1 Fixed and Relative Bouncing Results

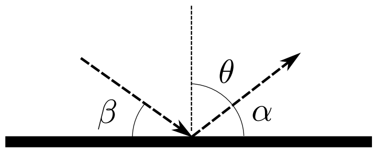

Figure 1: Angle of incidence β, angle of departure α and bouncing angle θ.

Definition 1. Fixed bouncing *consists in the robot driving in a straight line until encountering δP, then make the robot rotate until its heading is at an angle θ clockwise of the inward-facing boundary normal.*

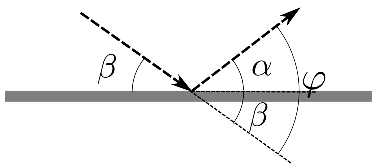

Figure 2: Angle of incidence β, angle of departure α and bouncing angle φ.

Definition 2. Relative bouncing *consists in the robot driving in a straight line until encountering δP, then make the robot rotate an angle φ from its current orientation.*

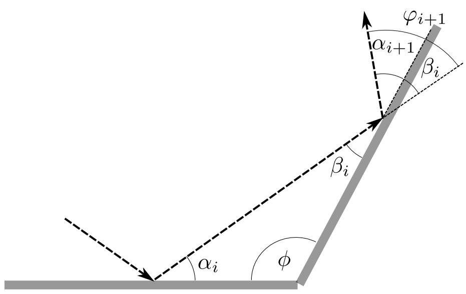

Figure 3: Equivalence between fixed bouncing and relative bouncing.

Proposition 1. *In every regular n-sided polygon with interior angle $\phi = (n-2)\pi/n$, there exist an angle φ that makes relative bouncing equivalent to fixed bouncing. Moreover, $\varphi = \pi - \phi$.*

Proof. We proceed by induction. Consider the angle of departure α_i after the robot bounces from δP for the *i-th* time. In the fixed bouncing scheme, the robot always bounces from δP at a fixed angle θ, which implies α_i to remain equal in each bounce, that is, $\alpha_{i+1} = \alpha_i$. Referring to Figure **??**, a departing angle α_i yields a angle of incidence β_i. If we require $\alpha_{i+1} = \alpha_i$, then an equivalent relative bouncing law should subsequently apply a bounce angle $\varphi_{i+1} = \alpha_{i+1} + \beta i = \alpha_i + \beta i$. From the triangle whose interior angles are α_i, βi and ϕ, we get that $\alpha_i + \beta i = \pi - \phi$, hence, $\varphi_{i+1} = \pi - \phi$, which is independent of index i. As the base case, consider that the robot encounters δP for the first time with an angle β_0; if the robot starts a relative bouncing scheme with an angle $\varphi_1 = \pi - \phi$ and keeps bouncing with $\varphi_i = \pi - \phi$ (assuming that this bouncing keeps the robot inside P), then it will produce a fixed bouncing behaviour with a fixed $\theta = \phi + \beta_0 - \pi/2$. The result follows. □

Corollary 1. *The angle $\varphi = \pi - \phi$ that makes relative bouncing equivalent to fixed bouncing is unique for a regular n-sided polygon, and for a given angle of incidence β_0 it yields at most one bouncing angle θ.*

2 Fixed Bouncing Results

2.1 Convergence Rate

Theorem 2. *Assume an adjacent edges fixed bouncing scheme, and assume that the robot starts over δP at distance $x_0 \in (0, l)$ from vertex v_i. When the robot encounters δP for the k^{th} time, then*

$$d({b_\theta^+}^k(x_0), x_{FP}) < c(\theta)^k l. \tag{1}$$

Proof. Consider two initial positions over a polygon edge, which have related distances x_0 and x_{FP} from vertex v_i, respectively. After iterating map $b_\theta^+(x)$ k times from both initial positions, the respective hit points over δP are distance $d({b_\theta^+}^k(x_0), {b_\theta^+}^k(x_{FP}))$ from each other, which is equal to distance $d({b_\theta^+}^k(x_0), x_{FP})$, as x_{FP} is a fixed point of function $b_\theta^+(x)$. Further developing $d({b_\theta^+}^k(x_0), x_{FP})$ yields

$$\begin{aligned}
d({b_\theta^+}^k(x_0), x_{FP}) &= \left|{b_\theta^+}^k(x_0) - x_{FP}\right| \\
&= \left|\sum_{i=1}^{k}(-l)(-c(\theta))^i + (-c(\theta))^k x_0 - x_{FP}\right| \\
&= \left|l + \frac{l(-c(\theta))^{k+1} - l}{1 + c(\theta)} + (-c(\theta))^k x_0 - \frac{lc(\theta)}{1 + c(\theta)}\right| \\
&= \left|(-c(\theta))^k \left[x_0 - \frac{lc(\theta)}{1 + c(\theta)}\right]\right| \\
&= c(\theta)^k d(x_0, x_{FP}).
\end{aligned}$$

