Bayesian Time-Series Econometrics

Book 2 - algebraic derivations

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First edition

Bayesian Time-Series Econometrics

Cover illustration: Thomas Bayes (d. 1761) in Terence O'Donnell, History of Life Insurance in Its Formative Years (Chicago: American Conservation Co:, 1936), p. 335.

To my wife, Mélanie.

To my sons, Tristan and Arnaud.

Contents

PART I

Bayesian statistics

CHAPTER 3

Three applied examples

derivations for equation (1.3.3)

$$f(y|p) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i} = p^{\sum_{i=1}^{n} y_i}(1-p)^{\sum_{i=1}^{n} 1-y_i} = p^m(1-p)^{n-m} \tag{a.1.3.1}$$

derivations for equation (1.3.5)

The derivative is given by:

$$\frac{dlog(f(y|p))}{dp} = \frac{m}{p} - \frac{n-m}{1-p} \tag{a.1.3.2}$$

Set the value to 0 and solve for p:

$$\begin{aligned}
&\frac{m}{p} - \frac{n-m}{1-p} = 0\\
\Leftrightarrow\ &\frac{m}{p} = \frac{n-m}{1-p}\\
\Leftrightarrow\ &m(1-p) = p(n-m)\\
\Leftrightarrow\ &m - mp = np - mp\\
\Leftrightarrow\ &m = np\\
\Leftrightarrow\ &p = \frac{m}{n}
\end{aligned} \tag{a.1.3.3}$$

derivations for equation (1.3.11)

$$f(y|p) = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} = \frac{\prod_{i=1}^{n} \lambda^{y_i}\ \prod_{i=1}^{n} e^{-\lambda}}{\prod_{i=1}^{n} y_i!} = \frac{\lambda^{\sum_{i=1}^{n} y_i}\ e^{-n\lambda}}{\prod_{i=1}^{n} y_i!} \tag{a.1.3.4}$$

derivations for equation (1.3.13)

The derivative is given by:

$$\frac{dlog(f(y|\lambda))}{d\lambda} = \frac{\sum_{i=1}^{n} y_i}{\lambda} - n \tag{a.1.3.5}$$

Set the value to 0 and solve for λ:

$$\begin{aligned}
&\frac{\sum_{i=1}^{n} y_i}{\lambda} - n = 0\\
\Leftrightarrow\ &\frac{\sum_{i=1}^{n} y_i}{\lambda} = n\\
\Leftrightarrow\ &\lambda = \frac{1}{n}\sum_{i=1}^{n} y_i
\end{aligned} \tag{a.1.3.6}$$

derivations for equation (1.3.16)

$$\begin{aligned} & \lambda^{\sum_{i=1}^n y_i}\ e^{-n\lambda} \times \lambda^{a-1}\ e^{-\lambda/b} \\ = \ & \lambda^{a+\sum_{i=1}^n y_i - 1}\ e^{-\lambda(n+1/b)} \\ = \ & \lambda^{a+\sum_{i=1}^n y_i - 1}\ e^{-\lambda/(n+1/b)^{-1}} \end{aligned} \tag{a.1.3.7}$$

Now:

$$(n+1/b)^{-1} = \frac{1}{n+1/b} = \frac{b}{bn+1} \tag{a.1.3.8}$$

Hence:

$$\begin{aligned} & \lambda^{\sum_{i=1}^n y_i}\ e^{-n\lambda} \times \lambda^{a-1}\ e^{-\lambda/b} \\ = \ & \lambda^{a+\sum_{i=1}^n y_i - 1}\ e^{-\lambda/\frac{b}{bn+1}} \end{aligned} \tag{a.1.3.9}$$

derivations for equation (1.3.19)

$$\begin{aligned} & f(y|\mu) \\ = \ & \prod_{i=1}^n (2\pi\sigma)^{-1/2} exp\left(-\frac{1}{2}\frac{(y_i-\mu)^2}{\sigma}\right) \\ = \ & \prod_{i=1}^n (2\pi\sigma)^{-1/2}\ \prod_{i=1}^n exp\left(-\frac{1}{2}\frac{(y_i-\mu)^2}{\sigma}\right) \\ = \ & (2\pi\sigma)^{-n/2}\ exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(y_i-\mu)^2}{\sigma}\right) \end{aligned} \tag{a.1.3.10}$$

derivations for equation (1.3.21)

The derivative is given by:

$$\frac{dlog(f(y|\mu))}{d\mu} = \sum_{i=1}^n \frac{(y_i-\mu)}{\sigma} \tag{a.1.3.11}$$

Set the value to 0 and solve for μ:

$$\begin{aligned} & \sum_{i=1}^n \frac{(y_i-\mu)}{\sigma} = 0 \\ \Leftrightarrow\ & \sum_{i=1}^n (y_i-\mu) = 0 \\ \Leftrightarrow\ & \sum_{i=1}^n y_i - n\mu = 0 \\ \Leftrightarrow\ & \sum_{i=1}^n y_i = n\mu \\ \Leftrightarrow\ & \mu = \frac{1}{n}\sum_{i=1}^n y_i \end{aligned} \tag{a.1.3.12}$$

derivations for equation (1.3.24)

First group the exponential terms:

$$\pi(\mu|y) \propto exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right)\times exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right) = exp\left(-\frac{1}{2}\left[\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}+\frac{(\mu-m)^2}{v}\right]\right) \tag{a.1.3.13}$$

Develop the term within the square bracket:

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}+\frac{(\mu-m)^2}{v}\\
=\ &\frac{1}{\sigma}\sum_{i=1}^{n}(y_i^2+\mu^2-2\mu y_i)+\frac{1}{v}(\mu^2+m^2-2\mu m)\\
=\ &\frac{1}{\sigma}\left(\sum_{i=1}^{n}y_i^2+n\mu^2-2\mu\sum_{i=1}^{n}y_i)\right)+\frac{1}{v}(\mu^2+m^2-2\mu m)
\end{aligned} \tag{a.1.3.14}$$

Group the terms:

$$= \mu^2\left(\frac{n}{\sigma}+\frac{1}{v}\right)-2\mu\left(\frac{1}{\sigma}\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{1}{\sigma}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v} \tag{a.1.3.15}$$

Set back in (a.1.3.13):

$$\pi(\mu|y) \propto exp\left(-\frac{1}{2}\left[\mu^2\left(\frac{n}{\sigma}+\frac{1}{v}\right)-2\mu\left(\frac{1}{\sigma}\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{1}{\sigma}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}\right]\right) \tag{a.1.3.16}$$

CHAPTER 4

Further aspects of Bayesian priors and posteriors

derivations for equation (1.4.5)

First group the terms:

$$\begin{aligned}
&\pi(\mu,\sigma|y)\\
\propto\ & \sigma^{-n/2}\, exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right)\times\sigma^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v\sigma}\right)\times\sigma^{-\alpha/2-1} exp\left(-\frac{\delta}{2\sigma}\right)\\
=\ & \sigma^{-(n+\alpha)/2-1}\times\sigma^{-1/2}\times exp\left(-\frac{1}{2\sigma}\left[\sum_{i=1}^{n}(y_i-\mu)^2+\frac{(\mu-m)^2}{v}+\delta\right]\right) \qquad \text{(a.1.4.1)}
\end{aligned}$$

Develop the term in the square bracket:

$$\begin{aligned}
&\sum_{i=1}^{n}(y_i-\mu)^2+\frac{(\mu-m)^2}{v}+\delta\\
=\ & \sum_{i=1}^{n}(y_i^2+\mu^2-2\mu y_i)+\frac{\mu^2}{v}+\frac{m^2}{v}-2\mu\frac{m}{v}+\delta\\
=\ & \sum_{i=1}^{n}y_i^2+n\mu^2-2\mu\sum_{i=1}^{n}y_i+\frac{\mu^2}{v}+\frac{m^2}{v}-2\mu\frac{m}{v}+\delta\\
=\ & \mu^2\left(n+\frac{1}{v}\right)-2\mu\left(\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\delta \qquad \text{(a.1.4.2)}
\end{aligned}$$

Complete the squares:

$$= \mu^2\left(n+\frac{1}{v}\right)-2\mu\frac{\bar{v}}{\bar{v}}\left(\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\delta+\frac{\bar{m}^2}{\bar{v}}-\frac{\bar{m}^2}{\bar{v}} \qquad \text{(a.1.4.3)}$$

Define:

$$\bar{v}=\left(n+\frac{1}{v}\right)^{-1} \qquad \bar{m}=\bar{v}\left(\sum_{i=1}^{n}y_i+\frac{m}{v}\right) \qquad \text{(a.1.4.4)}$$

Then (a.1.4.3) rewrites:

$$\begin{aligned}
=\ & \frac{\mu^2}{\bar{v}}+\frac{\bar{m}^2}{\bar{v}}-2\mu\frac{\bar{m}}{\bar{v}}+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\delta-\frac{\bar{m}^2}{\bar{v}}\\
=\ & \frac{(\mu-\bar{m})^2}{\bar{v}}+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\delta-\frac{\bar{m}^2}{\bar{v}} \qquad \text{(a.1.4.5)}
\end{aligned}$$

Substituting back (a.1.4.5) in (a.1.4.1) eventually yields:

$$\begin{aligned}
& \pi(\mu,\sigma|y) \\
\propto\ & \sigma^{-(n+\alpha)/2-1} \times \sigma^{-1/2} \times exp\left(-\frac{1}{2\sigma}\left[\frac{(\mu-\bar{m})^2}{\bar{v}} + \sum_{i=1}^{n} y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}}\right]\right) \\
=\ & \sigma^{-(n+\alpha)/2-1} \times \sigma^{-1/2} \times exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times exp\left(-\frac{1}{2\sigma}\left[\sum_{i=1}^{n} y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}}\right]\right) \\
=\ & \sigma^{-\bar{\alpha}/2-1} \times \sigma^{-1/2} \times exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times exp\left(-\frac{\bar{\delta}}{2\sigma}\right)
\end{aligned} \tag{a.1.4.6}$$

with:

$$\bar{\alpha} = n + \alpha \qquad \bar{\delta} = \sum_{i=1}^{n} y_i^2 + \frac{m^2}{v} + \delta - \frac{\bar{m}^2}{\bar{v}} \tag{a.1.4.7}$$

derivations for equation (1.4.10)

Rearrange the terms:

$$\begin{aligned}
& \pi(\mu|y) \\
\propto\ & \Gamma\left(\frac{\bar{\alpha}+1}{2}\right)\left(\frac{\bar{\delta}+(\mu-\bar{m})^2/\bar{v}}{2}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
\propto\ & \left(\frac{\bar{\delta}+(\mu-\bar{m})^2/\bar{v}}{2}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
\propto\ & \left(\bar{\delta}+\frac{(\mu-\bar{m})^2}{\bar{v}}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
=\ & \bar{\delta}\left(1+\frac{(\mu-\bar{m})^2}{\bar{\delta}\bar{v}}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
\propto\ & \left(1+\frac{(\mu-\bar{m})^2}{\bar{\delta}\bar{v}}\right)^{-\frac{\bar{\alpha}+1}{2}} \\
=\ & \left(1+\frac{1}{\bar{\alpha}}\frac{(\mu-\bar{m})^2}{\bar{\delta}\bar{v}/\bar{\alpha}}\right)^{-\frac{\bar{\alpha}+1}{2}}
\end{aligned} \tag{a.1.4.8}$$

derivations for equation (1.4.13)

Solve for the derivative:

$$\begin{aligned}
& 2\int(\hat{\theta}-\theta)\ \pi(\theta|y)d\theta = 0 \\
\Leftrightarrow\ & \int(\hat{\theta}-\theta)\ \pi(\theta|y)d\theta = 0 \\
\Leftrightarrow\ & \int\hat{\theta}\ \pi(\theta|y)d\theta - \int\theta\ \pi(\theta|y)d\theta = 0 \\
\Leftrightarrow\ & \hat{\theta}\int\pi(\theta|y)d\theta = \int\theta\ \pi(\theta|y)d\theta \\
\Leftrightarrow\ & \hat{\theta} = \int\theta\ \pi(\theta|y)d\theta
\end{aligned} \tag{a.1.4.9}$$

derivations for equation (1.4.16)

Rearrange the expression:

$$\begin{aligned} f(y) &= \int\int (2\pi)^{-n/2}\ (2\pi)^{-1/2}\ v^{-1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\ &\times \sigma^{-n/2} exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right) \times \sigma^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v\sigma}\right) \times \sigma^{-\alpha/2-1} exp\left(-\frac{\delta}{2\sigma}\right) d\mu d\sigma \end{aligned} \tag{a.1.4.10}$$

The second row can be recognised as equation (a.1.4.1). Using the same manipulations, one obtains equation (a.1.4.6), and thus the previous expression rewrites as:

$$\begin{aligned} f(y) &= \int\int (2\pi)^{-n/2}\ (2\pi)^{-1/2}\ v^{-1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\ &\times \sigma^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times \sigma^{-\bar{\alpha}/2-1} exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\mu d\sigma \end{aligned} \tag{a.1.4.11}$$

with $\bar{m}, \bar{v}, \bar{\alpha}$ and $\bar{\delta}$ defined as in (a.1.4.4) and (a.1.4.7). Now add multiplicative terms to obtain normal and inverse Gamma probability density functions, and take constants out of the integral:

$$\begin{aligned} f(y) &= (2\pi)^{-n/2}\ v^{-1/2}\ \bar{v}^{1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\ \frac{\Gamma(\bar{\alpha}/2)}{\bar{\delta}/2^{\bar{\alpha}/2}} \\ &\times \int\int (2\pi\bar{v}\sigma)^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1} exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\mu d\sigma \end{aligned} \tag{a.1.4.12}$$

The expression can simplify further. Consider only the constant on the first line:

$$\begin{aligned} & (2\pi)^{-n/2}\ v^{-1/2}\ \bar{v}^{1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\ \frac{\Gamma(\bar{\alpha}/2)}{\bar{\delta}/2^{\bar{\alpha}/2}} \\ =\ & 2^{-n/2}\pi^{-n/2}\ v^{-1/2}\ ((n+1/v)^{-1})^{1/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{2^{\bar{\alpha}/2}}{2^{\alpha/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\ =\ & 2^{-n/2}\pi^{-n/2}\ v^{-1/2}\ (n+1/v)^{-1/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{2^{(\alpha+n)/2}}{2^{\alpha/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\ =\ & \pi^{-n/2}\ (1+vn)^{-1/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \end{aligned} \tag{a.1.4.13}$$

Substitute back in (a.1.4.12):

$$\begin{aligned} f(y) &= \pi^{-n/2}\ (1+vn)^{-1/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\ &\times \int\int (2\pi\bar{v}\sigma)^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\sigma\bar{v}}\right) \times \frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1} exp\left(-\frac{\bar{\delta}}{2\sigma}\right) d\mu d\sigma \end{aligned} \tag{a.1.4.14}$$

derivations for equation (1.4.19)

Rearrange the expression:

$$\begin{aligned}
& \mathbb{P}(M_i|y) = \frac{f(y|M_i)\ \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow\ & \mathbb{P}(M_i|y) = \frac{f(y,M_i)\ /\ \pi(M_i)\ \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow\ & \mathbb{P}(M_i|y) = \frac{\int f(y,M_i,\theta_i)\ /\ \pi(M_i) d\theta_i\ \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow\ & \mathbb{P}(M_i|y) = \frac{\int \frac{f(y,M_i,\theta_i)}{\pi(M_i,\theta)}\ \frac{\pi(M_i,\theta)}{\pi(M_i)} d\theta_i\ \mathbb{P}(M_i)}{f(y)} \\
\Leftrightarrow\ & \mathbb{P}(M_i|y) = \frac{\int f(y|M_i,\theta_i)\ \pi(\theta|M_i) d\theta_i\ \mathbb{P}(M_i)}{f(y)}
\end{aligned} \tag{a.1.4.15}$$

derivations for equation (1.4.24)

Rearrange the expression to obtain:

$$\begin{aligned}
& f(\hat{y}|y) \\
= & \int\int \sigma^{-1/2} exp\left(-\frac{1}{2}\frac{(\hat{y}-\mu)^2}{\sigma}\right) \times \sigma^{-n/2}\ exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right) \\
& \quad \times \sigma^{-1/2} exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v\sigma}\right) \times \sigma^{-\alpha/2-1} exp\left(-\frac{\delta}{2\sigma}\right) d\mu d\sigma \\
= & \int\int \sigma^{-1/2} exp\left(-\frac{1}{2\sigma}\left[(\hat{y}-\mu)^2 + \sum_{i=1}^{n}(y_i-\mu)^2 + \frac{(\mu-m)^2}{v} + \delta\right]\right) \sigma^{-(\alpha+n+1)/2-1} d\mu d\sigma \\
= & \int\int \sigma^{-1/2} exp\left(-\frac{1}{2\sigma}\left[(\hat{y}-\mu)^2 + \sum_{i=1}^{n}(y_i-\mu)^2 + \frac{(\mu-m)^2}{v} + \delta\right]\right) \sigma^{-\hat{\alpha}/2-1} d\mu d\sigma
\end{aligned} \tag{a.1.4.16}$$

with:

$$\hat{\alpha} = \alpha + n + 1 \tag{a.1.4.17}$$

Consider the term in square brackets:

$$
\begin{aligned}
& (\hat{y}-\mu)^2+\sum_{i=1}^{n}(y_i-\mu)^2+\frac{(\mu-m)^2}{v}+\delta \\
= \; & \hat{y}^2+\mu^2-2\hat{y}\mu+\sum_{i=1}^{n}(y_i^2+\mu^2-2y_i\mu)+\frac{\mu^2}{v}+\frac{m^2}{v}-2\mu\frac{m}{v}+\delta \\
= \; & \hat{y}^2+\mu^2-2\hat{y}\mu+\sum_{i=1}^{n}y_i^2+n\mu^2-2\mu\sum_{i=1}^{n}y_i+\frac{\mu^2}{v}+\frac{m^2}{v}-2\mu\frac{m}{v}+\delta \\
= \; & \left(\delta+\hat{y}^2+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}\right)+\mu^2\left(1+n+\frac{1}{v}\right)-2\mu\left(\hat{y}+\sum_{i=1}^{n}y_i+\frac{m}{v}\right) \\
= \; & \left(\delta+\hat{y}^2+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}\right)+\mu^2\left(1+n+\frac{1}{v}\right)-2\mu\frac{\hat{v}}{\hat{v}}\left(\hat{y}+\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{\hat{m}^2}{\hat{v}}-\frac{\hat{m}^2}{\hat{v}} \\
= \; & \left(\delta+\hat{y}^2+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}-\frac{\hat{m}^2}{\hat{v}}\right)+\mu^2\left(1+n+\frac{1}{v}\right)-2\mu\frac{\hat{v}}{\hat{v}}\left(\hat{y}+\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{\hat{m}^2}{\hat{v}} \qquad \text{(a.1.4.18)}
\end{aligned}
$$

Define:

$$
\hat{\delta}=\left(\delta+\hat{y}^2+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}-\frac{\hat{m}^2}{\hat{v}}\right) \qquad \hat{v}=\left(1+n+\frac{1}{v}\right)^{-1} \qquad \hat{m}=\hat{v}\left(\hat{y}+\sum_{i=1}^{n}y_i+\frac{m}{v}\right) \qquad \text{(a.1.4.19)}
$$

Then (a.1.4.17) becomes:

$$
\begin{aligned}
= \; & \hat{\delta}+\frac{\mu^2}{\hat{v}}-2\mu\frac{\hat{m}}{\hat{v}}+\frac{\hat{m}^2}{\hat{v}} \\
= \; & \hat{\delta}+\frac{(\mu-\hat{m})^2}{\hat{v}} \qquad \text{(a.1.4.20)}
\end{aligned}
$$

Substitute back in (a.1.4.16):

$$
\begin{aligned}
& f(\hat{y}|y) \\
= \; & \int\int \sigma^{-1/2}exp\left(-\frac{1}{2\sigma}\left[\hat{\delta}+\frac{(\mu-\hat{m})^2}{\hat{v}}\right]\right)\sigma^{-\hat{\alpha}/2-1}d\mu d\sigma \\
= \; & \int\int \sigma^{-1/2}exp\left(-\frac{1}{2}\frac{(\mu-\hat{m})^2}{\hat{v}\sigma}\right)\times\sigma^{-\hat{\alpha}/2-1}exp\left(-\frac{\hat{\delta}}{2\sigma}\right)d\mu d\sigma \\
= \; & \int \sigma^{-\hat{\alpha}/2-1}exp\left(-\frac{\hat{\delta}}{2\sigma}\right)\int \sigma^{-1/2}exp\left(-\frac{1}{2}\frac{(\mu-\hat{m})^2}{\hat{v}\sigma}\right)d\mu d\sigma \qquad \text{(a.1.4.21)}
\end{aligned}
$$

The second integral contains the kernel of a normal distribution with mean $\hat{m}$ and variance $\hat{v}\sigma$. It thus integrates to a constant (not involving $\hat{y}$) and can be relegated to the normalization constant, yielding:

$$
\propto \int \sigma^{-(\hat{\alpha}+1)/2-1}exp\left(-\frac{\hat{\delta}}{2\sigma}\right)d\sigma \qquad \text{(a.1.4.22)}
$$

The remaining integral contains the kernel of an inverse Gamma distribution with shape $\hat{\alpha}$ and scale $\hat{\delta}$. It integrates to the reciprocal of the normalization constant of the inverse Gamma distribution (see book1, section 4.3), which does involve $\hat{y}$. The term must thus be retained, yielding:

$$
\begin{aligned}
& f(\hat{y}|y) \\
\propto\ & \Gamma(\hat{\alpha})\,(\hat{\delta}/2)^{-\hat{\alpha}/2} \\
\propto\ & (\hat{\delta}/2)^{-\hat{\alpha}/2} \\
\propto\ & (\hat{\delta})^{-\hat{\alpha}/2} \\
=\ & \left(\delta+\hat{y}^2+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\frac{\hat{m}^2}{\hat{v}}\right)^{-\hat{\alpha}/2} \\
=\ & \left(\delta+\hat{y}^2+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\left[\hat{y}+\sum_{i=1}^{n} y_i+\frac{m}{v}\right]^2\right)^{-\hat{\alpha}/2}
\end{aligned}
\tag{a.1.4.23}
$$

Define:

$$
\tilde{m}=\sum_{i=1}^{n} y_i+\frac{m}{v} \tag{a.1.4.24}
$$

Then (a.1.4.23) becomes:

$$
\begin{aligned}
=\ & \left(\delta+\hat{y}^2+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}[\hat{y}+\tilde{m}]^2\right)^{-\hat{\alpha}/2} \\
=\ & \left(\delta+\hat{y}^2+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\hat{y}^2-\hat{v}\tilde{m}^2-2\hat{v}\tilde{m}\hat{y}\right)^{-\hat{\alpha}/2} \\
=\ & \left(\delta+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\tilde{m}^2+\hat{y}^2(1-\hat{v})-2\hat{v}\tilde{m}\hat{y}\right)^{-\hat{\alpha}/2}
\end{aligned}
\tag{a.1.4.25}
$$

Complete the squares:

$$
\begin{aligned}
=\ & \left(\delta+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\tilde{m}^2+\hat{y}^2(1-\hat{v})-2\hat{v}\frac{\ddot{v}}{\ddot{v}}\tilde{m}\hat{y}+\frac{\ddot{m}^2}{\ddot{v}}-\frac{\ddot{m}^2}{\ddot{v}}\right)^{-\hat{\alpha}/2} \\
=\ & \left(\left[\delta+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\tilde{m}^2-\frac{\ddot{m}^2}{\ddot{v}}\right]+\hat{y}^2(1-\hat{v})-2\hat{v}\frac{\ddot{v}}{\ddot{v}}\tilde{m}\hat{y}+\frac{\ddot{m}^2}{\ddot{v}}\right)^{-\hat{\alpha}/2}
\end{aligned}
\tag{a.1.4.26}
$$

Define:

$$
\bar{\alpha}=\alpha+n \qquad \ddot{\delta}=\delta+\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\hat{v}\tilde{m}^2-\frac{\ddot{m}^2}{\ddot{v}} \qquad \ddot{v}=(1-\hat{v})^{-1} \qquad \ddot{m}=\hat{v}\ddot{v}\tilde{m} \tag{a.1.4.27}
$$

Then (a.1.4.26) becomes:

$$
\begin{aligned}
&= \left(\ddot{\delta}+\frac{\hat{y}^2}{\ddot{v}}-2\hat{y}\frac{\ddot{m}}{\ddot{v}}+\frac{\ddot{m}^2}{\ddot{v}}\right)^{-(\bar{\alpha}+1)/2} \\
&= \left(\ddot{\delta}+\frac{(\hat{y}-\ddot{m})^2}{\ddot{v}}\right)^{-(\bar{\alpha}+1)/2} \\
&= \ddot{\delta}^{-(\bar{\alpha}+1)/2}\left(1+\frac{(\hat{y}-\ddot{m})^2}{\ddot{\delta}\ddot{v}}\right)^{-(\bar{\alpha}+1)/2} \\
&\propto \left(1+\frac{(\hat{y}-\ddot{m})^2}{\ddot{\delta}\ddot{v}}\right)^{-(\bar{\alpha}+1)/2} \\
&= \left(1+\frac{1}{\bar{\alpha}}\,\frac{(\hat{y}-\ddot{m})^2}{\ddot{\delta}\ddot{v}/\bar{\alpha}}\right)^{-(\bar{\alpha}+1)/2}
\end{aligned}
\tag{a.1.4.28}
$$

Finally, reformulate all the messy terms:

$$
\begin{aligned}
\ddot{v} &= (1-\hat{v})^{-1}=\frac{1}{1-\hat{v}}=\frac{1}{1-\frac{1}{1+n+\frac{1}{v}}}=\frac{1}{\frac{1+n+\frac{1}{v}-1}{1+n+\frac{1}{v}}}=\frac{1}{\frac{n+\frac{1}{v}}{1+n+\frac{1}{v}}}=\frac{1+n+\frac{1}{v}}{n+\frac{1}{v}}=\frac{v+vn+1}{vn+1} \\
&=1+\frac{v}{vn+1}=1+\frac{1}{n+1/v}=1+\left(n+\frac{1}{v}\right)^{-1}=1+\bar{v}
\end{aligned}
\tag{a.1.4.29}
$$

with $\bar{v}$ defined as in (a.1.4.4).

