1 Fixed and Relative Bouncing Results

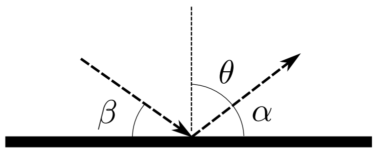

Figure 1: Angle of incidence β, angle of departure α and bouncing angle θ.

Definition 1. **Fixed bouncing** *consists in the robot driving in a straight line until encountering δP, then make the robot rotate until its heading is at an angle θ clockwise of the inward-facing boundary normal.*

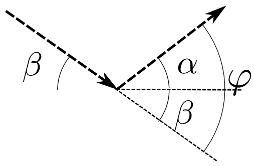

Figure 2: Angle of incidence β, angle of departure α and bouncing angle φ.

Definition 2. **Relative bouncing** *consists in the robot driving in a straight line until encountering δP, then make the robot rotate an angle φ from its current orientation.*

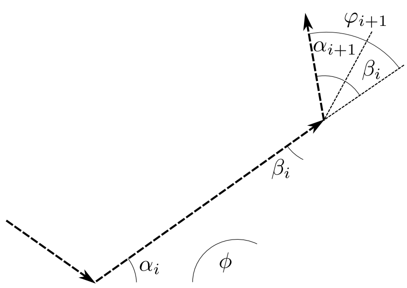

Figure 3: Equivalence between fixed bouncing and relative bouncing.

Proposition 1. *In every regular n-sided polygon with interior angle $\phi = (n-2)\pi/n$, there exist an angle φ that makes relative bouncing equivalent to fixed bouncing. Moreover, $\varphi = \pi - \phi$.*

Proof. We proceed by induction. Consider the angle of departure α_i after the robot bounces from δP for the *i-th* time. In the fixed bouncing scheme, the robot always bounces from δP at a fixed angle θ, which implies α_i to remain equal in each bounce, that is, $\alpha_{i+1} = \alpha_i$. Referring to Figure 3, a departing angle α_i yields a angle of incidence β_i. If we require $\alpha_{i+1} = \alpha_i$, then an equivalent relative bouncing law should subsequently apply a bounce angle $\varphi_{i+1} = \alpha_{i+1} + \beta i = \alpha_i + \beta i$. From the triangle whose interior angles are α_i, βi and ϕ, we get that $\alpha_i + \beta i = \pi - \phi$, hence, $\varphi_{i+1} = \pi - \phi$, which is independent of index i. As the base case, consider that the robot encounters δP for the first time with an angle β_0; if the robot starts a relative bouncing scheme with an angle $\varphi_1 = \pi - \phi$ and keeps bouncing with $\varphi_i = \pi - \phi$ (assuming that this bouncing keeps the robot inside P), then it will produce a fixed bouncing behaviour with a fixed $\theta = \phi + \beta_0 - \pi/2$. The result follows. □

Corollary 1. *The angle $\varphi = \pi - \phi$ that makes relative bouncing equivalent to fixed bouncing is unique for a regular n-sided polygon, and for a given angle of incidence β_0 it yields at most one bouncing angle θ.*

2 Fixed Bouncing Results

2.1 Convergence Rate

Theorem 2. *Assume an adjacent edges fixed bouncing scheme, and assume that the robot starts over δP at distance $x_0 \in (0, l)$ from vertex v_i. When the robot encounters δP for the k^{th} time, then*

$$d({b_\theta^+}^k(x_0), x_{FP}) < c(\theta)^k l. \tag{1}$$

Proof. Consider two initial positions over a polygon edge, which have related distances x_0 and x_{FP} from vertex v_i, respectively. After iterating map $b_\theta^+(x)$ k times from both initial positions, the respective hit points over δP are distance $d({b_\theta^+}^k(x_0), {b_\theta^+}^k(x_{FP}))$ from each other, which is equal to distance $d({b_\theta^+}^k(x_0), x_{FP})$, as x_{FP} is a fixed point of function $b_\theta^+(x)$. Further developing $d({b_\theta^+}^k(x_0), x_{FP})$ yields

$$\begin{aligned}
d({b_\theta^+}^k(x_0), x_{FP}) &= \left| {b_\theta^+}^k(x_0) - x_{FP} \right| \\
&= \left| \sum_{i=1}^{k} (-l)(-c(\theta))^i + (-c(\theta))^k x_0 - x_{FP} \right| \\
&= \left| l + \frac{l(-c(\theta))^{k+1} - l}{1 + c(\theta)} + (-c(\theta))^k x_0 - \frac{lc(\theta)}{1 + c(\theta)} \right| \\
&= \left| (-c(\theta))^k \left[x_0 - \frac{lc(\theta)}{1 + c(\theta)} \right] \right| \\
&= c(\theta)^k d(x_0, x_{FP}).
\end{aligned}$$

Considering in the last expression that x_0 can not be l distance apart from x_{FP}, Inequality (1) is obtained. □

Corollary 2. *If the initial distance x_0 is known, then $c(\theta)^k d(x_0, x_{FP})$ gives the exact bound to $d({b_\theta^+}^k(x_0), x_{FP})$.*

Corollary 3. *Consider a given $\epsilon < l$. Iteration k given by*

$$k = \left\lceil \frac{\log[\epsilon/l]}{\log[c(\theta)]} \right\rceil \tag{2}$$

yields $d(b_\theta^{+^k}(x_0), x_{FP}) \leq \epsilon$.

