \\
 & \propto \int \exp\left(-\frac{1}{2} [(\beta - \hat{b})'\hat{V}^{-1}(\beta - \hat{b}) + \sigma^{-1}\hat{y}'\hat{y} + \bar{b}'\bar{V}^{-1}\bar{b} - \hat{b}'\hat{V}^{-1}\hat{b}]\right) d\beta \\
 & = \int \exp\left(-\frac{1}{2}(\beta - \hat{b})'\hat{V}^{-1}(\beta - \hat{b})\right) \exp\left(-\frac{1}{2} [\sigma^{-1}\hat{y}'\hat{y} + \bar{b}'\bar{V}^{-1}\bar{b} - \hat{b}'\hat{V}^{-1}\hat{b}]\right) d\beta \\
 & = \exp\left(-\frac{1}{2} [\sigma^{-1}\hat{y}'\hat{y} + \bar{b}'\bar{V}^{-1}\bar{b} - \hat{b}'\hat{V}^{-1}\hat{b}]\right) \int \exp\left(-\frac{1}{2}(\beta - \hat{b})'\hat{V}^{-1}(\beta - \hat{b})\right) d\beta \\
 & \propto \exp\left(-\frac{1}{2} [\sigma^{-1}\hat{y}'\hat{y} + \bar{b}'\bar{V}^{-1}\bar{b} - \hat{b}'\hat{V}^{-1}\hat{b}]\right) \tag{a.3.10.5}
 \end{aligned}$$

Consider the term in square brackets:

$$\begin{aligned}
& \sigma^{-1} \hat{y}' \hat{y} + \bar{b}' \bar{V}^{-1} \bar{b} - \hat{b}' \hat{V}^{-1} \hat{b} \\
&= \sigma^{-1} \hat{y}' \hat{y} + \bar{b}' \bar{V}^{-1} \bar{b} - (\bar{V}^{-1} \bar{b} + \sigma^{-1} \hat{X}' \hat{y})' \hat{V} \hat{V}^{-1} \hat{V} (\bar{V}^{-1} \bar{b} + \sigma^{-1} \hat{X}' \hat{y}) \\
&= \sigma^{-1} \hat{y}' \hat{y} + \bar{b}' \bar{V}^{-1} \bar{b} - (\bar{V}^{-1} \bar{b} + \sigma^{-1} \hat{X}' \hat{y})' \hat{V} (\bar{V}^{-1} \bar{b} + \sigma^{-1} \hat{X}' \hat{y}) \\
&= \sigma^{-1} \hat{y}' \hat{y} + \bar{b}' \bar{V}^{-1} \bar{b} - \bar{b}' \bar{V}^{-1} \hat{V} \bar{V}^{-1} \bar{b} - \sigma^{-2} \hat{y}' \hat{X} \hat{V} \hat{X}' \hat{y} - 2 \sigma^{-1} \hat{y}' \hat{X} \hat{V} \bar{V}^{-1} \bar{b} \\
&= \hat{y}' (\sigma^{-1} I_m - \sigma^{-2} \hat{X} \hat{V} \hat{X}') \hat{y} - \bar{b}' (\bar{V}^{-1} - \bar{V}^{-1} \hat{V} \bar{V}^{-1}) \bar{b} - 2 \sigma^{-1} \hat{y}' \hat{X} \hat{V} \bar{V}^{-1} \bar{b}
\end{aligned} \tag{a.3.10.6}$$

In what follows, we make use of property m.13 (the Sherman-Woodbury-Morrison identity):
 $(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}$.

Consider the central part of the second term in (a.3.10.6). Rearrange and use the identity twice to obtain:

$$\begin{aligned}
& \bar{V}^{-1} - \bar{V}^{-1} \hat{V} \bar{V}^{-1} \\
&= \bar{V}^{-1} - \bar{V}^{-1} (\bar{V}^{-1} + \sigma^{-1} \hat{X}' \hat{X})^{-1} \bar{V}^{-1} \\
&= (\bar{V} + \sigma (\hat{X}' \hat{X})^{-1})^{-1} \\
&= \sigma^{-1} \hat{X}' \hat{X} - \sigma^{-1} \hat{X}' \hat{X} (\sigma^{-1} \hat{X}' \hat{X} + \bar{V}^{-1})^{-1} \sigma^{-1} \hat{X}' \hat{X} \\
&= \sigma^{-1} \hat{X}' \hat{X} - \sigma^{-2} \hat{X}' \hat{X} \hat{V} \hat{X}' \hat{X} \\
&= \hat{X}' (\sigma^{-1} I_m - \sigma^{-2} \hat{X} \hat{V} \hat{X}') \hat{X}
\end{aligned} \tag{a.3.10.7}$$

Consider finally the central part of the third term in (a.3.10.6). We note that $\hat{V}(\bar{V}^{-1} + \sigma^{-1} \hat{X}' \hat{X}) = I_m$ so $\hat{V} \bar{V}^{-1} = I_m - \hat{V} \sigma^{-1} \hat{X}' \hat{X}$. Following:

$$\hat{X} \hat{V} \bar{V}^{-1} = \hat{X} - \hat{X} \hat{V} \sigma^{-1} \hat{X}' \hat{X} = (I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}') \hat{X} \tag{a.3.10.8}$$

Substitute (a.3.10.7) and (a.3.10.8) back in (a.3.10.6) to obtain:

$$\begin{aligned}
&= \hat{y}' (\sigma^{-1} I_m - \sigma^{-2} \hat{X} \hat{V} \hat{X}') \hat{y} - \bar{b}' \hat{X}' (\sigma^{-1} I_m - \sigma^{-2} \hat{X} \hat{V} \hat{X}') \hat{X} \bar{b} - 2 \sigma^{-1} \hat{y}' (I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}') \hat{X} \bar{b} \\
&= \sigma^{-1} \hat{y}' (I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}') \hat{y} - \sigma^{-1} \bar{b}' \hat{X}' (I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}') \hat{X} \bar{b} - 2 \sigma^{-1} \hat{y}' (I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}') \hat{X} \bar{b} \\
&= \sigma^{-1} [\hat{y}' (I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}') \hat{y} - \bar{b}' \hat{X}' (I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}') \hat{X} \bar{b} - 2 \hat{y}' (I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}') \hat{X} \bar{b}] \\
&= \sigma^{-1} (\hat{y} - \hat{X} \bar{b})' (I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}') (\hat{y} - \hat{X} \bar{b})
\end{aligned} \tag{a.3.10.9}$$

We use one last time the Sherman-Woodbury-Morrison identity on the central term to obtain:

$$I_m - \sigma^{-1} \hat{X} \hat{V} \hat{X}' = I_m - \sigma^{-1} \hat{X} (\bar{V}^{-1} + \sigma^{-1} \hat{X}' \hat{X})^{-1} \hat{X}' = (I_m + \sigma^{-1} \hat{X} \bar{V} \hat{X}')^{-1} \tag{a.3.10.10}$$

Substituting back in (a.3.10.9):

$$\begin{aligned}
&= \sigma^{-1} (\hat{y} - \hat{X} \bar{b})' (I_m + \sigma^{-1} \hat{X} \bar{V} \hat{X}')^{-1} (\hat{y} - \hat{X} \bar{b}) \\
&= (\hat{y} - \hat{X} \bar{b})' (\sigma I_m + \hat{X} \bar{V} \hat{X}')^{-1} (\hat{y} - \hat{X} \bar{b})
\end{aligned} \tag{a.3.10.11}$$

Eventually substituting back in (a.3.10.5), we conclude:

$$f(\hat{y}|y) \propto \exp \left(-\frac{1}{2} (\hat{y} - \hat{X} \bar{b})' (\sigma I_m + \hat{X} \bar{V} \hat{X}')^{-1} (\hat{y} - \hat{X} \bar{b}) \right) \tag{a.3.10.12}$$

derivations for equation (3.10.6)

Rearrange the expression:

$$\begin{aligned}
& f(\hat{y}|y) \\
& \propto \int \int \sigma^{-m/2} \exp\left(-\frac{1}{2} \frac{(\hat{y} - \hat{X}\beta)'(\hat{y} - \hat{X}\beta)}{\sigma}\right) \exp\left(-\frac{1}{2} \frac{(y - X\beta)'(y - X\beta)}{\sigma}\right) \\
& \times \sigma^{-k/2} \exp\left(-\frac{1}{2}(\beta - b)'(\sigma V)^{-1}(\beta - b)\right) \times \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right) \\
& = \int \int \sigma^{-k/2} \exp\left(-\frac{1}{2\sigma} [(\hat{y} - \hat{X}\beta)'(\hat{y} - \hat{X}\beta) + (y - X\beta)'(y - X\beta) + (\beta - b)'(\sigma V)^{-1}(\beta - b) + \delta]\right) \\
& \times \sigma^{-(\alpha+n+m)/2-1} d\beta d\sigma \tag{a.3.10.13}
\end{aligned}$$

Consider the term in the square bracket:

$$\begin{aligned}
& (\hat{y} - \hat{X}\beta)'(\hat{y} - \hat{X}\beta) + (y - X\beta)'(y - X\beta) + (\beta - b)'(\sigma V)^{-1}(\beta - b) + \delta \\
& = \hat{y}'\hat{y} + \beta'\hat{X}'\hat{X}\beta - 2\beta'\hat{X}'\hat{y} + y'y + \beta'X'X\beta - 2\beta'X'y + \beta'V^{-1}\beta + b'V^{-1}b - 2\beta'V^{-1}b + \delta \\
& = (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b) + \beta'(V^{-1} + X'X + \hat{X}'\hat{X})\beta - 2\beta'(V^{-1}b + X'y + \hat{X}'\hat{y}) \\
& = (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b) + \beta'(V^{-1} + X'X + \hat{X}'\hat{X})\beta - 2\beta'\hat{V}^{-1}\hat{V}(V^{-1}b + X'y + \hat{X}'\hat{y}) + \hat{b}'\hat{V}^{-1}\hat{b} - \hat{b}'\hat{V}^{-1}\hat{b} \\
& = (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - \hat{b}'\hat{V}^{-1}\hat{b}) + \beta'(V^{-1} + X'X + \hat{X}'\hat{X})\beta - 2\beta'\hat{V}^{-1}\hat{V}(V^{-1}b + X'y + \hat{X}'\hat{y}) + \hat{b}'\hat{V}^{-1}\hat{b} \tag{a.3.10.14}
\end{aligned}$$

Define:

$$\hat{\delta} = \delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - \hat{b}'\hat{V}^{-1}\hat{b} \quad \hat{V} = (V^{-1} + X'X + \hat{X}'\hat{X})^{-1} \quad \hat{b} = \hat{V}(V^{-1}b + X'y + \hat{X}'\hat{y}) \tag{a.3.10.15}$$

Then (a.3.10.14) rewrites:

$$\begin{aligned}
& = \hat{\delta} + \beta'\hat{V}^{-1}\beta - 2\beta'\hat{V}^{-1}\hat{b} + \hat{b}'\hat{V}^{-1}\hat{b} \\
& = \hat{\delta} + (\beta - \hat{b})'\hat{V}^{-1}(\beta - \hat{b}) \tag{a.3.10.16}
\end{aligned}$$

Substitute back in (a.3.10.13):

$$\begin{aligned}
& f(\hat{y}|y) \\
& \propto \int \int \sigma^{-k/2} \exp\left(-\frac{1}{2\sigma} [\hat{\delta} + (\beta - \hat{b})'\hat{V}^{-1}(\beta - \hat{b})]\right) \sigma^{-(\alpha+n+m)/2-1} d\beta d\sigma \\
& = \int \int \sigma^{-k/2} \exp\left(-\frac{1}{2\sigma} (\beta - \hat{b})'\hat{V}^{-1}(\beta - \hat{b})\right) d\beta \sigma^{-(\alpha+n+m)/2-1} \exp\left(-\frac{\hat{\delta}}{2\sigma}\right) d\sigma \\
& = \int \int \sigma^{-k/2} \exp\left(-\frac{1}{2\sigma} (\beta - \hat{b})'\hat{V}^{-1}(\beta - \hat{b})\right) d\beta \sigma^{-(\bar{\alpha}+m)/2-1} \exp\left(-\frac{\hat{\delta}}{2\sigma}\right) d\sigma \tag{a.3.10.17}
\end{aligned}$$

with:

$$\hat{\alpha} = \alpha + n + m \tag{a.3.10.18}$$

The first term is the kernel of a multivariate normal distribution; integration hence yields a constant:

$$= \int \sigma^{-\hat{\alpha}/2-1} \exp\left(-\frac{\hat{\delta}}{2\sigma}\right) d\sigma \tag{a.3.10.19}$$

The remaining term is the kernel of an inverse gamma distribution; integration thus yields the reciprocal of the normalization constant:

$$\begin{aligned}
&= \Gamma(\hat{\alpha}/2)(\hat{\delta}/2)^{-\hat{\alpha}/2} \\
&\propto (\hat{\delta}/2)^{-\hat{\alpha}/2} \\
&\propto \hat{\delta}^{-\hat{\alpha}/2} \\
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - \hat{b}'\hat{V}^{-1}\hat{b})^{-\hat{\alpha}/2} \\
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - (V^{-1}b + X'y + \hat{X}'\hat{y})'\hat{V}^{-1}\hat{V}(V^{-1}b + X'y + \hat{X}'\hat{y}))^{-\hat{\alpha}/2} \\
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - (V^{-1}b + X'y + \hat{X}'\hat{y})'\hat{V}(V^{-1}b + X'y + \hat{X}'\hat{y}))^{-\hat{\alpha}/2} \tag{a.3.10.20}
\end{aligned}$$

Define:

$$\tilde{b} = V^{-1}b + X'y \tag{a.3.10.21}$$

Then (a.3.10.20) rewrites:

$$\begin{aligned}
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - (\tilde{b} + \hat{X}'\hat{y})'\hat{V}(\tilde{b} + \hat{X}'\hat{y}))^{-\hat{\alpha}/2} \\
&= (\delta + \hat{y}'\hat{y} + y'y + b'V^{-1}b - \tilde{b}'\hat{V}\tilde{b} - \hat{y}'\hat{X}\hat{V}\hat{X}'\hat{y} - 2\hat{y}'\hat{X}\hat{V}\tilde{b})^{-\hat{\alpha}/2} \\
&= (\delta + y'y + b'V^{-1}b + \hat{y}'(I_m - \hat{X}\hat{V}\hat{X}')\hat{y} - \tilde{b}'\hat{V}\tilde{b} - 2\hat{y}'\hat{X}\hat{V}\tilde{b})^{-\hat{\alpha}/2} \\
&= ([\delta + y'y + b'V^{-1}b - \tilde{b}'\hat{V}\tilde{b} - \hat{y}'\hat{V}^{-1}\hat{y}] + \hat{y}'(I_m - \hat{X}\hat{V}\hat{X}')\hat{y} - 2\hat{y}'\hat{V}^{-1}\hat{V}\hat{X}\hat{V}\tilde{b} + \hat{y}'\hat{V}^{-1}\hat{y})^{-\hat{\alpha}/2} \tag{a.3.10.22}
\end{aligned}$$

Define:

$$\ddot{\delta} = \delta + y'y + b'V^{-1}b - \tilde{b}'\hat{V}\tilde{b} - \hat{y}'\hat{V}^{-1}\hat{y} \quad \ddot{V} = (I_m - \hat{X}\hat{V}\hat{X}')^{-1} \quad \ddot{y} = \hat{V}\hat{X}\hat{V}\tilde{b} \tag{a.3.10.23}$$

Then (a.3.10.22) rewrites:

$$\begin{aligned}
&= (\ddot{\delta} + \hat{y}'\hat{V}^{-1}\hat{y} - 2\hat{y}'\hat{V}^{-1}\hat{y} + \hat{y}'\hat{V}^{-1}\hat{y})^{-\hat{\alpha}/2} \\
&= (\ddot{\delta} + (\hat{y} - \ddot{y})'\hat{V}^{-1}(\hat{y} - \ddot{y}))^{-\hat{\alpha}/2} \\
&= \ddot{\delta}^{-\hat{\alpha}/2}(1 + (\hat{y} - \ddot{y})'[\ddot{\delta}\hat{V}]^{-1}(\hat{y} - \ddot{y}))^{-\hat{\alpha}/2} \\
&\propto (1 + (\hat{y} - \ddot{y})'[\ddot{\delta}\hat{V}]^{-1}(\hat{y} - \ddot{y}))^{-\hat{\alpha}/2} \\
&= \left(1 + \frac{1}{\bar{\alpha}}(\hat{y} - \ddot{y})'[\ddot{\delta}\hat{V}/\bar{\alpha}]^{-1}(\hat{y} - \ddot{y})\right)^{-(\bar{\alpha}+m)/2} \tag{a.3.10.24}
\end{aligned}$$

Thus we finally conclude:

$$f(\hat{y}|y) \propto \left(1 + \frac{1}{\bar{\alpha}}(\hat{y} - \ddot{y})'[\ddot{\delta}\hat{V}/\bar{\alpha}]^{-1}(\hat{y} - \ddot{y})\right)^{-(\bar{\alpha}+m)/2} \tag{a.3.10.25}$$

Finally, reformulate the messy terms. First, reformulate \ddot{V} . For this, we make again use of property m.13 (the Sherman-Woodbury-Morrison identity): $(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}$.