Also:

$$
\frac{\hat{v}}{1-\hat{v}}=\frac{\frac{1}{1+n+\frac{1}{v}}}{1-\frac{1}{1+n+\frac{1}{v}}}=\frac{\frac{1}{1+n+\frac{1}{v}}}{\frac{1+n+\frac{1}{v}-1}{1+n+\frac{1}{v}}}=\frac{\frac{1}{1+n+\frac{1}{v}}}{\frac{n+\frac{1}{v}}{1+n+\frac{1}{v}}}=\frac{1}{n+\frac{1}{v}}=\left(n+\frac{1}{v}\right)^{-1}=\bar{v}
\tag{a.1.4.30}
$$

Then:

$$
\ddot{m}=\hat{v}\ddot{v}\tilde{m}=\frac{\hat{v}}{1-\hat{v}}\tilde{m}=\bar{v}\tilde{m}=\bar{v}\left(\sum_{i=1}^{n}y_i+\frac{m}{v}\right)=\bar{m}
\tag{a.1.4.31}
$$

with $\bar{m}$ defined as in (a.1.4.4).

Finally:

$$
\begin{aligned}
&\hat{v}\tilde{m}^2+\frac{\ddot{m}^2}{\ddot{v}}=\hat{v}\tilde{m}^2+(\hat{v}\ddot{v}\tilde{m})^2/\ddot{v}=\hat{v}\tilde{m}^2+\hat{v}^2\ddot{v}\tilde{m}^2=\hat{v}\tilde{m}^2(1+\hat{v}\ddot{v})=\hat{v}\tilde{m}^2\left(1+\frac{\hat{v}}{1-\hat{v}}\right) \\
&=\hat{v}\tilde{m}^2\left(\frac{1-\hat{v}+\hat{v}}{1-\hat{v}}\right)=\hat{v}\tilde{m}^2\left(\frac{1}{1-\hat{v}}\right)=\tilde{m}^2\left(\frac{\hat{v}}{1-\hat{v}}\right)=\tilde{m}^2\bar{v}=\tilde{m}\bar{m}=\frac{\bar{m}}{\bar{v}}\bar{m}=\frac{\bar{m}^2}{\bar{v}}
\end{aligned}
\tag{a.1.4.32}
$$

Substitute in (a.1.4.27) to obtain:

$$
\ddot{\delta}=\delta+\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}-\frac{\bar{m}^2}{\bar{v}}=\bar{\delta}
\tag{a.1.4.33}
$$

with $\bar{\delta}$ defined as in (a.1.4.7).

Substitute (a.1.4.29), (a.1.4.31) and (a.1.4.33) in (a.1.4.28) to eventually obtain:

$$
f(\hat{y}|y)\propto\left(1+\frac{1}{\bar{\alpha}}\,\frac{(\hat{y}-\bar{m})^2}{\bar{\delta}(1+\bar{v})/\bar{\alpha}}\right)^{-(\bar{\alpha}+1)/2}
\tag{a.1.4.34}
$$

CHAPTER 5

Properties of Bayesian estimates

derivations for equation (1.5.1)

The mean of a Beta distribution with shapes a and b is given by $\frac{a}{a+b}$. Given the posterior hyperparameters $\bar{\alpha} = \alpha + m$ and $\bar{\beta} = \beta + n - m$, the posterior mean writes as:

$$\begin{aligned}
& \mathbb{E}(p|y) \\
= \; & \frac{\bar{\alpha}}{\bar{\alpha}+\bar{\beta}} \\
= \; & \frac{\alpha+m}{\alpha+m+\beta+n-m} \\
= \; & \frac{\alpha+m}{\alpha+\beta+n} \\
= \; & \frac{\alpha}{\alpha+\beta+n} + \frac{m}{\alpha+\beta+n} \\
= \; & \frac{\alpha}{\alpha+\beta} \, \frac{\alpha+\beta}{\alpha+\beta+n} + \frac{m}{n} \, \frac{n}{\alpha+\beta+n} \\
= \; & \gamma \, \mathbb{E}(p) + (1-\gamma) \, \hat{p}
\end{aligned} \tag{a.1.5.1}$$

with:

$$\mathbb{E}(p) = \frac{\alpha}{\alpha+\beta} \qquad \hat{p} = \frac{m}{n} \qquad \gamma = \frac{\alpha+\beta}{\alpha+\beta+n} \tag{a.1.5.2}$$

derivations for equation (1.5.2)

The mean of a Gamma distribution with shape a and scale b is given by ab. Given the posterior hyperparameters $\bar{a} = a + \sum_{i=1}^{n} y_i$ and $\bar{b} = \frac{b}{bn+1}$, the posterior mean writes as:

$$\begin{aligned}
& \mathbb{E}(\lambda|y) \\
= \; & \frac{(a+\sum_{i=1}^{n} y_i)b}{bn+1} \\
= \; & \frac{ab}{bn+1} + \frac{b\sum_{i=1}^{n} y_i}{bn+1} \\
= \; & ab\left(\frac{1}{bn+1}\right) + \frac{\sum_{i=1}^{n} y_i}{n}\left(\frac{bn}{bn+1}\right) \\
= \; & \gamma \, \mathbb{E}(\lambda) + (1-\gamma) \, \hat{\lambda}
\end{aligned} \tag{a.1.5.3}$$

with:

$$\mathbb{E}(\lambda) = ab \qquad \hat{\lambda} = \frac{\sum_{i=1}^{n} y_i}{n} \qquad \gamma = \frac{1}{bn+1} \tag{a.1.5.4}$$

derivations for equation (1.5.3)

The mean of a normal distribution with mean μ and variance σ is given by μ. Given the posterior hyperparameters $\bar{v} = \left(\frac{n}{\sigma} + \frac{1}{v}\right)^{-1}$ and $\bar{m} = \bar{v}\left(\frac{1}{\sigma}\sum_{i=1}^{n} y_i + \frac{m}{v}\right)$, the posterior variance writes as:

$$\bar{v} = \left(\frac{n}{\sigma} + \frac{1}{v}\right)^{-1} = \frac{1}{n/\sigma + 1/v} = \frac{\sigma}{n + \sigma/v} \tag{a.1.5.5}$$

Then the posterior mean can be expressed as:

$$\begin{aligned}
& \mathbb{E}(\mu|y) \\
= \; & \frac{\sigma}{n+\sigma/v}\left(\frac{1}{\sigma}\sum_{i=1}^{n} y_i + \frac{m}{v}\right) \\
= \; & \frac{1}{n+\sigma/v}\left(\sum_{i=1}^{n} y_i\right) + \frac{\sigma}{n+\sigma/v}\left(\frac{m}{v}\right) \\
= \; & \frac{n}{n+\sigma/v}\left(\frac{\sum_{i=1}^{n} y_i}{n}\right) + \frac{\sigma/v}{n+\sigma/v}\, m \\
= \; & \frac{vn}{vn+\sigma}\left(\frac{\sum_{i=1}^{n} y_i}{n}\right) + \frac{\sigma}{vn+\sigma}\, m \\
= \; & \gamma\, \mathbb{E}(\mu) + (1-\gamma)\, \hat{\mu}
\end{aligned} \tag{a.1.5.6}$$

with:

$$\mathbb{E}(\mu) = m \qquad \hat{\mu} = \frac{\sum_{i=1}^{n} y_i}{n} \qquad \gamma = \frac{\sigma}{vn+\sigma} \tag{a.1.5.7}$$

PART II

Simulation methods

CHAPTER 6

The Gibbs sampling algorithm

derivations for equation (2.6.17)

Combine all the terms to obtain:

$$
\begin{aligned}
& f(y) \\
\approx\ & (2\pi\sigma)^{-n/2}\ \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right)\ \frac{(2\pi v)^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi\bar{v})^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & (2\pi)^{-n/2}\sigma^{-n/2}\ \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma}\right)\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-(\alpha+n)/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & (2\pi)^{-n/2}\ \exp\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\exp\left(-\frac{1}{2\sigma}\left[\delta+\sum_{i=1}^{n}(y_i-\mu)^2\right]\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & (2\pi)^{-n/2}\ \exp\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}} \\
=\ & 2^{-n/2}\pi^{-n/2}\ \exp\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta^{\alpha/2}}{\Gamma(\alpha/2)}}{\frac{\bar{\delta}^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}}\ \frac{2^{\bar{\alpha}/2}}{2^{\alpha/2}} \\
=\ & \pi^{-n/2}\ \exp\ \frac{v^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}\bar{v}^{-1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}\ \frac{\frac{\delta^{\alpha/2}}{\Gamma(\alpha/2)}}{\frac{\bar{\delta}^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}} \\
=\ & \pi^{-n/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\ \frac{\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}(v/\bar{v})^{1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)}
\end{aligned}
\tag{a.2.6.1}
$$

Now:

$$v/\bar{v} = v(n/\sigma + 1/v) = vn/\sigma + 1 \tag{a.2.6.2}$$

Hence:

$$f(y) \approx \pi^{-n/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\ \frac{\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)}{\frac{1}{J}\sum_{j=1}^{J}(1+vn/\sigma)^{1/2}\exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)} \tag{a.2.6.3}$$

CHAPTER 7

The Metropolis-Hastings algorithm

derivations for equation (2.7.7)

Rearrange:

$$
\begin{aligned}
& \pi(\mu|y,\lambda) \\
\propto\ & \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}\right)\times\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right) \\
=\ & \exp\left(-\frac{1}{2}\left[\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}+\frac{(\mu-m)^2}{v}\right]\right)
\end{aligned}
\qquad \text{(a.2.7.1)}
$$

Develop the term within the square bracket:

$$
\begin{aligned}
& \sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}+\frac{(\mu-m)^2}{v} \\
=\ & \frac{1}{\exp(\lambda)}\sum_{i=1}^{n}(y_i^2+\mu^2-2\mu y_i)+\frac{1}{v}(\mu^2+m^2-2\mu m) \\
=\ & \frac{1}{\exp(\lambda)}\left(\sum_{i=1}^{n}y_i^2+n\mu^2-2\mu\sum_{i=1}^{n}y_i)\right)+\frac{1}{v}(\mu^2+m^2-2\mu m)
\end{aligned}
\qquad \text{(a.2.7.2)}
$$

Group the terms and complete the squares:

$$
\begin{aligned}
=\ & \mu^2\left(\frac{n}{\exp(\lambda)}+\frac{1}{v}\right)-2\mu\left(\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v} \\
=\ & \mu^2\left(\frac{n}{\exp(\lambda)}+\frac{1}{v}\right)-2\mu\frac{\bar{v}}{\bar{v}}\left(\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i+\frac{m}{v}\right)+\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}+\frac{\bar{m}^2}{\bar{v}}-\frac{\bar{m}^2}{\bar{v}}
\end{aligned}
\qquad \text{(a.2.7.3)}
$$

Define:

$$
\bar{v}=\left(\frac{n}{\exp(\lambda)}+\frac{1}{v}\right)^{-1} \qquad \bar{m}=\bar{v}\left(\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i+\frac{m}{v}\right) \qquad \text{(a.2.7.4)}
$$

Then (a.2.7.3) rewrites:

$$
\begin{aligned}
=\ & \frac{\mu^2}{\bar{v}}+\frac{\bar{m}^2}{\bar{v}}-2\mu\frac{\bar{m}}{\bar{v}}+\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}-\frac{\bar{m}^2}{\bar{v}} \\
=\ & \frac{(\mu-\bar{m})^2}{\bar{v}}+\frac{1}{\exp(\lambda)}\sum_{i=1}^{n}y_i^2+\frac{m^2}{v}-\frac{\bar{m}^2}{\bar{v}}
\end{aligned}
\qquad \text{(a.2.7.5)}
$$

Substitute back in (a.2.7.1):

$$\begin{aligned}
& \pi(\mu|y,\lambda) \\
\propto\ & \exp\left(-\frac{1}{2}\left[\frac{(\mu-\bar{m})^2}{\bar{v}}+\frac{1}{\exp(\lambda)}\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\frac{\bar{m}^2}{\bar{v}}\right]\right) \\
=\ & \exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right)\ \exp\left(-\frac{1}{2}\left[\frac{1}{\exp(\lambda)}\sum_{i=1}^{n} y_i^2+\frac{m^2}{v}-\frac{\bar{m}^2}{\bar{v}}\right]\right) \\
\propto\ & \exp\left(-\frac{1}{2}\frac{(\mu-\bar{m})^2}{\bar{v}}\right) \qquad \text{(a.2.7.6)}
\end{aligned}$$

derivations for equation (2.7.14)

$$\begin{aligned}
& \alpha(\lambda^{(j-1)},\lambda^{(j)}) \\
=\ & \frac{\exp(\lambda^{(j)})^{-n/2}}{\exp(\lambda^{(j-1)})^{-n/2}}\ \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda^{(j)})}\right)}{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda^{(j-1)})}\right)}\ \frac{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j)}-g)^2}{z}\right)}{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j-1)}-g)^2}{z}\right)} \qquad \text{(a.2.7.7)}
\end{aligned}$$

Consider the first term:

$$\begin{aligned}
& \frac{\exp(\lambda^{(j)})^{-n/2}}{\exp(\lambda^{(j-1)})^{-n/2}} \\
=\ & \exp(\lambda^{(j)}-\lambda^{(j-1)})^{-n/2} \\
=\ & \exp\left(\frac{n}{2}(\lambda^{(j-1)}-\lambda^{(j)})\right) \qquad \text{(a.2.7.8)}
\end{aligned}$$

Consider the second term:

$$\begin{aligned}
& \frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda^{(j)})}\right)}{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda^{(j-1)})}\right)} \\
=\ & \frac{\exp\left(-\frac{1}{2}\exp(-\lambda^{(j)})\sum_{i=1}^{n}(y_i-\mu)^2\right)}{\exp\left(-\frac{1}{2}\exp(-\lambda^{(j-1)})\sum_{i=1}^{n}(y_i-\mu)^2\right)} \\
=\ & \exp\left(-\frac{1}{2}\left[\exp(-\lambda^{(j)})-\exp(-\lambda^{(j-1)})\right]\sum_{i=1}^{n}(y_i-\mu)^2\right) \\
=\ & \exp\left(\frac{1}{2}\left[\exp(-\lambda^{(j-1)})-\exp(-\lambda^{(j)})\right]\sum_{i=1}^{n}(y_i-\mu)^2\right) \qquad \text{(a.2.7.9)}
\end{aligned}$$

Consider the third term:

$$\begin{aligned}
& \frac{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j)}-g)^2}{z}\right)}{\exp\left(-\frac{1}{2}\frac{(\lambda^{(j-1)}-g)^2}{z}\right)} \\
=\ & \exp\left(-\frac{1}{2}\left[\frac{(\lambda^{(j)}-g)^2-(\lambda^{(j-1)}-g)^2}{z}\right]\right) \\
=\ & \exp\left(\frac{1}{2}\left[\frac{(\lambda^{(j-1)}-g)^2-(\lambda^{(j)}-g)^2}{z}\right]\right) \qquad \text{(a.2.7.10)}
\end{aligned}$$

Substitute back in (a.2.7.7):

$$
\begin{aligned}
&\alpha(\lambda^{(j-1)},\lambda^{(j)}) \\
= \;& \exp\left(\frac{1}{2}\left[\begin{array}{l} n(\lambda^{(j-1)}-\lambda^{(j)})+\left[\exp(-\lambda^{(j-1)})-\exp(-\lambda^{(j)})\right]\sum_{i=1}^{n}(y_i-\mu)^2 \\ +\dfrac{(\lambda^{(j-1)}-g)^2-(\lambda^{(j)}-g)^2}{z} \end{array}\right]\right)
\end{aligned}
\tag{a.2.7.11}
$$

derivations for equation (2.7.21)

Rearrange the expression:

$$
\begin{aligned}
&\frac{1}{f(y)} \\
\approx \;& \frac{1}{J}\sum_{j=1}^{J}\frac{g(\theta^{(j)})}{f(y|\mu^{(j)},\lambda^{(j)})\;\pi(\mu^{(j)})\;\pi(\lambda^{(j)})} \\
= \;& \frac{1}{J}\sum_{j=1}^{J}\frac{\mathbb{1}(\theta\in\hat{\Theta})\times\omega^{-1}(2\pi)^{-k/2}|\hat{\Sigma}|^{-1/2}\exp\left(-\frac{1}{2}(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right)\mathbb{1}(\theta\in\hat{\Theta})}{(2\pi\;\exp(\lambda))^{-n/2}\;\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}\right)\;(2\pi v)^{-1/2}\;\exp\left(-\frac{1}{2}\frac{(\mu-m)^2}{v}\right)\;(2\pi z)^{-1/2}\;\exp\left(-\frac{1}{2}\frac{(\lambda-g)^2}{z}\right)} \\
= \;& \mathbb{1}(\theta\in\hat{\Theta})\times\omega^{-1}(2\pi)^{(n+2-k)/2}|\hat{\Sigma}|^{-1/2}(vz)^{1/2} \\
& \times\frac{1}{J}\sum_{j=1}^{J}\exp\left(\frac{1}{2}\left[n\lambda+\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}+\frac{(\mu-m)^2}{v}+\frac{(\lambda-g)^2}{z}-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right]\right) \\
= \;& \mathbb{1}(\theta\in\hat{\Theta})\times(\omega J)^{-1}(2\pi)^{n/2}|\hat{\Sigma}|^{-1/2}(vz)^{1/2} \\
& \times\sum_{j=1}^{J}\exp\left(\frac{1}{2}\left[n\lambda+\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\exp(\lambda)}+\frac{(\mu-m)^2}{v}+\frac{(\lambda-g)^2}{z}-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right]\right)
\end{aligned}
\tag{a.2.7.12}
$$

CHAPTER 8

Mathematical theory

derivations for equation (2.8.11)

The definition of an invariant distribution implies that:

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4 & \cdots\end{pmatrix}\begin{pmatrix} p+q & r & 0 & 0 & 0 & \cdots \\ p & q & r & 0 & 0 & \cdots \\ 0 & p & q & r & 0 & \cdots \\ 0 & 0 & p & q & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4 & \cdots\end{pmatrix} \tag{a.2.8.1}$$

The product with the first column of P yields:

$$\begin{aligned} & \pi_1(p+q)+\pi_2 p = \pi_1 \\ \Leftrightarrow\ & \pi_2 p = \pi_1(1-p-q) \\ \Leftrightarrow\ & \pi_2 p = \pi_1 r \\ \Leftrightarrow\ & \pi_2 = (r/p)\pi_1 \end{aligned} \tag{a.2.8.2}$$

The second column yields:

$$\begin{aligned} & \pi_1 r+\pi_2 q+\pi_3 p = \pi_2 \\ \Leftrightarrow\ & \pi_1 r+\pi_3 p = \pi_2(1-q) \\ \Leftrightarrow\ & \pi_1 r+\pi_3 p = \pi_1(r/p)(1-q) \\ \Leftrightarrow\ & \pi_1+\pi_3(p/r) = \pi_1(1-q)/p \\ \Leftrightarrow\ & \pi_3(p/r) = \pi_1(1-q-p)/p \\ \Leftrightarrow\ & \pi_3(p/r) = \pi_1(r/p) \\ \Leftrightarrow\ & \pi_3(p/r) = \pi_2 \\ \Leftrightarrow\ & \pi_3 = (r/p)\pi_2 \\ \Leftrightarrow\ & \pi_3 = (r/p)^2\pi_1 \end{aligned} \tag{a.2.8.3}$$

The third column yields:

$$
\begin{aligned}
& \pi_2 r + \pi_3 q + \pi_4 p = \pi_3 \\
\Leftrightarrow\ & \pi_2 r + \pi_4 p = \pi_3(1-q) \\
\Leftrightarrow\ & \pi_2 r + \pi_4 p = \pi_2 (r/p)(1-q) \\
\Leftrightarrow\ & \pi_2 + \pi_4 (p/r) = \pi_2 (1-q)/p \\
\Leftrightarrow\ & \pi_4 (p/r) = \pi_2 (1-q-p)/p \\
\Leftrightarrow\ & \pi_4 (p/r) = \pi_2 (r/p) \\
\Leftrightarrow\ & \pi_4 (p/r) = \pi_3 \\
\Leftrightarrow\ & \pi_4 = (r/p)\pi_3 \\
\Leftrightarrow\ & \pi_4 = (r/p)^2 \pi_2 \\
\Leftrightarrow\ & \pi_4 = (r/p)^3 \pi_1 && \text{(a.2.8.4)}
\end{aligned}
$$

Continuing this way, one obtains in general that $\pi_j = (r/p)^{j-1}\pi_1$. If an invariant distribution exists, we must have $\pi_1 + \pi_2 + \pi_3 \cdots = 1$. Hence:

$$
\begin{aligned}
& \pi_1 + \pi_2 + \pi_3 + \cdots = 1 \\
\Leftrightarrow\ & \pi_1 + (r/p)\pi_1 + (r/p)^2 \pi_1 + \cdots = 1 \\
\Leftrightarrow\ & \pi_1 (1 + (r/p) + (r/p)^2 + \cdots) = 1 \\
\Leftrightarrow\ & \pi_1 \frac{1}{1 - r/p} = 1 \\
\Leftrightarrow\ & \pi_1 = 1 - r/p && \text{(a.2.8.5)}
\end{aligned}
$$

derivations for equation (2.8.16)

Start from the definition and rearrange:

$$
\begin{aligned}
\pi(y_t) &= \int \pi(y_{t-1})\ p(y_{t-1}, y_t)\ dy_{t-1} \\
&\propto \int \exp\left(-\frac{1}{2}\frac{(y_{t-1}-\mu)^2}{\sigma}\right)\ \exp\left(-\frac{1}{2}\frac{(y_t - c - \gamma y_{t-1})^2}{s}\right)\ dy_{t-1} \\
&= \int \exp\left(-\frac{1}{2}\left[\frac{(y_{t-1}-\mu)^2}{\sigma} + \frac{(y_t - c - \gamma y_{t-1})^2}{s}\right]\right)\ dy_{t-1} \\
&= \int \exp\left(-\frac{1}{2}\left[\frac{(y_{t-1}-\mu)^2(1-\gamma^2)}{s} + \frac{(y_t - c - \gamma y_{t-1})^2}{s}\right]\right)\ dy_{t-1} \\
&= \int \exp\left(-\frac{1}{2s}\left[(y_{t-1}-\mu)^2(1-\gamma^2) + (y_t - \mu(1-\gamma) - \gamma y_{t-1})^2\right]\right)\ dy_{t-1} && \text{(a.2.8.6)}
\end{aligned}
$$

Consider the term within the square brackets:

$$
\begin{aligned}
& (y_{t-1}-\mu)^2(1-\gamma^2) + (y_t - \mu(1-\gamma) - \gamma y_{t-1})^2 \\
=\ & (1-\gamma^2)y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2(1-\gamma^2)\mu y_{t-1} + y_t^2 + \mu^2(1-\gamma)^2 + \gamma^2 y_{t-1}^2 \\
-\ & 2\mu(1-\gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma(1-\gamma)y_{t-1} \\
=\ & y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2(1-\gamma^2)\mu y_{t-1} + y_t^2 + \mu^2(1-\gamma)^2 \\
-\ & 2\mu(1-\gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma(1-\gamma)y_{t-1} \\
=\ & y_{t-1}^2 + (1-\gamma^2)\mu^2 - 2\mu y_{t-1} + 2\gamma^2\mu y_{t-1} + y_t^2 + \mu^2(1-\gamma)^2 \\
-\ & 2\mu(1-\gamma)y_t - 2\gamma y_t y_{t-1} + 2\mu\gamma y_{t-1} - 2\mu\gamma^2 y_{t-1} && \text{(a.2.8.7)}
\end{aligned}
$$

$$
\begin{aligned}
&= y_{t-1}^2+(1-\gamma^2)\mu^2-2\mu y_{t-1}+y_t^2+\mu^2(1-\gamma)^2-2\mu(1-\gamma)y_t-2\gamma y_t y_{t-1}+2\mu\gamma y_{t-1}\\
&= y_{t-1}^2+(1-\gamma^2)\mu^2-2\mu y_{t-1}(1-\gamma)+y_t^2+\mu^2(1-\gamma)^2-2\mu(1-\gamma)y_t-2\gamma y_t y_{t-1}\\
&= y_{t-1}^2+(1-\gamma^2)\mu^2-2\mu y_{t-1}(1-\gamma)+y_t^2+\mu^2(1-\gamma)^2-2\mu(1-\gamma^2)y_t+2\mu(1-\gamma)\gamma y_t-2\gamma y_t y_{t-1}\\
&= y_{t-1}^2+(1-\gamma^2)\mu^2-2\mu y_{t-1}(1-\gamma)+(1-\gamma^2)y_t^2+\gamma^2 y_t^2+\mu^2(1-\gamma)^2\\
&- 2\mu(1-\gamma^2)y_t+2\mu(1-\gamma)\gamma y_t-2\gamma y_t y_{t-1}\\
&= (1-\gamma^2)y_t^2+(1-\gamma^2)\mu^2-2\mu(1-\gamma^2)y_t\\
&+ y_{t-1}^2+\mu^2(1-\gamma)^2+\gamma^2 y_t^2-2\gamma y_t y_{t-1}-2\mu y_{t-1}(1-\gamma)+2\mu(1-\gamma)\gamma y_t\\
&= (1-\gamma^2)y_t^2+(1-\gamma^2)\mu^2-2\mu(1-\gamma^2)y_t\\
&+ y_{t-1}^2+c^2+\gamma^2 y_t^2-2\gamma y_t y_{t-1}-2c y_{t-1}+2c\gamma y_t\\
&= (1-\gamma^2)(y_t-\mu)^2+(y_{t-1}-c-\gamma y_t)^2
\end{aligned}
\tag{a.2.8.8}
$$

Substitute back in (a.2.8.6):

$$\pi(y_t)=\int \exp\left(-\frac{1}{2s}\left[(1-\gamma^2)(y_t-\mu)^2+(y_{t-1}-c-\gamma y_t)^2\right]\right)\,dy_{t-1} \tag{a.2.8.9}$$

and this eventually reformulates as:

$$\pi(y_t)=\exp\left(-\frac{1}{2}\frac{(y_t-\mu)^2}{\sigma}\right)\int \exp\left(-\frac{1}{2}\frac{(y_{t-1}-c-\gamma y_t)^2}{s}\right)\,dy_{t-1} \tag{a.2.8.10}$$

PART III

Econometrics

CHAPTER 9

The linear regression model

derivations for equation (3.9.7)

Consider first β. To do so, rewrite the likelihood function as:

$$log(f(y|\beta,\sigma)) = -\frac{n}{2}\, log(2\pi) - \frac{n}{2}\, log(\sigma) - \frac{1}{2\sigma}(y'y + \beta'X'X\beta - 2\beta'X'y) \tag{a.3.9.1}$$

Then solve for the partial derivative:

$$\begin{aligned}
&\frac{\partial log(f(y|\beta,\sigma))}{\partial\beta} = 0\\
\Leftrightarrow\ & -\frac{1}{2\sigma}(2\beta'X'X - 2y'X) = 0\\
\Leftrightarrow\ & \beta'X'X - y'X = 0\\
\Leftrightarrow\ & \beta'X'X = y'X\\
\Leftrightarrow\ & X'X\beta = X'y\\
\Leftrightarrow\ & \beta = (X'X)^{-1}X'y
\end{aligned} \tag{a.3.9.2}$$