Proof. Equate the right term of Inequality (1) to ϵ, that is $c(\theta)^k l = \epsilon$, and solve for k. The resulting value of k is a real number; round it up to the nearest integer. $\square$

2.2 Time-varying Error

Theorem 3. *Consider that the robot bounces between adjacent edges with an angle θ perturbed by an $\epsilon(t)$, but that $\epsilon(t)$ is bounded such that $\theta \in [\theta_1, \theta_2]$ for all t. As $k \to \infty$, then*

$$b_{\theta\pm\epsilon(t)}^{+^k}(x) \in [\mathcal{B}_1, \mathcal{B}_2],$$

in which distances $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$\mathcal{B}_1 = \frac{a(\theta_1)c(\theta_1)c(\theta_2)}{1 - c(\theta_1)c(\theta_2)} \tag{3}$$

$$\mathcal{B}_2 = \frac{a(\theta_2)c(\theta_1)c(\theta_2)}{1 - c(\theta_1)c(\theta_2)} \tag{4}$$

with $c(\theta) = \cos(\theta)/\cos(\theta - \phi)$ and $a(\theta) = l/c(\theta) - l$.

2.3 Rectangle

Proposition 4. *In every rectangle with sides length l_1 and l_2, with $l_1 < l_2$, there exists a range for θ such that iterating $B_\theta(x)$ on any $x \in \delta P$ results in a stable limit cycle, which strikes adjacent edges at points that are at distance x_{FP_1} or x_{FP_2} from the nearest clockwise vertex. Distances x_{FP_1} and x_{FP_2} are given by*

$$x_{FP_1} = \begin{cases} \frac{c(\theta)l_2 - c^2(\theta)l_1}{1-c^2(\theta)}, & \gamma < \theta < \pi/2 \\ \frac{l_1 - c(\theta)l_2}{1-c^2(\theta)}, & -\pi/2 < \theta < -\gamma \end{cases} \tag{5}$$

$$x_{FP_2} = \begin{cases} \frac{c(\theta)l_1 - c^2(\theta)l_2}{1-c^2(\theta)}, & \gamma < \theta < \pi/2 \\ \frac{l_2 - c(\theta)l_1}{1-c^2(\theta)}, & -\pi/2 < \theta < -\gamma \end{cases} \tag{6}$$

in which $c(\theta) = cos(\theta)/cos(\theta - \phi)$, $\gamma = \pi/2 - \arctan(l_1/l_2)$.

Sketch of Proof. Constrain the robot to bounce counterclockwise, at an angle $0 < \theta < \pi/2$, and consider the bounce map that takes the robot from an edge length l_1 to the adjacent edge length l_2, and then, in a second bounce, takes the robot from the edge length l_2 to the next edge length l_1. Using law of sines and composing the two consecutive bounces, yields the bounce map

$$b_\theta^+(x) = c(\theta)l_2 - c^2(\theta)l_1 + c^2(\theta)x, \tag{7}$$

in which $c(\theta) = cos(\theta)/cos(\theta - \phi)$. The bounce map shown in Equation (7) is a contraction mapping if $|c(\theta)| < 1$, which by Banach fixed-point theorem, it has a unique fixed point.

Then, iterating the map b_θ^+ k times and taking the limit as $k \to \infty$, we can explicitly find the value of the fixed point, and thus the points on δP touched by the robot in its orbit. This, yields the first expression in Equation (5). The angle $\gamma = \pi/2 - \arctan(l_1/l_2)$ comes from restricting the bounces to hit adjacent edges.

Following the same procedure but staring the bouncing at an edge length l_2, and considering clockwise bounces, we obtain the remaining three expressions for all the remaining cases in Equations (5) and (6). □

Remark 1. *Fixed point x_{FP_1} corresponds to sides of length l_1, and fixed point x_{FP_2} corresponds to sides of length l_2.*

2.4 Convex Polygons

Consider a convex n-sided polygon and order its edges in a counterclockwise direction, with l_i, $i = 0, 1, ..., n-1$, as the length of edge e_i (Figure 4). Refer as ϕ_i to the interior angle formed by edges e_i and $e_{(i+1) \bmod n}$.

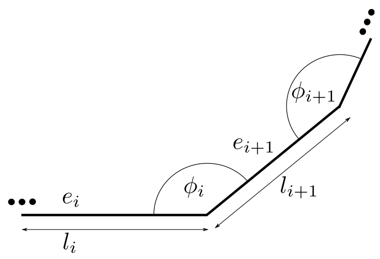

Figure 4: Convex polygon with edges e_i length l_i and interior angles ϕ_i.