Then, starting from (a.3.10.23):

$$\begin{aligned}
&\ddot{V} \\
&= (I_m - \hat{X}\hat{V}\hat{X}')^{-1} \\
&= (I_m - \hat{X}(V^{-1} + X'X + \hat{X}'\hat{X})^{-1}\hat{X}')^{-1} \\
&= I_m + \hat{X}(V^{-1} + X'X)^{-1}\hat{X}' \\
&= I_m + \hat{X}\bar{V}\hat{X}' \tag{a.3.10.26}
\end{aligned}$$

Now consider the term \ddot{y} . Start from:

$$\begin{aligned}
 & \dot{V}\hat{X}\hat{V} \\
 &= (I_m + \hat{X}\bar{V}\hat{X}')\hat{X}\hat{V} \\
 &= \hat{X}\hat{V} + \hat{X}\bar{V}\hat{X}'\hat{X}\hat{V} \\
 &= \hat{X}(\hat{V} + \bar{V}\hat{X}'\hat{X}\hat{V}) \\
 &= \hat{X}(I_m + \bar{V}\hat{X}'\hat{X})\hat{V}
 \end{aligned} \tag{a.3.10.27}$$

We then note that (a.3.10.15) implies:

$$\hat{V} = (V^{-1} + X'X + \hat{X}'\hat{X})^{-1} \Leftrightarrow \hat{V} = (\bar{V}^{-1} + \hat{X}'\hat{X})^{-1} \tag{a.3.10.28}$$

Hence:

$$\begin{aligned}
 &= \hat{X}(I_m + \bar{V}\hat{X}'\hat{X})(\bar{V}^{-1} + \hat{X}'\hat{X})^{-1} \\
 &= \hat{X}\bar{V}(\bar{V}^{-1} + \hat{X}'\hat{X})(\bar{V}^{-1} + \hat{X}'\hat{X})^{-1} \\
 &= \hat{X}\bar{V}
 \end{aligned} \tag{a.3.10.29}$$

Using this result in (a.3.10.23), and combining with definition (a.3.10.21), we obtain:

$$\ddot{y} = \dot{V}\hat{X}\hat{V}\tilde{b} = \hat{X}\bar{V}\tilde{b} = \hat{X}\bar{b} \tag{a.3.10.30}$$

Finally, reformulate $\ddot{\delta}$. First, note that:

$$\begin{aligned}
 & \tilde{b}'\hat{V}\tilde{b} + \dot{y}'\bar{V}^{-1}\ddot{y} \\
 &= \tilde{b}'\bar{V}^{-1}\hat{V}\bar{V}^{-1}\tilde{b} + \tilde{b}'\hat{X}'(I_m - \hat{X}\hat{V}\hat{X}')\hat{X}\bar{b} \\
 &= \tilde{b}'[(\hat{V}^{-1} - \hat{X}'\hat{X})'\hat{V}(\hat{V}^{-1} - \hat{X}'\hat{X}) + \hat{X}'\hat{X} - \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X}]\bar{b} \\
 &= \tilde{b}'[(\hat{V}^{-1} - \hat{X}'\hat{X})'(I_k - \hat{V}\hat{X}'\hat{X}) + \hat{X}'\hat{X} - \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X}]\bar{b} \\
 &= \tilde{b}'[\hat{V}^{-1} - \hat{X}'\hat{X} - \hat{X}'\hat{X} + \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X} + \hat{X}'\hat{X} - \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X}]\bar{b} \\
 &= \tilde{b}'[\hat{V}^{-1} - \hat{X}'\hat{X}]\bar{b} \\
 &= \tilde{b}'[\bar{V}^{-1} + \hat{X}'\hat{X} - \hat{X}'\hat{X}]\bar{b} \\
 &= \tilde{b}'\bar{V}^{-1}\bar{b}
 \end{aligned} \tag{a.3.10.31}$$

Substituting this in (a.3.10.23) to obtain:

$$\ddot{\delta} = \delta + y'y + b'V^{-1}b - \tilde{b}'\hat{V}\tilde{b} - \dot{y}'\bar{V}^{-1}\ddot{y} = \delta + y'y + b'V^{-1}b - \tilde{b}'\bar{V}^{-1}\bar{b} = \bar{\delta} \tag{a.3.10.32}$$

Eventually substituting for (a.3.10.26), (a.3.10.30) and (a.3.10.32) in (a.3.10.25) yields:

$$f(\hat{y}|y) \propto \left(1 + \frac{1}{\bar{\alpha}}(\hat{y} - \hat{X}\bar{b})'[\bar{\delta}(I_m + \hat{X}\bar{V}\hat{X}')/\bar{\alpha}]^{-1}(\hat{y} - \hat{X}\bar{b})\right)^{-(\bar{\alpha}+m)/2} \tag{a.3.10.33}$$

derivations for equation (3.10.10)

The log likelihood function is given by:

$$\log(f(y|\beta, \sigma)) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma) - \frac{1}{2} \frac{(y - X\beta)'(y - X\beta)}{\sigma} \quad (\text{a.3.10.34})$$

The function is estimated at the maximum likelihood values. Hence $\beta = \hat{\beta}$ and $\sigma = \hat{\sigma} = \frac{\hat{\epsilon}'\hat{\epsilon}}{n}$.

Substituting in (a.3.10.34):

$$\begin{aligned} & \log(f(y|\hat{\beta}, \hat{\sigma})) \\ &= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\hat{\sigma}) - \frac{1}{2} \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{\hat{\sigma}} \\ &= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\hat{\sigma}) - \frac{1}{2} \frac{n \hat{\epsilon}'\hat{\epsilon}}{\hat{\epsilon}'\hat{\epsilon}} \\ &= -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\hat{\sigma}) - \frac{n}{2} \\ &= -\frac{n}{2} [\log(2\pi) + \log(\hat{\sigma}) + 1] \end{aligned} \quad (\text{a.3.10.35})$$

Then AIC obtains as:

$$\begin{aligned} AIC &= 2k/n - 2 \hat{L}/n \\ &= 2k/n - 2 \left(-\frac{n}{2} [\log(2\pi) + \log(\hat{\sigma}) + 1] \right) / n \\ &= 2k/n + \log(2\pi) + \log(\hat{\sigma}) + 1 \end{aligned} \quad (\text{a.3.10.36})$$

Using similar calculations, BIC immediately obtains as:

$$BIC = k \log(n)/n + \log(2\pi) + \log(\hat{\sigma}) + 1 \quad (\text{a.3.10.37})$$

Removing the constants that make the value invariant to the number of coefficients:

$$AIC = 2k/n + \log(\hat{\sigma}) \quad BIC = k \log(n)/n + \log(\hat{\sigma}) \quad (\text{a.3.10.38})$$

derivations for equation (3.10.19)

Rearrange:

$$\begin{aligned}
 & f(y) \\
 &= \int (2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2} \frac{(y-X\beta)'(y-X\beta)}{\sigma}\right) \times (2\pi)^{-k/2} |V|^{-1/2} \exp\left(-\frac{1}{2} (\beta-b)'V^{-1}(\beta-b)\right) d\beta \\
 &= \int (2\pi)^{-(n+k)/2} \sigma^{-n/2} |V|^{-1/2} \times \exp\left(-\frac{1}{2} [(y-X\beta)'\sigma^{-1}(y-X\beta) + (\beta-b)'V^{-1}(\beta-b)]\right) d\beta
 \end{aligned} \tag{a.3.10.39}$$

Consider the term square brackets:

$$\begin{aligned}
 & (y-X\beta)'\sigma^{-1}(y-X\beta) + (\beta-b)'V^{-1}(\beta-b) \\
 &= y'\sigma^{-1}y + \beta'X'\sigma^{-1}X\beta - 2\beta'X'\sigma^{-1}y + \beta'V^{-1}\beta + b'V^{-1}b - 2\beta'V^{-1}b \\
 &= \beta'(V^{-1} + \sigma^{-1}X'X)\beta - 2\beta'(V^{-1}b + \sigma^{-1}X'y) + y'\sigma^{-1}y + b'V^{-1}b \\
 &= \beta'(V^{-1} + \sigma^{-1}X'X)\beta - 2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b + \sigma^{-1}X'y) + \bar{b}\bar{V}^{-1}\bar{b} + y'\sigma^{-1}y + b'V^{-1}b - \bar{b}\bar{V}^{-1}\bar{b}
 \end{aligned} \tag{a.3.10.40}$$

Define:

$$\bar{V} = (V^{-1} + \sigma^{-1}X'X)^{-1} \quad \bar{b} = \bar{V}(V^{-1}b + \sigma^{-1}X'y) \tag{a.3.10.41}$$

Then (a.3.10.40) reformulates:

$$\begin{aligned}
 &= \beta'\bar{V}^{-1}\beta - 2\beta'\bar{V}^{-1}\bar{b} + \bar{b}\bar{V}^{-1}\bar{b} + y'\sigma^{-1}y + b'V^{-1}b - \bar{b}\bar{V}^{-1}\bar{b} \\
 &= (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b}) + y'\sigma^{-1}y + b'V^{-1}b - \bar{b}\bar{V}^{-1}\bar{b}
 \end{aligned} \tag{a.3.10.42}$$

Substitute back in (a.3.10.39):

$$\begin{aligned}
 & f(y) \\
 &= \int (2\pi)^{-(n+k)/2} \sigma^{-n/2} |V|^{-1/2} \times \exp\left(-\frac{1}{2} [(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b}) + y'\sigma^{-1}y + b'V^{-1}b - \bar{b}\bar{V}^{-1}\bar{b}]\right) d\beta \\
 &= (2\pi)^{-(n+k)/2} \sigma^{-n/2} |V|^{-1/2} (2\pi)^{k/2} |\bar{V}|^{1/2} \times \exp\left(-\frac{1}{2} [y'\sigma^{-1}y + b'V^{-1}b - \bar{b}\bar{V}^{-1}\bar{b}]\right) \\
 &\times \int (2\pi)^{-k/2} |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right) d\beta \\
 &= (2\pi)^{-n/2} \sigma^{-n/2} |\bar{V}|^{1/2} |V|^{-1/2} \times \exp\left(-\frac{1}{2} [y'\sigma^{-1}y + b'V^{-1}b - \bar{b}\bar{V}^{-1}\bar{b}]\right) \\
 &\times \int (2\pi)^{-k/2} |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right) d\beta
 \end{aligned} \tag{a.3.10.43}$$

derivations for equation (3.10.21)

Consider the term:

$$\begin{aligned}
 & (2\pi)^{-n/2} \sigma^{-n/2} |\bar{V}|^{1/2} |V|^{-1/2} \\
 &= (2\pi)^{-n/2} \sigma^{-n/2} |(V^{-1} + \sigma^{-1} X'X)^{-1}|^{1/2} |V|^{-1/2} \\
 &= (2\pi)^{-n/2} \sigma^{-n/2} |V^{-1} + \sigma^{-1} X'X|^{-1/2} |V|^{-1/2} \\
 &= (2\pi)^{-n/2} \sigma^{-n/2} (|V| |V^{-1} + \sigma^{-1} X'X|)^{-1/2} \\
 &= (2\pi)^{-n/2} \sigma^{-n/2} |I_k + \sigma^{-1} V X'X|^{-1/2} \tag{a.3.10.44}
 \end{aligned}$$

Hence:

$$f(y) = (2\pi)^{-n/2} \sigma^{-n/2} |I_k + \sigma^{-1} V X'X|^{-1/2} \exp \left(-\frac{1}{2} [y' \sigma^{-1} y + b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b}] \right) \tag{a.3.10.45}$$

derivations for equation (3.10.23)

Rearrange the expression:

$$\begin{aligned}
 & f(y) \\
 &= \int \int (2\pi\sigma)^{-n/2} \exp \left(-\frac{1}{2} \frac{(y - X\beta)'(y - X\beta)}{\sigma} \right) \\
 &\times (2\pi)^{-k/2} |\sigma V|^{-1/2} \exp \left(-\frac{1}{2} (\beta - b)' (\sigma V)^{-1} (\beta - b) \right) \times \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \sigma^{-\alpha/2-1} \exp \left(-\frac{\delta}{2\sigma} \right) d\beta d\sigma \\
 &= \int \int (2\pi\sigma)^{-n/2} (2\pi)^{-k/2} |\sigma V|^{-1/2} \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \sigma^{-\alpha/2-1} \\
 &\times \exp \left(-\frac{1}{2\sigma} [(y - X\beta)'(y - X\beta) + (\beta - b)' V^{-1} (\beta - b) + \delta] \right) d\beta d\sigma \tag{a.3.10.46}
 \end{aligned}$$

Consider the term in square brackets and complete the squares:

$$\begin{aligned}
 & (y - X\beta)'(y - X\beta) + (\beta - b)' V^{-1} (\beta - b) + \delta \\
 &= y'y + \beta' X' X \beta - 2\beta' X'y + \beta' V^{-1} \beta + b' V^{-1} b - 2\beta' V^{-1} b + \delta \\
 &= \beta' (V^{-1} + X'X) \beta - 2\beta' (V^{-1} b + X'y) + \delta + y'y + b' V^{-1} b \\
 &= \beta' (V^{-1} + X'X) \beta - 2\beta' \bar{V}^{-1} \bar{V} (V^{-1} b + X'y) + \delta + y'y + b' V^{-1} b + \bar{b}' \bar{V}^{-1} \bar{b} - \bar{b}' \bar{V}^{-1} \bar{b} \tag{a.3.10.47}
 \end{aligned}$$

Define:

$$\bar{V} = (V^{-1} + X'X)^{-1} \quad \bar{b} = \bar{V} (V^{-1} b + X'y) \quad \bar{\delta} = \delta + y'y + b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} \tag{a.3.10.48}$$

Then (a.3.10.47) rewrites:

$$\begin{aligned}
 &= \beta' \bar{V}^{-1} \beta - 2\beta' \bar{V}^{-1} \bar{b} + \bar{b}' \bar{V}^{-1} \bar{b} + \bar{\delta} \\
 &= (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + \bar{\delta} \tag{a.3.10.49}
 \end{aligned}$$

Substituting back in (a.3.10.46):

$$\begin{aligned}
& f(y) \\
&= \int \int (2\pi\sigma)^{-n/2} (2\pi)^{-k/2} |\sigma V|^{-1/2} \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \sigma^{-\alpha/2-1} \\
&\times \exp\left(-\frac{1}{2\sigma} [(\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + \bar{\delta}]\right) d\beta d\sigma \\
&= \int \int (2\pi)^{-n/2} (2\pi)^{-k/2} |\sigma V|^{-1/2} \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\
&\times \sigma^{-(\alpha+n)/2-1} \exp\left(-\frac{1}{2\sigma} [(\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + \bar{\delta}]\right) d\beta d\sigma
\end{aligned} \tag{a.3.10.50}$$

define:

$$\bar{\alpha} = \alpha + n \tag{a.3.10.51}$$

Then (a.3.10.50) rewrites:

$$\begin{aligned}
&= \int \int (2\pi)^{-n/2} (2\pi)^{-k/2} |\sigma V|^{-1/2} \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\
&\times \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{1}{2\sigma} [(\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + \bar{\delta}]\right) d\beta d\sigma \\
&= (2\pi)^{-n/2} |\sigma V|^{-1/2} |\sigma \bar{V}|^{1/2} \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \frac{\Gamma(\bar{\alpha}/2)}{\bar{\delta}/2^{\bar{\alpha}/2}} \\
&\times \int \int (2\pi)^{-k/2} |\sigma \bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta - \bar{b})' (\sigma \bar{V})^{-1} (\beta - \bar{b})\right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\beta d\sigma \\
&= 2^{-n/2} \pi^{-n/2} |V|^{-1/2} |\bar{V}|^{1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{2^{(\alpha+n)/2}}{2^{\alpha/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
&\times \int \int (2\pi)^{-k/2} |\sigma \bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta - \bar{b})' (\sigma \bar{V})^{-1} (\beta - \bar{b})\right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\beta d\sigma \\
&= \pi^{-n/2} |V|^{-1/2} |\bar{V}|^{1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
&\times \int \int (2\pi)^{-k/2} |\sigma \bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta - \bar{b})' (\sigma \bar{V})^{-1} (\beta - \bar{b})\right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\beta d\sigma
\end{aligned} \tag{a.3.10.52}$$

derivations for equation (3.10.25)

Reformulate the expression:

$$\begin{aligned}
& \pi^{-n/2} |V|^{-1/2} |\bar{V}|^{1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= & \pi^{-n/2} |V|^{-1/2} |(V^{-1} + X'X)^{-1}|^{1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= & \pi^{-n/2} |V|^{-1/2} |(V^{-1} + X'X)|^{-1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= & \pi^{-n/2} |V(V^{-1} + X'X)|^{-1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= & \pi^{-n/2} |I_k + VX'X|^{-1/2} \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \tag{a.3.10.53}
\end{aligned}$$

derivations for equation (3.10.27)

Rearrange the expression:

$$\begin{aligned}
& f(y) \\
& \approx \frac{f(y|\beta^*, \sigma^*)\pi(\beta^*, \sigma^*)}{\pi(\sigma^*|y, \beta^*) \times \frac{1}{J} \sum_{j=1}^J \pi(\beta^*|\sigma^{(j)}, y)} \\
& = (2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2} \frac{(y - X\beta)'(y - X\beta)}{\sigma}\right) \times \frac{(2\pi)^{-k/2} |V|^{-1/2} \exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-k/2} |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)} \\
& \times \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)} \sigma^{-\bar{\alpha}/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
& = 2^{-n/2} \pi^{-n/2} \sigma^{-n/2} \exp\left(-\frac{1}{2} \frac{(y - X\beta)'(y - X\beta)}{\sigma}\right) \times \frac{(2\pi)^{-k/2} |V|^{-1/2} \exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-k/2} |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)} \\
& \times \frac{\Gamma(\bar{\alpha}/2) 2^{(\alpha+n)/2}}{\Gamma(\alpha/2) 2^{\alpha/2}} \frac{\delta^{\alpha/2} \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right)}{\bar{\delta}^{\bar{\alpha}/2} \sigma^{-(\alpha+n)/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
& = 2^{-n/2} \pi^{-n/2} \sigma^{-n/2} \times \frac{(2\pi)^{-k/2} |V|^{-1/2} \exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-k/2} |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)} \\
& \times \frac{\Gamma(\bar{\alpha}/2) 2^{(\alpha+n)/2}}{\Gamma(\alpha/2) 2^{\alpha/2}} \frac{\delta^{\alpha/2} \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta + (y - X\beta)'(y - X\beta)}{2\sigma}\right)}{\bar{\delta}^{\bar{\alpha}/2} \sigma^{-(\alpha+n)/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
& = 2^{-n/2} \pi^{-n/2} \sigma^{-n/2} \times \frac{(2\pi)^{-k/2} |V|^{-1/2} \exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-k/2} |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)} \\
& \times \frac{\Gamma(\bar{\alpha}/2) 2^{(\alpha+n)/2}}{\Gamma(\alpha/2) 2^{\alpha/2}} \frac{\delta^{\alpha/2} \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right)}{\bar{\delta}^{\bar{\alpha}/2} \sigma^{-(\alpha+n)/2-1} \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
& = \pi^{-n/2} \frac{|V|^{-1/2} \exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)} \frac{\Gamma(\bar{\alpha}/2) \delta^{\alpha/2}}{\Gamma(\alpha/2) \bar{\delta}^{\bar{\alpha}/2}} \\
& = \pi^{-n/2} \frac{\exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J |V|^{1/2} |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)} \frac{\Gamma(\bar{\alpha}/2) \delta^{\alpha/2}}{\Gamma(\alpha/2) \bar{\delta}^{\bar{\alpha}/2}} \\
& = \pi^{-n/2} \frac{\exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J |V|^{1/2} |V^{-1} + \sigma^{-1}X'X|^{-1/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)} \frac{\Gamma(\bar{\alpha}/2) \delta^{\alpha/2}}{\Gamma(\alpha/2) \bar{\delta}^{\bar{\alpha}/2}} \\
& = \pi^{-n/2} \frac{\exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J |V|^{1/2} |V^{-1} + \sigma^{-1}X'X|^{1/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)} \frac{\Gamma(\bar{\alpha}/2) \delta^{\alpha/2}}{\Gamma(\alpha/2) \bar{\delta}^{\bar{\alpha}/2}} \\
& = \pi^{-n/2} \frac{\exp\left(-\frac{1}{2}(\beta - b)'V^{-1}(\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J |I_k + \sigma^{-1}VX'X|^{1/2} \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)} \frac{\Gamma(\bar{\alpha}/2) \delta^{\alpha/2}}{\Gamma(\alpha/2) \bar{\delta}^{\bar{\alpha}/2}} \tag{a.3.10.54}
\end{aligned}$$

derivations for equation (3.10.29)