Hence the estimate is $\hat{\beta} = (X'X)^{-1}X'y$. Consider now σ. Solve for the partial derivative:

$$\begin{aligned}
&\frac{\partial log(f(y|\beta,\sigma))}{\partial\sigma} = 0\\
\Leftrightarrow\ & -\frac{n}{2}\frac{1}{\sigma} + \frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma^2} = 0\\
\Leftrightarrow\ & -n + \frac{(y-X\beta)'(y-X\beta)}{\sigma} = 0\\
\Leftrightarrow\ & \frac{(y-X\beta)'(y-X\beta)}{\sigma} = n\\
\Leftrightarrow\ & \sigma = \frac{(y-X\beta)'(y-X\beta)}{n}
\end{aligned} \tag{a.3.9.3}$$

This expression gives the optimum for any value of β. To obtain a global maximum we must choose the value of β that maximizes the likelihood, namely $\hat{\beta}$. Therefore, the estimate for σ is given by $\hat{\sigma} = (y - X\hat{\beta})'(y - X\hat{\beta})/n$.

derivations for equation (3.9.12)

Develop and group:

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right) \times \exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)\\
=\ & \exp\left(-\frac{1}{2}\left[(y-X\beta)'\sigma^{-1}(y-X\beta) + (\beta-b)'V^{-1}(\beta-b)\right]\right)
\end{aligned} \tag{a.3.9.4}$$

Consider the term in square brackets:

$$
\begin{aligned}
& (y-X\beta)'\sigma^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\
= \;& y'\sigma^{-1}y+\beta'X'\sigma^{-1}X\beta-2\beta'X'\sigma^{-1}y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b \\
= \;& \beta'(V^{-1}+\sigma^{-1}X'X)\beta-2\beta'(V^{-1}b+\sigma^{-1}X'y)+b'V^{-1}b+y'\sigma^{-1}y
\end{aligned} \tag{a.3.9.5}
$$

Substitute back in (a.3.9.4):

$$
= \exp\left(-\frac{1}{2}\left[\beta'(V^{-1}+\sigma^{-1}X'X)\beta-2\beta'(V^{-1}b+\sigma^{-1}X'y)+b'V^{-1}b+y'\sigma^{-1}y\right]\right) \tag{a.3.9.6}
$$

derivations for equation (3.9.23)

Group and rearrange:

$$
\begin{aligned}
& \pi(\beta,\sigma|y) \\
\propto \;& \sigma^{-n/2}\exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right)\times|\sigma V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'(\sigma V)^{-1}(\beta-b)\right) \\
\times \;& \sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right) \\
= \;& \sigma^{-k/2}\exp\left(-\frac{1}{2\sigma}\left[(y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\right]\right)\sigma^{-(\alpha+n)/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)
\end{aligned} \tag{a.3.9.7}
$$

Define:

$$
\bar{\alpha}=\alpha+n \tag{a.3.9.8}
$$

Then:

$$
= \sigma^{-k/2}\exp\left(-\frac{1}{2\sigma}\left[(y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\right]\right)\sigma^{-\bar{\alpha}/2-1}\exp\left(-\frac{\delta}{2\sigma}\right) \tag{a.3.9.9}
$$

Consider the term between the square brackets:

$$
\begin{aligned}
& (y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\
= \;& y'y+\beta'X'X\beta-2\beta'X'y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b \\
= \;& \beta'(V^{-1}+X'X)\beta-2\beta'(V^{-1}b+X'y)+y'y+b'V^{-1}b \\
= \;& \beta'(V^{-1}+X'X)\beta-2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+X'y)+y'y+b'V^{-1}b+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b}
\end{aligned} \tag{a.3.9.10}
$$

Define:

$$
\bar{V}=(V^{-1}+X'X)^{-1} \qquad \bar{b}=\bar{V}(V^{-1}b+X'y) \tag{a.3.9.11}
$$

Then (a.3.9.10) becomes:

$$
\begin{aligned}
= \;& \beta'\bar{V}^{-1}\beta-2\beta'\bar{V}^{-1}\bar{b}+y'y+b'V^{-1}b+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b} \\
= \;& (\beta'\bar{V}^{-1}\beta-2\beta'\bar{V}\bar{b}+\bar{b}'\bar{V}^{-1}\bar{b})+y'y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b} \\
= \;& (\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}
\end{aligned} \tag{a.3.9.12}
$$

Substitute back in (a.3.9.9):

$$
= \sigma^{-k/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'(\sigma\bar{V})^{-1}(\beta-\bar{b})\right)\sigma^{-\bar{\alpha}/2-1}\exp\left(-\frac{\delta+y'y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}}{2\sigma}\right) \tag{a.3.9.13}
$$

Define:

$$\bar{\delta} = \delta + y'y + b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} \tag{a.3.9.14}$$

Then (a.3.9.13) eventually rewrites:

$$\pi(\beta,\sigma|y) \propto \sigma^{-k/2} \ \exp\left(-\frac{1}{2}(\beta-\bar{b})'(\sigma\bar{V})^{-1}(\beta-\bar{b})\right) \ \sigma^{-\bar{\alpha}/2-1} \ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right) \tag{a.3.9.15}$$

derivations for equation (3.9.28)

Rearrange:

$$\begin{aligned}
& \Gamma\left(\frac{\bar{\alpha}+k}{2}\right)\left(\frac{\bar{\delta}+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})}{2}\right)^{-\frac{\bar{\alpha}+k}{2}} \\
\propto \ & \left(\frac{\bar{\delta}+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})}{2}\right)^{-\frac{\bar{\alpha}+k}{2}} \\
\propto \ & \left(\bar{\delta}+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)^{-\frac{\bar{\alpha}+k}{2}} \\
\propto \ & \left(1+(\beta-\bar{b})'(\bar{\delta}\bar{V})^{-1}(\beta-\bar{b})\right)^{-\frac{\bar{\alpha}+k}{2}} \\
\propto \ & \left(1+\frac{1}{\bar{\alpha}}(\beta-\bar{b})'(\bar{\delta}\bar{V}/\bar{\alpha})^{-1}(\beta-\bar{b})\right)^{-\frac{\bar{\alpha}+k}{2}}
\end{aligned} \tag{a.3.9.16}$$

derivations for equation (3.9.38)

Rearrange the likelihood function:

$$\begin{aligned}
& f(y|\beta,\sigma) \\
= \ & (2\pi)^{-n/2} \ |\sigma W|^{-1/2} \ \exp\left(-\frac{1}{2}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)\right) \\
= \ & (2\pi\sigma)^{-n/2} \ |W|^{-1/2} \ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'W^{-1}(y-X\beta)}{\sigma}\right)
\end{aligned} \tag{a.3.9.17}$$

This reformulates further as:

$$\begin{aligned}
= \ & (2\pi\sigma)^{-n/2} \ \left(\prod_{i=1}^{n} w_i^{-1/2}\right) \ \exp\left(-\frac{1}{2}\frac{(y-X\beta)' \ diag(\exp(-Z\gamma)) \ (y-X\beta)}{\sigma}\right) \\
= \ & (2\pi\sigma)^{-n/2} \ \left(\prod_{i=1}^{n} \exp(z_i'\gamma)\right)^{-1/2} \ \exp\left(-\frac{1}{2}\frac{(y-X\beta)' \ diag(\exp(-Z\gamma)) \ (y-X\beta)}{\sigma}\right) \\
= \ & (2\pi\sigma)^{-n/2} \ \left(\exp\left(\sum_{i=1}^{n} z_i'\gamma\right)\right)^{-1/2} \ \exp\left(-\frac{1}{2}\frac{(y-X\beta)' \ diag(\exp(-Z\gamma)) \ (y-X\beta)}{\sigma}\right) \\
= \ & (2\pi\sigma)^{-n/2} \ \left(\exp\left(1_n'Z\gamma\right)\right)^{-1/2} \ \exp\left(-\frac{1}{2}\frac{(y-X\beta)' \ diag(\exp(-Z\gamma)) \ (y-X\beta)}{\sigma}\right) \\
= \ & (2\pi\sigma)^{-n/2} \ \exp\left(-\frac{1}{2}1_n'Z\gamma\right) \ \exp\left(-\frac{1}{2}\frac{(y-X\beta)' \ diag(\exp(-Z\gamma)) \ (y-X\beta)}{\sigma}\right) \\
= \ & (2\pi\sigma)^{-n/2} \ \exp\left(-\frac{1}{2}\left[1_n'Z\gamma+(y-X\beta)' \ diag(\exp(-Z\gamma)) \ (y-X\beta)/\sigma\right]\right)
\end{aligned} \tag{a.3.9.18}$$

derivations for equation (3.9.44)

Rearrange the terms:

$$\begin{aligned}
&\pi(\beta|y,\sigma,w)\\
=&\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'W^{-1}(y-X\beta)}{\sigma}\right)\times\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)\\
=&\ \exp\left(-\frac{1}{2}\left[(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\right)\right]
\end{aligned} \tag{a.3.9.19}$$

Consider the term in square brackets and complete the squares:

$$\begin{aligned}
&(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\\
=&\ y'(\sigma W)^{-1}y+\beta'X'(\sigma W)^{-1}X\beta-2\beta'X'(\sigma W)^{-1}y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b\\
=&\ \beta'(V^{-1}+\sigma^{-1}X'W^{-1}X)\beta-2\beta'(V^{-1}b+\sigma^{-1}X'W^{-1}y)+y'(\sigma W)^{-1}y+b'V^{-1}b\\
=&\ \beta'(V^{-1}+\sigma^{-1}X'W^{-1}X)\beta-2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+\sigma^{-1}X'W^{-1}y)\\
&+y'(\sigma W)^{-1}y+b'V^{-1}b+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b}
\end{aligned} \tag{a.3.9.20}$$

Define:

$$\bar{V}=(V^{-1}+\sigma^{-1}X'W^{-1}X)^{-1} \qquad \bar{b}=\bar{V}(V^{-1}b+\sigma^{-1}X'W^{-1}y) \tag{a.3.9.21}$$

Then (a.3.9.20) rewrites:

$$\begin{aligned}
=&\ \beta'\bar{V}^{-1}\beta-2\beta'\bar{V}^{-1}\bar{b}+\bar{b}'\bar{V}^{-1}\bar{b}+y'(\sigma W)^{-1}y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}\\
=&\ (\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'(\sigma W)^{-1}y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}
\end{aligned} \tag{a.3.9.22}$$

Substitute back in (a.3.9.19):

$$\begin{aligned}
&\pi(\beta|y,\sigma,w)\\
=&\ \exp\left(-\frac{1}{2}\left[(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'(\sigma W)^{-1}y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}\right]\right)\\
=&\ \exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)\ \exp\left(-\frac{1}{2}\left[y'(\sigma W)^{-1}y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}\right]\right)\\
\propto&\ \exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)
\end{aligned} \tag{a.3.9.23}$$

derivations for equation (3.9.69)

Rearrange the terms:

$$\begin{aligned}
&\pi(\phi|y,\beta,\sigma)\\
\propto&\ \exp\left(-\frac{1}{2}\frac{(\varepsilon-E\phi)'(\varepsilon-E\phi)}{\sigma}\right)\times\exp\left(-\frac{1}{2}(\phi-p)'H^{-1}(\phi-p)\right)\\
=&\ \exp\left(-\frac{1}{2}\left[(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\phi-p)'H^{-1}(\phi-p)\right]\right)
\end{aligned} \tag{a.3.9.24}$$

Consider the term in square brackets and complete the squares:

$$
\begin{aligned}
& (\varepsilon - E\phi)'\sigma^{-1}(\varepsilon - E\phi) + (\phi - p)'H^{-1}(\phi - p) \\
= \; & \varepsilon'\sigma^{-1}\varepsilon + \phi'E'\sigma^{-1}E\phi - 2\phi'E'\sigma^{-1}\varepsilon + \phi'H^{-1}\phi + p'H^{-1}p - 2\phi'H^{-1}p \\
= \; & \phi'(H^{-1} + \sigma^{-1}E'E)\phi - 2\phi'(H^{-1}p + \sigma^{-1}E'\varepsilon) + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p \\
= \; & \phi'(H^{-1} + \sigma^{-1}E'E)\phi - 2\phi'\bar{H}^{-1}\bar{H}(H^{-1}p + \sigma^{-1}E'\varepsilon) + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p + \bar{p}'\bar{H}^{-1}\bar{p} - \bar{p}'\bar{H}^{-1}\bar{p}
\end{aligned} \tag{a.3.9.25}
$$

Define:

$$
\bar{H} = (H^{-1} + \sigma^{-1}E'E)^{-1} \qquad \bar{p} = \bar{H}(H^{-1}p + \sigma^{-1}E'\varepsilon) \tag{a.3.9.26}
$$

Then (a.3.9.25) becomes:

$$
\begin{aligned}
= \; & \phi'\bar{H}^{-1}\phi - 2\phi'\bar{H}^{-1}\bar{p} + \bar{p}'\bar{H}^{-1}\bar{p} + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p - \bar{p}'\bar{H}^{-1}\bar{p} \\
= \; & (\phi - \bar{p})'\bar{H}^{-1}(\phi - \bar{p}) + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p - \bar{p}'\bar{H}^{-1}\bar{p}
\end{aligned} \tag{a.3.9.27}
$$

Substitute back in (a.3.9.24) to obtain:

$$
\begin{aligned}
& \pi(\phi|y,\beta,\sigma) \\
= \; & \exp\left(-\frac{1}{2}\left[(\phi - \bar{p})'\bar{H}^{-1}(\phi - \bar{p}) + \varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p - \bar{p}'\bar{H}^{-1}\bar{p}\right]\right) \\
= \; & \exp\left(-\frac{1}{2}(\phi - \bar{p})'\bar{H}^{-1}(\phi - \bar{p})\right)\exp\left(-\frac{1}{2}\left[\varepsilon'\sigma^{-1}\varepsilon + p'H^{-1}p - \bar{p}'\bar{H}^{-1}\bar{p}\right]\right) \\
\propto \; & \exp\left(-\frac{1}{2}(\phi - \bar{p})'\bar{H}^{-1}(\phi - \bar{p})\right)
\end{aligned} \tag{a.3.9.28}
$$

CHAPTER 10

Applications with the linear regression model

derivations for equation (3.10.4)

Rearrange the expression:

$$\begin{aligned}
& f(\hat{y}|y) \\
\propto \ & \int \exp\left(-\frac{1}{2}\frac{(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)}{\sigma}\right) \exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right) d\beta \\
= \ & \int \exp\left(-\frac{1}{2}\left[\sigma^{-1}(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right]\right) d\beta
\end{aligned} \tag{a.3.10.1}$$

Consider the term in square brackets:

$$\begin{aligned}
& \sigma^{-1}(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)+(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b}) \\
= \ & \sigma^{-1}\hat{y}'\hat{y}+\sigma^{-1}\beta'\hat{X}'\hat{X}\beta-2\sigma^{-1}\beta'\hat{X}'\hat{y}+\beta'\bar{V}^{-1}\beta+\bar{b}'\bar{V}^{-1}\bar{b}-2\beta'\bar{V}^{-1}\bar{b} \\
= \ & \beta'(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})\beta-2\beta'(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y})+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b} \\
= \ & \beta'(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})\beta-2\beta'\hat{V}^{-1}\hat{V}(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y})+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}+\hat{b}'\hat{V}^{-1}\hat{b}-\hat{b}'\hat{V}^{-1}\hat{b}
\end{aligned} \tag{a.3.10.2}$$

Define:

$$\hat{V}=(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})^{-1} \qquad \hat{b}=\hat{V}(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y}) \tag{a.3.10.3}$$

Then (a.3.10.2) becomes:

$$\begin{aligned}
= \ & \beta'\hat{V}^{-1}\beta-2\beta'\hat{V}^{-1}\hat{b}+\hat{b}'\hat{V}^{-1}\hat{b}+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b} \\
= \ & (\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b}
\end{aligned} \tag{a.3.10.4}$$

Substituting back in (a.3.10.1):

$$\begin{aligned}
& f(\hat{y}|y) \\
\propto \ & \int \exp\left(-\frac{1}{2}\left[(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})+\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b}\right]\right) d\beta \\
= \ & \int \exp\left(-\frac{1}{2}(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right)\exp\left(-\frac{1}{2}\left[\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b}\right]\right) d\beta \\
= \ & \exp\left(-\frac{1}{2}\left[\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b}\right]\right) \int \exp\left(-\frac{1}{2}(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right) d\beta \\
\propto \ & \exp\left(-\frac{1}{2}\left[\sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b}\right]\right)
\end{aligned} \tag{a.3.10.5}$$

Consider the term in square brackets:

$$\begin{aligned}
& \sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\hat{b}'\hat{V}^{-1}\hat{b} \\
= & \ \sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y})'\hat{V}\hat{V}^{-1}\hat{V}(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y}) \\
= & \ \sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y})'\hat{V}(\bar{V}^{-1}\bar{b}+\sigma^{-1}\hat{X}'\hat{y}) \\
= & \ \sigma^{-1}\hat{y}'\hat{y}+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\hat{V}\bar{V}^{-1}\bar{b}-\sigma^{-2}\hat{y}'\hat{X}\hat{V}\hat{X}'\hat{y}-2\sigma^{-1}\hat{y}'\hat{X}\hat{V}\bar{V}^{-1}\bar{b} \\
= & \ \hat{y}'(\sigma^{-1}I_m-\sigma^{-2}\hat{X}\hat{V}\hat{X}')\hat{y}-\bar{b}'(\bar{V}^{-1}-\bar{V}^{-1}\hat{V}\bar{V}^{-1})\bar{b}-2\sigma^{-1}\hat{y}'\hat{X}\hat{V}\bar{V}^{-1}\bar{b}
\end{aligned} \tag{a.3.10.6}$$

In what follows, we make use of property m.13 (the Sherman-Woodbury-Morrison identity):
$(A+BDC)^{-1}=A^{-1}-A^{-1}B(D^{-1}+CA^{-1}B)^{-1}CA^{-1}$.

Consider the central part of the second term in (a.3.10.6). Rearrange and use the identity twice to obtain:

$$\begin{aligned}
& \bar{V}^{-1}-\bar{V}^{-1}\hat{V}\bar{V}^{-1} \\
= & \ \bar{V}^{-1}-\bar{V}^{-1}(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})^{-1}\bar{V}^{-1} \\
= & \ (\bar{V}+\sigma(\hat{X}'\hat{X})^{-1})^{-1} \\
= & \ \sigma^{-1}\hat{X}'\hat{X}-\sigma^{-1}\hat{X}'\hat{X}(\sigma^{-1}\hat{X}'\hat{X}+\bar{V}^{-1})^{-1}\sigma^{-1}\hat{X}'\hat{X} \\
= & \ \sigma^{-1}\hat{X}'\hat{X}-\sigma^{-2}\hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X} \\
= & \ \hat{X}'(\sigma^{-1}I_m-\sigma^{-2}\hat{X}\hat{V}\hat{X}')\hat{X}
\end{aligned} \tag{a.3.10.7}$$

Consider finally the contral part of the third term in (a.3.10.6). We note that $\hat{V}(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})=I_m$ so $\hat{V}\bar{V}^{-1}=I_m-\hat{V}\sigma^{-1}\hat{X}'\hat{X}$. Following:

$$\hat{X}\hat{V}\bar{V}^{-1}=\hat{X}-\hat{X}\hat{V}\sigma^{-1}\hat{X}'\hat{X}=(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X} \tag{a.3.10.8}$$

Substitute (a.3.10.7) and (a.3.10.8) back in (a.3.10.6) to obtain:

$$\begin{aligned}
= & \ \hat{y}'(\sigma^{-1}I_m-\sigma^{-2}\hat{X}\hat{V}\hat{X}')\hat{y}-\bar{b}'\hat{X}'(\sigma^{-1}I_m-\sigma^{-2}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b}-2\sigma^{-1}\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b} \\
= & \ \sigma^{-1}\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{y}-\sigma^{-1}\bar{b}'\hat{X}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b}-2\sigma^{-1}\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b} \\
= & \ \sigma^{-1}\left[\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{y}-\bar{b}'\hat{X}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b}-2\hat{y}'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')\hat{X}\bar{b}\right] \\
= & \ \sigma^{-1}(\hat{y}-\hat{X}\bar{b})'(I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}')(\hat{y}-\hat{X}\bar{b})
\end{aligned} \tag{a.3.10.9}$$

We use one last time the Sherman-Woodbury-Morrison identity on the central term to obtain:

$$I_m-\sigma^{-1}\hat{X}\hat{V}\hat{X}'=I_m-\sigma^{-1}\hat{X}(\bar{V}^{-1}+\sigma^{-1}\hat{X}'\hat{X})^{-1}\hat{X}'=(I_m+\sigma^{-1}\hat{X}\bar{V}\hat{X}')^{-1} \tag{a.3.10.10}$$

Substituting back in (a.3.10.9):

$$\begin{aligned}
= & \ \sigma^{-1}(\hat{y}-\hat{X}\bar{b})'(I_m+\sigma^{-1}\hat{X}\bar{V}\hat{X}')^{-1}(\hat{y}-\hat{X}\bar{b}) \\
= & \ (\hat{y}-\hat{X}\bar{b})'(\sigma I_m+\hat{X}\bar{V}\hat{X}')^{-1}(\hat{y}-\hat{X}\bar{b})
\end{aligned} \tag{a.3.10.11}$$

Eventually substituting back in (a.3.10.5), we conclude:

$$f(\hat{y}|y)\propto\exp\left(-\frac{1}{2}(\hat{y}-\hat{X}\bar{b})'(\sigma I_m+\hat{X}\bar{V}\hat{X}')^{-1}(\hat{y}-\hat{X}\bar{b})\right) \tag{a.3.10.12}$$

derivations for equation (3.10.6)

Rearrange the expression:

$$
\begin{aligned}
& f(\hat{y}|y) \\
\propto\ & \int\int \sigma^{-m/2} \exp\left(-\frac{1}{2}\frac{(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)}{\sigma}\right) \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right) \\
\times\ & \sigma^{-k/2} \exp\left(-\frac{1}{2}(\beta-b)'(\sigma V)^{-1}(\beta-b)\right) \times \sigma^{-\alpha/2-1} \exp\left(-\frac{\delta}{2\sigma}\right) \\
=\ & \int\int \sigma^{-k/2} \exp\left(-\frac{1}{2\sigma}\left[(\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)+(y-X\beta)'(y-X\beta)+(\beta-b)'(\sigma V)^{-1}(\beta-b)+\delta\right]\right) \\
\times\ & \sigma^{-(\alpha+n+m)/2-1} d\beta d\sigma
\end{aligned} \tag{a.3.10.13}
$$

Consider the term in the square bracket:

$$
\begin{aligned}
& (\hat{y}-\hat{X}\beta)'(\hat{y}-\hat{X}\beta)+(y-X\beta)'(y-X\beta)+(\beta-b)'(\sigma V)^{-1}(\beta-b)+\delta \\
=\ & \hat{y}'\hat{y}+\beta'\hat{X}'\hat{X}\beta-2\beta'\hat{X}'\hat{y}+y'y+\beta'X'X\beta-2\beta'X'y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b+\delta \\
=\ & (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b)+\beta'(V^{-1}+X'X+\hat{X}'\hat{X})\beta-2\beta'(V^{-1}b+X'y+\hat{X}'\hat{y}) \\
=\ & (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b)+\beta'(V^{-1}+X'X+\hat{X}'\hat{X})\beta-2\beta'\hat{V}^{-1}\hat{V}(V^{-1}b+X'y+\hat{X}'\hat{y})+\hat{b}'\hat{V}^{-1}\hat{b}-\hat{b}'\hat{V}^{-1}\hat{b} \\
=\ & (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b-\hat{b}'\hat{V}^{-1}\hat{b})+\beta'(V^{-1}+X'X+\hat{X}'\hat{X})\beta-2\beta'\hat{V}^{-1}\hat{V}(V^{-1}b+X'y+\hat{X}'\hat{y})+\hat{b}'\hat{V}^{-1}\hat{b}
\end{aligned} \tag{a.3.10.14}
$$

Define:

$$
\hat{\delta}=\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b-\hat{b}'\hat{V}^{-1}\hat{b} \qquad \hat{V}=(V^{-1}+X'X+\hat{X}'\hat{X})^{-1} \qquad \hat{b}=\hat{V}(V^{-1}b+X'y+\hat{X}'\hat{y}) \tag{a.3.10.15}
$$

Then (a.3.10.14) rewrites:

$$
\begin{aligned}
=\ & \hat{\delta}+\beta'\hat{V}^{-1}\beta-2\beta'\hat{V}^{-1}\hat{b}+\hat{b}'\hat{V}^{-1}\hat{b} \\
=\ & \hat{\delta}+(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})
\end{aligned} \tag{a.3.10.16}
$$

Substitute back in (a.3.10.13):

$$
\begin{aligned}
& f(\hat{y}|y) \\
\propto\ & \int\int \sigma^{-k/2} \exp\left(-\frac{1}{2\sigma}\left[\hat{\delta}+(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right]\right) \sigma^{-(\alpha+n+m)/2-1} d\beta d\sigma \\
=\ & \int\int \sigma^{-k/2} \exp\left(-\frac{1}{2\sigma}(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right) d\beta\ \sigma^{-(\alpha+n+m)/2-1} \exp\left(-\frac{\hat{\delta}}{2\sigma}\right) d\sigma \\
=\ & \int\int \sigma^{-k/2} \exp\left(-\frac{1}{2\sigma}(\beta-\hat{b})'\hat{V}^{-1}(\beta-\hat{b})\right) d\beta\ \sigma^{-(\hat{\alpha}+m)/2-1} \exp\left(-\frac{\hat{\delta}}{2\sigma}\right) d\sigma
\end{aligned} \tag{a.3.10.17}
$$

with:

$$
\hat{\alpha}=\alpha+n+m \tag{a.3.10.18}
$$

The first term is the kernel of a multivariate normal distribution; integration hence yields a constant:

$$
= \int \sigma^{-\hat{\alpha}/2-1} \exp\left(-\frac{\hat{\delta}}{2\sigma}\right) d\sigma \tag{a.3.10.19}
$$

The remaining term is the krenel of an inverse gamma distribution; integration thus yields the reciprocal of the normalization constant:

$$
\begin{aligned}
&= \Gamma(\hat{\alpha}/2)(\hat{\delta}/2)^{-\hat{\alpha}/2} \\
&\propto (\hat{\delta}/2)^{-\hat{\alpha}/2} \\
&\propto \hat{\delta}^{-\hat{\alpha}/2} \\
&= (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b-\hat{b}'\hat{V}^{-1}\hat{b})^{-\hat{\alpha}/2} \\
&= (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b-(V^{-1}b+X'y+\hat{X}'\hat{y})'\hat{V}'\hat{V}^{-1}\hat{V}(V^{-1}b+X'y+\hat{X}'\hat{y}))^{-\hat{\alpha}/2} \\
&= (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b-(V^{-1}b+X'y+\hat{X}'\hat{y})'\hat{V}(V^{-1}b+X'y+\hat{X}'\hat{y}))^{-\hat{\alpha}/2} \qquad \text{(a.3.10.20)}
\end{aligned}
$$