Considering in the last expression that x_0 can not be l distance apart from x_{FP}, Inequality (**??**) is obtained. □

Corollary 2. *If the initial distance x_0 is known, then $c(\theta)^k d(x_0, x_{FP})$ gives the exact bound to $d({b_\theta^+}^k(x_0), x_{FP})$.*

Corollary 3. *Consider a given $\epsilon < l$. Iteration k given by*

$$k = \left\lceil \frac{\log\left[\epsilon/l\right]}{\log\left[c(\theta)\right]} \right\rceil \tag{2}$$

yields $d({b_\theta^+}^k(x_0), x_{FP}) \leq \epsilon$.

Proof. Equate the right term of Inequality (**??**) to ϵ, that is $c(\theta)^k l = \epsilon$, and solve for k. The resulting value of k is a real number; round it up to the nearest integer. □

2.2 Time-varying Error

Theorem 3. *Consider that the robot bounces between adjacent edges with an angle θ perturbed by an $\epsilon(t)$, but that $\epsilon(t)$ is bounded such that $\theta \in [\theta_1, \theta_2]$ for all t. As $k \to \infty$, then*

$$ {b^+_{\theta \pm \epsilon(t)}}^k(x) \in [\mathcal{B}_1, \mathcal{B}_2], $$

in which distances $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$ \mathcal{B}_1 = \frac{a(\theta_1)c(\theta_1)c(\theta_2)}{1 - c(\theta_1)c(\theta_2)} \tag{3} $$

$$ \mathcal{B}_2 = \frac{a(\theta_2)c(\theta_1)c(\theta_2)}{1 - c(\theta_1)c(\theta_2)} \tag{4} $$

with $c(\theta) = \cos(\theta)/\cos(\theta - \phi)$ *and* $a(\theta) = l/c(\theta) - l$.

2.3 Rectangle

Proposition 4. *In every rectangle with sides length l_1 and l_2, with $l_1 < l_2$, there exists a range for θ such that iterating $B_\theta(x)$ on any $x \in \delta P$ results in a stable limit cycle, which strikes adjacent edges at points that are at distance x_{FP_1} or x_{FP_2} from the nearest clockwise vertex. Distances x_{FP_1} and x_{FP_2} are given by*

$$ x_{FP_1} = \begin{cases} \frac{c(\theta)l_2 - c^2(\theta)l_1}{1-c^2(\theta)}, & \gamma < \theta < \pi/2 \\ \frac{l_1 - c(\theta)l_2}{1-c^2(\theta)}, & -\pi/2 < \theta < -\gamma \end{cases} \tag{5} $$

$$ x_{FP_2} = \begin{cases} \frac{c(\theta)l_1 - c^2(\theta)l_2}{1-c^2(\theta)}, & \gamma < \theta < \pi/2 \\ \frac{l_2 - c(\theta)l_1}{1-c^2(\theta)}, & -\pi/2 < \theta < -\gamma \end{cases} \tag{6} $$

in which $c(\theta) = cos(\theta)/cos(\theta - \phi)$, $\gamma = \pi/2 - \arctan(l_1/l_2)$.

Sketch of Proof. Constrain the robot to bounce counterclockwise, at an angle $0 < \theta < \pi/2$, and consider the bounce map that takes the robot from an edge length l_1 to the adjacent edge length l_2, and then, in a second bounce, takes the robot from the edge length l_2 to the next edge length l_1. Using law of sines and composing the two consecutive bounces, yields the bounce map

$$ b^+_\theta(x) = c(\theta)l_2 - c^2(\theta)l_1 + c^2(\theta)x, \tag{7} $$

in which $c(\theta) = cos(\theta)/cos(\theta - \phi)$. The bounce map shown in Equation (**??**) is a contraction mapping if $|c(\theta)| < 1$, which by Banach fixed-point theorem, it has a unique fixed point.