Substitute for the functions and rearrange:

$$\begin{aligned}
& \frac{1}{f(y)} \\
& \approx \frac{1}{J} \sum_{j=1}^J \frac{g(\theta^{(j)})}{f(y|\beta^{(j)}, \sigma^{(j)}, \gamma^{(j)}) \pi(\beta^{(j)}) \pi(\sigma^{(j)}) \pi(\gamma^{(j)})} \\
& = \frac{1}{J} \sum_{j=1}^J \frac{\omega^{-1} (2\pi)^{-(k+h+1)/2} |\hat{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta})\right) \mathbb{1}(\theta \in \hat{\Theta})}{\left[(2\pi\sigma)^{-n/2} |W|^{-1/2} \exp\left(-\frac{1}{2} \frac{(y-X\beta)' W^{-1}(y-X\beta)}{\sigma}\right) \times (2\pi)^{-k/2} |V|^{-1/2} \exp\left(-\frac{1}{2}(\beta - b)' V^{-1}(\beta - b)\right) \right.} \\
& \quad \left. \times \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right) \times (2\pi)^{-h/2} |Q|^{-1/2} \exp\left(-\frac{1}{2}(\gamma - g)' Q^{-1}(\gamma - g)\right) \right] } \\
& = \frac{1}{J} \sum_{j=1}^J \mathbb{1}(\theta \in \hat{\Theta}) \omega^{-1} (2\pi)^{(n+k+h-(k+h+1))/2} |\hat{\Sigma}|^{-1/2} |W|^{1/2} |V|^{1/2} |Q|^{1/2} \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}} \sigma^{(\alpha+n)/2+1} \\
& \quad \times \exp\left(\frac{1}{2} \left[(y-X\beta)'(\sigma W)^{-1}(y-X\beta) + (\beta - b)' V^{-1}(\beta - b) + \delta \sigma^{-1} \right. \right. \\
& \quad \left. \left. + (\gamma - g)' Q^{-1}(\gamma - g) - (\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta}) \right] \right) \\
& = (\omega J)^{-1} (2\pi)^{(n-1)/2} |\hat{\Sigma}|^{-1/2} |V|^{1/2} |Q|^{1/2} \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}} \\
& \quad \times \sum_{j=1}^J \mathbb{1}(\theta \in \hat{\Theta}) |W|^{1/2} \sigma^{(\alpha+n)/2+1} \exp\left(\frac{1}{2} \left[(y-X\beta)'(\sigma W)^{-1}(y-X\beta) + (\beta - b)' V^{-1}(\beta - b) \right. \right. \\
& \quad \left. \left. + \delta \sigma^{-1} + (\gamma - g)' Q^{-1}(\gamma - g) - (\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta}) \right] \right) \tag{a.3.10.55}
\end{aligned}$$

Using logs on both sides yields:

$$\begin{aligned}
& -\log(f(y)) \approx \log\left((\omega J)^{-1} (2\pi)^{(n-1)/2} |\hat{\Sigma}|^{-1/2} |V|^{1/2} |Q|^{1/2} \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
& + \log\left(\sum_{j=1}^J \mathbb{1}(\theta \in \hat{\Theta}) |W|^{1/2} \sigma^{(\alpha+n)/2+1} \exp\left(\frac{1}{2} \left[(y-X\beta)'(\sigma W)^{-1}(y-X\beta) + (\beta - b)' V^{-1}(\beta - b) \right. \right. \right. \\
& \quad \left. \left. + \delta \sigma^{-1} + (\gamma - g)' Q^{-1}(\gamma - g) - (\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta}) \right] \right)\right) \tag{a.3.10.56}
\end{aligned}$$

or:

$$\begin{aligned}
& \log(f(y)) \approx -\log\left((\omega J)^{-1} (2\pi)^{(n-1)/2} |\hat{\Sigma}|^{-1/2} |V|^{1/2} |Q|^{1/2} \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
& -\log\left(\sum_{j=1}^J \mathbb{1}(\theta \in \hat{\Theta}) |W|^{1/2} \sigma^{(\alpha+n)/2+1} \exp\left(\frac{1}{2} \left[(y-X\beta)'(\sigma W)^{-1}(y-X\beta) + (\beta - b)' V^{-1}(\beta - b) \right. \right. \right. \\
& \quad \left. \left. + \delta \sigma^{-1} + (\gamma - g)' Q^{-1}(\gamma - g) - (\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta}) \right] \right)\right) \tag{a.3.10.57}
\end{aligned}$$

derivations for equation (3.10.32)

Substitute for the functions and rearrange:

$$\begin{aligned}
& \frac{1}{f(y)} \\
& \approx \frac{1}{J} \sum_{j=1}^J \frac{g(\theta^{(j)})}{f(y|\beta^{(j)}, \sigma^{(j)}, \phi^{(j)}) \pi(\beta^{(j)}) \pi(\sigma^{(j)}) \pi(\phi^{(j)})} \\
& = \frac{1}{J} \sum_{j=1}^J \frac{\omega^{-1} (2\pi)^{-(k+q+1)/2} |\hat{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2}(\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta})\right) \mathbb{1}(\theta \in \hat{\Theta})}{\left[(2\pi\sigma)^{-T/2} \exp\left(-\frac{1}{2}(\varepsilon - E\phi)' \sigma^{-1}(\varepsilon - E\phi)\right) \times (2\pi)^{-k/2} |V|^{-1/2} \exp\left(-\frac{1}{2}(\beta - b)' V^{-1}(\beta - b)\right) \right.} \\
& \quad \left. \times \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right) \times (2\pi)^{-q/2} |Z|^{-1/2} \exp\left(-\frac{1}{2}(\phi - p)' Z^{-1}(\phi - p)\right) \right] } \\
& = \frac{1}{J} \sum_{j=1}^J \mathbb{1}(\theta \in \hat{\Theta}) \omega^{-1} (2\pi)^{(T+k+q-(k+q+1))/2} |\hat{\Sigma}|^{-1/2} |V|^{1/2} |Z|^{1/2} \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}} \sigma^{(\alpha+T)/2+1} \\
& \quad \times \exp\left(\frac{1}{2} \left[(\varepsilon - E\phi)' \sigma^{-1}(\varepsilon - E\phi) + (\beta - b)' V^{-1}(\beta - b) \right. \right. \\
& \quad \left. \left. + \delta \sigma^{-1} + (\phi - p)' Z^{-1}(\phi - p) - (\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta}) \right] \right) \\
& = (\omega J)^{-1} (2\pi)^{(T-1)/2} |\hat{\Sigma}|^{-1/2} |V|^{1/2} |Z|^{1/2} \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}} \\
& \quad \times \sum_{j=1}^J \mathbb{1}(\theta \in \hat{\Theta}) \sigma^{(\alpha+T)/2+1} \exp\left(\frac{1}{2} \left[(\varepsilon - E\phi)' \sigma^{-1}(\varepsilon - E\phi) + (\beta - b)' V^{-1}(\beta - b) \right. \right. \\
& \quad \left. \left. + \delta \sigma^{-1} + (\phi - p)' Z^{-1}(\phi - p) - (\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta}) \right] \right)
\end{aligned} \tag{a.3.10.58}$$

Using logs on both sides yields:

$$\begin{aligned}
& -\log(f(y)) \approx \log\left((\omega J)^{-1} (2\pi)^{(T-1)/2} |\hat{\Sigma}|^{-1/2} |V|^{1/2} |Z|^{1/2} \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
& + \log\left(\sum_{j=1}^J \mathbb{1}(\theta \in \hat{\Theta}) \sigma^{(\alpha+T)/2+1} \exp\left(\frac{1}{2} \left[(\varepsilon - E\phi)' \sigma^{-1}(\varepsilon - E\phi) + (\beta - b)' V^{-1}(\beta - b) \right. \right. \right. \\
& \quad \left. \left. + \delta \sigma^{-1} + (\phi - p)' Z^{-1}(\phi - p) - (\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta}) \right] \right)\right)
\end{aligned} \tag{a.3.10.59}$$

or:

$$\begin{aligned}
& \log(f(y)) \approx -\log\left((\omega J)^{-1} (2\pi)^{(T-1)/2} |\hat{\Sigma}|^{-1/2} |V|^{1/2} |Z|^{1/2} \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
& -\log\left(\sum_{j=1}^J \mathbb{1}(\theta \in \hat{\Theta}) \sigma^{(\alpha+T)/2+1} \exp\left(\frac{1}{2} \left[(\varepsilon - E\phi)' \sigma^{-1}(\varepsilon - E\phi) + (\beta - b)' V^{-1}(\beta - b) \right. \right. \right. \\
& \quad \left. \left. + \delta \sigma^{-1} + (\phi - p)' Z^{-1}(\phi - p) - (\theta - \hat{\theta})' \hat{\Sigma}^{-1}(\theta - \hat{\theta}) \right] \right)\right)
\end{aligned} \tag{a.3.10.60}$$

PART IV

Vector autoregressions

Vector autoregressions

derivations for equation (4.11.9)

Consider first β . To do so, rewrite the likelihood function as:

$$\log(f(y|\beta, \Sigma)) = -\frac{nT}{2} \log(2\pi) - \frac{1}{2} \log(|\bar{\Sigma}|) - \frac{1}{2} (y' \bar{\Sigma}^{-1} y + \beta' \bar{X}' \bar{\Sigma}^{-1} \bar{X} \beta - 2\beta' \bar{X}' \bar{\Sigma}^{-1} y) \quad (\text{a.4.11.1})$$

Then solve for the partial derivative:

$$\begin{aligned} \frac{\partial \log(f(y|\beta, \Sigma))}{\partial \beta} &= 0 \\ \Leftrightarrow -\frac{1}{2} (2\beta' \bar{X}' \bar{\Sigma}^{-1} \bar{X} - 2y' \bar{\Sigma}^{-1} \bar{X}' y) &= 0 \\ \Leftrightarrow \beta' \bar{X}' \bar{\Sigma}^{-1} \bar{X} - y' \bar{\Sigma}^{-1} \bar{X}' y &= 0 \\ \Leftrightarrow \beta' \bar{X}' \bar{\Sigma}^{-1} \bar{X} &= y' \bar{\Sigma}^{-1} \bar{X}' y \\ \Leftrightarrow \bar{X}' \bar{\Sigma}^{-1} \bar{X} \beta &= \bar{X}' \bar{\Sigma}^{-1} y \\ \Leftrightarrow \beta &= (\bar{X}' \bar{\Sigma}^{-1} \bar{X})^{-1} \bar{X}' \bar{\Sigma}^{-1} y \end{aligned} \quad (\text{a.4.11.2})$$

The formula can simplify further. Note first that:

$$\begin{aligned} \bar{X}' \bar{\Sigma}^{-1} &= (I_n \otimes X)' (\Sigma \otimes I_T)^{-1} \\ &= (I_n \otimes X') (\Sigma^{-1} \otimes I_T) \\ &= \Sigma^{-1} \otimes X' \end{aligned} \quad (\text{a.4.11.3})$$

Substituting for (a.4.11.3) in (a.4.11.2):

$$\begin{aligned} \beta &= (\bar{X}' \bar{\Sigma}^{-1} \bar{X})^{-1} \bar{X}' \bar{\Sigma}^{-1} y \\ &= [(\Sigma^{-1} \otimes X') (I_n \otimes X)]^{-1} [(\Sigma^{-1} \otimes X') y] \\ &= (\Sigma^{-1} \otimes X' X)^{-1} (\Sigma^{-1} \otimes X') \text{vec}(Y) \\ &= (\Sigma \otimes (X' X)^{-1}) (\Sigma^{-1} \otimes X') \text{vec}(Y) \\ &= (I_n \otimes (X' X)^{-1} X') \text{vec}(Y) \\ &= \text{vec}((X' X)^{-1} X' Y) \\ &= \text{vec}(\hat{B}) \end{aligned} \quad (\text{a.4.11.4})$$

with:

$$\hat{B} = (X' X)^{-1} X' Y \quad (\text{a.4.11.5})$$

Hence the maximum likelihood estimate is $\hat{\beta} = \text{vec}(\hat{B})$.

To obtain the maximum likelihood estimate for Σ , it is convenient to use property d.7 that states the equivalence between the multivariate normal and matrix normal distributions. Doing so, the likelihood function of the VAR model rewrites as:

$$f(y|\beta, \Sigma) = (2\pi)^{-nT/2} |\Sigma|^{-T/2} \exp \left(-\frac{1}{2} \text{tr} [\Sigma^{-1} (Y - X\beta)' (Y - X\beta)] \right) \quad (\text{a.4.11.6})$$

And the log-likelihood becomes:

$$\log(f(y|\beta, \Sigma)) = -\frac{nT}{2} \log(2\pi) - \frac{T}{2} \log(|\Sigma|) - \frac{1}{2} \text{tr} [\Sigma^{-1} (Y - X\beta)' (Y - X\beta)] \quad (\text{a.4.11.7})$$

Then solve for the partial derivative:

$$\begin{aligned} \frac{\partial \log(f(y|\beta, \Sigma))}{\partial \Sigma} &= 0 \\ \Leftrightarrow -\frac{T}{2} \Sigma^{-1} - \frac{1}{2} (Y - X\beta)' (Y - X\beta) (-\Sigma^{-1} \Sigma^{-1}) &= 0 \\ \Leftrightarrow -T \Sigma^{-1} + (Y - X\beta)' (Y - X\beta) (\Sigma^{-1} \Sigma^{-1}) &= 0 \\ \Leftrightarrow T \Sigma^{-1} &= (Y - X\beta)' (Y - X\beta) (\Sigma^{-1} \Sigma^{-1}) \\ \Leftrightarrow T \Sigma &= (Y - X\beta)' (Y - X\beta) \\ \Leftrightarrow \Sigma &= \frac{1}{T} (Y - X\beta)' (Y - X\beta) \end{aligned} \quad (\text{a.4.11.8})$$

Replacing β with its maximum likelihood estimate $\hat{\beta}$, we conclude that the maximum likelihood estimate for Σ is $\hat{\Sigma} = \frac{1}{T} (Y - X\hat{\beta})' (Y - X\hat{\beta})$.

derivations for equation (4.11.14)

Group terms:

$$\begin{aligned} \pi(\beta|y) &\propto \exp \left(-\frac{1}{2} (y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) \right) \times \exp \left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b) \right) \\ &= \exp \left(-\frac{1}{2} [(y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) + (\beta - b)' V^{-1} (\beta - b)] \right) \end{aligned} \quad (\text{a.4.11.9})$$

Consider the terms in square brackets:

$$\begin{aligned} &(y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) + (\beta - b)' V^{-1} (\beta - b) \\ &= y' \bar{\Sigma}^{-1} y + \beta' \bar{X}' \bar{\Sigma}^{-1} \bar{X} \beta - 2\beta' \bar{X}' \bar{\Sigma}^{-1} y + \beta' V^{-1} \beta + b' V^{-1} b - 2\beta' V^{-1} b \\ &= \beta' (V^{-1} + \bar{X}' \bar{\Sigma}^{-1} \bar{X}) \beta - 2\beta' (V^{-1} b + \bar{X}' \bar{\Sigma}^{-1} y) + b' V^{-1} b + y' \bar{\Sigma}^{-1} y \end{aligned} \quad (\text{a.4.11.10})$$

Complete the squares:

$$= \beta' (V^{-1} + \bar{X}' \bar{\Sigma}^{-1} \bar{X}) \beta - 2\beta' \bar{V}^{-1} \bar{V} (V^{-1} b + \bar{X}' \bar{\Sigma}^{-1} y) + \bar{b}' \bar{V}^{-1} \bar{b} - \bar{b}' \bar{V}^{-1} \bar{b} + b' V^{-1} b + y' \bar{\Sigma}^{-1} y \quad (\text{a.4.11.11})$$

Define:

$$\bar{V} = (V^{-1} + \bar{X}' \bar{\Sigma}^{-1} \bar{X})^{-1} \quad \bar{b} = \bar{V} (V^{-1} b + \bar{X}' \bar{\Sigma}^{-1} y) \quad (\text{a.4.11.12})$$

Then (a.4.11.11) rewrites:

$$\begin{aligned}
&= \beta' \bar{V}^{-1} \beta - 2\beta' \bar{V}^{-1} \bar{b} + \bar{b}' \bar{V}^{-1} \bar{b} - \bar{b}' \bar{V}^{-1} \bar{b} + b' V^{-1} b + y' \bar{\Sigma}^{-1} y \\
&= (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + (b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} + y' \bar{\Sigma}^{-1} y)
\end{aligned} \tag{a.4.11.13}$$

Substitute (a.4.11.13) back in (a.4.11.9):

$$\begin{aligned}
&\pi(\beta|y) \\
&= \exp \left(-\frac{1}{2} [(\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + (b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} + y' \bar{\Sigma}^{-1} y)] \right) \\
&= \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right) \exp \left(-\frac{1}{2} (b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} + y' \bar{\Sigma}^{-1} y) \right) \\
&\propto \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right)
\end{aligned} \tag{a.4.11.14}$$

Where the last line obtains by noting that the second term in row 2 does not involve β and can hence be relegated to the normalization constant.