Define:

$$\tilde{b} = V^{-1}b + X'y \qquad \text{(a.3.10.21)}$$

Then (a.3.10.20) rewrites:

$$
\begin{aligned}
&= (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b-(\tilde{b}+\hat{X}'\hat{y})'\hat{V}(\tilde{b}+\hat{X}'\hat{y}))^{-\hat{\alpha}/2} \\
&= (\delta+\hat{y}'\hat{y}+y'y+b'V^{-1}b-\tilde{b}'\hat{V}\tilde{b}-\hat{y}'\hat{X}\hat{V}\hat{X}'\hat{y}-2\hat{y}'\hat{X}\hat{V}\tilde{b})^{-\hat{\alpha}/2} \\
&= (\delta+y'y+b'V^{-1}b+\hat{y}'(I_m-\hat{X}\hat{V}\hat{X}')\hat{y}-\tilde{b}'\hat{V}\tilde{b}-2\hat{y}'\hat{X}\hat{V}\tilde{b})^{-\hat{\alpha}/2} \\
&= ([\delta+y'y+b'V^{-1}b-\tilde{b}'\hat{V}\tilde{b}-\ddot{y}'\ddot{V}^{-1}\ddot{y}]+\hat{y}'(I_m-\hat{X}\hat{V}\hat{X}')\hat{y}-2\hat{y}'\ddot{V}^{-1}\ddot{V}\hat{X}\hat{V}\tilde{b}+\ddot{y}'\ddot{V}^{-1}\ddot{y})^{-\hat{\alpha}/2} \qquad \text{(a.3.10.22)}
\end{aligned}
$$

Define:

$$\ddot{\delta} = \delta+y'y+b'V^{-1}b-\tilde{b}'\hat{V}\tilde{b}-\ddot{y}'\ddot{V}^{-1}\ddot{y} \qquad \ddot{V} = (I_m-\hat{X}\hat{V}\hat{X}')^{-1} \qquad \ddot{y} = \ddot{V}\hat{X}\hat{V}\tilde{b} \qquad \text{(a.3.10.23)}$$

Then (a.3.10.22) rewrites:

$$
\begin{aligned}
&= (\ddot{\delta}+\hat{y}'\ddot{V}^{-1}\hat{y}-2\hat{y}'\ddot{V}^{-1}\ddot{y}+\ddot{y}'\ddot{V}^{-1}\ddot{y})^{-\hat{\alpha}/2} \\
&= (\ddot{\delta}+(\hat{y}-\ddot{y})'\ddot{V}^{-1}(\hat{y}-\ddot{y}))^{-\hat{\alpha}/2} \\
&= \ddot{\delta}^{-\hat{\alpha}/2}(1+(\hat{y}-\ddot{y})'[\ddot{\delta}\ddot{V}]^{-1}(\hat{y}-\ddot{y}))^{-\hat{\alpha}/2} \\
&\propto (1+(\hat{y}-\ddot{y})'[\ddot{\delta}\ddot{V}]^{-1}(\hat{y}-\ddot{y}))^{-\hat{\alpha}/2} \\
&= \left(1+\frac{1}{\bar{\alpha}}(\hat{y}-\ddot{y})'[\ddot{\delta}\ddot{V}/\bar{\alpha}]^{-1}(\hat{y}-\ddot{y})\right)^{-(\bar{\alpha}+m)/2} \qquad \text{(a.3.10.24)}
\end{aligned}
$$

Thus we finally conclude:

$$f(\hat{y}|y) \propto \left(1+\frac{1}{\bar{\alpha}}(\hat{y}-\ddot{y})'[\ddot{\delta}\ddot{V}/\bar{\alpha}]^{-1}(\hat{y}-\ddot{y})\right)^{-(\bar{\alpha}+m)/2} \qquad \text{(a.3.10.25)}$$

Finally, reformulate the messy terms. First, reformulate $\ddot{V}$. For this, we make again use of property m.13 (the Sherman-Woodbury-Morrison identity): $(A+BDC)^{-1} = A^{-1}-A^{-1}B(D^{-1}+CA^{-1}B)^{-1}CA^{-1}$.

Then, starting from (a.3.10.23):

$$
\begin{aligned}
&\ddot{V} \\
&= (I_m-\hat{X}\hat{V}\hat{X}')^{-1} \\
&= (I_m-\hat{X}(V^{-1}+X'X+\hat{X}'\hat{X})^{-1}\hat{X}')^{-1} \\
&= I_m+\hat{X}(V^{-1}+X'X)^{-1}\hat{X}' \\
&= I_m+\hat{X}\bar{V}\hat{X}' \qquad \text{(a.3.10.26)}
\end{aligned}
$$

Now consider the term $\ddot{y}$. Start from:

$$\begin{aligned} & \ddot{V}\hat{X}\hat{V} \\ = \; & (I_m + \hat{X}\bar{V}\hat{X}')\hat{X}\hat{V} \\ = \; & \hat{X}\hat{V} + \hat{X}\bar{V}\hat{X}'\hat{X}\hat{V} \\ = \; & \hat{X}(\hat{V} + \bar{V}\hat{X}'\hat{X}\hat{V}) \\ = \; & \hat{X}(I_m + \bar{V}\hat{X}'\hat{X})\hat{V} \end{aligned} \tag{a.3.10.27}$$

We then note that (a.3.10.15) implies:

$$\hat{V} = (V^{-1} + X'X + \hat{X}'\hat{X})^{-1} \Leftrightarrow \hat{V} = (\bar{V}^{-1} + \hat{X}'\hat{X})^{-1} \tag{a.3.10.28}$$

Hence:

$$\begin{aligned} = \; & \hat{X}(I_m + \bar{V}\hat{X}'\hat{X})(\bar{V}^{-1} + \hat{X}'\hat{X})^{-1} \\ = \; & \hat{X}\bar{V}(\bar{V}^{-1} + \hat{X}'\hat{X})(\bar{V}^{-1} + \hat{X}'\hat{X})^{-1} \\ = \; & \hat{X}\bar{V} \end{aligned} \tag{a.3.10.29}$$

Using this result in (a.3.10.23), and combining with definition (a.3.10.21), we obtain:

$$\ddot{y} = \ddot{V}\hat{X}\hat{V}\tilde{b} = \hat{X}\bar{V}\tilde{b} = \hat{X}\bar{b} \tag{a.3.10.30}$$

Finally, reformulate $\ddot{\delta}$. First, note that:

$$\begin{aligned} & \tilde{b}'\hat{V}\tilde{b} + \ddot{y}'\ddot{V}^{-1}\ddot{y} \\ = \; & \bar{b}'\bar{V}^{-1}\hat{V}\bar{V}^{-1}\bar{b} + \bar{b}'\hat{X}'(I_m - \hat{X}\hat{V}\hat{X}')\hat{X}\bar{b} \\ = \; & \bar{b}'[(\hat{V}^{-1} - \hat{X}'\hat{X})'\hat{V}(\hat{V}^{-1} - \hat{X}'\hat{X}) + \hat{X}'\hat{X} - \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X}]\bar{b} \\ = \; & \bar{b}'[(\hat{V}^{-1} - \hat{X}'\hat{X})'(I_k - \hat{V}\hat{X}'\hat{X}) + \hat{X}'\hat{X} - \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X}]\bar{b} \\ = \; & \bar{b}'[\hat{V}^{-1} - \hat{X}'\hat{X} - \hat{X}'\hat{X} + \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X} + \hat{X}'\hat{X} - \hat{X}'\hat{X}\hat{V}\hat{X}'\hat{X}]\bar{b} \\ = \; & \bar{b}'[\hat{V}^{-1} - \hat{X}'\hat{X}]\bar{b} \\ = \; & \bar{b}'[\bar{V}^{-1} + \hat{X}'\hat{X} - \hat{X}'\hat{X}]\bar{b} \\ = \; & \bar{b}'\bar{V}^{-1}\bar{b} \end{aligned} \tag{a.3.10.31}$$

Substituting this in (a.3.10.23) to obtain:

$$\ddot{\delta} \; = \; \delta + y'y + b'V^{-1}b - \tilde{b}'\hat{V}\tilde{b} - \ddot{y}'\ddot{V}^{-1}\ddot{y} \; = \; \delta + y'y + b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} \; = \; \bar{\delta} \tag{a.3.10.32}$$

Eventually substituting for (a.3.10.26), (a.3.10.30) and (a.3.10.32) in (a.3.10.25) yields:

$$f(\hat{y}|y) \propto \left(1 + \frac{1}{\bar{\alpha}}(\hat{y} - \hat{X}\bar{b})'[\bar{\delta}(I_m + \hat{X}\bar{V}\hat{X}')/\bar{\alpha}]^{-1}(\hat{y} - \hat{X}\bar{b})\right)^{-(\bar{\alpha}+m)/2} \tag{a.3.10.33}$$

derivations for equation (3.10.10)

The log likelihood function is given by:

$$\log(f(y|\beta,\sigma)) = -\frac{n}{2}\ \log(2\pi) - \frac{n}{2}\ \log(\sigma) - \frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma} \tag{a.3.10.34}$$

The function is estimated at the maximum likelihood values. Hence $\beta = \hat{\beta}$ and $\sigma = \hat{\sigma} = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}$.

Substituting in (a.3.10.34):

$$\begin{aligned}
& \log(f(y|\hat{\beta},\hat{\sigma})) \\
= & -\frac{n}{2}\ \log(2\pi) - \frac{n}{2}\ \log(\hat{\sigma}) - \frac{1}{2}\frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{\hat{\sigma}} \\
= & -\frac{n}{2}\ \log(2\pi) - \frac{n}{2}\ \log(\hat{\sigma}) - \frac{1}{2}\frac{n\ \hat{\varepsilon}'\hat{\varepsilon}}{\hat{\varepsilon}'\hat{\varepsilon}} \\
= & -\frac{n}{2}\ \log(2\pi) - \frac{n}{2}\ \log(\hat{\sigma}) - \frac{n}{2} \\
= & -\frac{n}{2}\ [\log(2\pi) + \ \log(\hat{\sigma}) + 1]
\end{aligned} \tag{a.3.10.35}$$

Then AIC obtains as:

$$\begin{aligned}
& AIC = 2k/n - 2\ \hat{L}/n \\
= & \ 2k/n - 2\left(-\frac{n}{2}\ [\log(2\pi) + \ \log(\hat{\sigma}) + 1]\right)/n \\
= & \ 2k/n\ +\ \log(2\pi)\ +\ \log(\hat{\sigma}) + 1
\end{aligned} \tag{a.3.10.36}$$

Using similar calculations, BIC immediately obtains as:

$$BIC = k\ \log(n)/n\ +\ \log(2\pi)\ +\ \log(\hat{\sigma}) + 1 \tag{a.3.10.37}$$

Removing the constants that make the value invariant to the number of coefficients:

$$AIC = 2k/n + \ \log(\hat{\sigma}) \qquad BIC = k\ \log(n)/n\ +\ \log(\hat{\sigma}) \tag{a.3.10.38}$$

derivations for equation (3.10.19)

Rearrange:

$$
\begin{aligned}
& f(y) \\
= & \int (2\pi\sigma)^{-n/2} \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right) \times (2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right) d\beta \\
= & \int (2\pi)^{-(n+k)/2}\, \sigma^{-n/2}\, |V|^{-1/2} \times \exp\left(-\frac{1}{2}\left[(y-X\beta)'\sigma^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\right]\right) d\beta
\end{aligned}
\tag{a.3.10.39}
$$

Consider the term square brackets:

$$
\begin{aligned}
& (y-X\beta)'\sigma^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b) \\
= & \; y'\sigma^{-1}y+\beta'X'\sigma^{-1}X\beta-2\beta'X'\sigma^{-1}y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b \\
= & \; \beta'(V^{-1}+\sigma^{-1}X'X)\beta-2\beta'(V^{-1}b+\sigma^{-1}X'y)+y'\sigma^{-1}y+b'V^{-1}b \\
= & \; \beta'(V^{-1}+\sigma^{-1}X'X)\beta-2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+\sigma^{-1}X'y)+\bar{b}\bar{V}^{-1}\bar{b}+y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}
\end{aligned}
\tag{a.3.10.40}
$$

Define:

$$
\bar{V}=(V^{-1}+\sigma^{-1}X'X)^{-1} \qquad \bar{b}=\bar{V}(V^{-1}b+\sigma^{-1}X'y) \tag{a.3.10.41}
$$

Then (a.3.10.40) reformulates:

$$
\begin{aligned}
= & \; \beta'\bar{V}^{-1}\beta-2\beta'\bar{V}^{-1}\bar{b}+\bar{b}\bar{V}^{-1}\bar{b}+y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b} \\
= & \; (\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}
\end{aligned}
\tag{a.3.10.42}
$$

Substitute back in (a.3.10.39):

$$
\begin{aligned}
& f(y) \\
= & \int (2\pi)^{-(n+k)/2}\, \sigma^{-n/2}\, |V|^{-1/2} \times \exp\left(-\frac{1}{2}\left[(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}\right]\right) d\beta \\
= & \; (2\pi)^{-(n+k)/2}\, \sigma^{-n/2}\, |V|^{-1/2}(2\pi)^{k/2}|\bar{V}|^{1/2} \times \exp\left(-\frac{1}{2}\left[y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}\right]\right) \\
\times & \int (2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right) d\beta \\
= & \; (2\pi)^{-n/2}\, \sigma^{-n/2}|\bar{V}|^{1/2}|V|^{-1/2} \times \exp\left(-\frac{1}{2}\left[y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}\right]\right) \\
\times & \int (2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right) d\beta
\end{aligned}
\tag{a.3.10.43}
$$

derivations for equation (3.10.21)

Consider the term:

$$
\begin{aligned}
& (2\pi)^{-n/2}\ \sigma^{-n/2}|\bar{V}|^{1/2}|V|^{-1/2} \\
= \ & (2\pi)^{-n/2}\ \sigma^{-n/2}|(V^{-1}+\sigma^{-1}X'X)^{-1}|^{1/2}|V|^{-1/2} \\
= \ & (2\pi)^{-n/2}\ \sigma^{-n/2}|V^{-1}+\sigma^{-1}X'X|^{-1/2}|V|^{-1/2} \\
= \ & (2\pi)^{-n/2}\ \sigma^{-n/2}(|V||V^{-1}+\sigma^{-1}X'X|)^{-1/2} \\
= \ & (2\pi)^{-n/2}\ \sigma^{-n/2}|I_k+\sigma^{-1}VX'X|^{-1/2}
\end{aligned}
\tag{a.3.10.44}
$$

Hence:

$$
f(y) = (2\pi)^{-n/2}\ \sigma^{-n/2}|I_k+\sigma^{-1}VX'X|^{-1/2}\exp\left(-\frac{1}{2}\left[y'\sigma^{-1}y+b'V^{-1}b-\bar{b}\bar{V}^{-1}\bar{b}\right]\right)
\tag{a.3.10.45}
$$

derivations for equation (3.10.23)

Rearrange the expression:

$$
\begin{aligned}
& f(y) \\
= \ & \int\!\!\int (2\pi\sigma)^{-n/2}\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right) \\
\times \ & (2\pi)^{-k/2}|\sigma V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'(\sigma V)^{-1}(\beta-b)\right)\times\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)d\beta d\sigma \\
= \ & \int\!\!\int (2\pi\sigma)^{-n/2}\ (2\pi)^{-k/2}|\sigma V|^{-1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1} \\
\times \ & \exp\left(-\frac{1}{2\sigma}\left[(y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)+\delta\right]\right)d\beta d\sigma
\end{aligned}
\tag{a.3.10.46}
$$

Consider the term in square brackets and complete the squares:

$$
\begin{aligned}
& (y-X\beta)'(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)+\delta \\
= \ & y'y+\beta'X'X\beta-2\beta'X'y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b+\delta \\
= \ & \beta'(V^{-1}+X'X)\beta-2\beta'(V^{-1}b+X'y)+\delta+y'y+b'V^{-1}b \\
= \ & \beta'(V^{-1}+X'X)\beta-2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+X'y)+\delta+y'y+b'V^{-1}b+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b}
\end{aligned}
\tag{a.3.10.47}
$$

Define:

$$
\bar{V} = (V^{-1}+X'X)^{-1} \qquad \bar{b} = \bar{V}(V^{-1}b+X'y) \qquad \bar{\delta} = \delta+y'y+b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}
\tag{a.3.10.48}
$$

Then (a.3.10.47) rewrites:

$$
\begin{aligned}
= \ & \beta'\bar{V}^{-1}\beta-2\beta'\bar{V}^{-1}\bar{b}+\bar{b}'\bar{V}^{-1}\bar{b}+\bar{\delta} \\
= \ & (\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+\bar{\delta}
\end{aligned}
\tag{a.3.10.49}
$$

Substituting back in (a.3.10.46):

$$
\begin{aligned}
& f(y) \\
= \;& \int\int (2\pi\sigma)^{-n/2}\ (2\pi)^{-k/2}|\sigma V|^{-1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1} \\
\times \;& \exp\left(-\frac{1}{2\sigma}\left[(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+\bar{\delta}\right]\right)d\beta d\sigma \\
= \;& \int\int (2\pi)^{-n/2}\ (2\pi)^{-k/2}|\sigma V|^{-1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\
\times \;& \sigma^{-(\alpha+n)/2-1}\exp\left(-\frac{1}{2\sigma}\left[(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+\bar{\delta}\right]\right)d\beta d\sigma
\end{aligned}
\tag{a.3.10.50}
$$

define:

$$
\bar{\alpha} = \alpha + n \tag{a.3.10.51}
$$

Then (a.3.10.50) rewrites:

$$
\begin{aligned}
= \;& \int\int (2\pi)^{-n/2}\ (2\pi)^{-k/2}|\sigma V|^{-1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)} \\
\times \;& \sigma^{-\bar{\alpha}/2-1}\exp\left(-\frac{1}{2\sigma}\left[(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+\bar{\delta}\right]\right)d\beta d\sigma \\
= \;& (2\pi)^{-n/2}\ |\sigma V|^{-1/2}|\sigma\bar{V}|^{1/2}\ \frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\ \frac{\Gamma(\bar{\alpha}/2)}{\bar{\delta}/2^{\bar{\alpha}/2}} \\
\times \;& \int\int (2\pi)^{-k/2}|\sigma\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'(\sigma\bar{V})^{-1}(\beta-\bar{b})\right)\times\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1}\exp\left(-\frac{\bar{\delta}}{2\sigma}\right)d\beta d\sigma \\
= \;& 2^{-n/2}\pi^{-n/2}\ |V|^{-1/2}|\bar{V}|^{1/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{2^{(\alpha+n)/2}}{2^{\alpha/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
\times \;& \int\int (2\pi)^{-k/2}|\sigma\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'(\sigma\bar{V})^{-1}(\beta-\bar{b})\right)\times\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1}\exp\left(-\frac{\bar{\delta}}{2\sigma}\right)d\beta d\sigma \\
= \;& \pi^{-n/2}\ |V|^{-1/2}|\bar{V}|^{1/2}\ \frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
\times \;& \int\int (2\pi)^{-k/2}|\sigma\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'(\sigma\bar{V})^{-1}(\beta-\bar{b})\right)\times\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1}\exp\left(-\frac{\bar{\delta}}{2\sigma}\right)d\beta d\sigma
\end{aligned}
\tag{a.3.10.52}
$$

derivations for equation (3.10.25)

Reformulate the expression:

$$
\begin{aligned}
& \pi^{-n/2}\,|V|^{-1/2}|\bar{V}|^{1/2}\,\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\,\frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= \; & \pi^{-n/2}\,|V|^{-1/2}|(V^{-1}+X'X)^{-1}|^{1/2}\,\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\,\frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= \; & \pi^{-n/2}\,|V|^{-1/2}|(V^{-1}+X'X)|^{-1/2}\,\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\,\frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= \; & \pi^{-n/2}\,|V(V^{-1}+X'X)|^{-1/2}\,\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\,\frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \\
= \; & \pi^{-n/2}\,|I_k+VX'X|^{-1/2}\,\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}}\,\frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)} \qquad \text{(a.3.10.53)}
\end{aligned}
$$

derivations for equation (3.10.27)

Rearrange the expression:

$$
\begin{aligned}
& f(y) \\
\approx\ & \frac{f(y|\beta^*,\sigma^*)\pi(\beta^*,\sigma^*)}{\pi(\sigma^*|y,\beta^*)\times\frac{1}{J}\sum_{j=1}^{J}\pi(\beta^*|\sigma^{(j)},y)} \\
=\ & (2\pi\sigma)^{-n/2}\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right)\times\frac{(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \\
\times\ & \frac{\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)}{\frac{\bar{\delta}/2^{\bar{\alpha}/2}}{\Gamma(\bar{\alpha}/2)}\sigma^{-\bar{\alpha}/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & 2^{-n/2}\pi^{-n/2}\sigma^{-n/2}\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'(y-X\beta)}{\sigma}\right)\times\frac{(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \\
\times\ & \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{2^{(\alpha+n)/2}}{2^{\alpha/2}}\frac{\delta^{\alpha/2}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)}{\bar{\delta}^{\bar{\alpha}/2}\ \sigma^{-(\alpha+n)/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & 2^{-n/2}\pi^{-n/2}\sigma^{-n/2}\ \times\frac{(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \\
\times\ & \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{2^{(\alpha+n)/2}}{2^{\alpha/2}}\frac{\delta^{\alpha/2}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta+(y-X\beta)'(y-X\beta)}{2\sigma}\right)}{\bar{\delta}^{\bar{\alpha}/2}\ \sigma^{-(\alpha+n)/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & 2^{-n/2}\pi^{-n/2}\sigma^{-n/2}\ \times\frac{(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-k/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \\
\times\ & \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{2^{(\alpha+n)/2}}{2^{\alpha/2}}\frac{\delta^{\alpha/2}\sigma^{-\alpha/2-1}\exp\left(-\frac{\bar{\delta}}{2\sigma}\right)}{\bar{\delta}^{\bar{\alpha}/2}\ \sigma^{-(\alpha+n)/2-1}\ \exp\left(-\frac{\bar{\delta}}{2\sigma}\right)} \\
=\ & \pi^{-n/2}\frac{|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \\
=\ & \pi^{-n/2}\frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|V|^{1/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \\
=\ & \pi^{-n/2}\frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|V|^{1/2}|(V^{-1}+\sigma^{-1}X'X)^{-1}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \\
=\ & \pi^{-n/2}\frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|V|^{1/2}|V^{-1}+\sigma^{-1}X'X|^{1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \\
=\ & \pi^{-n/2}\frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|I_k+\sigma^{-1}VX'X|^{1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma(\bar{\alpha}/2)}{\Gamma(\alpha/2)}\frac{\delta^{\alpha/2}}{\bar{\delta}^{\bar{\alpha}/2}} \qquad \text{(a.3.10.54)}
\end{aligned}
$$

derivations for equation (3.10.29)

Substitute for the functions and rearrange:

$$
\begin{aligned}
&\frac{1}{f(y)} \\
\approx\ & \frac{1}{J}\sum_{j=1}^{J} \frac{g(\theta^{(j)})}{f(y|\beta^{(j)},\sigma^{(j)},\gamma^{(j)})\ \pi(\beta^{(j)})\ \pi(\sigma^{(j)})\ \pi(\gamma^{(j)})} \\
=\ & \frac{1}{J}\sum_{j=1}^{J} \frac{\omega^{-1}(2\pi)^{-(k+h+1)/2}|\hat{\Sigma}|^{-1/2}\exp\left(-\frac{1}{2}(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right)\mathbb{1}(\theta\in\hat{\Theta})}{\left[\begin{array}{l}(2\pi\sigma)^{-n/2}\ |W|^{-1/2}\ \exp\left(-\frac{1}{2}\frac{(y-X\beta)'W^{-1}(y-X\beta)}{\sigma}\right)\times(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)\\ \times\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)\times(2\pi)^{-h/2}|Q|^{-1/2}\exp\left(-\frac{1}{2}(\gamma-g)'Q^{-1}(\gamma-g)\right)\end{array}\right]} \\
=\ & \frac{1}{J}\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ \ \omega^{-1}(2\pi)^{(n+k+h-(k+h+1))/2}\ |\hat{\Sigma}|^{-1/2}\ |W|^{1/2}\ |V|^{1/2}\ |Q|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\ \ \sigma^{(\alpha+n)/2+1} \\
&\times\exp\left(\frac{1}{2}\left[\begin{array}{l}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)+\delta\sigma^{-1}\\ +(\gamma-g)'Q^{-1}(\gamma-g)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right) \\
=\ & (\omega J)^{-1}(2\pi)^{(n-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Q|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}} \\
&\times\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ |W|^{1/2}\ \sigma^{(\alpha+n)/2+1}\ \exp\left(\frac{1}{2}\left[\begin{array}{l}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\gamma-g)'Q^{-1}(\gamma-g)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)
\end{aligned}
\tag{a.3.10.55}
$$

Using logs on both sides yields:

$$
\begin{aligned}
&-log(f(y))\approx log\left((\omega J)^{-1}(2\pi)^{(n-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Q|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
&+log\left(\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ |W|^{1/2}\ \sigma^{(\alpha+n)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\gamma-g)'Q^{-1}(\gamma-g)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)\right)
\end{aligned}
\tag{a.3.10.56}
$$

or:

$$
\begin{aligned}
&log(f(y))\approx -log\left((\omega J)^{-1}(2\pi)^{(n-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Q|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
&-log\left(\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ |W|^{1/2}\ \sigma^{(\alpha+n)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(y-X\beta)'(\sigma W)^{-1}(y-X\beta)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\gamma-g)'Q^{-1}(\gamma-g)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)\right)
\end{aligned}
\tag{a.3.10.57}
$$

derivations for equation (3.10.32)