Lemma 1. *Assume that the robot bounces at angle θ between adjacent edges e_i and $e_{(i+1) \bmod n}$. Define the constrained bounce map $b_\theta^+ : (0, l_i) \to (0, l_i)$, such that it makes the robot bounce m times from e_i back to e_i, and maps x, the robot's distance from vertex v_i, to $b_\theta^+(x)$ distance again from v_i. This mapping function is given by*

$$b_\theta^+(x) = \sum_{j=0}^{m-1} (-1)^{m-1-j} l_{(i+j) \bmod n} \prod_{k=j}^{m-1} \mathrm{C}_{(i+k) \bmod n} + (-1)^m x \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \tag{8}$$

in which $\mathrm{C}_j = \cos(\theta)/\cos(\theta - \phi_j)$.

Proof. If the robot bounces at fixed angle θ, the map between edges e_i and $e_{(i+1) \bmod n}$ is given by $\mathbf{b}(l_i, \phi_i) = \frac{\cos(\theta)}{\cos(\theta - \phi_i)}(l_i - x)$. Let $\mathrm{C}_i = \cos(\theta - \phi_i)$. Bouncing m times through consecutive edges corresponds to composing $\mathbf{b}(l_i, \phi_i)$ $m-1$ times, which yields

$$\mathbf{b}(l_{i+m-1}, \phi_{i+m-1}) \circ \mathbf{b}(l_{i+m-2}, \phi_{i+m-2}) \circ ... \circ \mathbf{b}(l_i, \phi_i)$$

$$= \mathrm{C}_{i+m-1}(l_{i+m-1} - (\mathrm{C}_{i+m-2}(l_{i+m-2} - ... - \mathrm{C}_i(l_i - x)...)))$$
$$= \mathrm{C}_{i+m-1} l_{i+m-1} - \mathrm{C}_{i+m-1}\mathrm{C}_{i+m-2} l_{i+m-2} + ... \mathrm{C}_{i+m-1} ... \mathrm{C}_{i+1}\mathrm{C}_i(l_i - x_i)$$

Grouping the terms in the last expression yields Equation (8).

□

Remark 2. *Note that $m \in \{1, 2..., n\}$. In the case of regular n-sided polygons, setting $m = 1$ correctly models a bounce map between consecutive edges, as $l_i = l_j$ and $\phi_i = \phi_j$, $\forall i, j \in \{0, 1, .., n-1\}$.*

Lemma 2. *If $\left| \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \right| < 1$, then $b_\theta^+(x)$ is a contraction mapping and has a unique fixed point.*

Proof. In order for $b_\theta^+(x)$ to be a contraction mapping with a unique fixed point, we check that $d(b_\theta^+(x), b_\theta^+(y)) \leq kd(x, y)$ for all $x, y \in (0, l)$ and some nonnegative real number $0 \leq k < 1$.

$$d(b_\theta^+(x), b_\theta^+(y))$$

$$\begin{aligned}
&= \Bigg| \sum_{j=0}^{m-1} (-1)^{m-1-j} l_{(i+j) \bmod n} \prod_{k=j}^{m-1} \mathrm{C}_{(i+k) \bmod n} + (-1)^m x \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \\
&- \sum_{j=0}^{m-1} (-1)^{m-1-j} l_{(i+j) \bmod n} \prod_{k=j}^{m-1} \mathrm{C}_{(i+k) \bmod n} - (-1)^m y \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \Bigg| \\
&= \left| (-1)^m \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} (x - y) \right| \\
&= \left| \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \right| d(x, y).
\end{aligned}$$

Thus if $\left| \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \right| < 1$, then b_θ^+ is a contraction mapping, and by the Banach fixed-point theorem, it has a unique fixed point. □

Theorem 5. *In every convex n-sided polygon, there exists a range for θ such that iterating $b_\theta^+(x)$ results in a stable limit cycle of period n, which strikes e_i at point distance x_{FP} from vertex v_i. Distance x_{FP} is given by*

$$x_{FP} = \frac{\sum_{j=0}^{m-1} (-1)^{m-1-j} l_{(i+j) \bmod n} \prod_{k=j}^{m-1} \mathrm{C}_{(i+k) \bmod n}}{1 - (-1)^m \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n}} \tag{9}$$

in which $\mathrm{C}_j = cos(\theta)/cos(\theta - \phi_j)$, and $\theta \in (\min\{\phi_i/2\}, \pi/2)$.

Proof. The function b_θ^+ given in Lemma 1 describes a robot that bounces at angle θ between adjacent edges e_i and $e_{(i+1) \bmod n}$ in a convex n-gon, beginning its trajectory at a point $p \in P$ which is at a distance x from v_i. If the robot is constrained to hit adjacent walls, then θ must be greater than $\min\{\phi_i/2\}$ and less than $\pi/2$. Setting $\theta \in (\min\{\phi_i/2\}, \pi/2)$ also yields $\left| \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \right| < 1$, hence, by Lemma 2, $b_\theta^+(x)$ has a unique fixed point, which is an orbit of B_θ since the robot keeps contacting e_i at the same distance from v_i every time it returns to it.

To get the fixed point x_{FP} equate Equation (8) to x, and solve for x yielding Equation (9). The result follows. □