The terms in (a.4.11.12) simplify. Note first that:

$$\begin{aligned}
&\bar{X}' \bar{\Sigma}^{-1} \bar{X} \\
&= (I_n \otimes X)' (\Sigma \otimes I_T)^{-1} (I_n \otimes X) \\
&= (I_n \otimes X') (\Sigma^{-1} \otimes I_T) (I_n \otimes X) \\
&= (\Sigma^{-1} \otimes X') (I_n \otimes X) \\
&= \Sigma^{-1} \otimes X' X
\end{aligned} \tag{a.4.11.15}$$

Similarly:

$$\begin{aligned}
&\bar{X}' \bar{\Sigma}^{-1} y \\
&= (I_n \otimes X)' (\Sigma \otimes I_T)^{-1} \text{vec}(Y) \\
&= (I_n \otimes X') (\Sigma^{-1} \otimes I_T) \text{vec}(Y) \\
&= (\Sigma^{-1} \otimes X') \text{vec}(Y) \\
&= \text{vec}(X' Y \Sigma^{-1})
\end{aligned} \tag{a.4.11.16}$$

Then (a.4.11.12) rewrites:

$$\bar{V} = (V^{-1} + \Sigma^{-1} \otimes X' X)^{-1} \quad \bar{b} = \bar{V} (V^{-1} b + \text{vec}(X' Y \Sigma^{-1})) \tag{a.4.11.17}$$

derivations for equation (4.11.23)

Start from the vectorized likelihood function:

$$f(y|\beta, \Sigma) = (2\pi)^{-nT/2} |\bar{\Sigma}|^{-1/2} \exp \left(-\frac{1}{2} (y - \bar{X} \beta)' \bar{\Sigma}^{-1} (y - \bar{X} \beta) \right) \tag{a.4.11.18}$$

Use then property d.7 that establishes the equivalence between the multivariate normal and matrix normal distributions to reformulate the likelihood in vectorized form as:

$$f(y|\mathcal{B}, \Sigma) = (2\pi)^{-nT/2} |\Sigma|^{-T/2} \exp \left(-\frac{1}{2} \text{tr} [\Sigma^{-1} (Y - X \mathcal{B})' (Y - X \mathcal{B})] \right) \tag{a.4.11.19}$$

Consider the quadratic term:

$$\begin{aligned}
& (Y - X\beta)'(Y - X\beta) \\
&= Y'Y + \beta'X'XB - 2\beta'X'Y \\
&= Y'Y + \beta'X'XB - 2\beta'X'Y + 2\hat{\beta}X'Y - 2\hat{\beta}X'Y \\
&= Y'Y + \beta'X'XB - 2\beta'(X'X)(X'X)^{-1}X'Y + 2\hat{\beta}(X'X)(X'X)^{-1}X'Y - 2\hat{\beta}X'Y \\
&= Y'Y + \beta'X'XB - 2\beta'(X'X)\hat{\beta} + 2\hat{\beta}(X'X)\hat{\beta} - 2\hat{\beta}X'Y \\
&= Y'Y + \beta'X'XB - 2\beta'(X'X)\hat{\beta} + \hat{\beta}(X'X)\hat{\beta} + \hat{\beta}(X'X)\hat{\beta} - 2\hat{\beta}X'Y \\
&= (\beta'(X'X)\hat{\beta} + \hat{\beta}(X'X)\hat{\beta} - 2\beta'(X'X)\hat{\beta}) + (Y'Y + \hat{\beta}(X'X)\hat{\beta} - 2\hat{\beta}X'Y) \\
&= (B - \hat{B})'(X'X)(B - \hat{B}) + (Y - X\hat{B})'(Y - X\hat{B})
\end{aligned} \tag{a.4.11.20}$$

Hence (a.4.11.19) rewrites:

$$\begin{aligned}
f(y|\beta, \Sigma) &= (2\pi)^{-nT/2} |\Sigma|^{-T/2} \exp \left(-\frac{1}{2} \text{tr} [\Sigma^{-1} (B - \hat{B})'(X'X)(B - \hat{B}) + \Sigma^{-1} (Y - X\hat{B})'(Y - X\hat{B})] \right) \\
&= (2\pi)^{-nT/2} |\Sigma|^{-T/2} \exp \left(-\frac{1}{2} \text{tr} [\Sigma^{-1} (B - \hat{B})'(X'X)(B - \hat{B})] \right) \\
&\quad \times \exp \left(-\frac{1}{2} \text{tr} [\Sigma^{-1} (Y - X\hat{B})'(Y - X\hat{B})] \right)
\end{aligned} \tag{a.4.11.21}$$

Also, it follows from property m.55 that:

$$\text{tr} [\Sigma^{-1} (B - \hat{B})'(X'X)(B - \hat{B})] = (\beta - \hat{\beta})' (\Sigma \otimes (X'X)^{-1})^{-1} (\beta - \hat{\beta}) \tag{a.4.11.22}$$

Substituting back in (a.4.11.21) yields:

$$\begin{aligned}
f(y|\beta, \Sigma) &= (2\pi)^{-nT/2} |\Sigma|^{-T/2} \exp \left(-\frac{1}{2} (\beta - \hat{\beta})' (\Sigma \otimes (X'X)^{-1})^{-1} (\beta - \hat{\beta}) \right) \\
&\quad \times \exp \left(-\frac{1}{2} \text{tr} [\Sigma^{-1} (Y - X\hat{B})'(Y - X\hat{B})] \right)
\end{aligned} \tag{a.4.11.23}$$

derivations for equation (4.11.32)

Start from the joint posterior:

$$\begin{aligned}
\pi(\beta, \Sigma|y) &\propto |\Sigma|^{-T/2} \exp \left(-\frac{1}{2} (\beta - \hat{\beta})' (\Sigma \otimes (X'X)^{-1})^{-1} (\beta - \hat{\beta}) \right) \\
&\quad \times \exp \left(-\frac{1}{2} \text{tr} [\Sigma^{-1} (Y - X\hat{B})'(Y - X\hat{B})] \right) \\
&\quad \times |\Sigma \otimes W|^{-1/2} \exp \left(-\frac{1}{2} (\beta - b)' (\Sigma \otimes W)^{-1} (\beta - b) \right) \\
&\quad \times |\Sigma|^{-(\alpha+n+1)/2} \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} S \} \right)
\end{aligned} \tag{a.4.11.24}$$

Note first that:

$$|\Sigma \otimes W|^{-1/2} = |\Sigma|^{-k/2} |W|^{-n/2} \tag{a.4.11.25}$$

Hence, substituting back in (a.4.11.24) and rearranging:

$$\begin{aligned}
\pi(\beta, \Sigma|y) &\propto \exp\left(-\frac{1}{2}(\beta - \hat{\beta})' (\Sigma \otimes (X'X)^{-1})^{-1} (\beta - \hat{\beta})\right) \\
&\times |\Sigma|^{-T/2} \exp\left(-\frac{1}{2} \text{tr} [\Sigma^{-1} (Y - X\hat{B})' (Y - X\hat{B})]\right) \\
&\times |\Sigma|^{-k/2} \exp\left(-\frac{1}{2}(\beta - b)' (\Sigma \otimes W)^{-1} (\beta - b)\right) \\
&\times |\Sigma|^{-(\alpha+n+1)/2} \exp\left(-\frac{1}{2} \text{tr} \{\Sigma^{-1} S\}\right)
\end{aligned} \tag{a.4.11.26}$$

After regrouping, one obtains:

$$\begin{aligned}
\pi(\beta, \Sigma|y) &\propto |\Sigma|^{-k/2} \exp\left(-\frac{1}{2} \left[(\beta - b)' (\Sigma \otimes W)^{-1} (\beta - b) + (\beta - \hat{\beta})' (\Sigma \otimes (X'X)^{-1})^{-1} (\beta - \hat{\beta}) \right]\right) \\
&\times |\Sigma|^{-(\alpha+T+n+1)/2} \exp\left(-\frac{1}{2} \text{tr} \{\Sigma^{-1} [S + (Y - X\hat{B})' (Y - X\hat{B})]\}\right)
\end{aligned} \tag{a.4.11.27}$$

Consider the term within the curly brackets in the first row:

$$\begin{aligned}
&(\beta - b)' (\Sigma \otimes W)^{-1} (\beta - b) + (\beta - \hat{\beta})' (\Sigma \otimes (X'X)^{-1})^{-1} (\beta - \hat{\beta}) \\
&= \text{tr} \{\Sigma^{-1} (\mathcal{B} - B)' W^{-1} (\mathcal{B} - B)\} + \text{tr} \{\Sigma^{-1} (\mathcal{B} - \hat{B})' (X'X) (\mathcal{B} - \hat{B})\} \\
&= \text{tr} \{\Sigma^{-1} [(\mathcal{B} - B)' W^{-1} (\mathcal{B} - B) + (\mathcal{B} - \hat{B})' (X'X) (\mathcal{B} - \hat{B})]\} \\
&= \text{tr} \{\Sigma^{-1} [(\mathcal{B} - B)' W^{-1} (\mathcal{B} - B) + (\mathcal{B} - \hat{B})' (X'X) (\mathcal{B} - \hat{B})]\} \\
&= \text{tr} \{\Sigma^{-1} [\mathcal{B}' W^{-1} \mathcal{B} + \mathcal{B}' W^{-1} B - 2 \mathcal{B}' W^{-1} B + \mathcal{B}' (X'X) \mathcal{B} + \hat{B}' (X'X) \hat{B} - 2 \mathcal{B}' (X'X) \hat{B}]\} \\
&= \text{tr} \{\Sigma^{-1} [\mathcal{B}' (W^{-1} + X'X) \mathcal{B} - 2 \mathcal{B}' (W^{-1} B + X'X \hat{B}) + \mathcal{B}' W^{-1} B + \hat{B}' (X'X) \hat{B}]\}
\end{aligned} \tag{a.4.11.28}$$

Complete the squares:

$$= \text{tr} \{\Sigma^{-1} [\mathcal{B}' (W^{-1} + X'X) \mathcal{B} - 2 \mathcal{B}' \bar{W}^{-1} \bar{W} (W^{-1} B + X'X \hat{B}) + \bar{B}' \bar{W}^{-1} \bar{B} - \bar{B}' \bar{W}^{-1} \bar{B} + B' W^{-1} B + \hat{B}' (X'X) \hat{B}]\} \tag{a.4.11.29}$$

Define:

$$\bar{W} = (W^{-1} + X'X)^{-1} \quad \bar{B} = \bar{W} (W^{-1} B + X'X \hat{B}) \tag{a.4.11.30}$$

then the expression becomes:

$$\begin{aligned}
&= \text{tr} \{\Sigma^{-1} [\mathcal{B}' \bar{W}^{-1} \mathcal{B} - 2 \mathcal{B}' \bar{W}^{-1} \bar{B} + \bar{B}' \bar{W}^{-1} \bar{B} - \bar{B}' \bar{W}^{-1} \bar{B} + B' W^{-1} B + \hat{B}' (X'X) \hat{B}]\} \\
&= \text{tr} \{\Sigma^{-1} [(\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) - \bar{B}' \bar{W}^{-1} \bar{B} + B' W^{-1} B + \hat{B}' (X'X) \hat{B}]\} \\
&= \text{tr} \{\Sigma^{-1} (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B})\} + \text{tr} \{\Sigma^{-1} [B' W^{-1} B + \hat{B}' (X'X) \hat{B} - \bar{B}' \bar{W}^{-1} \bar{B}]\}
\end{aligned} \tag{a.4.11.31}$$

Substituting (a.4.11.31) in (a.4.11.27) yields:

$$\begin{aligned}
\pi(\beta, \Sigma|y) &\propto |\Sigma|^{-k/2} \exp \left(-\frac{1}{2} \left[\text{tr} \{ \Sigma^{-1} (\mathcal{B}' - \bar{\mathcal{B}})' \bar{W}^{-1} (\mathcal{B}' - \bar{\mathcal{B}}) \} + \text{tr} \{ \Sigma^{-1} [B' W^{-1} B + \hat{B}' (X' X) \hat{B} - \bar{B}' \bar{W}^{-1} \bar{B}] \} \right] \right) \\
&\quad \times |\Sigma|^{-(\alpha+T+n+1)/2} \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} [S + (Y - X \hat{B})' (Y - X \hat{B})] \} \right) \\
&= |\Sigma|^{-k/2} \exp \left(-\frac{1}{2} \left[\text{tr} \{ \Sigma^{-1} (\mathcal{B}' - \bar{\mathcal{B}})' \bar{W}^{-1} (\mathcal{B}' - \bar{\mathcal{B}}) \} \right] \right) \\
&\quad \times |\Sigma|^{-(\alpha+T+n+1)/2} \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} [S + (Y - X \hat{B})' (Y - X \hat{B}) + B' W^{-1} B + \hat{B}' (X' X) \hat{B} - \bar{B}' \bar{W}^{-1} \bar{B}] \} \right)
\end{aligned} \tag{a.4.11.32}$$

Define:

$$\bar{\alpha} = \alpha + T \quad \bar{S} = S + (Y - X \hat{B})' (Y - X \hat{B}) + B' W^{-1} B + \hat{B}' (X' X) \hat{B} - \bar{B}' \bar{W}^{-1} \bar{B} \tag{a.4.11.33}$$

Then the expression becomes:

$$\begin{aligned}
\pi(\beta, \Sigma|y) &\propto |\Sigma|^{-k/2} \exp \left(-\frac{1}{2} \left[\text{tr} \{ \Sigma^{-1} (\mathcal{B}' - \bar{\mathcal{B}})' \bar{V}^{-1} (\mathcal{B}' - \bar{\mathcal{B}}) \} \right] \right) \\
&\quad \times |\Sigma|^{-(\bar{\alpha}+n+1)/2} \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} \bar{S} \} \right)
\end{aligned} \tag{a.4.11.34}$$

Some of the expressions simplify. Consider first (a.4.11.30):

$$\bar{B} = \bar{W} (W^{-1} B + X' X \hat{B}) = \bar{W} (W^{-1} B + X' X (X' X)^{-1} X' Y) = \bar{W} (W^{-1} B + X' Y) \tag{a.4.11.35}$$

Consider then:

$$\begin{aligned}
&(Y - X \hat{B})' (Y - X \hat{B}) + \hat{B}' X' X \hat{B} \\
&= Y' Y + \hat{B}' X' X \hat{B} - \hat{B}' X' Y - Y' X \hat{B} + \hat{B}' X' X \hat{B} \\
&= Y' Y + 2 \hat{B}' X' X \hat{B} - \hat{B}' X' Y - Y' X \hat{B} \\
&= Y' Y + 2 Y' X (X' X)^{-1} X' X (X' X)^{-1} X' Y - Y' X (X' X)^{-1} X' Y - Y' X (X' X)^{-1} X' Y \\
&= Y' Y + 2 Y' X (X' X)^{-1} X' Y - Y' X (X' X)^{-1} X' Y - Y' X (X' X)^{-1} X' Y \\
&= Y' Y
\end{aligned} \tag{a.4.11.36}$$

Substitute back in (a.4.11.33):

$$\bar{S} = S + Y' Y + B' W^{-1} B - \bar{B}' \bar{W}^{-1} \bar{B} \tag{a.4.11.37}$$

derivations for equation (4.11.37)

Start from the initial equation and reformulate:

$$\begin{aligned}
& \pi(\mathcal{B}|y) \\
& \propto \left| \bar{S} + (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) \right|^{-\frac{\bar{\alpha}+k}{2}} \\
& = \left| \bar{S} \{ I_n + \bar{S}^{-1} (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) \} \right|^{-\frac{\bar{\alpha}+k}{2}} \\
& = \left| \bar{S} \right|^{-\frac{\bar{\alpha}+k}{2}} \left| I_n + \bar{S}^{-1} (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) \right|^{-\frac{\bar{\alpha}+k}{2}} \\
& = \left| I_n + \bar{S}^{-1} (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) \right|^{-\frac{\bar{\alpha}+k}{2}} \\
& = \left| I_n + \bar{S}^{-1} (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) \right|^{-\frac{\alpha+T+k}{2}} \\
& = \left| I_n + \bar{S}^{-1} (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) \right|^{-\frac{(\alpha+T-n+1)+k+n-1}{2}}
\end{aligned} \tag{a.4.11.38}$$

Define:

$$\hat{\alpha} = \alpha + T - n + 1 \tag{a.4.11.39}$$

Then:

$$\begin{aligned}
& = \left| I_n + \bar{S}^{-1} (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) \right|^{-\frac{\hat{\alpha}+k+n-1}{2}} \\
& = \left| I_n + \frac{1}{\hat{\alpha}} (\bar{S}/\hat{\alpha})^{-1} (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) \right|^{-\frac{\hat{\alpha}+k+n-1}{2}}
\end{aligned} \tag{a.4.11.40}$$

Define:

$$\hat{S} = \bar{S}/\hat{\alpha} \tag{a.4.11.41}$$

Then:

$$\pi(\mathcal{B}|y) \propto \left| I_n + \frac{1}{\hat{\alpha}} \hat{S}^{-1} (\mathcal{B}' - \bar{B})' \bar{W}^{-1} (\mathcal{B}' - \bar{B}) \right|^{-\frac{\hat{\alpha}+k+n-1}{2}} \tag{a.4.11.42}$$

derivations for equation (4.11.45)

Note that:

$$|\bar{\Sigma}|^{-1/2} = |\Sigma \otimes I_T|^{-1/2} = |\Sigma|^{-T/2} |I_T|^{-n/2} = |\Sigma|^{-T/2} \tag{a.4.11.43}$$

Also:

$$(y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) = (y - (I_n \otimes X)\beta)' (\Sigma \otimes I_T)^{-1} (y - (I_n \otimes X)\beta) = tr \{ \Sigma^{-1} (Y - X\mathcal{B})' (Y - X\mathcal{B}) \} \tag{a.4.11.44}$$

Then substituting in the original expression:

$$\begin{aligned}
\pi(\Sigma|y, \beta) &\propto |\Sigma|^{-T/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}(Y - X\beta)'(Y - X\beta)\}\right) \times |\Sigma|^{-(\alpha+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}S\}\right) \\
&= |\Sigma|^{-(\alpha+T+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}[S + (Y - X\beta)'(Y - X\beta)]\}\right) \\
&= |\Sigma|^{-(\bar{\alpha}+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}\bar{S}\}\right)
\end{aligned} \tag{a.4.11.45}$$

with:

$$\bar{\alpha} = \alpha + T \quad \bar{S} = S + (Y - X\beta)'(Y - X\beta) \tag{a.4.11.46}$$

derivations for equation (4.11.49)

Start from Bayes rule and rearrange:

$$\begin{aligned}
&\pi(\beta, \Sigma|y) \\
&\propto f(y|\beta, \Sigma)\pi(\beta)\pi(\Sigma) \\
&\propto |\Sigma|^{-T/2} \exp\left(-\frac{1}{2}(\beta - \hat{\beta})'(\Sigma \otimes (X'X)^{-1})(\beta - \hat{\beta})\right) \\
&\times \exp\left(-\frac{1}{2} \text{tr}[\Sigma^{-1}(Y - X\hat{\beta})'(Y - X\hat{\beta})]\right) \times |\Sigma|^{-(\alpha+1)/2} \\
&= |\Sigma|^{-k/2} \exp\left(-\frac{1}{2}(\beta - \hat{\beta})'(\Sigma \otimes (X'X)^{-1})(\beta - \hat{\beta})\right) \\
&\times |\Sigma|^{-(T-k+\alpha+1)/2} \exp\left(-\frac{1}{2} \text{tr}[\Sigma^{-1}(Y - X\hat{\beta})'(Y - X\hat{\beta})]\right) \\
&= |\Sigma|^{-k/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}(\mathcal{B}' - \hat{B})'(X'X)(\mathcal{B}' - \hat{B})\}\right) \\
&\times |\Sigma|^{-(T-k+n+3)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}(Y - X\hat{\beta})'(Y - X\hat{\beta})\}\right) \\
&= |\Sigma|^{-k/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}(\mathcal{B}' - \hat{B})'\hat{W}^{-1}(\mathcal{B}' - \hat{B})\}\right) \\
&\times |\Sigma|^{-(\hat{\alpha}+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}\hat{S}\}\right)
\end{aligned} \tag{a.4.11.47}$$

with:

$$\hat{W} = (X'X)^{-1} \quad \hat{\alpha} = T - k + 2 \quad \hat{S} = (Y - X\hat{\beta})'(Y - X\hat{\beta}) \tag{a.4.11.48}$$

derivations for equation (4.11.52)