Substitute for the functions and rearrange:

$$
\begin{aligned}
&\frac{1}{f(y)} \\
\approx\ & \frac{1}{J}\sum_{j=1}^{J}\frac{g(\theta^{(j)})}{f(y|\beta^{(j)},\sigma^{(j)},\phi^{(j)})\ \pi(\beta^{(j)})\ \pi(\sigma^{(j)})\ \pi(\phi^{(j)})} \\
=\ & \frac{1}{J}\sum_{j=1}^{J}\frac{\omega^{-1}(2\pi)^{-(k+q+1)/2}|\hat{\Sigma}|^{-1/2}\exp\left(-\frac{1}{2}(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\right)\mathbb{1}(\theta\in\hat{\Theta})}{\left[\begin{array}{l}(2\pi\sigma)^{-T/2}\ \exp\left(-\frac{1}{2}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)\right)\times(2\pi)^{-k/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)\\ \times\frac{\delta/2^{\alpha/2}}{\Gamma(\alpha/2)}\sigma^{-\alpha/2-1}\exp\left(-\frac{\delta}{2\sigma}\right)\times(2\pi)^{-q/2}|Z|^{-1/2}\exp\left(-\frac{1}{2}(\phi-p)'Z^{-1}(\phi-p)\right)\end{array}\right]} \\
=\ & \frac{1}{J}\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\ \ \omega^{-1}(2\pi)^{(T+k+q-(k+q+1))/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Z|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\ \ \sigma^{(\alpha+T)/2+1} \\
&\times\exp\left(\frac{1}{2}\left[\begin{array}{l}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\phi-p)'Z^{-1}(\phi-p)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right) \\
=\ & (\omega J)^{-1}(2\pi)^{(T-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Z|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}} \\
&\times\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\sigma^{(\alpha+T)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\phi-p)'Z^{-1}(\phi-p)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)
\end{aligned}
\tag{a.3.10.58}
$$

Using logs on both sides yields:

$$
\begin{aligned}
&-log(f(y))\approx log\left((\omega J)^{-1}(2\pi)^{(T-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Z|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
&+log\left(\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\sigma^{(\alpha+T)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\phi-p)'Z^{-1}(\phi-p)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)\right)
\end{aligned}
\tag{a.3.10.59}
$$

or:

$$
\begin{aligned}
&log(f(y))\approx -log\left((\omega J)^{-1}(2\pi)^{(T-1)/2}\ |\hat{\Sigma}|^{-1/2}\ |V|^{1/2}\ |Z|^{1/2}\ \ \frac{\Gamma(\alpha/2)}{\delta/2^{\alpha/2}}\right) \\
&-log\left(\sum_{j=1}^{J}\mathbb{1}(\theta\in\hat{\Theta})\sigma^{(\alpha+T)/2+1}\exp\left(\frac{1}{2}\left[\begin{array}{l}(\varepsilon-E\phi)'\sigma^{-1}(\varepsilon-E\phi)+(\beta-b)'V^{-1}(\beta-b)\\ +\delta\sigma^{-1}+(\phi-p)'Z^{-1}(\phi-p)-(\theta-\hat{\theta})'\hat{\Sigma}^{-1}(\theta-\hat{\theta})\end{array}\right]\right)\right)
\end{aligned}
\tag{a.3.10.60}
$$

PART IV

Vector autoregressions

CHAPTER 11

Vector autoregressions

derivations for equation (4.11.9)

Consider first β. To do so, rewrite the likelihood function as:

$$\log(f(y|\beta,\Sigma)) = -\frac{nT}{2}\ \log(2\pi) - \frac{1}{2}\ \log(|\bar{\Sigma}|) - \frac{1}{2}(y'\bar{\Sigma}^{-1}y + \beta'\bar{X}'\bar{\Sigma}^{-1}\bar{X}\beta - 2\beta'\bar{X}'\bar{\Sigma}^{-1}y) \qquad \text{(a.4.11.1)}$$

Then solve for the partial derivative:

$$\begin{aligned}
& \frac{\partial log(f(y|\beta,\Sigma))}{\partial\beta} = 0 \\
\Leftrightarrow\ & -\frac{1}{2}(2\beta'\bar{X}'\bar{\Sigma}^{-1}\bar{X} - 2y'\bar{\Sigma}^{-1}\bar{X}'y) = 0 \\
\Leftrightarrow\ & \beta'\bar{X}'\bar{\Sigma}^{-1}\bar{X} - y'\bar{\Sigma}^{-1}\bar{X} = 0 \\
\Leftrightarrow\ & \beta'\bar{X}'\bar{\Sigma}^{-1}\bar{X} = y'\bar{\Sigma}^{-1}\bar{X} \\
\Leftrightarrow\ & \bar{X}'\bar{\Sigma}^{-1}\bar{X}\beta = \bar{X}'\bar{\Sigma}^{-1}y \\
\Leftrightarrow\ & \beta = (\bar{X}'\bar{\Sigma}^{-1}\bar{X})^{-1}\bar{X}'\bar{\Sigma}^{-1}y
\end{aligned} \qquad \text{(a.4.11.2)}$$

The formula can simplify further. Note first that:

$$\begin{aligned}
& \bar{X}'\bar{\Sigma}^{-1} \\
=\ & (I_n \otimes X)'(\Sigma \otimes I_T)^{-1} \\
=\ & (I_n \otimes X')(\Sigma^{-1} \otimes I_T) \\
=\ & \Sigma^{-1} \otimes X'
\end{aligned} \qquad \text{(a.4.11.3)}$$

Substituting for (a.4.11.3) in (a.4.11.2):

$$\begin{aligned}
& \beta \\
=\ & (\bar{X}'\bar{\Sigma}^{-1}\bar{X})^{-1}\bar{X}'\bar{\Sigma}^{-1}y \\
=\ & \left[(\Sigma^{-1} \otimes X')(I_n \otimes X)\right]^{-1}\left[(\Sigma^{-1} \otimes X')y\right] \\
=\ & (\Sigma^{-1} \otimes X'X)^{-1}\ (\Sigma^{-1} \otimes X')\ vec(Y) \\
=\ & (\Sigma \otimes (X'X)^{-1})\ (\Sigma^{-1} \otimes X')\ vec(Y) \\
=\ & (I_n \otimes (X'X)^{-1}X')\ vec(Y) \\
=\ & vec((X'X)^{-1}X'Y) \\
=\ & vec(\hat{\mathcal{B}})
\end{aligned} \qquad \text{(a.4.11.4)}$$

with:

$$\hat{\mathcal{B}} = (X'X)^{-1}X'Y \qquad \text{(a.4.11.5)}$$

Hence the maximum likelihood estimate is $\hat{\beta} = vec(\hat{\mathcal{B}})$.

To obtain the maximum likelihood estimate for Σ, it is convenient to use property d.7 that states the equivalence between the multivariate normal and matrix normal distributions. Doing so, the likelihood function of the VAR model rewrites as:

$$f(y|\beta,\Sigma) = (2\pi)^{-nT/2}|\Sigma|^{-T/2}exp\left(-\frac{1}{2}tr\left[\Sigma^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})\right]\right) \tag{a.4.11.6}$$

And the log-likelihood becomes:

$$\log(f(y|\beta,\Sigma) = -\frac{nT}{2}\log(2\pi) - \frac{T}{2}\log(|\Sigma|) - \frac{1}{2}tr\left[\Sigma^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})\right] \tag{a.4.11.7}$$

Then solve for the partial derivative:

$$\begin{aligned}
& \frac{\partial log(f(y|\beta,\Sigma))}{\partial\Sigma} = 0 \\
\Leftrightarrow\ & -\frac{T}{2}\Sigma^{-1} - \frac{1}{2}(Y-X\mathcal{B})'(Y-X\mathcal{B})(-\Sigma^{-1}\Sigma^{-1}) = 0 \\
\Leftrightarrow\ & -T\Sigma^{-1} + (Y-X\mathcal{B})'(Y-X\mathcal{B})(\Sigma^{-1}\Sigma^{-1}) = 0 \\
\Leftrightarrow\ & T\Sigma^{-1} = (Y-X\mathcal{B})'(Y-X\mathcal{B})(\Sigma^{-1}\Sigma^{-1}) \\
\Leftrightarrow\ & T\Sigma = (Y-X\mathcal{B})'(Y-X\mathcal{B}) \\
\Leftrightarrow\ & \Sigma = \frac{1}{T}(Y-X\mathcal{B})'(Y-X\mathcal{B})
\end{aligned} \tag{a.4.11.8}$$

Replacing $\mathcal{B}$ with its maximum likelihood estimate $\hat{\mathcal{B}}$, we conclude that the maximum likelihood estimate for Σ is $\hat{\Sigma} = \frac{1}{T}(Y-X\hat{\mathcal{B}})'(Y-X\hat{\mathcal{B}})$.

derivations for equation (4.11.14)

Group terms:

$$\begin{aligned}
& \pi(\beta|y) \\
\propto\ & \exp\left(-\frac{1}{2}(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta)\right) \times \exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right) \\
=\ & \exp\left(-\frac{1}{2}\left[(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta) + (\beta-b)'V^{-1}(\beta-b)\right]\right)
\end{aligned} \tag{a.4.11.9}$$

Consider the terms in square brackets:

$$\begin{aligned}
& (y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta) + (\beta-b)'V^{-1}(\beta-b) \\
=\ & y'\bar{\Sigma}^{-1}y + \beta'\bar{X}'\bar{\Sigma}^{-1}\bar{X}\beta - 2\beta'\bar{X}'\bar{\Sigma}^{-1}y + \beta'V^{-1}\beta + b'V^{-1}b - 2\beta'V^{-1}b \\
=\ & \beta'(V^{-1}+\bar{X}'\bar{\Sigma}^{-1}\bar{X})\beta - 2\beta'(V^{-1}b + \bar{X}'\bar{\Sigma}^{-1}y) + b'V^{-1}b + y'\bar{\Sigma}^{-1}y
\end{aligned} \tag{a.4.11.10}$$

Complete the squares:

$$= \beta'(V^{-1}+\bar{X}'\bar{\Sigma}^{-1}\bar{X})\beta - 2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+\bar{X}'\bar{\Sigma}^{-1}y) + \bar{b}'\bar{V}^{-1}\bar{b} - \bar{b}'\bar{V}^{-1}\bar{b} + b'V^{-1}b + y'\bar{\Sigma}^{-1}y \tag{a.4.11.11}$$

Define:

$$\bar{V} = (V^{-1}+\bar{X}'\bar{\Sigma}^{-1}\bar{X})^{-1} \qquad \bar{b} = \bar{V}(V^{-1}b+\bar{X}'\bar{\Sigma}^{-1}y) \tag{a.4.11.12}$$

Then (a.4.11.11) rewrites:

$$\begin{aligned}&= \beta'\bar{V}^{-1}\beta - 2\beta'\bar{V}^{-1}\bar{b} + \bar{b}'\bar{V}^{-1}\bar{b} - \bar{b}'\bar{V}^{-1}\bar{b} + b'V^{-1}b + y'\bar{\Sigma}^{-1}y \\ &= (\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b}) + (b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} + y'\bar{\Sigma}^{-1}y)\end{aligned} \tag{a.4.11.13}$$

Substitute (a.4.11.13) back in (a.4.11.9):

$$\begin{aligned}&\pi(\beta|y) \\ &= \exp\left(-\frac{1}{2}\left[(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b}) + (b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} + y'\bar{\Sigma}^{-1}y)\right]\right) \\ &= \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)\exp\left(-\frac{1}{2}(b'V^{-1}b - \bar{b}'\bar{V}^{-1}\bar{b} + y'\bar{\Sigma}^{-1}y)\right) \\ &\propto \exp\left(-\frac{1}{2}(\beta - \bar{b})'\bar{V}^{-1}(\beta - \bar{b})\right)\end{aligned} \tag{a.4.11.14}$$

Where the last line obtains by noting that the second term in row 2 does not involve β and can hence be relegated to the normalization constant.

The terms in (a.4.11.12) simplify. Note first that:

$$\begin{aligned}&\bar{X}'\bar{\Sigma}^{-1}\bar{X} \\ &= (I_n \otimes X)'(\Sigma \otimes I_T)^{-1}(I_n \otimes X) \\ &= (I_n \otimes X')(\Sigma^{-1} \otimes I_T)(I_n \otimes X) \\ &= (\Sigma^{-1} \otimes X')(I_n \otimes X) \\ &= \Sigma^{-1} \otimes X'X\end{aligned} \tag{a.4.11.15}$$

Similarly:

$$\begin{aligned}&\bar{X}'\bar{\Sigma}^{-1}y \\ &= (I_n \otimes X)'(\Sigma \otimes I_T)^{-1}vec(Y) \\ &= (I_n \otimes X')(\Sigma^{-1} \otimes I_T)vec(Y) \\ &= (\Sigma^{-1} \otimes X')vec(Y) \\ &= vec(X'Y\Sigma^{-1})\end{aligned} \tag{a.4.11.16}$$

Then (a.4.11.12) rewrites:

$$\bar{V} = (V^{-1} + \Sigma^{-1} \otimes X'X)^{-1} \qquad \bar{b} = \bar{V}(V^{-1}b + vec(X'Y\Sigma^{-1})) \tag{a.4.11.17}$$

derivations for equation (4.11.23)

Start from the vectorized likelihood function:

$$f(y|\beta,\Sigma) = (2\pi)^{-nT/2}|\bar{\Sigma}|^{-1/2}exp\left(-\frac{1}{2}(y - \bar{X}\beta)'\bar{\Sigma}^{-1}(y - \bar{X}\beta)\right) \tag{a.4.11.18}$$

Use then property d.7 that establishes the equivalence between the multivariate normal and matrix normal distributions to reformulate the likelihood in vectorized form as:

$$f(y|\mathcal{B},\Sigma) = (2\pi)^{-nT/2}|\Sigma|^{-T/2}exp\left(-\frac{1}{2}\,tr\left[\Sigma^{-1}(Y - X\mathcal{B})'(Y - X\mathcal{B})\right]\right) \tag{a.4.11.19}$$

Consider the quadratic term:

$$\begin{aligned}
& (Y-X\mathcal{B})'(Y-X\mathcal{B}) \\
= \;& Y'Y+B'X'XB-2B'X'Y \\
= \;& Y'Y+B'X'XB-2B'X'Y+2\hat{B}X'Y-2\hat{B}X'Y \\
= \;& Y'Y+B'X'XB-2B'(X'X)(X'X)^{-1}X'Y+2\hat{B}(X'X)(X'X)^{-1}X'Y-2\hat{B}X'Y \\
= \;& Y'Y+B'X'XB-2B'(X'X)\hat{B}+2\hat{B}(X'X)\hat{B}-2\hat{B}X'Y \\
= \;& Y'Y+B'X'XB-2B'(X'X)\hat{B}+\hat{B}(X'X)\hat{B}+\hat{B}(X'X)\hat{B}-2\hat{B}X'Y \\
= \;& (B'(X'X)B+\hat{B}(X'X)\hat{B}-2B'(X'X)\hat{B})+(Y'Y+\hat{B}(X'X)\hat{B}-2\hat{B}X'Y) \\
= \;& (B-\hat{B})'(X'X)(B-\hat{B})+(Y-X\hat{B})'(Y-X\hat{B})
\end{aligned} \tag{a.4.11.20}$$

Hence (a.4.11.19) rewrites:

$$\begin{aligned}
f(y|\mathcal{B},\Sigma) &= (2\pi)^{-nT/2}|\Sigma|^{-T/2}exp\left(-\frac{1}{2}\,tr\left[\Sigma^{-1}(B-\hat{B})'(X'X)(B-\hat{B})+\Sigma^{-1}(Y-X\hat{B})'(Y-X\hat{B})\right]\right) \\
&= (2\pi)^{-nT/2}|\Sigma|^{-T/2}exp\left(-\frac{1}{2}\,tr\left[\Sigma^{-1}(B-\hat{B})'(X'X)(B-\hat{B})\right]\right) \\
&\quad\times exp\left(-\frac{1}{2}\,tr\left[\Sigma^{-1}(Y-X\hat{B})'(Y-X\hat{B})\right]\right)
\end{aligned} \tag{a.4.11.21}$$

Also, it follows from property m.55 that:

$$tr\left[\Sigma^{-1}(B-\hat{B})'(X'X)(B-\hat{B})\right]=(\beta-\hat{\beta})'(\Sigma\otimes(X'X)^{-1})^{-1}(\beta-\hat{\beta}) \tag{a.4.11.22}$$

Substituting back in (a.4.11.21) yields:

$$\begin{aligned}
f(y|\beta,\Sigma) = \;& (2\pi)^{-nT/2}|\Sigma|^{-T/2}exp\left(-\frac{1}{2}(\beta-\hat{\beta})'(\Sigma\otimes(X'X)^{-1})^{-1}(\beta-\hat{\beta})\right) \\
& \times exp\left(-\frac{1}{2}\,tr\left[\Sigma^{-1}(Y-X\hat{B})'(Y-X\hat{B})\right]\right)
\end{aligned} \tag{a.4.11.23}$$

derivations for equation (4.11.32)

Start from the joint posterior:

$$\begin{aligned}
\pi(\beta,\Sigma|y) &\propto |\Sigma|^{-T/2}exp\left(-\frac{1}{2}(\beta-\hat{\beta})'\;(\Sigma\otimes(X'X)^{-1})^{-1}\;(\beta-\hat{\beta})\right) \\
&\quad\times exp\left(-\frac{1}{2}\,tr\left[\Sigma^{-1}(Y-X\hat{B})'(Y-X\hat{B})\right]\right) \\
&\quad\times|\Sigma\otimes W|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'(\Sigma\otimes W)^{-1}(\beta-b)\right) \\
&\quad\times|\Sigma|^{-(\alpha+n+1)/2}exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}S\right\}\right)
\end{aligned} \tag{a.4.11.24}$$

Note first that:

$$|\Sigma\otimes W|^{-1/2}=|\Sigma|^{-k/2}|W|^{-n/2} \tag{a.4.11.25}$$

Hence, substituting back in (a.4.11.24) and rearranging:

$$\begin{aligned}\pi(\beta,\Sigma|y) \;&\propto exp\left(-\frac{1}{2}(\beta-\hat{\beta})'\;(\Sigma\otimes(X'X)^{-1})^{-1}\;(\beta-\hat{\beta})\right)\\ &\times|\Sigma|^{-T/2}exp\left(-\frac{1}{2}\;tr\left[\Sigma^{-1}(Y-X\hat{B})'(Y-X\hat{B})\right]\right)\\ &\times|\Sigma|^{-k/2}\exp\left(-\frac{1}{2}(\beta-b)'(\Sigma\otimes W)^{-1}(\beta-b)\right)\\ &\times|\Sigma|^{-(\alpha+n+1)/2}exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}S\right\}\right)\end{aligned} \quad (a.4.11.26)$$

After regrouping, one obtains:

$$\begin{aligned}\pi(\beta,\Sigma|y) \;&\propto|\Sigma|^{-k/2}exp\left(-\frac{1}{2}\left[(\beta-b)'(\Sigma\otimes W)^{-1}(\beta-b)+(\beta-\hat{\beta})'\;(\Sigma\otimes(X'X)^{-1})^{-1}\;(\beta-\hat{\beta})\right]\right)\\ &\times|\Sigma|^{-(\alpha+T+n+1)/2}exp\left(-\frac{1}{2}\;tr\left\{\Sigma^{-1}\left[S+(Y-X\hat{B})'(Y-X\hat{B})\right]\right\}\right)\end{aligned} \quad (a.4.11.27)$$

Consider the term within the curly brackets in the first row:

$$\begin{aligned}&(\beta-b)'(\Sigma\otimes W)^{-1}(\beta-b)+(\beta-\hat{\beta})'\;(\Sigma\otimes(X'X)^{-1})^{-1}\;(\beta-\hat{\beta})\\ =\;& tr\left\{\Sigma^{-1}(\mathcal{B}-B)'W^{-1}(\mathcal{B}-B)\right\}+tr\left\{\Sigma^{-1}(\mathcal{B}-\hat{B})'(X'X)(\mathcal{B}-\hat{B})\right\}\\ =\;& tr\left\{\Sigma^{-1}\left[(\mathcal{B}-B)'W^{-1}(\mathcal{B}-B)+(\mathcal{B}-\hat{B})'(X'X)(\mathcal{B}-\hat{B})\right]\right\}\\ =\;& tr\left\{\Sigma^{-1}\left[(\mathcal{B}-B)'W^{-1}(\mathcal{B}-B)+(\mathcal{B}-\hat{B})'(X'X)(\mathcal{B}-\hat{B})\right]\right\}\\ =\;& tr\left\{\Sigma^{-1}\left[\mathcal{B}'W^{-1}\mathcal{B}+B'W^{-1}B-2\mathcal{B}'W^{-1}B+\mathcal{B}'(X'X)\mathcal{B}+\hat{B}'(X'X)\hat{B}-2\mathcal{B}'(X'X)\hat{B}\right]\right\}\\ =\;& tr\left\{\Sigma^{-1}\left[\mathcal{B}'(W^{-1}+X'X)\mathcal{B}-2\mathcal{B}'(W^{-1}B+X'X\hat{B})+B'W^{-1}B+\hat{B}'(X'X)\hat{B}\right]\right\}\end{aligned} \quad (a.4.11.28)$$

Complete the squares:

$$=\;tr\left\{\Sigma^{-1}\left[\mathcal{B}'(W^{-1}+X'X)\mathcal{B}-2\mathcal{B}'\bar{W}^{-1}\bar{W}(W^{-1}B+X'X\hat{B})+\bar{B}'\bar{W}^{-1}\bar{B}-\bar{B}'\bar{W}^{-1}\bar{B}+B'W^{-1}B+\hat{B}'(X'X)\hat{B}\right]\right\} \quad (a.4.11.29)$$

Define:

$$\bar{W}=(W^{-1}+X'X)^{-1} \qquad \bar{B}=\bar{W}(W^{-1}B+X'X\hat{B}) \quad (a.4.11.30)$$

then the expression becomes:

$$\begin{aligned}=\;& tr\left\{\Sigma^{-1}\left[\mathcal{B}'\bar{W}^{-1}\mathcal{B}-2\mathcal{B}'\bar{W}^{-1}\bar{B}+\bar{B}'\bar{W}^{-1}\bar{B}-\bar{B}'\bar{W}^{-1}\bar{B}+B'W^{-1}B+\hat{B}'(X'X)\hat{B}\right]\right\}\\ =\;& tr\left\{\Sigma^{-1}\left[(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})-\bar{B}'\bar{W}^{-1}\bar{B}+B'W^{-1}B+\hat{B}'(X'X)\hat{B}\right]\right\}\\ =\;& tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}+tr\left\{\Sigma^{-1}\left[B'W^{-1}B+\hat{B}'(X'X)\hat{B}-\bar{B}'\bar{W}^{-1}\bar{B}\right]\right\}\end{aligned} \quad (a.4.11.31)$$

Substituting (a.4.11.31) in (a.4.11.27) yields:

$$\begin{aligned}\pi(\beta,\Sigma|y) &\propto |\Sigma|^{-k/2} exp\left(-\frac{1}{2}\left[tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}+tr\left\{\Sigma^{-1}\left[B'W^{-1}B+\hat{B}'(X'X)\hat{B}-\bar{B}'\bar{W}^{-1}\bar{B}\right]\right\}\right]\right)\\ &\times|\Sigma|^{-(\alpha+T+n+1)/2} exp\left(-\frac{1}{2}\,tr\left\{\Sigma^{-1}\left[S+(Y-X\hat{B})'(Y-X\hat{B})\right]\right\}\right)\\ &=|\Sigma|^{-k/2} exp\left(-\frac{1}{2}\left[tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}\right]\right)\\ &\times|\Sigma|^{-(\alpha+T+n+1)/2} exp\left(-\frac{1}{2}\,tr\left\{\Sigma^{-1}\left[S+(Y-X\hat{B})'(Y-X\hat{B})+B'W^{-1}B+\hat{B}'(X'X)\hat{B}-\bar{B}'\bar{W}^{-1}\bar{B}\right]\right\}\right)\end{aligned} \tag{a.4.11.32}$$

Define:

$$\bar{\alpha}=\alpha+T \qquad \bar{S}=S+(Y-X\hat{B})'(Y-X\hat{B})+B'W^{-1}B+\hat{B}'(X'X)\hat{B}-\bar{B}'\bar{W}^{-1}\bar{B} \tag{a.4.11.33}$$

Then the expression becomes:

$$\begin{aligned}\pi(\beta,\Sigma|y) &\propto |\Sigma|^{-k/2} exp\left(-\frac{1}{2}\left[tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{V}^{-1}(\mathcal{B}'-\bar{B})\right\}\right]\right)\\ &\times|\Sigma|^{-(\bar{\alpha}+n+1)/2} exp\left(-\frac{1}{2}\,tr\left\{\Sigma^{-1}\bar{S}\right\}\right)\end{aligned} \tag{a.4.11.34}$$

Some of the expressions simplify. Consider first (a.4.11.30):

$$\bar{B}=\bar{W}(W^{-1}B+X'X\hat{B})=\bar{W}(W^{-1}B+X'X(X'X)^{-1}X'Y)=\bar{W}(W^{-1}B+X'Y) \tag{a.4.11.35}$$

Consider then:

$$\begin{aligned}&(Y-X\hat{B})'(Y-X\hat{B})+\hat{B}'X'X\hat{B}\\ =\;&Y'Y+\hat{B}'X'X\hat{B}-\hat{B}'X'Y-Y'X\hat{B}+\hat{B}'X'X\hat{B}\\ =\;&Y'Y+2\hat{B}'X'X\hat{B}-\hat{B}'X'Y-Y'X\hat{B}\\ =\;&Y'Y+2Y'X(X'X)^{-1}X'X(X'X)^{-1}X'Y-Y'X(X'X)^{-1}X'Y-Y'X(X'X)^{-1}X'Y\\ =\;&Y'Y+2Y'X(X'X)^{-1}X'Y-Y'X(X'X)^{-1}X'Y-Y'X(X'X)^{-1}X'Y\\ =\;&Y'Y\end{aligned} \tag{a.4.11.36}$$

Substitute back in (a.4.11.33):

$$\bar{S}=S+Y'Y+B'W^{-1}B-\bar{B}'\bar{W}^{-1}\bar{B} \tag{a.4.11.37}$$

derivations for equation (4.11.37)

Start from the initial equation and reformulate:

$$\begin{aligned}
& \pi(\mathcal{B}\,|y) \\
\propto\; & \left|\bar{S}+(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right|^{-\frac{\bar{\alpha}+k}{2}} \\
=\; & \left|\bar{S}\left\{I_n+\bar{S}^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}\right|^{-\frac{\bar{\alpha}+k}{2}} \\
=\; & \left|\bar{S}\right|^{-\frac{\bar{\alpha}+k}{2}}\left|I_n+\bar{S}^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right|^{-\frac{\bar{\alpha}+k}{2}} \\
=\; & \left|I_n+\bar{S}^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right|^{-\frac{\bar{\alpha}+k}{2}} \\
=\; & \left|I_n+\bar{S}^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right|^{-\frac{\alpha+T+k}{2}} \\
=\; & \left|I_n+\bar{S}^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right|^{-\frac{(\alpha+T-n+1)+k+n-1}{2}}
\end{aligned} \tag{a.4.11.38}$$

