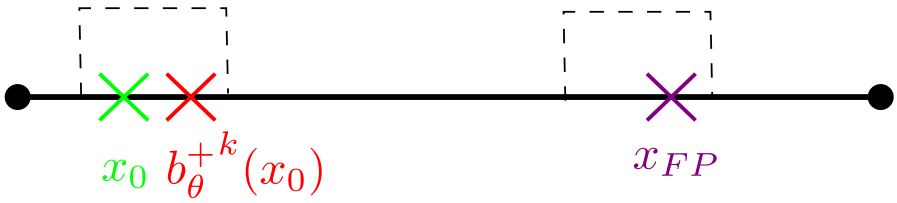
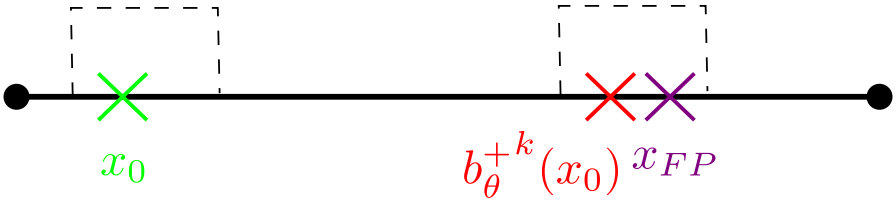
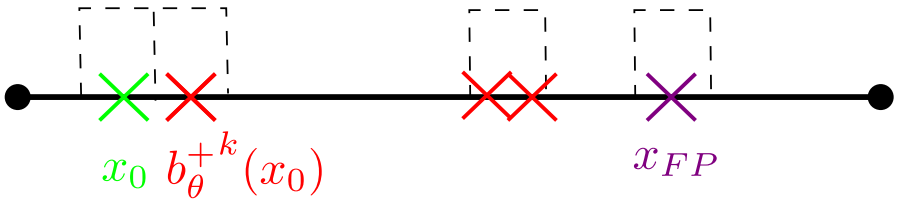


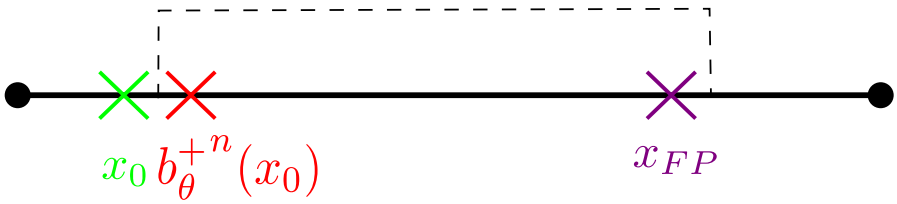
(a) Slow convergence: finds a cycle that is not a periodic orbit.



(b) Fast convergence: finds the real periodic orbit.



(c) Increasing resolution can still lead to false cycle detection as we approach the fixed point.



(d) Decreasing resolution can lead to finding correct cycle, with lower spatial resolution.

Figure 1: Resolution analysis.

We explore the trade-off between spatial resolution and correctness of cycle detection with a discretized approach to fixed-angle bouncing.

In Figure 1 we show an edge length l for a given n-gon (polygon with n sides). We assume that the robot hits that edge for the first time at point x_0 and that there exists a stable limit cycle of period n that hits the edge at point x_{FP}. Let $b_\theta^{+k}(x_0)$ be the hit point on the polygon boundary after iterating k times the bouncing map $b_\theta^{+}(x_0)$, starting from x_0. Finally, we show as dotted squares the cells for a given discretization.

In regular polygons, for certain intervals of θ, only one stable limit cycle exists. When the discretized limit-cycle finding algorithm finds more than one periodic orbit, only one of them can be the true limit cycle. The other orbits appear as an artefact of the current discretization resolution. For certain values of θ, the convergence rate towards x_{FP} is quite small, resulting in successive collision points $b_\theta^{+k}(x_0)$ that slowly move away from x_0. For example, consider Figure 1a and assume that $b_\theta^{+k}(x_0)$ has already returned to the edge containing the start point x_0. For a given resolution and a certain θ, x_0 and $b_\theta^{+k}(x_0)$ can fall in the same cell z, which would result into an apparent periodic orbit, but z is indeed a transient cell. Conversely, in Figure 1b, the algorithm has iterated enough (or convergence is fast enough) such that $b_\theta^{+k}(x_0)$

and x_{FP} are in the same cell z, so z is truly a persistent cell (all future bounces on this edge will be in z).

To avoid the issue present in Figure 1a, the discretization resolution could be increased such that x_0 and $b_\theta^{+k}(x_0)$ fall into different cells z and w, just as seen in Figure 1c, therefore, z and w would be classified as transient cells. Nonetheless, with this finer discretization it is still possible that as $b_\theta^{+k}(x_0)$ keeps iterating, two consecutive collision points $b_\theta^{+jn}(x_0)$ and $b_\theta^{+(j+1)n}(x_0)$ will fall in the same cell, and we still detect a spurious periodic cycle. The resolution could again be increased but this also has the inconvenience that as the cells become smaller, the algorithm will need more iterations in order to find the real stable limit cycle. And there is no clear way to choose a discretization fine enough to guarantee no false cycles..

Based on the last analysis it is desirable to have large enough cells to avoid over iterating the algorithm, but we want to have the smallest cells possible (to have good spatial resolution of the location of the limit cycle in the workspace).

Inequality (1) can be used to compute an actual bound for the distance between $b_\theta^{+k}(x_0)$ and x_{FP} as a function of k. We set $k = n$, as the minimum k that returns the robot to the edge where x_0 is located, opening the possibility of closing a cycle. Thus, the quantity $c(\theta)^n l$ is an upper bound for the distance between $b_\theta^{+n}(x_0)$ (the first collision point where the robot returns to the edge it started on) and x_{FP}. If chosen as the discretization length, we are guaranteed to detect only one cycle (the correct one).

For an example, suppose that in Figure 1b, the point $b_\theta^{+k}(x_0)$ has $k = n$. If that is the case, it can be seen that the shown cell size is enough to contain distance $d(b_\theta^{+n}(x_0), x_{FP})$, so the algorithm would not identify spurious periodic orbits and would rapidly find the persistent cell that contains x_{FP}. On the other hand, if the distance between $b_\theta^{+n}(x_0)$ and x_{FP} is larger, it would require bigger cells just as shown in Figure 1d. In either case, selecting the cell size according to distance $d(b_\theta^{+n}(x_0), x_{FP})$ (actually according to its bound $c(\theta)^n l$), makes the algorithm identify a single limit cycle with fewer iterations.

If we want better spatial resolution, we can ignore any limit cycles detected in the first jn collisions, and choose a discretization of $c(\theta)^{jn} d(x_0, x_{FP})$. This lets us tune the spatial resolution to be finer without detecting false cycles.

Caveats: This is only for regular polygons, and assumes that the discretization is rectilinear to the polygon boundary (which may only be true for one side). Uneven overlaps between the polygon boundary and the discretization may still cause false cycles to be found.

Theorem 1 *Assume an adjacent edges fixed bouncing scheme, and assume that the robot starts over δP at distance $x_0 \in (0, l)$ from vertex v_i. When the robot encounters δP for the k^{th} time, then*

$$d(b_\theta^{+k}(x_0), x_{FP}) < c(\theta)^k l. \tag{1}$$

Corollary 1 *If the initial distance x_0 is known, then $c(\theta)^k d(x_0, x_{FP})$ gives the exact bound to $d(b_\theta^{+k}(x_0), x_{FP})$.*