Start from the joint posterior, group the terms and integrate:

$$\pi(\mathcal{B}|y) = \int \pi(\mathcal{B}, \Sigma|y) d\Sigma \propto \int |\Sigma|^{-(\hat{\alpha}+k+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}[\hat{S} + (\mathcal{B}' - \hat{B})'\hat{W}^{-1}(\mathcal{B}' - \hat{B})]\}\right) d\Sigma \tag{a.4.11.49}$$

This is the kernel of an inverse Wishart distribution with degrees of freedom $(\hat{\alpha} + k)$ and scale $\hat{S} + (\mathcal{B}' - \hat{B})'\hat{W}^{-1}(\mathcal{B}' - \hat{B})$, and integration yields the reciprocal of the normalization constant of the distribution. Hence:

$$\pi(\mathcal{B}|y) \propto \Gamma_n \left(\frac{\hat{\alpha} + k}{2} \right) 2^{(\hat{\alpha} + k)n/2} |\hat{S} + (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B})|^{-\frac{\hat{\alpha} + k}{2}} \propto |\hat{S} + (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B})|^{-\frac{\hat{\alpha} + k}{2}} \quad (\text{a.4.11.50})$$

Rearrange:

$$\begin{aligned} \pi(\mathcal{B}|y) &\propto |\hat{S} + (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B})|^{-\frac{\hat{\alpha} + k}{2}} \\ &= |\hat{S} \{I_n + \hat{S}^{-1} (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B})\}|^{-\frac{\hat{\alpha} + k}{2}} \\ &= |\hat{S}|^{-\frac{\hat{\alpha} + k}{2}} |I_n + \hat{S}^{-1} (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B})|^{-\frac{\hat{\alpha} + k}{2}} \\ &= |I_n + \hat{S}^{-1} (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B})|^{-\frac{\hat{\alpha} + k}{2}} \\ &= |I_n + \hat{S}^{-1} (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B})|^{-\frac{T+2}{2}} \\ &= |I_n + \hat{S}^{-1} (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B})|^{-\frac{(T-n-k+3)+k+n-1}{2}} \end{aligned} \quad (\text{a.4.11.51})$$

Define:

$$\tilde{\alpha} = T - n - k + 3 \quad (\text{a.4.11.52})$$

Then:

$$\begin{aligned} &= |I_n + \hat{S}^{-1} (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B})|^{-\frac{\tilde{\alpha} + k + n - 1}{2}} \\ &= \left| I_n + \frac{1}{\tilde{\alpha}} (\hat{S}/\tilde{\alpha})^{-1} (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B}) \right|^{-\frac{\tilde{\alpha} + k + n - 1}{2}} \end{aligned} \quad (\text{a.4.11.53})$$

Define:

$$\tilde{S} = \hat{S}/\tilde{\alpha} \quad (\text{a.4.11.54})$$

Then:

$$\pi(\mathcal{B}|y) \propto \left| I_n + \frac{1}{\tilde{\alpha}} \tilde{S}^{-1} (\mathcal{B}' - \hat{B})' \hat{W}^{-1} (\mathcal{B}' - \hat{B}) \right|^{-\frac{\tilde{\alpha} + k + n - 1}{2}} \quad (\text{a.4.11.55})$$

derivations for equation (4.11.68)

Start from the original likelihood function:

$$f(y|\beta, \Sigma) = (2\pi)^{-nT/2} |\bar{\Sigma}|^{-1/2} \exp \left(-\frac{1}{2} (y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) \right) \quad (\text{a.4.11.56})$$

Consider first the determinant term in (a.4.11.56) and rearrange:

$$\begin{aligned}
& |\bar{\Sigma}|^{-1/2} \\
&= |\Sigma \otimes I_T|^{-1/2} \\
&= |\Sigma|^{-T/2} |I_T|^{-k/2} \\
&= |\Sigma|^{-T/2} \\
&= |\Phi^{-1} \Lambda \Phi^{-1'}|^{-T/2} \\
&= |\Phi^{-1}|^{-T/2} |\Lambda|^{-T/2} |\Phi^{-1'}|^{-T/2} \\
&= |\Phi|^{T/2} |\Lambda|^{-T/2} |\Phi'|^{T/2} \\
&= |\Lambda|^{-T/2} \\
&= \prod_{i=1}^n \lambda_i^{-T/2} \tag{a.4.11.57}
\end{aligned}$$

Consider then the quadratic form in (a.4.11.56):

$$\begin{aligned}
& (y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) \\
&= (y - (I_n \otimes X)\beta)' (\Sigma \otimes I_T)^{-1} (y - (I_n \otimes X)\beta) \\
&= tr [\Sigma^{-1} (Y - X\mathcal{B})' I_T^{-1} (Y - X\mathcal{B})] \\
&= tr [\Sigma^{-1} (Y - X\mathcal{B})' (Y - X\mathcal{B})] \\
&= tr [\Sigma^{-1} \mathcal{E}' \mathcal{E}] \\
&= tr [\mathcal{E} \Sigma^{-1} \mathcal{E}'] \\
&= tr \left[\begin{pmatrix} \varepsilon_1' \Sigma^{-1} \varepsilon_1 & \varepsilon_1' \Sigma^{-1} \varepsilon_2 & \cdots & \varepsilon_1' \Sigma^{-1} \varepsilon_T \\ \varepsilon_2' \Sigma^{-1} \varepsilon_1 & \varepsilon_2' \Sigma^{-1} \varepsilon_2 & \cdots & \varepsilon_2' \Sigma^{-1} \varepsilon_T \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_T' \Sigma^{-1} \varepsilon_1 & \varepsilon_T' \Sigma^{-1} \varepsilon_2 & \cdots & \varepsilon_T' \Sigma^{-1} \varepsilon_T \end{pmatrix} \right] \\
&= \sum_{t=1}^T \varepsilon_t' \Sigma^{-1} \varepsilon_t \\
&= \sum_{t=1}^T \varepsilon_t' (\Phi^{-1} \Lambda \Phi^{-1'})^{-1} \varepsilon_t \\
&= \sum_{t=1}^T \varepsilon_t' (\Phi' \Lambda^{-1} \Phi) \varepsilon_t \\
&= \sum_{t=1}^T (\Phi \varepsilon_t)' \Lambda^{-1} (\Phi \varepsilon_t) \tag{a.4.11.58}
\end{aligned}$$

Consider the first term in the quadratic form:

$$\begin{aligned}
& \Phi \varepsilon_t \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \phi_{21} & 1 & \ddots & \cdots \\ \vdots & \ddots & \ddots & 0 \\ \phi_{n1} & \cdots & \phi_{n(n-1)} & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{n,t} \end{pmatrix} \\
&= \begin{pmatrix} \varepsilon_{1,t} \\ \phi_2 \varepsilon_{-2,t} + \varepsilon_{2,t} \\ \vdots \\ \phi_n \varepsilon_{-n,t} + \varepsilon_{n,t} \end{pmatrix} \tag{a.4.11.59}
\end{aligned}$$

with:

$$\varepsilon_{-i,t} = (\varepsilon_{1,t} \ \varepsilon_{2,t} \ \cdots \ \varepsilon_{i-1,t})' \tag{a.4.11.60}$$

Substitute (a.4.11.59) back in the quadratic form (a.4.11.58):

$$\begin{aligned}
& (\Phi \varepsilon_t)' \Lambda^{-1} (\Phi \varepsilon_t) \\
&= (\varepsilon_{1,t} \ \phi_2 \varepsilon_{-2,t} + \varepsilon_{2,t} \ \cdots \ \phi_n \varepsilon_{-n,t} + \varepsilon_{n,t}) \begin{pmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^{-1} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \phi_2 \varepsilon_{-2,t} + \varepsilon_{2,t} \\ \vdots \\ \phi_n \varepsilon_{-n,t} + \varepsilon_{n,t} \end{pmatrix} \\
&= (\lambda_1^{-1} \varepsilon_{1,t} \ \lambda_2^{-1} (\phi_2 \varepsilon_{-2,t} + \varepsilon_{2,t}) \ \cdots \ \lambda_n^{-1} (\phi_n \varepsilon_{-n,t} + \varepsilon_{n,t})) \begin{pmatrix} \varepsilon_{1,t} \\ \phi_2 \varepsilon_{-2,t} + \varepsilon_{2,t} \\ \vdots \\ \phi_n \varepsilon_{-n,t} + \varepsilon_{n,t} \end{pmatrix} \\
&= \sum_{i=1}^n \lambda_i^{-1} (\varepsilon_{i,t} + \phi_i \varepsilon_{-i,t})^2 \tag{a.4.11.61}
\end{aligned}$$

Substitute (a.4.11.61) back in (a.4.11.58) to obtain:

$$\begin{aligned}
& \sum_{t=1}^T (\Phi \varepsilon_t)' \Lambda^{-1} (\Phi \varepsilon_t) \\
&= \sum_{t=1}^T \sum_{i=1}^n \lambda_i^{-1} (\varepsilon_{i,t} + \phi_i \varepsilon_{-i,t})^2 \\
&= \sum_{i=1}^n \lambda_i^{-1} \left(\sum_{t=1}^T (\varepsilon_{i,t} + \phi_i \varepsilon_{-i,t})^2 \right) \tag{a.4.11.62}
\end{aligned}$$

Then, note that:

$$\begin{aligned}
& \sum_{t=1}^T (\varepsilon_{i,t} + \phi_i \varepsilon_{-i,t})^2 \\
&= (\varepsilon_{i,1} + \phi_i' \varepsilon_{-i,1} \quad \varepsilon_{i,2} + \phi_i' \varepsilon_{-i,2} \quad \cdots \quad \varepsilon_{i,T} + \phi_i' \varepsilon_{-i,T}) \begin{pmatrix} \varepsilon_{i,1} + \phi_i' \varepsilon_{-i,1} \\ \varepsilon_{i,2} + \phi_i' \varepsilon_{-i,2} \\ \vdots \\ \varepsilon_{i,T} + \phi_i' \varepsilon_{-i,T} \end{pmatrix} \\
&= (\varepsilon_{i,1} + \varepsilon_{-i,1}' \phi_i \quad \varepsilon_{i,2} + \varepsilon_{-i,2}' \phi_i \quad \cdots \quad \varepsilon_{i,T} + \varepsilon_{-i,T}' \phi_i) \begin{pmatrix} \varepsilon_{i,1} + \varepsilon_{-i,1}' \phi_i \\ \varepsilon_{i,2} + \varepsilon_{-i,2}' \phi_i \\ \vdots \\ \varepsilon_{i,T} + \varepsilon_{-i,T}' \phi_i \end{pmatrix} \\
&= (\mathcal{E}_i + \mathcal{E}_{-i}' \phi_i)' (\mathcal{E}_i + \mathcal{E}_{-i} \phi_i) \tag{a.4.11.63}
\end{aligned}$$

with:

$$\mathcal{E}_{-i} = (\mathcal{E}_1 \quad \mathcal{E}_2 \quad \cdots \quad \mathcal{E}_{i-1}) \tag{a.4.11.64}$$

Also, noting that $\mathcal{E}_i = Y_i - X\beta_i$:

$$\sum_{t=1}^T (\varepsilon_{i,t} + \phi_i \varepsilon_{-i,t})^2 = (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i)' (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i) \tag{a.4.11.65}$$

Substitute (a.4.11.65) back in (a.4.11.62):

$$\sum_{t=1}^T (\Phi \varepsilon_t)' \Lambda^{-1} (\Phi \varepsilon_t) = \sum_{i=1}^n \lambda_i^{-1} (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i)' (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i) \tag{a.4.11.66}$$

Therefore, substituting (a.4.11.66) back in (a.4.11.58):

$$(y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) = \sum_{i=1}^n \lambda_i^{-1} (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i)' (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i) \tag{a.4.11.67}$$

Finally, replacing (a.4.11.57) and (a.4.11.67) in (a.4.11.56) yields:

$$f(y|\beta, \lambda, \phi) = (2\pi)^{-nT/2} \left(\prod_{i=1}^n \lambda_i^{-T/2} \right) \exp \left(-\frac{1}{2} \sum_{i=1}^n \lambda_i^{-1} (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i)' (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i) \right) \tag{a.4.11.68}$$

derivations for equation (4.11.75)

Start from:

$$\pi(\beta_i|y, \beta_{-i}) \propto \exp \left(-\frac{1}{2} \lambda_i^{-1} (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i)' (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i) \right) \times \exp \left(-\frac{1}{2} (\beta_i - b_i)' V_i^{-1} (\beta_i - b_i) \right) \tag{a.4.11.69}$$

Develop the first quadratic form:

$$\begin{aligned}
& (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i)' (Y_i - X\beta_i + \mathcal{E}_{-i} \phi_i) \\
&= Y_i' Y_i + \beta_i' X' X \beta_i + \phi_i' \mathcal{E}_{-i}' \mathcal{E}_{-i} \phi_i - 2\beta_i' X' (Y_i + \mathcal{E}_{-i} \phi_i) + 2Y_i' \mathcal{E}_{-i} \phi_i \tag{a.4.11.70}
\end{aligned}$$

Develop the second quadratic form:

$$\begin{aligned}
& (\beta_i - b_i)' V_i^{-1} (\beta_i - b_i) \\
&= \beta_i' V_i^{-1} \beta_i + b_i' V_i^{-1} b_i - 2\beta_i' V_i^{-1} b_i
\end{aligned} \tag{a.4.11.71}$$

Substitute (a.4.11.70) and (a.4.11.71) back in (a.4.11.69) to obtain:

$$\begin{aligned}
& \pi(\beta_i | y, \beta_{-i}) \\
& \propto \exp \left(-\frac{1}{2} \lambda_i^{-1} (Y_i' Y_i + \beta_i' X' X \beta_i + \phi_i' \mathcal{E}_{-i}' \mathcal{E}_{-i} \phi_i - 2\beta_i' X' (Y_i + \mathcal{E}_{-i} \phi_i) + 2Y_i' \mathcal{E}_{-i} \phi_i) \right) \\
& \times \exp \left(-\frac{1}{2} (\beta_i' V_i^{-1} \beta_i + b_i' V_i^{-1} b_i - 2\beta_i' V_i^{-1} b_i) \right) \\
&= \exp \left(-\frac{1}{2} (\lambda_i^{-1} \beta_i' X' X \beta_i - 2\lambda_i^{-1} \beta_i' X' (Y_i + \mathcal{E}_{-i} \phi_i) + \beta_i' V_i^{-1} \beta_i - 2\beta_i' V_i^{-1} b_i) \right) \\
& \times \exp \left(-\frac{1}{2} (\lambda_i^{-1} Y_i' Y_i + \lambda_i^{-1} \phi_i' \mathcal{E}_{-i}' \mathcal{E}_{-i} \phi_i + 2\lambda_i^{-1} Y_i' \mathcal{E}_{-i} \phi_i + b_i' V_i^{-1} b_i) \right) \\
& \propto \exp \left(-\frac{1}{2} (\lambda_i^{-1} \beta_i' X' X \beta_i - 2\lambda_i^{-1} \beta_i' X' (Y_i + \mathcal{E}_{-i} \phi_i) + \beta_i' V_i^{-1} \beta_i - 2\beta_i' V_i^{-1} b_i) \right) \\
& \propto \exp \left(-\frac{1}{2} [\beta_i' (\lambda_i^{-1} X' X + V_i^{-1}) \beta_i - 2\beta_i' (V_i^{-1} b_i + \lambda_i^{-1} X' [Y_i + \mathcal{E}_{-i} \phi_i])] \right)
\end{aligned} \tag{a.4.11.72}$$

Consider the term within the square brackets and complete the squares:

$$\begin{aligned}
& \beta_i' (\lambda_i^{-1} X' X + V_i^{-1}) \beta_i - 2\beta_i' (V_i^{-1} b_i + \lambda_i^{-1} X' [Y_i + \mathcal{E}_{-i} \phi_i]) \\
&= \beta_i' (\lambda_i^{-1} X' X + V_i^{-1}) \beta_i - 2\beta_i' \bar{V}_i^{-1} \bar{V}_i (V_i^{-1} b_i + \lambda_i^{-1} X' [Y_i + \mathcal{E}_{-i} \phi_i]) + \bar{b}_i' \bar{V}_i^{-1} \bar{b}_i - \bar{b}_i' \bar{V}_i^{-1} \bar{b}_i
\end{aligned} \tag{a.4.11.73}$$

Define:

$$\bar{V}_i = (\lambda_i^{-1} X' X + V_i^{-1})^{-1} \quad \bar{b}_i = \bar{V}_i (V_i^{-1} b_i + \lambda_i^{-1} X' [Y_i + \mathcal{E}_{-i} \phi_i]) \tag{a.4.11.74}$$

Then (a.4.11.73) rewrites:

$$\begin{aligned}
& \beta_i' (\lambda_i^{-1} X' X + V_i^{-1}) \beta_i - 2\beta_i' (V_i^{-1} b_i + \lambda_i^{-1} X' [Y_i + \mathcal{E}_{-i} \phi_i]) \\
&= \beta_i' \bar{V}_i^{-1} \beta_i - 2\beta_i' \bar{V}_i^{-1} \bar{b}_i + \bar{b}_i' \bar{V}_i^{-1} \bar{b}_i - \bar{b}_i' \bar{V}_i^{-1} \bar{b}_i \\
&= (\beta_i - \bar{b}_i)' \bar{V}_i^{-1} (\beta_i - \bar{b}_i) - \bar{b}_i' \bar{V}_i^{-1} \bar{b}_i
\end{aligned} \tag{a.4.11.75}$$

Substitute (a.4.11.75) back in (a.4.11.72) to obtain:

$$\begin{aligned}
& \pi(\beta_i | y, \beta_{-i}) \\
& \propto \exp \left(-\frac{1}{2} [(\beta_i - \bar{b}_i)' \bar{V}_i^{-1} (\beta_i - \bar{b}_i) - \bar{b}_i' \bar{V}_i^{-1} \bar{b}_i] \right) \\
&= \exp \left(-\frac{1}{2} (\beta_i - \bar{b}_i)' \bar{V}_i^{-1} (\beta_i - \bar{b}_i) \right) \exp \left(-\frac{1}{2} (-\bar{b}_i' \bar{V}_i^{-1} \bar{b}_i) \right) \\
& \propto \exp \left(-\frac{1}{2} (\beta_i - \bar{b}_i)' \bar{V}_i^{-1} (\beta_i - \bar{b}_i) \right)
\end{aligned} \tag{a.4.11.76}$$

derivations for equation (4.11.81)