Define:

$$\hat{\alpha}=\alpha+T-n+1 \tag{a.4.11.39}$$

Then:

$$\begin{aligned}
=\; & \left|I_n+\bar{S}^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right|^{-\frac{\hat{\alpha}+k+n-1}{2}} \\
=\; & \left|I_n+\frac{1}{\hat{\alpha}}(\bar{S}/\hat{\alpha})^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right|^{-\frac{\hat{\alpha}+k+n-1}{2}}
\end{aligned} \tag{a.4.11.40}$$

Define:

$$\hat{S}=\bar{S}/\hat{\alpha} \tag{a.4.11.41}$$

Then:

$$\pi(\mathcal{B}\,|y)\propto\left|I_n+\frac{1}{\hat{\alpha}}\hat{S}^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right|^{-\frac{\hat{\alpha}+k+n-1}{2}} \tag{a.4.11.42}$$

derivations for equation (4.11.45)

Note that:

$$|\bar{\Sigma}|^{-1/2}=|\Sigma\otimes I_T|^{-1/2}=|\Sigma|^{-T/2}|I_T|^{-n/2}=|\Sigma|^{-T/2} \tag{a.4.11.43}$$

Also:

$$(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta)=(y-(I_n\otimes X)\beta)'(\Sigma\otimes I_T)^{-1}(y-(I_n\otimes X)\beta)=tr\left\{\Sigma^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})\right\} \tag{a.4.11.44}$$

Then substituting in the original expression:

$$\begin{aligned}\pi(\Sigma|y,\beta) &\propto |\Sigma|^{-T/2} exp\left(-\frac{1}{2} tr\left\{\Sigma^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})\right\}\right) \times |\Sigma|^{-(\alpha+n+1)/2} exp\left(-\frac{1}{2} tr\left\{\Sigma^{-1}S\right\}\right) \\ &= |\Sigma|^{-(\alpha+T+n+1)/2} \; exp\left(-\frac{1}{2} tr\left\{\Sigma^{-1}\left[S+(Y-X\mathcal{B})'(Y-X\mathcal{B})\right]\right\}\right) \\ &= |\Sigma|^{-(\bar{\alpha}+n+1)/2} \; exp\left(-\frac{1}{2} tr\left\{\Sigma^{-1}\bar{S}\right\}\right) \end{aligned} \tag{a.4.11.45}$$

with:

$$\bar{\alpha} = \alpha + T \qquad \bar{S} = S + (Y-X\mathcal{B})'(Y-X\mathcal{B}) \tag{a.4.11.46}$$

derivations for equation (4.11.49)

Start from Bayes rule and rearrange:

$$\begin{aligned} & \pi(\beta,\Sigma|y) \\ \propto \; & f(y|\beta,\Sigma)\pi(\beta)\pi(\Sigma) \\ \propto \; & |\Sigma|^{-T/2} exp\left(-\frac{1}{2}(\beta-\hat{\beta})' \; (\Sigma\otimes(X'X)^{-1})^{-1} \; (\beta-\hat{\beta})\right) \\ \times \; & exp\left(-\frac{1}{2} \; tr\left[\Sigma^{-1}(Y-X\hat{B})'(Y-X\hat{B})\right]\right) \times |\Sigma|^{-(\alpha+1)/2} \\ = \; & |\Sigma|^{-k/2} \; exp\left(-\frac{1}{2}(\beta-\hat{\beta})' \; (\Sigma\otimes(X'X)^{-1})^{-1} \; (\beta-\hat{\beta})\right) \\ \times \; & |\Sigma|^{-(T-k+\alpha+1)/2} \; exp\left(-\frac{1}{2} \; tr\left[\Sigma^{-1}(Y-X\hat{B})'(Y-X\hat{B})\right]\right) \\ = \; & |\Sigma|^{-k/2} \; exp\left(-\frac{1}{2} \; tr\left\{\Sigma^{-1}(\mathcal{B}'-\hat{B})'(X'X)(\mathcal{B}'-\hat{B})\right\}\right) \\ \times \; & |\Sigma|^{-(T-k+n+3)/2} \; exp\left(-\frac{1}{2} \; tr\left\{\Sigma^{-1}(Y-X\hat{B})'(Y-X\hat{B})\right\}\right) \\ = \; & |\Sigma|^{-k/2} \; exp\left(-\frac{1}{2} \; tr\left\{\Sigma^{-1}(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right\}\right) \\ \times \; & |\Sigma|^{-(\hat{\alpha}+n+1)/2} \; exp\left(-\frac{1}{2} \; tr\left\{\Sigma^{-1}\hat{S}\right\}\right) \end{aligned} \tag{a.4.11.47}$$

with:

$$\hat{W} = (X'X)^{-1} \qquad \hat{\alpha} = T-k+2 \qquad \hat{S} = (Y-X\hat{B})'(Y-X\hat{B}) \tag{a.4.11.48}$$

derivations for equation (4.11.52)

Start from the joint posterior, group the terms and integrate:

$$\pi(\mathcal{B}|y) = \int \pi(\mathcal{B},\Sigma|y)d\Sigma \propto \int |\Sigma|^{-(\hat{\alpha}+k+n+1)/2} \; exp\left(-\frac{1}{2} \; tr\left\{\Sigma^{-1}\left[\hat{S}+(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right]\right\}\right) d\Sigma \tag{a.4.11.49}$$

This is the kernel of an inverse Wishart distribution with degrees of freedom $(\hat{\alpha}+k)$ and scale $\hat{S}+(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})$, and integration yields the reciprocal of the normalization constant of the distribution. Hence:

$$\pi(\mathcal{B}|y) \;\propto\; \Gamma_n\left(\frac{\hat{\alpha}+k}{2}\right)2^{(\hat{\alpha}+k)n/2}\left|\hat{S}+(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{\hat{\alpha}+k}{2}} \;\propto\; \left|\hat{S}+(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{\hat{\alpha}+k}{2}} \tag{a.4.11.50}$$

Rearrange:

$$\begin{aligned}
&\pi(\mathcal{B}|y)\\
\propto\; & \left|\hat{S}+(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{\hat{\alpha}+k}{2}}\\
=\; & \left|\hat{S}\left\{I_n+\hat{S}^{-1}(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right\}\right|^{-\frac{\hat{\alpha}+k}{2}}\\
=\; & \left|\hat{S}\right|^{-\frac{\hat{\alpha}+k}{2}}\left|I_n+\hat{S}^{-1}(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{\hat{\alpha}+k}{2}}\\
=\; & \left|I_n+\hat{S}^{-1}(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{\hat{\alpha}+k}{2}}\\
=\; & \left|I_n+\hat{S}^{-1}(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{T+2}{2}}\\
=\; & \left|I_n+\hat{S}^{-1}(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{(T-n-k+3)+k+n-1}{2}}
\end{aligned} \tag{a.4.11.51}$$

Define:

$$\tilde{\alpha}=T-n-k+3 \tag{a.4.11.52}$$

Then:

$$\begin{aligned}
=\; & \left|I_n+\hat{S}^{-1}(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{\tilde{\alpha}+k+n-1}{2}}\\
=\; & \left|I_n+\frac{1}{\tilde{\alpha}}(\hat{S}/\tilde{\alpha})^{-1}(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{\tilde{\alpha}+k+n-1}{2}}
\end{aligned} \tag{a.4.11.53}$$

Define:

$$\tilde{S}=\hat{S}/\tilde{\alpha} \tag{a.4.11.54}$$

Then:

$$\pi(\mathcal{B}|y)\propto\left|I_n+\frac{1}{\tilde{\alpha}}\tilde{S}^{-1}(\mathcal{B}'-\hat{B})'\hat{W}^{-1}(\mathcal{B}'-\hat{B})\right|^{-\frac{\tilde{\alpha}+k+n-1}{2}} \tag{a.4.11.55}$$

derivations for equation (4.11.68)

Start from the original likelihood function:

$$f(y|\beta,\Sigma)=(2\pi)^{-nT/2}|\bar{\Sigma}|^{-1/2}exp\left(-\frac{1}{2}(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta)\right) \tag{a.4.11.56}$$

Consider first the determinant term in (a.4.11.56) and rearrange:

$$\begin{aligned}
& |\bar{\Sigma}|^{-1/2} \\
= \ & |\Sigma \otimes I_T|^{-1/2} \\
= \ & |\Sigma|^{-T/2} |I_T|^{-k/2} \\
= \ & |\Sigma|^{-T/2} \\
= \ & |\Phi^{-1} \Lambda \Phi^{-1\prime}|^{-T/2} \\
= \ & |\Phi^{-1}|^{-T/2} |\Lambda|^{-T/2} |\Phi^{-1\prime}|^{-T/2} \\
= \ & |\Phi|^{T/2} |\Lambda|^{-T/2} |\Phi^{\prime}|^{T/2} \\
= \ & |\Lambda|^{-T/2} \\
= \ & \prod_{i=1}^{n} \lambda_i^{-T/2}
\end{aligned} \tag{a.4.11.57}$$

Consider then the quadratic form in (a.4.11.56):

$$\begin{aligned}
& (y - \bar{X}\beta)^{\prime} \bar{\Sigma}^{-1} (y - \bar{X}\beta) \\
= \ & (y - (I_n \otimes X)\beta)^{\prime} (\Sigma \otimes I_T)^{-1} (y - (I_n \otimes X)\beta) \\
= \ & tr \left[\Sigma^{-1} (Y - X\mathcal{B})^{\prime} I_T^{-1} (Y - X\mathcal{B}) \right] \\
= \ & tr \left[\Sigma^{-1} (Y - X\mathcal{B})^{\prime} (Y - X\mathcal{B}) \right] \\
= \ & tr \left[\Sigma^{-1} \mathcal{E}^{\prime} \mathcal{E} \right] \\
= \ & tr \left[\mathcal{E} \Sigma^{-1} \mathcal{E}^{\prime} \right] \\
= \ & tr \left[\begin{pmatrix} \varepsilon_1^{\prime} \Sigma^{-1} \varepsilon_1 & \varepsilon_1^{\prime} \Sigma^{-1} \varepsilon_2 & \cdots & \varepsilon_1^{\prime} \Sigma^{-1} \varepsilon_T \\ \varepsilon_2^{\prime} \Sigma^{-1} \varepsilon_1 & \varepsilon_2^{\prime} \Sigma^{-1} \varepsilon_2 & \cdots & \varepsilon_2^{\prime} \Sigma^{-1} \varepsilon_T \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_T^{\prime} \Sigma^{-1} \varepsilon_1 & \varepsilon_T^{\prime} \Sigma^{-1} \varepsilon_2 & \cdots & \varepsilon_T^{\prime} \Sigma^{-1} \varepsilon_T \end{pmatrix} \right] \\
= \ & \sum_{t=1}^{T} \varepsilon_t^{\prime} \Sigma^{-1} \varepsilon_t \\
= \ & \sum_{t=1}^{T} \varepsilon_t^{\prime} (\Phi^{-1} \Lambda \Phi^{-1\prime})^{-1} \varepsilon_t \\
= \ & \sum_{t=1}^{T} \varepsilon_t^{\prime} (\Phi^{\prime} \Lambda^{-1} \Phi) \varepsilon_t \\
= \ & \sum_{t=1}^{T} (\Phi \varepsilon_t)^{\prime} \Lambda^{-1} (\Phi \varepsilon_t)
\end{aligned} \tag{a.4.11.58}$$

Consider the first term in the quadratic form:

$$\begin{aligned}
&\Phi\ \varepsilon_t \\
= &\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \phi_{21} & 1 & \ddots & \cdots \\ \vdots & \ddots & \ddots & 0 \\ \phi_{n1} & \cdots & \phi_{n(n-1)} & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{n,t} \end{pmatrix} \\
= &\begin{pmatrix} \varepsilon_{1,t} \\ \phi_2 \varepsilon_{-2,t} + \varepsilon_{2,t} \\ \vdots \\ \phi_n \varepsilon_{-n,t} + \varepsilon_{n,t} \end{pmatrix} \qquad \text{(a.4.11.59)}
\end{aligned}$$

with:

$$\varepsilon_{-i,t} = (\varepsilon_{1,t}\ \varepsilon_{2,t}\ \cdots\ \varepsilon_{i-1,t})' \qquad \text{(a.4.11.60)}$$

Substitute (a.4.11.59) back in the quadratic form (a.4.11.58):

$$\begin{aligned}
&(\Phi\varepsilon_t)'\Lambda^{-1}(\Phi\varepsilon_t) \\
= &\begin{pmatrix} \varepsilon_{1,t} & \phi_2\varepsilon_{-2,t} + \varepsilon_{2,t} & \cdots & \phi_n\varepsilon_{-n,t} + \varepsilon_{n,t} \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n^{-1} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \phi_2 \varepsilon_{-2,t} + \varepsilon_{2,t} \\ \vdots \\ \phi_n \varepsilon_{-n,t} + \varepsilon_{n,t} \end{pmatrix} \\
= &\begin{pmatrix} \lambda_1^{-1}\ \varepsilon_{1,t} & \lambda_2^{-1}(\phi_2\varepsilon_{-2,t} + \varepsilon_{2,t}) & \cdots & \lambda_n^{-1}(\phi_n\varepsilon_{-n,t} + \varepsilon_{n,t}) \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t} \\ \phi_2 \varepsilon_{-2,t} + \varepsilon_{2,t} \\ \vdots \\ \phi_n \varepsilon_{-n,t} + \varepsilon_{n,t} \end{pmatrix} \\
= &\sum_{i=1}^{n} \lambda_i^{-1}(\varepsilon_{i,t} + \phi_i \varepsilon_{-i,t})^2 \qquad \text{(a.4.11.61)}
\end{aligned}$$

Substitute (a.4.11.61) back in (a.4.11.58) to obtain:

$$\begin{aligned}
&\sum_{t=1}^{T} (\Phi\varepsilon_t)'\Lambda^{-1}(\Phi\varepsilon_t) \\
= &\sum_{t=1}^{T}\sum_{i=1}^{n} \lambda_i^{-1}(\varepsilon_{i,t} + \phi_i \varepsilon_{-i,t})^2 \\
= &\sum_{i=1}^{n} \lambda_i^{-1} \left(\sum_{t=1}^{T} (\varepsilon_{i,t} + \phi_i \varepsilon_{-i,t})^2 \right) \qquad \text{(a.4.11.62)}
\end{aligned}$$

Then, note that:

$$\begin{aligned}
&\sum_{t=1}^{T}(\varepsilon_{i,t}+\phi_i\varepsilon_{-i,t})^2 \\
=\ & (\varepsilon_{i,1}+\phi_i'\varepsilon_{-i,1} \quad \varepsilon_{i,2}+\phi_i'\varepsilon_{-i,2} \quad \cdots \quad \varepsilon_{i,T}+\phi_i'\varepsilon_{-i,T})\begin{pmatrix}\varepsilon_{i,1}+\phi_i'\varepsilon_{-i,1}\\ \varepsilon_{i,2}+\phi_i'\varepsilon_{-i,2}\\ \vdots \\ \varepsilon_{i,T}+\phi_i'\varepsilon_{-i,T}\end{pmatrix} \\
=\ & (\varepsilon_{i,1}+\varepsilon_{-i,1}'\phi_i \quad \varepsilon_{i,2}+\varepsilon_{-i,2}'\phi_i \quad \cdots \quad \varepsilon_{i,T}+\varepsilon_{-i,T}'\phi_i)\begin{pmatrix}\varepsilon_{i,1}+\varepsilon_{-i,1}'\phi_i\\ \varepsilon_{i,2}+\varepsilon_{-i,2}'\phi_i\\ \vdots \\ \varepsilon_{i,T}+\varepsilon_{-i,T}'\phi_i\end{pmatrix} \\
=\ & (\mathcal{E}_i+\mathcal{E}_{-i}\,\phi_i)'(\mathcal{E}_i+\mathcal{E}_{-i}\,\phi_i)
\end{aligned} \tag{a.4.11.63}$$

with:

$$\mathcal{E}_{-i}=(\mathcal{E}_1\ \mathcal{E}_2\ \cdots\ \mathcal{E}_{i-1}) \tag{a.4.11.64}$$

Also, noting that $\mathcal{E}_i=Y_i-X\beta_i$:

$$\sum_{t=1}^{T}(\varepsilon_{i,t}+\phi_i\varepsilon_{-i,t})^2=(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i)'(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i) \tag{a.4.11.65}$$

Substitute (a.4.11.65) back in (a.4.11.62):

$$\sum_{t=1}^{T}(\Phi\varepsilon_t)'\Lambda^{-1}(\Phi\varepsilon_t)=\sum_{i=1}^{n}\lambda_i^{-1}(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i)'(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i) \tag{a.4.11.66}$$

Therefore, substituting (a.4.11.66) back in (a.4.11.58):

$$(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta)=\sum_{i=1}^{n}\lambda_i^{-1}(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i)'(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i) \tag{a.4.11.67}$$

Finally, replacing (a.4.11.57) and (a.4.11.67) in (a.4.11.56) yields:

$$f(y|\beta,\lambda,\phi)=(2\pi)^{-nT/2}\left(\prod_{i=1}^{n}\lambda_i^{-T/2}\right)exp\left(-\frac{1}{2}\sum_{i=1}^{n}\lambda_i^{-1}(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i)'(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i)\right) \tag{a.4.11.68}$$

derivations for equation (4.11.75)

Start from:

$$\pi(\beta_i|y,\beta_{-i})\propto exp\left(-\frac{1}{2}\lambda_i^{-1}(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i)'(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i)\right)\times exp\left(-\frac{1}{2}(\beta_i-b_i)'V_i^{-1}(\beta_i-b_i)\right) \tag{a.4.11.69}$$

Develop the first quadratic form:

$$\begin{aligned}
&(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i)'(Y_i-X\beta_i+\mathcal{E}_{-i}\,\phi_i) \\
=\ & Y_i'Y_i+\beta_i'X'X\beta_i+\phi_i'\,\mathcal{E}_{-i}'\,\mathcal{E}_{-i}\,\phi_i-2\beta_i'X'(Y_i+\mathcal{E}_{-i}\,\phi_i)+2Y_i'\,\mathcal{E}_{-i}\,\phi_i
\end{aligned} \tag{a.4.11.70}$$

Develop the second qudratic form:

$$
\begin{aligned}
& (\beta_i - b_i)' V_i^{-1} (\beta_i - b_i) \\
= \; & \beta_i' V_i^{-1} \beta_i + b_i' V_i^{-1} b_i - 2\beta_i' V_i^{-1} b_i
\end{aligned} \tag{a.4.11.71}
$$

Substitute (a.4.11.70) and (a.4.11.71) back in (a.4.11.69) to obtain:

$$
\begin{aligned}
& \pi(\beta_i | y, \beta_{-i}) \\
\propto \; & exp\left(-\frac{1}{2}\lambda_i^{-1}\left(Y_i'Y_i + \beta_i'X'X\beta_i + \phi_i'\mathcal{E}'_{-i}\mathcal{E}_{-i}\phi_i - 2\beta_i'X'(Y_i + \mathcal{E}_{-i}\phi_i) + 2Y_i'\mathcal{E}_{-i}\phi_i\right)\right) \\
\times \; & exp\left(-\frac{1}{2}\left(\beta_i'V_i^{-1}\beta_i + b_i'V_i^{-1}b_i - 2\beta_i'V_i^{-1}b_i\right)\right) \\
= \; & exp\left(-\frac{1}{2}\left(\lambda_i^{-1}\beta_i'X'X\beta_i - 2\lambda_i^{-1}\beta_i'X'(Y_i + \mathcal{E}_{-i}\phi_i) + \beta_i'V_i^{-1}\beta_i - 2\beta_i'V_i^{-1}b_i\right)\right) \\
\times \; & exp\left(-\frac{1}{2}\left(\lambda_i^{-1}Y_i'Y_i + \lambda_i^{-1}\phi_i'\mathcal{E}'_{-i}\mathcal{E}_{-i}\phi_i + 2\lambda_i^{-1}Y_i'\mathcal{E}_{-i}\phi_i + b_i'V_i^{-1}b_i\right)\right) \\
\propto \; & exp\left(-\frac{1}{2}\left(\lambda_i^{-1}\beta_i'X'X\beta_i - 2\lambda_i^{-1}\beta_i'X'(Y_i + \mathcal{E}_{-i}\phi_i) + \beta_i'V_i^{-1}\beta_i - 2\beta_i'V_i^{-1}b_i\right)\right) \\
\propto \; & exp\left(-\frac{1}{2}\left[\beta_i'(\lambda_i^{-1}X'X + V_i^{-1})\beta_i - 2\beta_i'(V_i^{-1}b_i + \lambda_i^{-1}X'[Y_i + \mathcal{E}_{-i}\phi_i])\right]\right)
\end{aligned} \tag{a.4.11.72}
$$

Consider the term within the square brackets and complete the squares:

$$
\begin{aligned}
& \beta_i'(\lambda_i^{-1}X'X + V_i^{-1})\beta_i - 2\beta_i'(V_i^{-1}b_i + \lambda_i^{-1}X'[Y_i + \mathcal{E}_{-i}\phi_i]) \\
= \; & \beta_i'(\lambda_i^{-1}X'X + V_i^{-1})\beta_i - 2\beta_i'\bar{V}_i^{-1}\bar{V}_i(V_i^{-1}b_i + \lambda_i^{-1}X'[Y_i + \mathcal{E}_{-i}\phi_i]) + \bar{b}_i'\bar{V}_i^{-1}\bar{b}_i - \bar{b}_i'\bar{V}_i^{-1}\bar{b}_i
\end{aligned} \tag{a.4.11.73}
$$

Define:

$$
\bar{V}_i = (\lambda_i^{-1}X'X + V_i^{-1})^{-1} \qquad \bar{b}_i = \bar{V}_i(V_i^{-1}b_i + \lambda_i^{-1}X'[Y_i + \mathcal{E}_{-i}\phi_i]) \tag{a.4.11.74}
$$

Then (a.4.11.73) rewrites:

$$
\begin{aligned}
& \beta_i'(\lambda_i^{-1}X'X + V_i^{-1})\beta_i - 2\beta_i'(V_i^{-1}b_i + \lambda_i^{-1}X'[Y_i + \mathcal{E}_{-i}\phi_i]) \\
= \; & \beta_i'\bar{V}_i^{-1}\beta_i - 2\beta_i'\bar{V}_i^{-1}\bar{b}_i + \bar{b}_i'\bar{V}_i^{-1}\bar{b}_i - \bar{b}_i'\bar{V}_i^{-1}\bar{b}_i \\
= \; & (\beta_i - \bar{b}_i)'\bar{V}_i^{-1}(\beta_i - \bar{b}_i) - \bar{b}_i'\bar{V}_i^{-1}\bar{b}_i
\end{aligned} \tag{a.4.11.75}
$$

Substitute (a.4.11.75) back in (a.4.11.72) to obtain:

$$
\begin{aligned}
& \pi(\beta_i | y, \beta_{-i}) \\
\propto \; & exp\left(-\frac{1}{2}\left[(\beta_i - \bar{b}_i)'\bar{V}_i^{-1}(\beta_i - \bar{b}_i) - \bar{b}_i'\bar{V}_i^{-1}\bar{b}_i\right]\right) \\
= \; & exp\left(-\frac{1}{2}(\beta_i - \bar{b}_i)'\bar{V}_i^{-1}(\beta_i - \bar{b}_i)\right) \; exp\left(-\frac{1}{2}(-\bar{b}_i'\bar{V}_i^{-1}\bar{b}_i)\right) \\
\propto \; & exp\left(-\frac{1}{2}(\beta_i - \bar{b}_i)'\bar{V}_i^{-1}(\beta_i - \bar{b}_i)\right)
\end{aligned} \tag{a.4.11.76}
$$

derivations for equation (4.11.81)

Rearrange the initial formula:

$$\begin{aligned}
&\pi(\phi_i|y,\phi_{-i}) \\
\propto\ & exp\left(-\frac{1}{2}\lambda_i^{-1}(Y_i - X\beta_i + \mathcal{E}_{-i}\,\phi_i)'(Y_i - X\beta_i + \mathcal{E}_{-i}\,\phi_i)\right) \times \exp\left(-\frac{1}{2}\tau^{-1}\phi_i'\phi_i\right) \\
=\ & exp\left(-\frac{1}{2}\left[\lambda_i^{-1}(Y_i - X\beta_i + \mathcal{E}_{-i}\,\phi_i)'(Y_i - X\beta_i + \mathcal{E}_{-i}\,\phi_i) + \tau^{-1}\phi_i'\phi_i\right]\right) \qquad \text{(a.4.11.77)}
\end{aligned}$$

Consider the term within the square brackets:

$$\begin{aligned}
&\lambda_i^{-1}(Y_i - X\beta_i + \mathcal{E}_{-i}\,\phi_i)'(Y_i - X\beta_i + \mathcal{E}_{-i}\,\phi_i) + \tau^{-1}\phi_i'\phi_i \\
=\ & \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i + \phi_i'\,\mathcal{E}'_{-i}\,\mathcal{E}_{-i}\,\phi_i + 2\phi_i'\,\mathcal{E}'_{-i}[Y_i - X\beta_i] - 2Y_i'X\beta_i) + \tau^{-1}\phi_i'\phi_i \\
=\ & \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i + \phi_i'\,\mathcal{E}'_{-i}\,\mathcal{E}_{-i}\,\phi_i - 2\phi_i'(-\mathcal{E}'_{-i}[Y_i - X\beta_i]) - 2Y_i'X\beta_i) + \tau^{-1}\phi_i'\phi_i \\
=\ & \phi_i'(\tau^{-1}I_{i-1} + \lambda_i^{-1}\,\mathcal{E}'_{-i}\,\mathcal{E}_{-i})\phi_i - 2\phi_i'(-\lambda_i^{-1}\,\mathcal{E}'_{-i}[Y_i - X\beta_i]) + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) \qquad \text{(a.4.11.78)}
\end{aligned}$$

Complete the squares:

$$\begin{aligned}
=\ & \phi_i'(\tau^{-1}I_{i-1} + \lambda_i^{-1}\,\mathcal{E}'_{-i}\,\mathcal{E}_{-i})\phi_i - 2\phi_i'\bar{Z}_i^{-1}\bar{Z}_i(-\lambda_i^{-1}\,\mathcal{E}'_{-i}[Y_i - X\beta_i]) + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) \\
+\ & \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i \qquad \text{(a.4.11.79)}
\end{aligned}$$

Define:

$$\bar{Z}_i = (\tau^{-1}I_{i-1} + \lambda_i^{-1}\,\mathcal{E}'_{-i}\,\mathcal{E}_{-i})^{-1} \qquad \bar{f}_i = \bar{Z}_i(-\lambda_i^{-1}\,\mathcal{E}'_{-i}[Y_i - X\beta_i]) \qquad \text{(a.4.11.80)}$$

Then (a.4.11.79) rewrites:

$$\begin{aligned}
=\ & \phi_i'\bar{Z}_i^{-1}\phi_i - 2\phi_i'\bar{Z}_i^{-1}\bar{f}_i + \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i \\
=\ & (\phi_i - \bar{f}_i)'\bar{Z}_i^{-1}(\phi_i - \bar{f}_i) + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i \qquad \text{(a.4.11.81)}
\end{aligned}$$

Substitute (a.4.11.81) back in (a.4.11.77):

$$\begin{aligned}
&\pi(\phi_i|y,\phi_{-i}) \\
\propto\ & exp\left(-\frac{1}{2}\left[(\phi_i - \bar{f}_i)'\bar{Z}_i^{-1}(\phi_i - \bar{f}_i) + \lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i\right]\right) \\
=\ & exp\left(-\frac{1}{2}(\phi_i - \bar{f}_i)'\bar{Z}_i^{-1}(\phi_i - \bar{f}_i)\right) exp\left(-\frac{1}{2}\left[\lambda_i^{-1}(Y_i'Y_i + \beta_i'X'X\beta_i - 2Y_i'X\beta_i) - \bar{f}_i'\bar{Z}_i^{-1}\bar{f}_i\right]\right) \\
\propto\ & exp\left(-\frac{1}{2}(\phi_i - \bar{f}_i)'\bar{Z}_i^{-1}(\phi_i - \bar{f}_i)\right) \qquad \text{(a.4.11.82)}
\end{aligned}$$

Eventually, notice that:

$$\bar{f}_i = \bar{Z}_i(-\lambda_i^{-1}\,\mathcal{E}'_{-i}[Y_i - X\beta_i]) = \bar{Z}_i(-\lambda_i^{-1}\,\mathcal{E}'_{-i}\,\mathcal{E}_i) \qquad \text{(a.4.11.83)}$$

CHAPTER 12

Further aspects of Bayesian vector autoregressions

derivations for equation (4.12.3)

Consider the general VAR model and reformulate it:

$$\begin{aligned}
& y_t = Cz_t + A_1 y_{t-1} + \cdots + A_p y_{t-p} + \varepsilon_t \\
\Leftrightarrow\ & y_t - y_{t-1} = Cz_t + A_1 y_{t-1} + \cdots + A_p y_{t-p} - y_{t-1} + \varepsilon_t \\
\Leftrightarrow\ & \Delta y_t = Cz_t + \sum_{i=1}^{p} A_i y_{t-i} - y_{t-1} + \varepsilon_t \\
\Leftrightarrow\ & \Delta y_t = Cz_t + \sum_{i=1}^{p} \left(A_i y_{t-i} + \sum_{j=i+1}^{p} A_j y_{t-i} - \sum_{j=i+1}^{p} A_j y_{t-i} \right) - y_{t-1} + \varepsilon_t \\
\Leftrightarrow\ & \Delta y_t = Cz_t + \left(\sum_{i=1}^{p} A_i - I \right) y_{t-1} - \sum_{i=1}^{p-1} \left(\sum_{j=i+1}^{p} A_j \right) (y_{t-i} - y_{t-i-1}) + \varepsilon_t \\
\Leftrightarrow\ & \Delta y_t = Cz_t + \left(\sum_{i=1}^{p} A_i - I \right) y_{t-1} + \sum_{i=1}^{p-1} B_i \Delta y_{t-i} + \varepsilon_t
\end{aligned} \tag{a.4.12.1}$$

with:

$$B_i = - \sum_{j=i+1}^{p} A_j \tag{a.4.12.2}$$

derivations for equation (4.12.17)

Start from:

$$\begin{aligned}
& Y_{lrp} = X_{lrp} \mathcal{B} + \mathcal{E}_{lrp} \\
\Leftrightarrow\ & Y_{lrp} = (0_{n\times m} \quad \mathbf{1}'_p \otimes Y_{lrp}) \begin{pmatrix} C' \\ A'_1 \\ \vdots \\ A'_p \end{pmatrix} + \mathcal{E}_{lrp} \\
\Leftrightarrow\ & Y_{lrp} = Y_{lrp} A'_1 + \cdots + Y_{lrp} A'_p + \mathcal{E}_{lrp} \\
\Leftrightarrow\ & Y'_{lrp} = A_1 Y'_{lrp} + \cdots + A_p Y'_{lrp} + \mathcal{E}'_{lrp} \\
\Leftrightarrow\ & (A_1 + \cdots + A_p - I) Y'_{lrp} = -\mathcal{E}'_{lrp} \\
\Leftrightarrow\ & (A_1 + \cdots + A_p - I) \left[diag(H\,\bar{y}/\pi_7)\, H^{-1\prime} \right]' = -\mathcal{E}'_{lrp} \\
\Leftrightarrow\ & (A_1 + \cdots + A_p - I) H^{-1}\ diag(H\bar{y}/\pi_7) = -\mathcal{E}'_{lrp} \\
\Leftrightarrow\ & (A_1 + \cdots + A_p - I) H^{-1} = -\ diag(\pi_7/H\bar{y})\ \ \mathcal{E}'_{lrp}
\end{aligned} \tag{a.4.12.3}$$

derivations for equation (4.12.20)

Focus on the term within the integral and rearrange:

$$\begin{aligned}
& (2\pi)^{-nT/2}|\bar{\Sigma}|^{-1/2}exp\left(-\frac{1}{2}(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta)\right)\times(2\pi)^{-q/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right) \\
= \; & (2\pi)^{-nT/2}|\bar{\Sigma}|^{-1/2}(2\pi)^{-q/2}|V|^{-1/2}exp\left(-\frac{1}{2}\left[(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta)+(\beta-b)'V^{-1}(\beta-b)\right]\right)
\end{aligned} \tag{a.4.12.4}$$

Also:

$$|\bar{\Sigma}|^{-1/2}=|\Sigma\otimes I_T|^{-1/2}=|\Sigma|^{-T/2}|I_T|^{-n/2}=|\Sigma|^{-T/2} \tag{a.4.12.5}$$

And:

$$\begin{aligned}
& (y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta)+(\beta-b)'V^{-1}(\beta-b) \\
= \; & y'\bar{\Sigma}^{-1}y+\beta'\bar{X}'\bar{\Sigma}^{-1}\bar{X}\beta-2\beta'\bar{X}'\bar{\Sigma}^{-1}y+\beta'V^{-1}\beta+b'V^{-1}b-2\beta'V^{-1}b \\
= \; & \beta'(V^{-1}+\bar{X}'\bar{\Sigma}^{-1}\bar{X})\beta-2\beta'(V^{-1}b+\bar{X}'\bar{\Sigma}^{-1}y)+b'V^{-1}b+y'\bar{\Sigma}^{-1}y
\end{aligned} \tag{a.4.12.6}$$

Complete the squares:

$$= \beta'(V^{-1}+\bar{X}'\bar{\Sigma}^{-1}\bar{X})\beta-2\beta'\bar{V}^{-1}\bar{V}(V^{-1}b+\bar{X}'\bar{\Sigma}^{-1}y)+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b}+b'V^{-1}b+y'\bar{\Sigma}^{-1}y \tag{a.4.12.7}$$

Define:

$$\bar{V}=(V^{-1}+\bar{X}'\bar{\Sigma}^{-1}\bar{X})^{-1} \qquad \bar{b}=\bar{V}(V^{-1}b+\bar{X}'\bar{\Sigma}^{-1}y) \tag{a.4.12.8}$$

Then (a.4.12.7) rewrites:

$$\begin{aligned}
= \; & \beta'\bar{V}^{-1}\beta-2\beta'\bar{V}^{-1}\bar{b}+\bar{b}'\bar{V}^{-1}\bar{b}-\bar{b}'\bar{V}^{-1}\bar{b}+b'V^{-1}b+y'\bar{\Sigma}^{-1}y \\
= \; & (\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+(b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}+y'\bar{\Sigma}^{-1}y)
\end{aligned} \tag{a.4.12.9}$$

The terms in (a.4.12.8) simplify. Note first that:

$$\bar{X}'\bar{\Sigma}^{-1}\bar{X}=(I_n\otimes X)'(\Sigma\otimes I_T)^{-1}(I_n\otimes X)=(I_n\otimes X')(\Sigma^{-1}\otimes I_T)(I_n\otimes X)=(\Sigma^{-1}\otimes X')(I_n\otimes X)=\Sigma^{-1}\otimes X'X \tag{a.4.12.10}$$

Similarly:

$$\bar{X}'\bar{\Sigma}^{-1}y=(I_n\otimes X)'(\Sigma\otimes I_T)^{-1}vec(Y)=(I_n\otimes X')(\Sigma^{-1}\otimes I_T)vec(Y)=(\Sigma^{-1}\otimes X')vec(Y)=vec(X'Y\Sigma^{-1}) \tag{a.4.12.11}$$

Substituting (a.4.12.5) and (a.4.12.9) back in (a.4.12.4), the term within the integral rewrites:

$$\begin{aligned}
& (2\pi)^{-nT/2}|\Sigma|^{-T/2}(2\pi)^{-q/2}|V|^{-1/2}exp\left(-\frac{1}{2}\left[(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+(b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}+y'\bar{\Sigma}^{-1}y)\right]\right) \\
= \; & (2\pi)^{-nT/2}|\Sigma|^{-T/2}(2\pi)^{-q/2}|V|^{-1/2}|\bar{V}|^{1/2}|\bar{V}|^{-1/2} \\
\times \; & exp\left(-\frac{1}{2}\left[(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})+(b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}+y'\bar{\Sigma}^{-1}y)\right]\right) \\
= \; & (2\pi)^{-nT/2}|\Sigma|^{-T/2}|V|^{-1/2}|\bar{V}|^{1/2}exp\left(-\frac{1}{2}(b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}+y'\bar{\Sigma}^{-1}y)\right) \\
\times \; & (2\pi)^{-q/2}\bar{V}^{-1}exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)
\end{aligned} \tag{a.4.12.12}$$

The expression simplifies further. Note that:

$$\begin{aligned}
& |V|^{-1/2}|\bar{V}|^{1/2} \\
= \; & |V|^{-1/2}|(V^{-1}+\Sigma^{-1}\otimes X'X)^{-1}|^{1/2} \\
= \; & |V|^{-1/2}|(V^{-1}+\Sigma^{-1}\otimes X'X)|^{-1/2} \\
= \; & |V\ (V^{-1}+\Sigma^{-1}\otimes X'X)|^{-1/2} \\
= \; & |I+V(\Sigma^{-1}\otimes X'X)|^{-1/2}
\end{aligned} \tag{a.4.12.13}$$

Also:

$$\begin{aligned}
& y'\bar{\Sigma}^{-1}y \\
= \; & y'(\Sigma\otimes I_T)^{-1}y \\
= \; & y'(\Sigma^{-1}\otimes I_T)y \\
= \; & y'\ vec(Y\Sigma^{-1}) \\
= \; & vec(Y)'\ vec(Y\Sigma^{-1}) \\
= \; & tr(Y'Y\Sigma^{-1})
\end{aligned} \tag{a.4.12.14}$$

Thus, substituting again (a.4.12.13) and (a.4.12.14) back in (a.4.12.12), the term within the integral eventually rewrites as:

$$\begin{aligned}
& (2\pi)^{-nT/2}|\Sigma|^{-T/2}|I+V(\Sigma^{-1}\otimes X'X)|^{-1/2} exp\left(-\frac{1}{2}\left[b'V^{-1}b-\bar{b}'\bar{V}^{-1}\bar{b}+tr(Y'Y\Sigma^{-1})\right]\right) \\
\times \; & (2\pi)^{-q/2}\bar{V}^{-1} exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)
\end{aligned} \tag{a.4.12.15}$$

derivations for equation (4.12.23)

Focus on the term within the integral and rearrange:

$$\begin{aligned}
& (2\pi)^{-nT/2}|\Sigma|^{-T/2} exp\left(-\frac{1}{2}(\beta-\hat{\beta})'\ (\Sigma\otimes(X'X)^{-1})^{-1}\ (\beta-\hat{\beta})\right) \\
\times \; & exp\left(-\frac{1}{2}\ tr\left[\Sigma^{-1}(Y-X\hat{B})'(Y-X\hat{B})\right]\right) \\
\times \; & (2\pi)^{-q/2}|\Sigma\otimes W|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'(\Sigma\otimes W)^{-1}(\beta-b)\right) \\
\times \; & \frac{2^{-\alpha n/2}}{\Gamma_n\left(\frac{\alpha}{2}\right)}|S|^{\alpha/2}|\Sigma|^{-(\alpha+n+1)/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}S\right\}\right) \\
= \; & (2\pi)^{-nT/2}\ \frac{2^{-\alpha n/2}}{\Gamma_n\left(\frac{\alpha}{2}\right)}\ |S|^{\alpha/2} \\
\times \; & (2\pi)^{-nk/2}|\Sigma\otimes W|^{-1/2} exp\left(-\frac{1}{2}\left[(\beta-b)'(\Sigma\otimes W)^{-1}(\beta-b)+(\beta-\hat{\beta})'\ (\Sigma\otimes(X'X)^{-1})^{-1}\ (\beta-\hat{\beta})\right]\right) \\
\times \; & |\Sigma|^{-(\alpha+T+n+1)/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}\left[S+(Y-X\hat{B})'(Y-X\hat{B})\right]\right\}\right)
\end{aligned} \tag{a.4.12.16}$$

Note first that:

$$|\Sigma\otimes W|^{-1/2}=|\Sigma|^{-k/2}|W|^{-n/2} \tag{a.4.12.17}$$

Then focus on the first term between square brackets to obtain:

$$\begin{aligned}
& (\beta-b)'(\Sigma\otimes W)^{-1}(\beta-b)+(\beta-\hat{\beta})'\ (\Sigma\otimes(X'X)^{-1})^{-1}\ (\beta-\hat{\beta}) \\
=\ & tr\left\{\Sigma^{-1}(\mathcal{B}-B)'W^{-1}(\mathcal{B}-B)\right\}+tr\left\{\Sigma^{-1}(\mathcal{B}-\hat{B})'(X'X)(\mathcal{B}-\hat{B})\right\} \\
=\ & tr\left\{\Sigma^{-1}\left[(\mathcal{B}-B)'W^{-1}(\mathcal{B}-B)+(\mathcal{B}-\hat{B})'(X'X)(\mathcal{B}-\hat{B})\right]\right\} \\
=\ & tr\left\{\Sigma^{-1}\left[(\mathcal{B}-B)'W^{-1}(\mathcal{B}-B)+(\mathcal{B}-\hat{B})'(X'X)(\mathcal{B}-\hat{B})\right]\right\} \\
=\ & tr\left\{\Sigma^{-1}\left[\mathcal{B}'W^{-1}\mathcal{B}+B'W^{-1}B-2\mathcal{B}'W^{-1}B+\mathcal{B}'(X'X)\mathcal{B}+\hat{B}'(X'X)\hat{B}-2\mathcal{B}'(X'X)\hat{B}\right]\right\} \\
=\ & tr\left\{\Sigma^{-1}\left[\mathcal{B}'(W^{-1}+X'X)\mathcal{B}-2\mathcal{B}'(W^{-1}B+X'X\hat{B})+B'W^{-1}B+\hat{B}'(X'X)\hat{B}\right]\right\} \qquad \text{(a.4.12.18)}
\end{aligned}$$

Complete the squares:

$$= tr\left\{\Sigma^{-1}\left[\mathcal{B}'(W^{-1}+X'X)\mathcal{B}-2\mathcal{B}'\bar{W}^{-1}\bar{W}(W^{-1}B+X'X\hat{B})+\bar{B}'\bar{W}^{-1}\bar{B}-\bar{B}'\bar{W}^{-1}\bar{B}+B'W^{-1}B+\hat{B}'(X'X)\hat{B}\right]\right\} \qquad \text{(a.4.12.19)}$$

Define:

$$\bar{W}=(W^{-1}+X'X)^{-1} \qquad \bar{B}=\bar{W}(W^{-1}B+X'X\hat{B}) \qquad \text{(a.4.12.20)}$$

then the expression becomes:

$$\begin{aligned}
=\ & tr\left\{\Sigma^{-1}\left[\mathcal{B}'\bar{W}^{-1}\mathcal{B}-2\mathcal{B}'\bar{W}^{-1}\bar{B}+\bar{B}'\bar{W}^{-1}\bar{B}-\bar{B}'\bar{W}^{-1}\bar{B}+B'W^{-1}B+\hat{B}'(X'X)\hat{B}\right]\right\} \\
=\ & tr\left\{\Sigma^{-1}\left[(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})-\bar{B}'\bar{W}^{-1}\bar{B}+B'W^{-1}B+\hat{B}'(X'X)\hat{B}\right]\right\} \\
=\ & tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}+tr\left\{\Sigma^{-1}\left[B'W^{-1}B+\hat{B}'(X'X)\hat{B}-\bar{B}'\bar{W}^{-1}\bar{B}\right]\right\} \qquad \text{(a.4.12.21)}
\end{aligned}$$

Substituting (a.4.12.17) and (a.4.12.21) in (a.4.12.16) yields:

$$\begin{aligned}
& (2\pi)^{-nT/2}\ |W|^{-n/2}\ \frac{2^{-\alpha n/2}}{\Gamma_n\left(\frac{\alpha}{2}\right)}\ |S|^{\alpha/2} \\
\times\ & (2\pi)^{-nk/2}|\Sigma|^{-k/2}exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}\right) \\
\times\ & |\Sigma|^{-(\alpha+T+n+1)/2}exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}\left[S+(Y-X\hat{B})'(Y-X\hat{B})+B'W^{-1}B+\hat{B}'(X'X)\hat{B}-\bar{B}'\bar{W}^{-1}\bar{B}\right]\right\}\right) \qquad \text{(a.4.12.22)}
\end{aligned}$$

Define:

$$\bar{\alpha}=\alpha+T \qquad \bar{S}=S+(Y-X\hat{B})'(Y-X\hat{B})+B'W^{-1}B+\hat{B}'(X'X)\hat{B}-\bar{B}'\bar{W}^{-1}\bar{B} \qquad \text{(a.4.12.23)}$$

Then the expression becomes:

$$\begin{aligned}
& (2\pi)^{-nT/2}\ |W|^{-n/2}\ \frac{2^{-\alpha n/2}}{\Gamma_n\left(\frac{\alpha}{2}\right)}\ |S|^{\alpha/2} \\
\times\ & (2\pi)^{-nk/2}|\Sigma|^{-k/2}exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}\right) \\
\times\ & |\Sigma|^{-(\bar{\alpha}+n+1)/2}exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}\bar{S}\right\}\right) \qquad \text{(a.4.12.24)}
\end{aligned}$$

And finally, this rewrites:

$$
\begin{aligned}
& (2\pi)^{-nT/2} \;\; |W|^{-n/2}|\bar{W}|^{n/2} \;\; \frac{2^{\bar{\alpha}n/2}}{2^{\alpha n/2}} \;\; \frac{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}{\Gamma_n\left(\frac{\alpha}{2}\right)} \;\; |S|^{\alpha/2}|\bar{S}|^{-\bar{\alpha}/2} \\
\times \;\; & (2\pi)^{-nk/2}|\Sigma|^{-k/2}|\bar{W}|^{-n/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}\right) \\
\times \;\; & \frac{2^{-\bar{\alpha}n/2}}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}|\bar{S}|^{\bar{\alpha}/2}|\Sigma|^{-(\bar{\alpha}+n+1)/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}\bar{S}\right\}\right)
\end{aligned}
\tag{a.4.12.25}
$$

Go on simplifying. Note that:

$$
(2\pi)^{-nT/2}\frac{2^{\bar{\alpha}n/2}}{2^{\alpha n/2}} = 2^{-nT/2} \;\; \pi^{-nT/2} \;\; \frac{2^{(\alpha+T)n/2}}{2^{\alpha n/2}} = \pi^{-nT/2}
\tag{a.4.12.26}
$$

Also:

$$
\begin{aligned}
& |W|^{-n/2}|\bar{W}|^{n/2} \\
= \;\; & |W|^{-n/2}|(W^{-1}+X'X)^{-1}|^{n/2} \\
= \;\; & |W|^{-n/2}|(W^{-1}+X'X)|^{-n/2} \\
= \;\; & |W(W^{-1}+X'X)|^{-n/2} \\
= \;\; & |I+WX'X|^{-n/2}
\end{aligned}
\tag{a.4.12.27}
$$

Consider then:

$$
\begin{aligned}
& (Y-X\hat{B})'(Y-X\hat{B})+\hat{B}'X'X\hat{B} \\
= \;\; & Y'Y+\hat{B}'X'X\hat{B}-\hat{B}'X'Y-Y'X\hat{B}+\hat{B}'X'X\hat{B} \\
= \;\; & Y'Y+2\hat{B}'X'X\hat{B}-\hat{B}'X'Y-Y'X\hat{B} \\
= \;\; & Y'Y+2Y'X(X'X)^{-1}X'X(X'X)^{-1}X'Y-Y'X(X'X)^{-1}X'Y-Y'X(X'X)^{-1}X'Y \\
= \;\; & Y'Y+2Y'X(X'X)^{-1}X'Y-Y'X(X'X)^{-1}X'Y-Y'X(X'X)^{-1}X'Y \\
= \;\; & Y'Y
\end{aligned}
\tag{a.4.12.28}
$$

Substitute back in (a.4.12.23):

$$
\bar{S} = S+Y'Y+B'W^{-1}B-\bar{B}'\bar{W}^{-1}\bar{B}
\tag{a.4.12.29}
$$

Then:

$$
\begin{aligned}
= \;\; & |S|^{\alpha/2}|\bar{S}|^{-\bar{\alpha}/2} \\
= \;\; & |S|^{\alpha/2}|S+Y'Y+B'W^{-1}B-\bar{B}'\bar{W}^{-1}\bar{B}|^{-\bar{\alpha}/2} \\
= \;\; & |S|^{-T/2}|S|^{(\alpha+T)/2}|S+Y'Y+B'W^{-1}B-\bar{B}'\bar{W}^{-1}\bar{B}|^{-\bar{\alpha}/2} \\
= \;\; & |S|^{-T/2}|S|^{\bar{\alpha}/2}|S+Y'Y+B'W^{-1}B-\bar{B}'\bar{W}^{-1}\bar{B}|^{-\bar{\alpha}/2} \\
= \;\; & |S|^{-T/2}|S^{-1}|^{-\bar{\alpha}/2}|S+Y'Y+B'W^{-1}B-\bar{B}'\bar{W}^{-1}\bar{B}|^{-\bar{\alpha}/2} \\
= \;\; & |S|^{-T/2}|S^{-1}(S+Y'Y+B'W^{-1}B-\bar{B}'\bar{W}^{-1}\bar{B})|^{-\bar{\alpha}/2} \\
= \;\; & |S|^{-T/2}|I+S^{-1}(Y'Y+B'W^{-1}B-\bar{B}'\bar{W}^{-1}\bar{B})|^{-\bar{\alpha}/2} \\
= \;\; & |S|^{-T/2}|I+S^{-1}(\bar{S}-S)|^{-\bar{\alpha}/2}
\end{aligned}
\tag{a.4.12.30}
$$

Substituting back (a.4.12.26), (a.4.12.27) and (a.4.12.30) in (a.4.12.25) finally yields:

$$\begin{aligned} & \pi^{-nT/2} \ |I+WX'X|^{-n/2} \ |S|^{-T/2}|I+S^{-1}(\bar{S}-S)|^{-\bar{\alpha}/2} \ \frac{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}{\Gamma_n\left(\frac{\alpha}{2}\right)} \\ \times & \ (2\pi)^{-nk/2}|\Sigma|^{-k/2}|\bar{W}|^{-n/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}\right) \\ \times & \ \frac{2^{-\bar{\alpha}n/2}}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}|\bar{S}|^{\bar{\alpha}/2}|\Sigma|^{-(\bar{\alpha}+n+1)/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}\bar{S}\right\}\right) \end{aligned} \tag{a.4.12.31}$$

Therefore, substituting back in the integrals, we obtain:

$$\begin{aligned} f(y) \ &= \pi^{-nT/2} \ |I+WX'X|^{-n/2} \ |S|^{-T/2}|I+S^{-1}(\bar{S}-S)|^{-\bar{\alpha}/2} \ \frac{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}{\Gamma_n\left(\frac{\alpha}{2}\right)} \\ &\times \int\int (2\pi)^{-nk/2}|\Sigma|^{-k/2}|\bar{W}|^{-n/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}(\mathcal{B}'-\bar{B})'\bar{W}^{-1}(\mathcal{B}'-\bar{B})\right\}\right) d\beta \\ &\times \frac{2^{-\bar{\alpha}n/2}}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}|\bar{S}|^{\bar{\alpha}/2}|\Sigma|^{-(\bar{\alpha}+n+1)/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}\bar{S}\right\}\right) d\sigma \end{aligned} \tag{a.4.12.32}$$

derivations for equation (4.12.26)

$$\begin{aligned} & f(y) \\ \approx & \ \frac{f(y|\beta^*,\Sigma^*)\pi(\beta^*,\Sigma^*)}{\pi(\Sigma^*|y,\beta^*) \times \frac{1}{J}\sum_{j=1}^{J}\pi(\beta^*|\Sigma^{(j)},y)} \\ = & \ (2\pi)^{-nT/2}|\bar{\Sigma}|^{-1/2} exp\left(-\frac{1}{2}(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta)\right) \\ \times & \ \frac{(2\pi)^{-q/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-q/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n\left(\frac{\alpha}{2}\right)}|S|^{\alpha/2}|\Sigma|^{-(\alpha+n+1)/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}S\right\}\right)}{\frac{2^{-\bar{\alpha}n/2}}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}|\bar{S}|^{\bar{\alpha}/2}|\Sigma|^{-(\bar{\alpha}+n+1)/2} \ exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}\bar{S}\right\}\right)} \end{aligned} \tag{a.4.12.33}$$

Now, note that:

$$|\bar{\Sigma}|^{-1/2} = |\Sigma \otimes I_T|^{-1/2} = |\Sigma|^{-T/2}|I_T|^{-n/2} = |\Sigma|^{-T/2} \tag{a.4.12.34}$$

Also:

$$\begin{aligned} & (y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta) \\ = & \ (y-(I_n \otimes X)\beta)'(\Sigma \otimes I_T)^{-1}(y-(I_n \otimes X)\beta) \\ = & \ tr\left\{\Sigma^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})\right\} \end{aligned} \tag{a.4.12.35}$$

Substituting back (a.4.12.34) and (a.4.12.35) in (a.4.12.33):

$$\begin{aligned} & f(y) \\ \approx & \ (2\pi)^{-nT/2}|\Sigma|^{-T/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})\right\}\right) \\ \times & \ \frac{(2\pi)^{-q/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-q/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n\left(\frac{\alpha}{2}\right)}|S|^{\alpha/2}|\Sigma|^{-(\alpha+n+1)/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}S\right\}\right)}{\frac{2^{-\bar{\alpha}n/2}}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}|\bar{S}|^{\bar{\alpha}/2}|\Sigma|^{-(\bar{\alpha}+n+1)/2} \ exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}\bar{S}\right\}\right)} \end{aligned} \tag{a.4.12.36}$$