Rearrange the initial formula:

$$\begin{aligned}
& \pi(\phi_i|y, \phi_{-i}) \\
& \propto \exp\left(-\frac{1}{2}\lambda_i^{-1}(Y_i - X\beta_i + \mathcal{E}_{-i}\phi_i)'(Y_i - X\beta_i + \mathcal{E}_{-i}\phi_i)\right) \times \exp\left(-\frac{1}{2}\tau^{-1}\phi_i'\phi_i\right) \\
& = \exp\left(-\frac{1}{2}\left[\lambda_i^{-1}(Y_i - X\beta_i + \mathcal{E}_{-i}\phi_i)'(Y_i - X\beta_i + \mathcal{E}_{-i}\phi_i) + \tau^{-1}\phi_i'\phi_i\right]\right) \quad (\text{a.4.11.77})
\end{aligned}$$

Consider the term within the square brackets:

$$\begin{aligned}
& \lambda_i^{-1}(Y_i - X\beta_i + \mathcal{E}_{-i}\phi_i)'(Y_i - X\beta_i + \mathcal{E}_{-i}\phi_i) + \tau^{-1}\phi_i'\phi_i \\
& = \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i + \phi_i'\mathcal{E}_{-i}'\mathcal{E}_{-i}\phi_i + 2\phi_i'\mathcal{E}_{-i}'[Y_i - X\beta_i] - 2Y_i'X\beta_i) + \tau^{-1}\phi_i'\phi_i \\
& = \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i + \phi_i'\mathcal{E}_{-i}'\mathcal{E}_{-i}\phi_i - 2\phi_i'(-\mathcal{E}_{-i}'[Y_i - X\beta_i]) - 2Y_i'X\beta_i) + \tau^{-1}\phi_i'\phi_i \\
& = \phi_i'(\tau^{-1}I_{i-1} + \lambda_i^{-1}\mathcal{E}_{-i}'\mathcal{E}_{-i})\phi_i - 2\phi_i'(-\lambda_i^{-1}\mathcal{E}_{-i}'[Y_i - X\beta_i]) + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) \quad (\text{a.4.11.78})
\end{aligned}$$

Complete the squares:

$$\begin{aligned}
& = \phi_i'(\tau^{-1}I_{i-1} + \lambda_i^{-1}\mathcal{E}_{-i}'\mathcal{E}_{-i})\phi_i - 2\phi_i'\bar{Z}_i^{-1}\bar{Z}_i(-\lambda_i^{-1}\mathcal{E}_{-i}'[Y_i - X\beta_i]) + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) \\
& + \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i \quad (\text{a.4.11.79})
\end{aligned}$$

Define:

$$\bar{Z}_i = (\tau^{-1}I_{i-1} + \lambda_i^{-1}\mathcal{E}_{-i}'\mathcal{E}_{-i})^{-1} \quad \bar{f}_i = \bar{Z}_i(-\lambda_i^{-1}\mathcal{E}_{-i}'[Y_i - X\beta_i]) \quad (\text{a.4.11.80})$$

Then (a.4.11.79) rewrites:

$$\begin{aligned}
& = \phi_i'\bar{Z}_i^{-1}\phi_i - 2\phi_i'\bar{Z}_i^{-1}\bar{f}_i + \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i \\
& = (\phi_i - \bar{f}_i)'\bar{Z}_i^{-1}(\phi_i - \bar{f}_i) + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i \quad (\text{a.4.11.81})
\end{aligned}$$

Substitute (a.4.11.81) back in (a.4.11.77):

$$\begin{aligned}
& \pi(\phi_i|y, \phi_{-i}) \\
& \propto \exp\left(-\frac{1}{2}\left[(\phi_i - \bar{f}_i)'\bar{Z}_i^{-1}(\phi_i - \bar{f}_i) + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i\right]\right) \\
& = \exp\left(-\frac{1}{2}(\phi_i - \bar{f}_i)'\bar{Z}_i^{-1}(\phi_i - \bar{f}_i)\right) \exp\left(-\frac{1}{2}\left[\lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i\right]\right) \\
& \propto \exp\left(-\frac{1}{2}(\phi_i - \bar{f}_i)'\bar{Z}_i^{-1}(\phi_i - \bar{f}_i)\right) \quad (\text{a.4.11.82})
\end{aligned}$$

Eventually, notice that:

$$\bar{f}_i = \bar{Z}_i(-\lambda_i^{-1}\mathcal{E}_{-i}'[Y_i - X\beta_i]) = \bar{Z}_i(-\lambda_i^{-1}\mathcal{E}_{-i}'\mathcal{E}_i) \quad (\text{a.4.11.83})$$

Further aspects of Bayesian vector autoregressions

derivations for equation (4.12.3)

Consider the general VAR model and reformulate it:

$$\begin{aligned}
 y_t &= Cz_t + A_1 y_{t-1} + \dots + A_p y_{t-p} + \varepsilon_t \\
 \Leftrightarrow y_t - y_{t-1} &= Cz_t + A_1 y_{t-1} + \dots + A_p y_{t-p} - y_{t-1} + \varepsilon_t \\
 \Leftrightarrow \Delta y_t &= Cz_t + \sum_{i=1}^p A_i y_{t-i} - y_{t-1} + \varepsilon_t \\
 \Leftrightarrow \Delta y_t &= Cz_t + \sum_{i=1}^p \left(A_i y_{t-i} + \sum_{j=i+1}^p A_j y_{t-i} - \sum_{j=i+1}^p A_j y_{t-i} \right) - y_{t-1} + \varepsilon_t \\
 \Leftrightarrow \Delta y_t &= Cz_t + \left(\sum_{i=1}^p A_i - I \right) y_{t-1} - \sum_{i=1}^{p-1} \left(\sum_{j=i+1}^p A_j \right) (y_{t-i} - y_{t-i-1}) + \varepsilon_t \\
 \Leftrightarrow \Delta y_t &= Cz_t + \left(\sum_{i=1}^p A_i - I \right) y_{t-1} + \sum_{i=1}^{p-1} B_i \Delta y_{t-i} + \varepsilon_t
 \end{aligned} \tag{a.4.12.1}$$

with:

$$B_i = - \sum_{j=i+1}^p A_j \tag{a.4.12.2}$$

derivations for equation (4.12.17)

Start from:

$$\begin{aligned}
 Y_{lrp} &= X_{lrp} \mathcal{B} + \mathcal{E}_{lrp} \\
 \Leftrightarrow Y_{lrp} &= (0_{n \times m} \quad \mathbf{1}'_p \otimes Y_{lrp}) \begin{pmatrix} C' \\ A'_1 \\ \vdots \\ A'_p \end{pmatrix} + \mathcal{E}_{lrp} \\
 \Leftrightarrow Y_{lrp} &= Y_{lrp} A'_1 + \dots + Y_{lrp} A'_p + \mathcal{E}_{lrp} \\
 \Leftrightarrow Y'_{lrp} &= A_1 Y'_{lrp} + \dots + A_p Y'_{lrp} + \mathcal{E}'_{lrp} \\
 \Leftrightarrow (A_1 + \dots + A_p - I) Y'_{lrp} &= -\mathcal{E}'_{lrp} \\
 \Leftrightarrow (A_1 + \dots + A_p - I) [diag(H \bar{y} / \pi_7) H^{-1}]' &= -\mathcal{E}'_{lrp} \\
 \Leftrightarrow (A_1 + \dots + A_p - I) H^{-1} diag(H \bar{y} / \pi_7) &= -\mathcal{E}'_{lrp} \\
 \Leftrightarrow (A_1 + \dots + A_p - I) H^{-1} &= -diag(\pi_7 / H \bar{y}) \mathcal{E}'_{lrp}
 \end{aligned} \tag{a.4.12.3}$$

derivations for equation (4.12.20)

Focus on the term within the integral and rearrange:

$$\begin{aligned}
 & (2\pi)^{-nT/2} |\bar{\Sigma}|^{-1/2} \exp \left(-\frac{1}{2} (y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) \right) \times (2\pi)^{-q/2} |V|^{-1/2} \exp \left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b) \right) \\
 &= (2\pi)^{-nT/2} |\bar{\Sigma}|^{-1/2} (2\pi)^{-q/2} |V|^{-1/2} \exp \left(-\frac{1}{2} [(y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) + (\beta - b)' V^{-1} (\beta - b)] \right)
 \end{aligned} \tag{a.4.12.4}$$

Also:

$$|\bar{\Sigma}|^{-1/2} = |\Sigma \otimes I_T|^{-1/2} = |\Sigma|^{-T/2} |I_T|^{-n/2} = |\Sigma|^{-T/2} \tag{a.4.12.5}$$

And:

$$\begin{aligned}
 & (y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) + (\beta - b)' V^{-1} (\beta - b) \\
 &= y' \bar{\Sigma}^{-1} y + \beta' \bar{X}' \bar{\Sigma}^{-1} \bar{X} \beta - 2\beta' \bar{X}' \bar{\Sigma}^{-1} y + \beta' V^{-1} \beta + b' V^{-1} b - 2\beta' V^{-1} b \\
 &= \beta' (V^{-1} + \bar{X}' \bar{\Sigma}^{-1} \bar{X}) \beta - 2\beta' (V^{-1} b + \bar{X}' \bar{\Sigma}^{-1} y) + b' V^{-1} b + y' \bar{\Sigma}^{-1} y
 \end{aligned} \tag{a.4.12.6}$$

Complete the squares:

$$= \beta' (V^{-1} + \bar{X}' \bar{\Sigma}^{-1} \bar{X}) \beta - 2\beta' \bar{V}^{-1} \bar{V} (V^{-1} b + \bar{X}' \bar{\Sigma}^{-1} y) + \bar{b}' \bar{V}^{-1} \bar{b} - \bar{b}' \bar{V}^{-1} \bar{b} + b' V^{-1} b + y' \bar{\Sigma}^{-1} y \tag{a.4.12.7}$$

Define:

$$\bar{V} = (V^{-1} + \bar{X}' \bar{\Sigma}^{-1} \bar{X})^{-1} \quad \bar{b} = \bar{V} (V^{-1} b + \bar{X}' \bar{\Sigma}^{-1} y) \tag{a.4.12.8}$$

Then (a.4.12.7) rewrites:

$$\begin{aligned}
 &= \beta' \bar{V}^{-1} \beta - 2\beta' \bar{V}^{-1} \bar{b} + \bar{b}' \bar{V}^{-1} \bar{b} - \bar{b}' \bar{V}^{-1} \bar{b} + b' V^{-1} b + y' \bar{\Sigma}^{-1} y \\
 &= (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + (b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} + y' \bar{\Sigma}^{-1} y)
 \end{aligned} \tag{a.4.12.9}$$

The terms in (a.4.12.8) simplify. Note first that:

$$\bar{X}' \bar{\Sigma}^{-1} \bar{X} = (I_n \otimes X)' (\Sigma \otimes I_T)^{-1} (I_n \otimes X) = (I_n \otimes X') (\Sigma^{-1} \otimes I_T) (I_n \otimes X) = (\Sigma^{-1} \otimes X') (I_n \otimes X) = \Sigma^{-1} \otimes X' X \tag{a.4.12.10}$$

Similarly:

$$\bar{X}' \bar{\Sigma}^{-1} y = (I_n \otimes X)' (\Sigma \otimes I_T)^{-1} \text{vec}(Y) = (I_n \otimes X') (\Sigma^{-1} \otimes I_T) \text{vec}(Y) = (\Sigma^{-1} \otimes X') \text{vec}(Y) = \text{vec}(X' Y \Sigma^{-1}) \tag{a.4.12.11}$$

Substituting (a.4.12.5) and (a.4.12.9) back in (a.4.12.4), the term within the integral rewrites:

$$\begin{aligned}
 & (2\pi)^{-nT/2} |\Sigma|^{-T/2} (2\pi)^{-q/2} |V|^{-1/2} \exp \left(-\frac{1}{2} [(\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + (b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} + y' \bar{\Sigma}^{-1} y)] \right) \\
 &= (2\pi)^{-nT/2} |\Sigma|^{-T/2} (2\pi)^{-q/2} |V|^{-1/2} |\bar{V}|^{1/2} |\bar{V}|^{-1/2} \\
 &\times \exp \left(-\frac{1}{2} [(\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) + (b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} + y' \bar{\Sigma}^{-1} y)] \right) \\
 &= (2\pi)^{-nT/2} |\Sigma|^{-T/2} |V|^{-1/2} |\bar{V}|^{1/2} \exp \left(-\frac{1}{2} (b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} + y' \bar{\Sigma}^{-1} y) \right) \\
 &\times (2\pi)^{-q/2} \bar{V}^{-1} \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right)
 \end{aligned} \tag{a.4.12.12}$$

The expression simplifies further. Note that:

$$\begin{aligned}
& |V|^{-1/2} |\bar{V}|^{1/2} \\
&= |V|^{-1/2} |(V^{-1} + \Sigma^{-1} \otimes X'X)^{-1}|^{1/2} \\
&= |V|^{-1/2} |(V^{-1} + \Sigma^{-1} \otimes X'X)|^{-1/2} \\
&= |V (V^{-1} + \Sigma^{-1} \otimes X'X)|^{-1/2} \\
&= |I + V(\Sigma^{-1} \otimes X'X)|^{-1/2}
\end{aligned} \tag{a.4.12.13}$$

Also:

$$\begin{aligned}
& y' \bar{\Sigma}^{-1} y \\
&= y' (\Sigma \otimes I_T)^{-1} y \\
&= y' (\Sigma^{-1} \otimes I_T) y \\
&= y' \text{vec}(Y \Sigma^{-1}) \\
&= \text{vec}(Y)' \text{vec}(Y \Sigma^{-1}) \\
&= \text{tr}(Y' Y \Sigma^{-1})
\end{aligned} \tag{a.4.12.14}$$

Thus, substituting again (a.4.12.13) and (a.4.12.14) back in (a.4.12.12), the term within the integral eventually rewrites as:

$$\begin{aligned}
& (2\pi)^{-nT/2} |\Sigma|^{-T/2} |I + V(\Sigma^{-1} \otimes X'X)|^{-1/2} \exp \left(-\frac{1}{2} [b' V^{-1} b - \bar{b}' \bar{V}^{-1} \bar{b} + \text{tr}(Y' Y \Sigma^{-1})] \right) \\
& \times (2\pi)^{-q/2} \bar{V}^{-1} \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right)
\end{aligned} \tag{a.4.12.15}$$

derivations for equation (4.12.23)

Focus on the term within the integral and rearrange:

$$\begin{aligned}
& (2\pi)^{-nT/2} |\Sigma|^{-T/2} \exp \left(-\frac{1}{2} (\beta - \hat{\beta})' (\Sigma \otimes (X'X)^{-1})^{-1} (\beta - \hat{\beta}) \right) \\
& \times \exp \left(-\frac{1}{2} \text{tr} [\Sigma^{-1} (Y - X\hat{B})' (Y - X\hat{B})] \right) \\
& \times (2\pi)^{-q/2} |\Sigma \otimes W|^{-1/2} \exp \left(-\frac{1}{2} (\beta - b)' (\Sigma \otimes W)^{-1} (\beta - b) \right) \\
& \times \frac{2^{-\alpha n/2}}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2} |\Sigma|^{-(\alpha+n+1)/2} \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} S \} \right) \\
&= (2\pi)^{-nT/2} \frac{2^{-\alpha n/2}}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2} \\
& \times (2\pi)^{-nk/2} |\Sigma \otimes W|^{-1/2} \exp \left(-\frac{1}{2} [(\beta - b)' (\Sigma \otimes W)^{-1} (\beta - b) + (\beta - \hat{\beta})' (\Sigma \otimes (X'X)^{-1})^{-1} (\beta - \hat{\beta})] \right) \\
& \times |\Sigma|^{-(\alpha+T+n+1)/2} \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} [S + (Y - X\hat{B})' (Y - X\hat{B})] \} \right)
\end{aligned} \tag{a.4.12.16}$$

Note first that:

$$|\Sigma \otimes W|^{-1/2} = |\Sigma|^{-k/2} |W|^{-n/2} \tag{a.4.12.17}$$

Then focus on the first term between square brackets to obtain:

$$\begin{aligned}
& (\beta - b)'(\Sigma \otimes W)^{-1}(\beta - b) + (\beta - \hat{\beta})'(\Sigma \otimes (X'X)^{-1})^{-1}(\beta - \hat{\beta}) \\
&= tr\{\Sigma^{-1}(\mathcal{B} - B)'W^{-1}(\mathcal{B} - B)\} + tr\{\Sigma^{-1}(\mathcal{B} - \hat{B})'(X'X)(\mathcal{B} - \hat{B})\} \\
&= tr\{\Sigma^{-1}[(\mathcal{B} - B)'W^{-1}(\mathcal{B} - B) + (\mathcal{B} - \hat{B})'(X'X)(\mathcal{B} - \hat{B})]\} \\
&= tr\{\Sigma^{-1}[(\mathcal{B} - B)'W^{-1}(\mathcal{B} - B) + (\mathcal{B} - \hat{B})'(X'X)(\mathcal{B} - \hat{B})]\} \\
&= tr\{\Sigma^{-1}[\mathcal{B}'W^{-1}\mathcal{B} + B'W^{-1}B - 2\mathcal{B}'W^{-1}B + \mathcal{B}'(X'X)\mathcal{B} + \hat{B}'(X'X)\hat{B} - 2\mathcal{B}'(X'X)\hat{B}]\} \\
&= tr\{\Sigma^{-1}[\mathcal{B}'(W^{-1} + X'X)\mathcal{B} - 2\mathcal{B}'(W^{-1}B + X'X\hat{B}) + B'W^{-1}B + \hat{B}'(X'X)\hat{B}]\} \tag{a.4.12.18}
\end{aligned}$$

Complete the squares:

$$= tr\{\Sigma^{-1}[\mathcal{B}'(W^{-1} + X'X)\mathcal{B} - 2\mathcal{B}'\bar{W}^{-1}\bar{W}(W^{-1}B + X'X\hat{B}) + \bar{B}'\bar{W}^{-1}\bar{B} - \bar{B}'\bar{W}^{-1}\bar{B} + B'W^{-1}B + \hat{B}'(X'X)\hat{B}]\} \tag{a.4.12.19}$$

Define:

$$\bar{W} = (W^{-1} + X'X)^{-1} \quad \bar{B} = \bar{W}(W^{-1}B + X'X\hat{B}) \tag{a.4.12.20}$$

then the expression becomes:

$$\begin{aligned}
&= tr\{\Sigma^{-1}[\mathcal{B}'\bar{W}^{-1}\mathcal{B} - 2\mathcal{B}'\bar{W}^{-1}\bar{B} + \bar{B}'\bar{W}^{-1}\bar{B} - \bar{B}'\bar{W}^{-1}\bar{B} + B'W^{-1}B + \hat{B}'(X'X)\hat{B}]\} \\
&= tr\{\Sigma^{-1}[(\mathcal{B}' - \bar{B})'\bar{W}^{-1}(\mathcal{B}' - \bar{B}) - \bar{B}'\bar{W}^{-1}\bar{B} + B'W^{-1}B + \hat{B}'(X'X)\hat{B}]\} \\
&= tr\{\Sigma^{-1}(\mathcal{B}' - \bar{B})'\bar{W}^{-1}(\mathcal{B}' - \bar{B})\} + tr\{\Sigma^{-1}[B'W^{-1}B + \hat{B}'(X'X)\hat{B} - \bar{B}'\bar{W}^{-1}\bar{B}]\} \tag{a.4.12.21}
\end{aligned}$$

Substituting (a.4.12.17) and (a.4.12.21) in (a.4.12.16) yields:

$$\begin{aligned}
& (2\pi)^{-nT/2} |W|^{-n/2} \frac{2^{-\alpha n/2}}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2} \\
& \times (2\pi)^{-nk/2} |\Sigma|^{-k/2} \exp\left(-\frac{1}{2} tr\{\Sigma^{-1}(\mathcal{B}' - \bar{B})'\bar{W}^{-1}(\mathcal{B}' - \bar{B})\}\right) \\
& \times |\Sigma|^{-(\alpha + T + n + 1)/2} \exp\left(-\frac{1}{2} tr\{\Sigma^{-1}[S + (Y - X\hat{B})'(Y - X\hat{B}) + B'W^{-1}B + \hat{B}'(X'X)\hat{B} - \bar{B}'\bar{W}^{-1}\bar{B}]\}\right) \tag{a.4.12.22}
\end{aligned}$$

Define:

$$\bar{\alpha} = \alpha + T \quad \bar{S} = S + (Y - X\hat{B})'(Y - X\hat{B}) + B'W^{-1}B + \hat{B}'(X'X)\hat{B} - \bar{B}'\bar{W}^{-1}\bar{B} \tag{a.4.12.23}$$

Then the expression becomes:

$$\begin{aligned}
& (2\pi)^{-nT/2} |W|^{-n/2} \frac{2^{-\alpha n/2}}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2} \\
& \times (2\pi)^{-nk/2} |\Sigma|^{-k/2} \exp\left(-\frac{1}{2} tr\{\Sigma^{-1}(\mathcal{B}' - \bar{B})'\bar{W}^{-1}(\mathcal{B}' - \bar{B})\}\right) \\
& \times |\Sigma|^{-(\bar{\alpha} + n + 1)/2} \exp\left(-\frac{1}{2} tr\{\Sigma^{-1}\bar{S}\}\right) \tag{a.4.12.24}
\end{aligned}$$

And finally, this rewrites:

$$\begin{aligned}
& (2\pi)^{-nT/2} |W|^{-n/2} |\bar{W}|^{n/2} \frac{2^{\bar{\alpha}n/2}}{2^{\alpha n/2}} \frac{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}{\Gamma_n\left(\frac{\alpha}{2}\right)} |S|^{\alpha/2} |\bar{S}|^{-\bar{\alpha}/2} \\
& \times (2\pi)^{-nk/2} |\Sigma|^{-k/2} |\bar{W}|^{-n/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}(\mathcal{B}' - \bar{\mathcal{B}})' \bar{W}^{-1}(\mathcal{B}' - \bar{\mathcal{B}})\}\right) \\
& \times \frac{2^{-\bar{\alpha}n/2}}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)} |\bar{S}|^{\bar{\alpha}/2} |\Sigma|^{-(\bar{\alpha}+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1} \bar{S}\}\right)
\end{aligned} \tag{a.4.12.25}$$

Go on simplifying. Note that:

$$(2\pi)^{-nT/2} \frac{2^{\bar{\alpha}n/2}}{2^{\alpha n/2}} = 2^{-nT/2} \pi^{-nT/2} \frac{2^{(\alpha+T)n/2}}{2^{\alpha n/2}} = \pi^{-nT/2} \tag{a.4.12.26}$$

Also:

$$\begin{aligned}
& |W|^{-n/2} |\bar{W}|^{n/2} \\
& = |W|^{-n/2} |(W^{-1} + X'X)^{-1}|^{n/2} \\
& = |W|^{-n/2} |(W^{-1} + X'X)|^{-n/2} \\
& = |W(W^{-1} + X'X)|^{-n/2} \\
& = |I + WX'X|^{-n/2}
\end{aligned} \tag{a.4.12.27}$$

Consider then:

$$\begin{aligned}
& (Y - X\hat{B})'(Y - X\hat{B}) + \hat{B}'X'X\hat{B} \\
& = Y'Y + \hat{B}'X'X\hat{B} - \hat{B}'X'Y - Y'X\hat{B} + \hat{B}'X'X\hat{B} \\
& = Y'Y + 2\hat{B}'X'X\hat{B} - \hat{B}'X'Y - Y'X\hat{B} \\
& = Y'Y + 2Y'X(X'X)^{-1}X'X(X'X)^{-1}X'Y - Y'X(X'X)^{-1}X'Y - Y'X(X'X)^{-1}X'Y \\
& = Y'Y + 2Y'X(X'X)^{-1}X'Y - Y'X(X'X)^{-1}X'Y - Y'X(X'X)^{-1}X'Y \\
& = Y'Y
\end{aligned} \tag{a.4.12.28}$$

Substitute back in (a.4.12.23):

$$\bar{S} = S + Y'Y + B'W^{-1}B - \bar{B}'\bar{W}^{-1}\bar{B} \tag{a.4.12.29}$$

Then:

$$\begin{aligned}
& = |S|^{\alpha/2} |\bar{S}|^{-\bar{\alpha}/2} \\
& = |S|^{\alpha/2} |S + Y'Y + B'W^{-1}B - \bar{B}'\bar{W}^{-1}\bar{B}|^{-\bar{\alpha}/2} \\
& = |S|^{-T/2} |S|^{(\alpha+T)/2} |S + Y'Y + B'W^{-1}B - \bar{B}'\bar{W}^{-1}\bar{B}|^{-\bar{\alpha}/2} \\
& = |S|^{-T/2} |S|^{\bar{\alpha}/2} |S + Y'Y + B'W^{-1}B - \bar{B}'\bar{W}^{-1}\bar{B}|^{-\bar{\alpha}/2} \\
& = |S|^{-T/2} |S^{-1}|^{-\bar{\alpha}/2} |S + Y'Y + B'W^{-1}B - \bar{B}'\bar{W}^{-1}\bar{B}|^{-\bar{\alpha}/2} \\
& = |S|^{-T/2} |S^{-1}(S + Y'Y + B'W^{-1}B - \bar{B}'\bar{W}^{-1}\bar{B})|^{-\bar{\alpha}/2} \\
& = |S|^{-T/2} |I + S^{-1}(Y'Y + B'W^{-1}B - \bar{B}'\bar{W}^{-1}\bar{B})|^{-\bar{\alpha}/2} \\
& = |S|^{-T/2} |I + S^{-1}(\bar{S} - S)|^{-\bar{\alpha}/2}
\end{aligned} \tag{a.4.12.30}$$

Substituting back (a.4.12.26), (a.4.12.27) and (a.4.12.30) in (a.4.12.25) finally yields:

$$\begin{aligned} & \pi^{-nT/2} |I + WX'X|^{-n/2} |S|^{-T/2} |I + S^{-1}(\bar{S} - S)|^{-\bar{\alpha}/2} \frac{\Gamma_n(\frac{\bar{\alpha}}{2})}{\Gamma_n(\frac{\alpha}{2})} \\ & \times (2\pi)^{-nk/2} |\Sigma|^{-k/2} |\bar{W}|^{-n/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}(\mathcal{B}' - \bar{B})' \bar{W}^{-1}(\mathcal{B}' - \bar{B})\}\right) \\ & \times \frac{2^{-\bar{\alpha}n/2}}{\Gamma_n(\frac{\bar{\alpha}}{2})} |\bar{S}|^{\bar{\alpha}/2} |\Sigma|^{-(\bar{\alpha}+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1} \bar{S}\}\right) \end{aligned} \quad (\text{a.4.12.31})$$

Therefore, substituting back in the integrals, we obtain:

$$\begin{aligned} f(y) &= \pi^{-nT/2} |I + WX'X|^{-n/2} |S|^{-T/2} |I + S^{-1}(\bar{S} - S)|^{-\bar{\alpha}/2} \frac{\Gamma_n(\frac{\bar{\alpha}}{2})}{\Gamma_n(\frac{\alpha}{2})} \\ & \times \int \int (2\pi)^{-nk/2} |\Sigma|^{-k/2} |\bar{W}|^{-n/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1}(\mathcal{B}' - \bar{B})' \bar{W}^{-1}(\mathcal{B}' - \bar{B})\}\right) d\beta \\ & \times \frac{2^{-\bar{\alpha}n/2}}{\Gamma_n(\frac{\bar{\alpha}}{2})} |\bar{S}|^{\bar{\alpha}/2} |\Sigma|^{-(\bar{\alpha}+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1} \bar{S}\}\right) d\sigma \end{aligned} \quad (\text{a.4.12.32})$$

derivations for equation (4.12.26)

$$\begin{aligned} & f(y) \\ & \approx \frac{f(y|\beta^*, \Sigma^*) \pi(\beta^*, \Sigma^*)}{\pi(\Sigma^*|y, \beta^*) \times \frac{1}{J} \sum_{j=1}^J \pi(\beta^*|\Sigma^{(j)}, y)} \\ & = (2\pi)^{-nT/2} |\bar{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2} (y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta)\right) \\ & \times \frac{(2\pi)^{-q/2} |V|^{-1/2} \exp\left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-q/2} |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b})\right)} \times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2} |\Sigma|^{-(\alpha+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1} S\}\right)}{\frac{2^{-\bar{\alpha}n/2}}{\Gamma_n(\frac{\bar{\alpha}}{2})} |\bar{S}|^{\bar{\alpha}/2} |\Sigma|^{-(\bar{\alpha}+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1} \bar{S}\}\right)} \end{aligned} \quad (\text{a.4.12.33})$$

Now, note that:

$$|\bar{\Sigma}|^{-1/2} = |\Sigma \otimes I_T|^{-1/2} = |\Sigma|^{-T/2} |I_T|^{-n/2} = |\Sigma|^{-T/2} \quad (\text{a.4.12.34})$$

Also:

$$\begin{aligned} & (y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) \\ & = (y - (I_n \otimes X)\beta)' (\Sigma \otimes I_T)^{-1} (y - (I_n \otimes X)\beta) \\ & = \text{tr}\{\Sigma^{-1} (Y - XB)' (Y - XB)\} \end{aligned} \quad (\text{a.4.12.35})$$

Substituting back (a.4.12.34) and (a.4.12.35) in (a.4.12.33):

$$\begin{aligned} & f(y) \\ & \approx (2\pi)^{-nT/2} |\Sigma|^{-T/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1} (Y - XB)' (Y - XB)\}\right) \\ & \times \frac{(2\pi)^{-q/2} |V|^{-1/2} \exp\left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b)\right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-q/2} |\bar{V}|^{-1/2} \exp\left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b})\right)} \times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2} |\Sigma|^{-(\alpha+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1} S\}\right)}{\frac{2^{-\bar{\alpha}n/2}}{\Gamma_n(\frac{\bar{\alpha}}{2})} |\bar{S}|^{\bar{\alpha}/2} |\Sigma|^{-(\bar{\alpha}+n+1)/2} \exp\left(-\frac{1}{2} \text{tr}\{\Sigma^{-1} \bar{S}\}\right)} \end{aligned} \quad (\text{a.4.12.36})$$

Continue with the definitions of $\bar{\alpha}$ and \bar{S} :

$$\begin{aligned}
&= 2^{-nT/2} \pi^{-nT/2} |\Sigma|^{-T/2} \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} (Y - X \mathcal{B})' (Y - X \mathcal{B}) \} \right) \\
&\times \frac{(2\pi)^{-q/2} |V|^{-1/2} \exp \left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b) \right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-q/2} |\bar{V}|^{-1/2} \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right)} \\
&\times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2} |\Sigma|^{-(\alpha+n+1)/2} \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} S \} \right)}{\frac{2^{-(\alpha+T)n/2}}{\Gamma_n(\frac{\alpha}{2})} |\bar{S}|^{\bar{\alpha}/2} |\Sigma|^{-(\alpha+T+n+1)/2} \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} [S + (Y - X \mathcal{B})' (Y - X \mathcal{B})] \} \right)} \\
&= 2^{-nT/2} \pi^{-nT/2} |\Sigma|^{-T/2} \\
&\times \frac{(2\pi)^{-q/2} |V|^{-1/2} \exp \left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b) \right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-q/2} |\bar{V}|^{-1/2} \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right)} \times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2} |\Sigma|^{-(\alpha+n+1)/2}}{\frac{2^{-(\alpha+T)n/2}}{\Gamma_n(\frac{\alpha}{2})} |\bar{S}|^{\bar{\alpha}/2} |\Sigma|^{-(\alpha+T+n+1)/2}} \\
&= 2^{-nT/2} \pi^{-nT/2} \times \frac{(2\pi)^{-q/2} |V|^{-1/2} \exp \left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b) \right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-q/2} |\bar{V}|^{-1/2} \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right)} \times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2}}{\frac{2^{-(\alpha+T)n/2}}{\Gamma_n(\frac{\alpha}{2})} |\bar{S}|^{\bar{\alpha}/2}} \\
&= \pi^{-nT/2} \times \frac{(2\pi)^{-q/2} |V|^{-1/2} \exp \left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b) \right)}{\frac{1}{J} \sum_{j=1}^J (2\pi)^{-q/2} |\bar{V}|^{-1/2} \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right)} \times \frac{\frac{1}{\Gamma_n(\frac{\alpha}{2})} |S|^{\alpha/2}}{\frac{1}{\Gamma_n(\frac{\alpha}{2})} |\bar{S}|^{\bar{\alpha}/2}} \\
&= \pi^{-nT/2} \frac{\exp \left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b) \right)}{\frac{1}{J} \sum_{j=1}^J |V|^{1/2} |\bar{V}|^{-1/2} \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right)} \frac{\Gamma_n(\frac{\alpha}{2})}{\Gamma_n(\frac{\alpha}{2})} \frac{|S|^{\alpha/2}}{|\bar{S}|^{\bar{\alpha}/2}} \tag{a.4.12.37}
\end{aligned}$$

Note that:

$$\begin{aligned}
&= |S|^{\alpha/2} |\bar{S}|^{-\bar{\alpha}/2} \\
&= |S|^{\alpha/2} |S + (Y - X \mathcal{B})' (Y - X \mathcal{B})|^{-\bar{\alpha}/2} \\
&= |S|^{-T/2} |S|^{(\alpha+T)/2} |S + (Y - X \mathcal{B})' (Y - X \mathcal{B})|^{-\bar{\alpha}/2} \\
&= |S|^{-T/2} |S|^{\bar{\alpha}/2} |S + (Y - X \mathcal{B})' (Y - X \mathcal{B})|^{-\bar{\alpha}/2} \\
&= |S|^{-T/2} |S^{-1}|^{-\bar{\alpha}/2} |S + (Y - X \mathcal{B})' (Y - X \mathcal{B})|^{-\bar{\alpha}/2} \\
&= |S|^{-T/2} |S^{-1} (S + (Y - X \mathcal{B})' (Y - X \mathcal{B}))|^{-\bar{\alpha}/2} \\
&= |S|^{-T/2} |I + S^{-1} (Y - X \mathcal{B})' (Y - X \mathcal{B})|^{-\bar{\alpha}/2} \\
&= |S|^{-T/2} |I + S^{-1} (\bar{S} - S)|^{-\bar{\alpha}/2} \tag{a.4.12.38}
\end{aligned}$$

Also:

$$\begin{aligned}
&= |V|^{1/2} |\bar{V}|^{-1/2} \\
&= |V|^{1/2} |(V^{-1} + \Sigma^{-1} \otimes X'X)^{-1}|^{-1/2} \\
&= |V|^{1/2} |V^{-1} + \Sigma^{-1} \otimes X'X|^{1/2} \\
&= |I + V(\Sigma^{-1} \otimes X'X)|^{1/2} \tag{a.4.12.39}
\end{aligned}$$

Substituting (a.4.12.38) and (a.4.12.39) back in (a.4.12.37):

$$\begin{aligned}
&= \pi^{-nT/2} \frac{\Gamma_n(\frac{\alpha}{2})}{\Gamma_n(\frac{\alpha}{2})} |S|^{-T/2} |I + S^{-1} (\bar{S} - S)|^{-\bar{\alpha}/2} \\
&\times \frac{\exp \left(-\frac{1}{2} (\beta - b)' V^{-1} (\beta - b) \right)}{\frac{1}{J} \sum_{j=1}^J |I + V(\Sigma^{-1} \otimes X'X)|^{1/2} \exp \left(-\frac{1}{2} (\beta - \bar{b})' \bar{V}^{-1} (\beta - \bar{b}) \right)} \tag{a.4.12.40}
\end{aligned}$$

derivations for equation (4.12.29)

Use simple back recursion to obtain:

$$\begin{aligned}
\gamma_t &= \mu_t + F\gamma_{t-1} + \xi_t \\
\Leftrightarrow \gamma_t &= \mu_t + F(\mu_{t-1} + F\gamma_{t-2} + \xi_{t-1}) + \xi_t \\
\Leftrightarrow \gamma_t &= \mu_t + F\mu_{t-1} + F^2\gamma_{t-2} + \xi_t + F\xi_{t-1} \\
\Leftrightarrow \gamma_t &= \mu_t + F\mu_{t-1} + F^2(\mu_{t-2} + F\gamma_{t-3} + \xi_{t-2}) + \xi_t + F\xi_{t-1} \\
\Leftrightarrow \gamma_t &= \mu_t + F\mu_{t-1} + F^2\mu_{t-2} + F^3\gamma_{t-3} + \xi_t + F\xi_{t-1} + F^2\xi_{t-2}
\end{aligned} \tag{a.4.12.41}$$

Going on this way:

$$\gamma_t = \sum_{i=0}^j F^i \mu_{t-i} + F^j \gamma_{t-j} + \sum_{i=0}^j F^i \xi_{t-i} \tag{a.4.12.42}$$

derivations for equation (4.12.30)

Consider the general formulation of the VAR model:

$$y_t = Cz_t + A_1 y_{t-1} + \dots + A_p y_{t-p} + \varepsilon_t \tag{a.4.12.43}$$

Taking expectations on both sides, noting that $\mathbb{E}(y_t) = \mathbb{E}(y_{t-1}) = \dots = \mathbb{E}(y_{t-p}) = \mu$ by stationarity, and that $\mathbb{E}(z_t) = z_t$ and $\mathbb{E}(\varepsilon_t) = 0$, one obtains:

$$\mu = Cz_t + A_1 \mu + \dots + A_p \mu \tag{a.4.12.44}$$

Rearranging:

$$(I - A_1 - \dots - A_p)\mu = Cz_t \tag{a.4.12.45}$$

Which eventually yields:

$$\mu = (I - A_1 - \dots - A_p)^{-1} C z_t \tag{a.4.12.46}$$

derivations for equation (4.12.34)