Continue with the definitions of $\bar{\alpha}$ and $\bar{S}$:

$$= 2^{-nT/2}\pi^{-nT/2}|\Sigma|^{-T/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})\right\}\right)$$

$$\times \frac{(2\pi)^{-q/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-q/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}$$

$$\times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n\left(\frac{\alpha}{2}\right)}|S|^{\alpha/2}|\Sigma|^{-(\alpha+n+1)/2} exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}S\right\}\right)}{\frac{2^{-(\alpha+T)n/2}}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}|\bar{S}|^{\bar{\alpha}/2}|\Sigma|^{-(\alpha+T+n+1)/2}\ \ exp\left(-\frac{1}{2}tr\left\{\Sigma^{-1}\left[S+(Y-X\mathcal{B})'(Y-X\mathcal{B})\right]\right\}\right)}$$

$$= 2^{-nT/2}\pi^{-nT/2}|\Sigma|^{-T/2}$$

$$\times \frac{(2\pi)^{-q/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-q/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n\left(\frac{\alpha}{2}\right)}|S|^{\alpha/2}|\Sigma|^{-(\alpha+n+1)/2}}{\frac{2^{-(\alpha+T)n/2}}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}|\bar{S}|^{\bar{\alpha}/2}|\Sigma|^{-(\alpha+T+n+1)/2}}$$

$$= 2^{-nT/2}\pi^{-nT/2} \times \frac{(2\pi)^{-q/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-q/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \times \frac{\frac{2^{-\alpha n/2}}{\Gamma_n\left(\frac{\alpha}{2}\right)}|S|^{\alpha/2}}{\frac{2^{-(\alpha+T)n/2}}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}|\bar{S}|^{\bar{\alpha}/2}}$$

$$= \pi^{-nT/2} \times \frac{(2\pi)^{-q/2}|V|^{-1/2}\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}(2\pi)^{-q/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \times \frac{\frac{1}{\Gamma_n\left(\frac{\alpha}{2}\right)}|S|^{\alpha/2}}{\frac{1}{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}|\bar{S}|^{\bar{\alpha}/2}}$$

$$= \pi^{-nT/2}\ \frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|V|^{1/2}|\bar{V}|^{-1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)}\ \frac{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}{\Gamma_n\left(\frac{\alpha}{2}\right)}\ \frac{|S|^{\alpha/2}}{|\bar{S}|^{\bar{\alpha}/2}} \qquad \text{(a.4.12.37)}$$

Note that:

$$\begin{aligned}
&= |S|^{\alpha/2}|\bar{S}|^{-\bar{\alpha}/2}\\
&= |S|^{\alpha/2}|S+(Y-X\mathcal{B})'(Y-X\mathcal{B})|^{-\bar{\alpha}/2}\\
&= |S|^{-T/2}|S|^{(\alpha+T)/2}|S+(Y-X\mathcal{B})'(Y-X\mathcal{B})|^{-\bar{\alpha}/2}\\
&= |S|^{-T/2}|S|^{\bar{\alpha}/2}|S+(Y-X\mathcal{B})'(Y-X\mathcal{B})|^{-\bar{\alpha}/2}\\
&= |S|^{-T/2}|S^{-1}|^{-\bar{\alpha}/2}|S+(Y-X\mathcal{B})'(Y-X\mathcal{B})|^{-\bar{\alpha}/2}\\
&= |S|^{-T/2}|S^{-1}(S+(Y-X\mathcal{B})'(Y-X\mathcal{B})|^{-\bar{\alpha}/2}\\
&= |S|^{-T/2}|I+S^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})|^{-\bar{\alpha}/2}\\
&= |S|^{-T/2}|I+S^{-1}(\bar{S}-S)|^{-\bar{\alpha}/2} && \text{(a.4.12.38)}
\end{aligned}$$

Also:

$$\begin{aligned}
&= |V|^{1/2}|\bar{V}|^{-1/2}\\
&= |V|^{1/2}|(V^{-1}+\Sigma^{-1}\otimes X'X)^{-1}|^{-1/2}\\
&= |V|^{1/2}|V^{-1}+\Sigma^{-1}\otimes X'X|^{1/2}\\
&= |I+V(\Sigma^{-1}\otimes X'X)|^{1/2} && \text{(a.4.12.39)}
\end{aligned}$$

Substituting (a.4.12.38) and (a.4.12.39) back in (a.4.12.37):

$$= \pi^{-nT/2}\ \frac{\Gamma_n\left(\frac{\bar{\alpha}}{2}\right)}{\Gamma_n\left(\frac{\alpha}{2}\right)}\ |S|^{-T/2}|I+S^{-1}(\bar{S}-S)|^{-\bar{\alpha}/2}$$

$$\times \frac{\exp\left(-\frac{1}{2}(\beta-b)'V^{-1}(\beta-b)\right)}{\frac{1}{J}\sum_{j=1}^{J}|I+V(\Sigma^{-1}\otimes X'X)|^{1/2}\exp\left(-\frac{1}{2}(\beta-\bar{b})'\bar{V}^{-1}(\beta-\bar{b})\right)} \qquad \text{(a.4.12.40)}$$

derivations for equation (4.12.29)

Use simple back recursion to obtain:

$$\begin{aligned}
&\gamma_t = \mu_t + F\gamma_{t-1} + \xi_t \\
\Leftrightarrow\ &\gamma_t = \mu_t + F(\mu_{t-1} + F\gamma_{t-2} + \xi_{t-1}) + \xi_t \\
\Leftrightarrow\ &\gamma_t = \mu_t + F\mu_{t-1} + F^2\gamma_{t-2} + \xi_t + F\xi_{t-1} \\
\Leftrightarrow\ &\gamma_t = \mu_t + F\mu_{t-1} + F^2(\mu_{t-2} + F\gamma_{t-3} + \xi_{t-2}) + \xi_t + F\xi_{t-1} \\
\Leftrightarrow\ &\gamma_t = \mu_t + F\mu_{t-1} + F^2\mu_{t-2} + F^3\gamma_{t-3} + \xi_t + F\xi_{t-1} + F^2\xi_{t-2}
\end{aligned} \tag{a.4.12.41}$$

Going on this way:

$$\gamma_t = \sum_{i=0}^{j} F^i\mu_{t-i} + F^j\gamma_{t-j} + \sum_{i=0}^{j} F^i\xi_{t-i} \tag{a.4.12.42}$$

derivations for equation (4.12.30)

Consider the general formulation of the VAR model:

$$y_t = Cz_t + A_1y_{t-1} + \cdots + A_py_{t-p} + \varepsilon_t \tag{a.4.12.43}$$

Taking expectations on both sides, noting that $\mathbb{E}(y_t) = \mathbb{E}(y_{t-1}) = \cdots = \mathbb{E}(y_{t-p}) = \mu$ by stationarity, and that $\mathbb{E}(z_t) = z_t$ and $\mathbb{E}(\varepsilon_t) = 0$, one obtains:

$$\mu = Cz_t + A_1\mu + \cdots + A_p\mu \tag{a.4.12.44}$$

Rearranging:

$$(I - A_1 - \cdots - A_p)\mu = Cz_t \tag{a.4.12.45}$$

Which eventually yields:

$$\mu = (I - A_1 - \cdots - A_p)^{-1}C\, z_t \tag{a.4.12.46}$$

derivations for equation (4.12.34)

Rearrange:

$$\begin{aligned}
&exp\left(-\frac{1}{2}(\beta-\hat{\beta})'\ (\Sigma\otimes(X'X)^{-1})^{-1}\ (\beta-\hat{\beta})\right)\times\exp\left(-\frac{1}{2}(\beta-b)'(\Sigma\otimes W)^{-1}(\beta-b)\right) \\
=\ &exp\left(-\frac{1}{2}tr\{\Sigma^{-1}(\mathcal{B}-\hat{B})'\ (X'X)\ (\mathcal{B}-\hat{B})\}\right)\times\exp\left(-\frac{1}{2}tr\{\Sigma^{-1}(\mathcal{B}-B)'\ W^{-1}\ (\mathcal{B}-B)\}\right) \\
=\ &exp\left(-\frac{1}{2}tr\{\Sigma^{-1}[(\mathcal{B}-\hat{B})'\ (X'X)\ (\mathcal{B}-\hat{B})+(\mathcal{B}-B)'\ W^{-1}\ (\mathcal{B}-B)]\}\right)
\end{aligned} \tag{a.4.12.47}$$

Consider the terms in square brackets and complete the squares:

$$\begin{aligned}
&(\mathcal{B}-\hat{B})'(X'X)(\mathcal{B}-\hat{B})+(\mathcal{B}-B)'W^{-1}(\mathcal{B}-B) \\
=\ &\mathcal{B}'(X'X)\mathcal{B}+\hat{B}'(X'X)\hat{B}-2\mathcal{B}'(X'X)\hat{B}+\mathcal{B}'W^{-1}\mathcal{B}+B'W^{-1}B-2\mathcal{B}W^{-1}B \\
=\ &\mathcal{B}'(W^{-1}+X'X)\mathcal{B}-2\mathcal{B}'(W^{-1}B+X'X\hat{B})+\hat{B}'(X'X)\hat{B}+B'W^{-1}B \\
=\ &\mathcal{B}'(W^{-1}+X'X)\mathcal{B}-2\mathcal{B}'\bar{W}^{-1}\bar{W}(W^{-1}B+X'X\hat{B})+\bar{B}'\bar{W}^{-1}\bar{B}-\bar{B}'\bar{W}^{-1}\bar{B}+\hat{B}'(X'X)\hat{B}+B'W^{-1}B
\end{aligned} \tag{a.4.12.48}$$

Define:

$$\bar{W} = (W^{-1}+X'X)^{-1} \qquad\qquad \bar{B} = \bar{W}(W^{-1}B+X'X\hat{B}) \tag{a.4.12.49}$$

Then:

$$
\begin{aligned}
&= \mathcal{B}'\bar{W}^{-1}\mathcal{B} - 2\mathcal{B}'\bar{W}^{-1}\bar{B} + \bar{B}'\bar{W}^{-1}\bar{B} - \bar{B}'\bar{W}^{-1}\bar{B} + \hat{B}'(X'X)\hat{B} + B'W^{-1}B \\
&= (\mathcal{B}-\bar{B})'\bar{W}^{-1}(\mathcal{B}-\bar{B}) - \bar{B}'\bar{W}^{-1}\bar{B} + \hat{B}'(X'X)\hat{B} + B'W^{-1}B \qquad \text{(a.4.12.50)}
\end{aligned}
$$

Substitute back in (a.4.12.47):

$$
\begin{aligned}
&= exp\left(-\frac{1}{2}tr\{\Sigma^{-1}[(\mathcal{B}-\bar{B})'\bar{W}^{-1}(\mathcal{B}-\bar{B}) - \bar{B}'\bar{W}^{-1}\bar{B} + \hat{B}'(X'X)\hat{B} + B'W^{-1}B]\}\right) \\
&= exp\left(-\frac{1}{2}tr\{\Sigma^{-1}(\mathcal{B}-\bar{B})'\bar{W}^{-1}(\mathcal{B}-\bar{B})\}\right) exp\left(-\frac{1}{2}tr\{\Sigma^{-1}[-\bar{B}'\bar{W}^{-1}\bar{B} + \hat{B}'(X'X)\hat{B} + B'W^{-1}B]\}\right) \\
&\propto exp\left(-\frac{1}{2}tr\{\Sigma^{-1}(\mathcal{B}-\bar{B})'\bar{W}^{-1}(\mathcal{B}-\bar{B})\}\right) \qquad \text{(a.4.12.51)}
\end{aligned}
$$

Finally, note again that:

$$
\bar{B} = \bar{W}(W^{-1}B + X'X\hat{B}) = \bar{W}(W^{-1}B + X'X(X'X)^{1}X'Y) = \bar{W}(W^{-1}B + X'Y) \qquad \text{(a.4.12.52)}
$$

derivations for equation (4.13.17)

The log-likelihood is given by:

$$
\log(f(y|\beta,\Sigma)) = -\frac{nT}{2}\log(2\pi) - \frac{1}{2}\log(|\bar{\Sigma}|) - \frac{1}{2}(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta) \qquad \text{(a.4.12.53)}
$$

Then note that:

$$
\log(|\bar{\Sigma}|) = \log(|\Sigma \otimes I_T|) = \log(|\Sigma|^T |I_T|^n) = \log(|\Sigma|^T) = T\log(|\Sigma|) \qquad \text{(a.4.12.54)}
$$

Also:

$$
(y-\bar{X}\beta)'\bar{\Sigma}^{-1}(y-\bar{X}\beta) = (y-(I_n\otimes X)\beta)'(\Sigma\otimes I_T)^{-1}(y-(I_n\otimes X)\beta) = tr\left\{\Sigma^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})\right\} \qquad \text{(a.4.12.55)}
$$

Substituting back (a.4.12.54) and (a.4.12.55) in (a.4.12.53):

$$
\log(f(y|\beta,\Sigma)) = -\frac{nT}{2}\log(2\pi) - \frac{T}{2}\log(|\Sigma|) - \frac{1}{2}tr\left\{\Sigma^{-1}(Y-X\mathcal{B})'(Y-X\mathcal{B})\right\} \qquad \text{(a.4.12.56)}
$$

The function is estimated at the maximum likelihood values. Hence $\mathcal{B} = \hat{\mathcal{B}}$ and $\Sigma = \hat{\Sigma} = \mathcal{E}'\mathcal{E}/T$. Substituting in (a.4.12.56):

$$
\begin{aligned}
&\log(f(y|\beta,\Sigma)) \\
&= -\frac{nT}{2}\log(2\pi) - \frac{T}{2}\log(|\hat{\Sigma}|) - \frac{1}{2}tr\left\{\hat{\Sigma}^{-1}(Y-X\hat{\mathcal{B}})'(Y-X\hat{\mathcal{B}})\right\} \\
&= -\frac{nT}{2}\log(2\pi) - \frac{T}{2}\log(|\hat{\Sigma}|) - \frac{1}{2}tr\left\{\hat{\Sigma}^{-1}\hat{\Sigma}T\right\} \\
&= -\frac{nT}{2}\log(2\pi) - \frac{T}{2}\log(|\hat{\Sigma}|) - \frac{T}{2} \\
&= -\frac{nT}{2}\log(2\pi) - \frac{T}{2}(1+\log(|\hat{\Sigma}|)) \\
&= -\frac{T}{2}(n\log(2\pi) + (1+\log(|\hat{\Sigma}|))) \qquad \text{(a.4.12.57)}
\end{aligned}
$$

Following, the AIC obtains as:

$$
\begin{aligned}
& AIC \\
= \; & 2q/T - 2\,\hat{L}/T \\
= \; & 2q/T - 2/T\; \left[-\frac{T}{2}\; (n\log(2\pi) + (1 + \log(|\hat{\Sigma}|))) \right] \\
= \; & 2q/T + n\; \log(2\pi) + 1 + \log(|\hat{\Sigma}|)
\end{aligned} \tag{a.4.12.58}
$$

It follows immediatey that the BIC is given by:

$$
\begin{aligned}
& BIC \\
= \; & q\log(T)/T - 2\,\hat{L}/T \\
= \; & q\log(T)/T - 2/T\; \left[-\frac{T}{2}\; (n\log(2\pi) + (1 + \log(|\hat{\Sigma}|))) \right] \\
= \; & q\log(T)/T + n\; \log(2\pi) + 1 + \log(|\hat{\Sigma}|)
\end{aligned} \tag{a.4.12.59}
$$

And the Hannan-Quinn criterion is given by:

$$
\begin{aligned}
& HQ \\
= \; & 2q\; \log(\log(T))/T - 2\,\hat{L}/T \\
= \; & 2q\; \log(\log(T))/T - 2/T\; \left[-\frac{T}{2}\; (n\log(2\pi) + (1 + \log(|\hat{\Sigma}|))) \right] \\
= \; & 2q\; \log(\log(T))/T + n\; \log(2\pi) + 1 + \log(|\hat{\Sigma}|)
\end{aligned} \tag{a.4.12.60}
$$

Removing the constants makes the value invariant to the number of endogenous variables:

$$
AIC = 2q/T + \log(|\hat{\Sigma}|) \qquad BIC = q\; \log(T)/T + \log(|\hat{\Sigma}|) \qquad HQ = 2q\; \log(\log(T))/T + \log(|\hat{\Sigma}|) \tag{a.4.12.61}
$$

CHAPTER 14

Bayesian VAR: advanced applications

derivations for equation (4.14.18)

The distribution for the conditional forecasts is given by:

$$R\,\hat{y}_{T+1:T+h} \sim N(\bar{y},\Omega) \tag{a.4.14.1}$$

The distribution for the unconditional forecasts is given by:

$$\hat{y}_{T+1:T+h} \sim N(f_{T+1:T+h},\ M(I_h \otimes \Gamma)M') \tag{a.4.14.2}$$

Using the selection matrix R on the unconditional forecasts, we find that the distribution of the forecasts for the variables on which conditions apply is given by:

$$R\,\hat{y}_{T+1:T+h} \sim N(R\,f_{T+1:T+h},\ D(I_h \otimes \Gamma)D') \tag{a.4.14.3}$$

with D is a $k \times nh$ matrix such that $D = RM$. Now choose any $(nh-k)\times nh$ matrix $\hat{D}$ such that the rows of $\hat{D}$ form an orthonormal basis for the nullspace of D. This implies that:

$$\hat{D}\hat{D}' = I_{nh-k} \qquad D\hat{D}' = 0_{k\times(nh-k)} \qquad \hat{D}D' = 0_{(nh-k)\times k} \tag{a.4.14.4}$$

Now define the random variable z as:

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} D \\ \hat{D} \end{pmatrix} M^{-1}\hat{y}_{T+1:T+h} = \begin{pmatrix} R\,\hat{y}_{T+1:T+h} \\ \hat{D}M^{-1}\hat{y}_{T+1:T+h} \end{pmatrix} \tag{a.4.14.5}$$

Thus, the distribution of z is given by:

$$z \sim N\left[\begin{pmatrix} Rf_{T+1:T+h} \\ \hat{D}M^{-1}f_{T+1:T+h} \end{pmatrix} \ ,\ \begin{pmatrix} D(I_h\otimes\Gamma)D' & 0_{k\times(nh-k)} \\ 0_{(nh-k)\times k} & \hat{D}(I_h\otimes\Gamma)\hat{D}' \end{pmatrix}\right] \tag{a.4.14.6}$$

The distribution of the conditions is given by (a.4.14.1). Because z_1 and z_2 are independent, we can simply substitute for (a.4.14.1) in (a.4.14.6) to obtain:

$$z \sim N\left[\begin{pmatrix} \bar{y} \\ \hat{D}M^{-1}f_{T+1:T+h} \end{pmatrix} \ ,\ \begin{pmatrix} \Omega & 0_{k\times(nh-k)} \\ 0_{(nh-k)\times k} & \hat{D}(I_h\otimes\Gamma)\hat{D}' \end{pmatrix}\right] \tag{a.4.14.7}$$

(a.4.14.5) permits to recover $\hat{y}_{T+1:T+h}$ from:

$$\hat{y}_{T+1:T+h} = M\begin{pmatrix} D \\ \hat{D} \end{pmatrix}^{-1} z \tag{a.4.14.8}$$

Since D has full row rank, then its generalised inverse D^* is such that $DD^* = I_k$. Hence:

$$\begin{pmatrix} D \\ \hat{D} \end{pmatrix}^{-1} = \begin{pmatrix} D^* & \hat{D}' \end{pmatrix} \tag{a.4.14.9}$$

which also implies that $\hat{D}D^* = 0$. Thus:

$$\hat{y}_{T+1:T+h} = M\begin{pmatrix} D^* & \hat{D}' \end{pmatrix} z \tag{a.4.14.10}$$

Eventually combining (a.4.14.10) with (a.4.14.7), we obtain the restricted distribution of $\hat{y}_{T+1:T+h}$ as $\hat{y}_{T+1:T+h} \sim N(\hat{\mu}, \hat{\Omega})$, with:

$$\hat{\mu} = M(D^{*}\bar{y} + \hat{D}'\hat{D}M^{-1}f_{T+1:T+h}) \qquad \hat{\Omega} = M(D^{*}\Omega D^{*\prime} + \hat{D}'\hat{D}(I_h \otimes \Gamma)\hat{D}'\hat{D})M' \tag{a.4.14.11}$$

Since $\hat{y}_{T+1:T+h} = f_{T+1:T+h} + M\ \xi_{T+1:T+h}$, it follows that $\xi_{T+1:T+h} = M^{-1}(\hat{y}_{T+1:T+h} - f_{T+1:T+h})$. Thus, from (a.4.14.11), the restricted distribution of the shocks $\xi_{T+1:T+h}$ is given by $\xi_{T+1:T+h} \sim N(\bar{\mu}, \bar{\Omega})$, with:

$$\bar{\mu} = D^{*}\bar{y} + \hat{D}'\hat{D}M^{-1}f_{T+1:T+h} - M^{-1}f_{T+1:T+h} \qquad \bar{\Omega} = D^{*}\Omega D^{*\prime} + \hat{D}'\hat{D}(I_h \otimes \Gamma)\hat{D}'\hat{D} \tag{a.4.14.12}$$

Also, it follows directly from post-multiplication of (a.4.14.9) that $D^{*}D + \hat{D}'\hat{D} = I_{nh}$, so that $\bar{\mu}$ in (a.4.14.12) rewrites:

$$\bar{\mu} = D^{*}\bar{y} - D^{*}DM^{-1}f_{T+1:T+h} = D^{*}\bar{y} - D^{*}Rf_{T+1:T+h} = D^{*}(\bar{y} - Rf_{T+1:T+h}) \tag{a.4.14.13}$$

And $\bar{\Omega}$ in (a.4.14.12) rewrites:

$$\bar{\Omega} = D^{*}\Omega D^{*\prime} + (I_{nh} - D^{*}D)(I_h \otimes \Gamma)(I_{nh} - D^{*}D) \tag{a.4.14.14}$$

derivations for equation (4.14.20)

Start from $\hat{y}_{T+1:T+h} = f_{T+1:T+h} + M\ \xi_{T+1:T+h}$. This implies that $\hat{y}_{T+1:T+h} \sim N(\hat{\mu}, \hat{\Omega})$, with:

$$\hat{\mu} = f_{T+1:T+h} + M\ \bar{\mu} \qquad \hat{\Omega} = M\bar{\Omega}M' \tag{a.4.14.15}$$

From (a.4.14.13), the first expression rewrites as:

$$\hat{\mu} = f_{T+1:T+h} + M\ \bar{\mu} = f_{T+1:T+h} + MD^{*}(\bar{y} - Rf_{T+1:T+h}) \tag{a.4.14.16}$$

And from (a.4.14.14), the second term obtains directly as:

$$\bar{\Omega} = M\ [D^{*}\Omega D^{*\prime} + (I_{nh} - D^{*}D)(I_h \otimes \Gamma)(I_{nh} - D^{*}D)]\ M' \tag{a.4.14.17}$$

derivations for equation (4.14.45)

We show that $\bar{H}_0^{-1}$ is given by:

$$\bar{H}_0^{-1} = \begin{pmatrix} H_0^{-1} & 0_{n\times k} \\ -\Gamma_{0,2}^{-1}\,\Gamma_{0,1}\,H_0^{-1} & \Gamma_{0,2}^{-1} \end{pmatrix} \tag{a.4.14.18}$$

Indeed:

$$\begin{aligned} & \bar{H}_0\bar{H}_0^{-1} \\ = & \begin{pmatrix} H_0 & 0_{n\times k} \\ \Gamma_{0,1} & \Gamma_{0,2} \end{pmatrix} \begin{pmatrix} H_0^{-1} & 0_{n\times k} \\ -\Gamma_{0,2}^{-1}\Gamma_{0,1}H_0^{-1} & \Gamma_{0,2}^{-1} \end{pmatrix} \\ = & \begin{pmatrix} H_0H_0^{-1} & 0_{n\times k} \\ \Gamma_{0,1}H_0^{-1} - \Gamma_{0,2}\Gamma_{0,2}^{-1}\Gamma_{0,1}H_0^{-1} & \Gamma_{0,2}\Gamma_{0,2}^{-1} \end{pmatrix} \\ = & \begin{pmatrix} I_n & 0_{n\times k} \\ 0_{k\times n} & I_k \end{pmatrix} \\ = & I_{\bar{n}} \end{aligned} \tag{a.4.14.19}$$

derivations for equation (4.14.46)

Start from the stacked SVAR formulation and develop:

$$
\begin{aligned}
& \bar{H}_0 \bar{y}_t = \bar{H}_+ \bar{x}_t + \bar{\xi}_t \\
\Leftrightarrow\ & \bar{H}_0^{-1} \bar{H}_0 \bar{y}_t = \bar{H}_0^{-1} \bar{H}_+ \bar{x}_t + \bar{H}_0^{-1} \bar{\xi}_t \\
\Leftrightarrow\ & \bar{y}_t = \bar{H}_0^{-1} \bar{H}_+ \bar{x}_t + \bar{H}_0^{-1} \bar{\xi}_t \\
\Leftrightarrow\ & \begin{pmatrix} y_t \\ r_t \end{pmatrix} = \bar{H}_0^{-1} \bar{H}_+ \bar{x}_t + \begin{pmatrix} H_0^{-1} & 0_{n \times k} \\ -\Gamma_{0,2}^{-1}\, \Gamma_{0,1}\, H_0^{-1} & \Gamma_{0,2}^{-1} \end{pmatrix} \begin{pmatrix} \xi_t \\ \nu_t \end{pmatrix}
\end{aligned}
\tag{a.4.14.20}
$$

Consider the lower block for m_t:

$$
r_t = \bar{H}_0^{-1} \bar{H}_+ \bar{x}_t - \Gamma_{0,2}^{-1}\, \Gamma_{0,1}\, H_0^{-1} \xi_t + \Gamma_{0,2}^{-1} \nu_t \tag{a.4.14.21}
$$

It then follows that:

$$
\begin{aligned}
& \mathbb{E}(r_t \xi_t') \\
=\ & \bar{H}_0^{-1} \bar{H}_+\, \mathbb{E}(\bar{x}_t \xi_t') - \Gamma_{0,2}^{-1}\, \Gamma_{0,1}\, H_0^{-1}\, \mathbb{E}(\xi_t \xi_t') + \Gamma_{0,2}^{-1}\, \mathbb{E}(\nu_t \xi_t') \\
=\ & -\Gamma_{0,2}^{-1}\, \Gamma_{0,1}\, H_0^{-1}
\end{aligned}
\tag{a.4.14.22}
$$

where we have used a standard exogeneity assumption $Ex(\xi_t | \bar{x}_t) = 0$, and the fact that ν_t and ξ_t are uncorrelated.

Bibliography