Rearrange:

$$\begin{aligned}
&\exp\left(-\frac{1}{2}(\beta - \hat{\beta})'(\Sigma \otimes (X'X)^{-1})^{-1}(\beta - \hat{\beta})\right) \times \exp\left(-\frac{1}{2}(\beta - b)'(\Sigma \otimes W)^{-1}(\beta - b)\right) \\
&= \exp\left(-\frac{1}{2}\text{tr}\{\Sigma^{-1}(\mathcal{B} - \hat{\mathcal{B}})'(X'X)(\mathcal{B} - \hat{\mathcal{B}})\}\right) \times \exp\left(-\frac{1}{2}\text{tr}\{\Sigma^{-1}(\mathcal{B} - B)'W^{-1}(\mathcal{B} - B)\}\right) \\
&= \exp\left(-\frac{1}{2}\text{tr}\{\Sigma^{-1}[(\mathcal{B} - \hat{\mathcal{B}})'(X'X)(\mathcal{B} - \hat{\mathcal{B}}) + (\mathcal{B} - B)'W^{-1}(\mathcal{B} - B)]\}\right)
\end{aligned} \tag{a.4.12.47}$$

Consider the terms in square brackets and complete the squares:

$$\begin{aligned}
&(\mathcal{B} - \hat{\mathcal{B}})'(X'X)(\mathcal{B} - \hat{\mathcal{B}}) + (\mathcal{B} - B)'W^{-1}(\mathcal{B} - B) \\
&= B'(X'X)\mathcal{B} + \hat{B}'(X'X)\hat{B} - 2B'(X'X)\hat{B} + B'W^{-1}\mathcal{B} + B'W^{-1}B - 2B'W^{-1}B \\
&= B'(W^{-1} + X'X)\mathcal{B} - 2B'(W^{-1}B + X'X\hat{B}) + \hat{B}'(X'X)\hat{B} + B'W^{-1}B \\
&= B'(W^{-1} + X'X)\mathcal{B} - 2B'\bar{W}^{-1}\bar{W}(W^{-1}B + X'X\hat{B}) + \bar{B}'\bar{W}^{-1}\bar{B} - \bar{B}'\bar{W}^{-1}\bar{B} + \hat{B}'(X'X)\hat{B} + B'W^{-1}B
\end{aligned} \tag{a.4.12.48}$$

Define:

$$\bar{W} = (W^{-1} + X'X)^{-1} \quad \bar{B} = \bar{W}(W^{-1}B + X'X\hat{B}) \tag{a.4.12.49}$$

Then:

$$\begin{aligned}
&= \mathcal{B}' \bar{W}^{-1} \mathcal{B} - 2 \mathcal{B}' \bar{W}^{-1} \bar{B} + \bar{B}' \bar{W}^{-1} \bar{B} - \bar{B}' \bar{W}^{-1} \bar{B} + \hat{B}' (X'X) \hat{B} + B' W^{-1} B \\
&= (\mathcal{B} - \bar{B})' \bar{W}^{-1} (\mathcal{B} - \bar{B}) - \bar{B}' \bar{W}^{-1} \bar{B} + \hat{B}' (X'X) \hat{B} + B' W^{-1} B
\end{aligned} \tag{a.4.12.50}$$

Substitute back in (a.4.12.47):

$$\begin{aligned}
&= \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} [(\mathcal{B} - \bar{B})' \bar{W}^{-1} (\mathcal{B} - \bar{B}) - \bar{B}' \bar{W}^{-1} \bar{B} + \hat{B}' (X'X) \hat{B} + B' W^{-1} B] \} \right) \\
&= \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} (\mathcal{B} - \bar{B})' \bar{W}^{-1} (\mathcal{B} - \bar{B}) \} \right) \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} [-\bar{B}' \bar{W}^{-1} \bar{B} + \hat{B}' (X'X) \hat{B} + B' W^{-1} B] \} \right) \\
&\propto \exp \left(-\frac{1}{2} \text{tr} \{ \Sigma^{-1} (\mathcal{B} - \bar{B})' \bar{W}^{-1} (\mathcal{B} - \bar{B}) \} \right)
\end{aligned} \tag{a.4.12.51}$$

Finally, note again that:

$$\bar{B} = \bar{W} (W^{-1} B + X'X \hat{B}) = \bar{W} (W^{-1} B + X'X (X'X)^{-1} X'Y) = \bar{W} (W^{-1} B + X'Y) \tag{a.4.12.52}$$

derivations for equation (4.13.17)

The log-likelihood is given by:

$$\log(f(y|\beta, \Sigma)) = -\frac{nT}{2} \log(2\pi) - \frac{1}{2} \log(|\bar{\Sigma}|) - \frac{1}{2} (y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) \tag{a.4.12.53}$$

Then note that:

$$\log(|\bar{\Sigma}|) = \log(|\Sigma \otimes I_T|) = \log(|\Sigma|^T |I_T|^n) = \log(|\Sigma|^T) = T \log(|\Sigma|) \tag{a.4.12.54}$$

Also:

$$(y - \bar{X}\beta)' \bar{\Sigma}^{-1} (y - \bar{X}\beta) = (y - (I_n \otimes X)\beta)' (\Sigma \otimes I_T)^{-1} (y - (I_n \otimes X)\beta) = \text{tr} \{ \Sigma^{-1} (Y - X\mathcal{B})' (Y - X\mathcal{B}) \} \tag{a.4.12.55}$$

Substituting back (a.4.12.54) and (a.4.12.55) in (a.4.12.53):

$$\log(f(y|\beta, \Sigma)) = -\frac{nT}{2} \log(2\pi) - \frac{T}{2} \log(|\Sigma|) - \frac{1}{2} \text{tr} \{ \Sigma^{-1} (Y - X\mathcal{B})' (Y - X\mathcal{B}) \} \tag{a.4.12.56}$$

The function is estimated at the maximum likelihood values. Hence $\mathcal{B} = \hat{\mathcal{B}}$ and $\Sigma = \hat{\Sigma} = \mathcal{E}' \mathcal{E} / T$. Substituting in (a.4.12.56):

$$\begin{aligned}
&\log(f(y|\beta, \Sigma)) \\
&= -\frac{nT}{2} \log(2\pi) - \frac{T}{2} \log(|\hat{\Sigma}|) - \frac{1}{2} \text{tr} \{ \hat{\Sigma}^{-1} (Y - X\hat{\mathcal{B}})' (Y - X\hat{\mathcal{B}}) \} \\
&= -\frac{nT}{2} \log(2\pi) - \frac{T}{2} \log(|\hat{\Sigma}|) - \frac{1}{2} \text{tr} \{ \hat{\Sigma}^{-1} \hat{\Sigma} T \} \\
&= -\frac{nT}{2} \log(2\pi) - \frac{T}{2} \log(|\hat{\Sigma}|) - \frac{T}{2} \\
&= -\frac{nT}{2} \log(2\pi) - \frac{T}{2} (1 + \log(|\hat{\Sigma}|)) \\
&= -\frac{T}{2} (n \log(2\pi) + (1 + \log(|\hat{\Sigma}|)))
\end{aligned} \tag{a.4.12.57}$$

Following, the AIC obtains as:

$$\begin{aligned}
& AIC \\
&= 2q/T - 2\hat{L}/T \\
&= 2q/T - 2/T \left[-\frac{T}{2} (n \log(2\pi) + (1 + \log(|\hat{\Sigma}|))) \right] \\
&= 2q/T + n \log(2\pi) + 1 + \log(|\hat{\Sigma}|) \tag{a.4.12.58}
\end{aligned}$$

It follows immediately that the BIC is given by:

$$\begin{aligned}
& BIC \\
&= q \log(T)/T - 2\hat{L}/T \\
&= q \log(T)/T - 2/T \left[-\frac{T}{2} (n \log(2\pi) + (1 + \log(|\hat{\Sigma}|))) \right] \\
&= q \log(T)/T + n \log(2\pi) + 1 + \log(|\hat{\Sigma}|) \tag{a.4.12.59}
\end{aligned}$$

And the Hannan-Quinn criterion is given by:

$$\begin{aligned}
& HQ \\
&= 2q \log(\log(T))/T - 2\hat{L}/T \\
&= 2q \log(\log(T))/T - 2/T \left[-\frac{T}{2} (n \log(2\pi) + (1 + \log(|\hat{\Sigma}|))) \right] \\
&= 2q \log(\log(T))/T + n \log(2\pi) + 1 + \log(|\hat{\Sigma}|) \tag{a.4.12.60}
\end{aligned}$$

Removing the constants makes the value invariant to the number of endogenous variables:

$$AIC = 2q/T + \log(|\hat{\Sigma}|) \quad BIC = q \log(T)/T + \log(|\hat{\Sigma}|) \quad HQ = 2q \log(\log(T))/T + \log(|\hat{\Sigma}|) \tag{a.4.12.61}$$

Bayesian VAR: advanced applications

derivations for equation (4.14.18)

The distribution for the conditional forecasts is given by:

$$R \hat{y}_{T+1:T+h} \sim N(\bar{y}, \Omega) \quad (\text{a.4.14.1})$$

The distribution for the unconditional forecasts is given by:

$$\hat{y}_{T+1:T+h} \sim N(f_{T+1:T+h}, M(I_h \otimes \Gamma)M') \quad (\text{a.4.14.2})$$

Using the selection matrix R on the unconditional forecasts, we find that the distribution of the forecasts for the variables on which conditions apply is given by:

$$R \hat{y}_{T+1:T+h} \sim N(R f_{T+1:T+h}, D(I_h \otimes \Gamma)D') \quad (\text{a.4.14.3})$$

with D is a $k \times nh$ matrix such that $D = RM$. Now choose any $(nh - k) \times nh$ matrix \hat{D} such that the rows of \hat{D} form an orthonormal basis for the nullspace of D . This implies that:

$$\hat{D}\hat{D}' = I_{nh-k} \quad D\hat{D}' = 0_{k \times (nh-k)} \quad \hat{D}D' = 0_{(nh-k) \times k} \quad (\text{a.4.14.4})$$

Now define the random variable z as:

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} D \\ \hat{D} \end{pmatrix} M^{-1} \hat{y}_{T+1:T+h} = \begin{pmatrix} R \hat{y}_{T+1:T+h} \\ \hat{D} M^{-1} \hat{y}_{T+1:T+h} \end{pmatrix} \quad (\text{a.4.14.5})$$

Thus, the distribution of z is given by:

$$z \sim N \left[\begin{pmatrix} R f_{T+1:T+h} \\ \hat{D} M^{-1} f_{T+1:T+h} \end{pmatrix}, \begin{pmatrix} D(I_h \otimes \Gamma)D' & 0_{k \times (nh-k)} \\ 0_{(nh-k) \times k} & \hat{D}(I_h \otimes \Gamma)\hat{D}' \end{pmatrix} \right] \quad (\text{a.4.14.6})$$

The distribution of the conditions is given by (a.4.14.1). Because z_1 and z_2 are independent, we can simply substitute for (a.4.14.1) in (a.4.14.6) to obtain:

$$z \sim N \left[\begin{pmatrix} \bar{y} \\ \hat{D} M^{-1} f_{T+1:T+h} \end{pmatrix}, \begin{pmatrix} \Omega & 0_{k \times (nh-k)} \\ 0_{(nh-k) \times k} & \hat{D}(I_h \otimes \Gamma)\hat{D}' \end{pmatrix} \right] \quad (\text{a.4.14.7})$$

(a.4.14.5) permits to recover $\hat{y}_{T+1:T+h}$ from:

$$\hat{y}_{T+1:T+h} = M \begin{pmatrix} D \\ \hat{D} \end{pmatrix}^{-1} z \quad (\text{a.4.14.8})$$

Since D has full row rank, then its generalised inverse D^* is such that $DD^* = I_k$. Hence:

$$\begin{pmatrix} D \\ \hat{D} \end{pmatrix}^{-1} = (D^* \quad \hat{D}') \quad (\text{a.4.14.9})$$

which also implies that $\hat{D}D^* = 0$. Thus:

$$\hat{y}_{T+1:T+h} = M (D^* \quad \hat{D}') z \quad (\text{a.4.14.10})$$

Eventually combining (a.4.14.10) with (a.4.14.7), we obtain the restricted distribution of $\hat{y}_{T+1:T+h}$ as $\hat{y}_{T+1:T+h} \sim N(\hat{\mu}, \hat{\Omega})$, with:

$$\hat{\mu} = M(D^* \bar{y} + \hat{D}' \hat{D} M^{-1} f_{T+1:T+h}) \quad \hat{\Omega} = M(D^* \Omega D^{*'} + \hat{D}' \hat{D} (I_h \otimes \Gamma) \hat{D}' \hat{D}) M' \quad (\text{a.4.14.11})$$

Since $\hat{y}_{T+1:T+h} = f_{T+1:T+h} + M \xi_{T+1:T+h}$, it follows that $\xi_{T+1:T+h} = M^{-1}(\hat{y}_{T+1:T+h} - f_{T+1:T+h})$. Thus, from (a.4.14.11), the restricted distribution of the shocks $\xi_{T+1:T+h}$ is given by $\xi_{T+1:T+h} \sim N(\bar{\mu}, \bar{\Omega})$, with:

$$\bar{\mu} = D^* \bar{y} + \hat{D}' \hat{D} M^{-1} f_{T+1:T+h} - M^{-1} f_{T+1:T+h} \quad \bar{\Omega} = D^* \Omega D^{*'} + \hat{D}' \hat{D} (I_h \otimes \Gamma) \hat{D}' \hat{D} \quad (\text{a.4.14.12})$$

Also, it follows directly from post-multiplication of (a.4.14.9) that $D^* D + \hat{D}' \hat{D} = I_{nh}$, so that $\bar{\mu}$ in (a.4.14.12) rewrites:

$$\bar{\mu} = D^* \bar{y} - D^* D M^{-1} f_{T+1:T+h} = D^* \bar{y} - D^* R f_{T+1:T+h} = D^* (\bar{y} - R f_{T+1:T+h}) \quad (\text{a.4.14.13})$$

And $\bar{\Omega}$ in (a.4.14.12) rewrites:

$$\bar{\Omega} = D^* \Omega D^{*'} + (I_{nh} - D^* D)(I_h \otimes \Gamma)(I_{nh} - D^* D) \quad (\text{a.4.14.14})$$

derivations for equation (4.14.20)

Start from $\hat{y}_{T+1:T+h} = f_{T+1:T+h} + M \xi_{T+1:T+h}$. This implies that $\hat{y}_{T+1:T+h} \sim N(\hat{\mu}, \hat{\Omega})$, with:

$$\hat{\mu} = f_{T+1:T+h} + M \bar{\mu} \quad \hat{\Omega} = M \bar{\Omega} M' \quad (\text{a.4.14.15})$$

From (a.4.14.13), the first expression rewrites as:

$$\hat{\mu} = f_{T+1:T+h} + M \bar{\mu} = f_{T+1:T+h} + M D^* (\bar{y} - R f_{T+1:T+h}) \quad (\text{a.4.14.16})$$

And from (a.4.14.14), the second term obtains directly as:

$$\bar{\Omega} = M [D^* \Omega D^{*'} + (I_{nh} - D^* D)(I_h \otimes \Gamma)(I_{nh} - D^* D)] M' \quad (\text{a.4.14.17})$$

derivations for equation (4.14.45)

We show that \bar{H}_0^{-1} is given by:

$$\bar{H}_0^{-1} = \begin{pmatrix} H_0^{-1} & 0_{n \times k} \\ -\Gamma_{0,2}^{-1} \Gamma_{0,1} H_0^{-1} & \Gamma_{0,2}^{-1} \end{pmatrix} \quad (\text{a.4.14.18})$$

Indeed:

$$\begin{aligned} & \bar{H}_0 \bar{H}_0^{-1} \\ &= \begin{pmatrix} H_0 & 0_{n \times k} \\ \Gamma_{0,1} & \Gamma_{0,2} \end{pmatrix} \begin{pmatrix} H_0^{-1} & 0_{n \times k} \\ -\Gamma_{0,2}^{-1} \Gamma_{0,1} H_0^{-1} & \Gamma_{0,2}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} H_0 H_0^{-1} & 0_{n \times k} \\ \Gamma_{0,1} H_0^{-1} - \Gamma_{0,2} \Gamma_{0,2}^{-1} \Gamma_{0,1} H_0^{-1} & \Gamma_{0,2} \Gamma_{0,2}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I_n & 0_{n \times k} \\ 0_{k \times n} & I_k \end{pmatrix} \\ &= I_{\bar{n}} \end{aligned} \quad (\text{a.4.14.19})$$

derivations for equation (4.14.46)

Start from the stacked SVAR formulation and develop:

$$\begin{aligned}
 \bar{H}_0 \bar{y}_t &= \bar{H}_+ \bar{x}_t + \bar{\xi}_t \\
 \Leftrightarrow \bar{H}_0^{-1} \bar{H}_0 \bar{y}_t &= \bar{H}_0^{-1} \bar{H}_+ \bar{x}_t + \bar{H}_0^{-1} \bar{\xi}_t \\
 \Leftrightarrow \bar{y}_t &= \bar{H}_0^{-1} \bar{H}_+ \bar{x}_t + \bar{H}_0^{-1} \bar{\xi}_t \\
 \Leftrightarrow \begin{pmatrix} y_t \\ r_t \end{pmatrix} &= \bar{H}_0^{-1} \bar{H}_+ \bar{x}_t + \begin{pmatrix} H_0^{-1} & 0_{n \times k} \\ -\Gamma_{0,2}^{-1} \Gamma_{0,1} & H_0^{-1} \end{pmatrix} \begin{pmatrix} \xi_t \\ v_t \end{pmatrix}
 \end{aligned} \tag{a.4.14.20}$$

Consider the lower block for m_t :

$$r_t = \bar{H}_0^{-1} \bar{H}_+ \bar{x}_t - \Gamma_{0,2}^{-1} \Gamma_{0,1} H_0^{-1} \xi_t + \Gamma_{0,2}^{-1} v_t \tag{a.4.14.21}$$

It then follows that:

$$\begin{aligned}
 &\mathbb{E}(r_t \xi_t') \\
 &= \bar{H}_0^{-1} \bar{H}_+ \mathbb{E}(\bar{x}_t \xi_t') - \Gamma_{0,2}^{-1} \Gamma_{0,1} H_0^{-1} \mathbb{E}(\xi_t \xi_t') + \Gamma_{0,2}^{-1} \mathbb{E}(v_t \xi_t') \\
 &= -\Gamma_{0,2}^{-1} \Gamma_{0,1} H_0^{-1}
 \end{aligned} \tag{a.4.14.22}$$

where we have used a standard exogeneity assumption $Ex(\xi_t | \bar{x}_t) = 0$, and the fact that v_t and ξ_t are uncorrelated.

Bibliography