Then, iterating the map b^+_θ k times and taking the limit as $k \to \infty$, we can explicitly find the value of the fixed point, and thus the points on δP touched by the robot in its orbit. This, yields the first expression in Equation (**??**). The angle $\gamma = \pi/2 - \arctan(l_1/l_2)$ comes from restricting the bounces to hit adjacent edges.

Following the same procedure but staring the bouncing at an edge length l_2, and considering clockwise bounces, we obtain the remaining three expressions for all the remaining cases in Equations (**??**) and (**??**). □

Remark 1. *Fixed point x_{FP_1} corresponds to sides of length l_1, and fixed point x_{FP_2} corresponds to sides of length l_2.*

2.4 Convex Polygons

Consider a convex n-sided polygon and order its edges in a counterclockwise direction, with l_i, $i = 0, 1, ..., n-1$, as the length of edge e_i (Figure **??**). Refer as ϕ_i to the interior angle formed by edges e_i and $e_{(i+1) \bmod n}$.

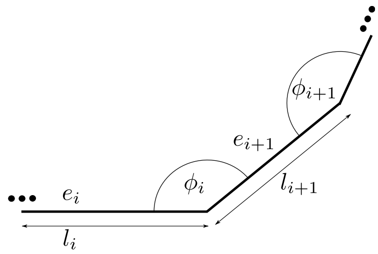

Figure 4: Convex polygon with edges e_i length l_i and interior angles ϕ_i.

Lemma 1. *Assume that the robot bounces at angle θ between adjacent edges e_i and $e_{(i+1) \bmod n}$. Define the constrained bounce map $b^+_\theta : (0, l_i) \to (0, l_i)$, such that it makes the robot bounce m times from e_i back to e_i, and maps x, the robot's distance from vertex v_i, to $b^+_\theta(x)$ distance again from v_i. This mapping function is given by*

$$b^+_\theta(x) = \sum_{j=0}^{m-1} (-1)^{m-1-j} l_{(i+j) \bmod n} \prod_{k=j}^{m-1} \mathrm{C}_{(i+k) \bmod n} + (-1)^m x \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \tag{8}$$

in which $\mathrm{C}_j = \cos(\theta)/\cos(\theta - \phi_j)$.

Proof. If the robot bounces at fixed angle θ, the map between edges e_i and $e_{(i+1) \bmod n}$ is given by $\mathbf{b}(l_i, \phi_i) = \frac{\cos(\theta)}{\cos(\theta - \phi_i)}(l_i - x)$. Let $\mathrm{C}_i = \cos(\theta - \phi_i)$. Bouncing m times through consecutive edges corresponds to composing $\mathbf{b}(l_i, \phi_i)$ $m-1$ times, which yields

$$\mathbf{b}(l_{i+m-1}, \phi_{i+m-1}) \circ \mathbf{b}(l_{i+m-2}, \phi_{i+m-2}) \circ ... \circ \mathbf{b}(l_i, \phi_i)$$

$$= \mathrm{C}_{i+m-1}(l_{i+m-1} - (\mathrm{C}_{i+m-2}(l_{i+m-2} - ... - \mathrm{C}_i(l_i - x)...)))$$
$$= \mathrm{C}_{i+m-1} l_{i+m-1} - \mathrm{C}_{i+m-1}\mathrm{C}_{i+m-2} l_{i+m-2} + ...\mathrm{C}_{i+m-1}...\mathrm{C}_{i+1}\mathrm{C}_i(l_i - x_i)$$

Grouping the terms in the last expression yields Equation (**??**).

□

Remark 2. *Note that $m \in \{1, 2..., n\}$. In the case of regular n-sided polygons, setting $m = 1$ correctly models a bounce map between consecutive edges, as $l_i = l_j$ and $\phi_i = \phi_j$, $\forall i, j \in \{0, 1, .., n-1\}$.*

Lemma 2. *If $\left| \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \right| < 1$, then $b_\theta^+(x)$ is a contraction mapping and has a unique fixed point.*

Proof. In order for $b_\theta^+(x)$ to be a contraction mapping with a unique fixed point, we check that $d(b_\theta^+(x), b_\theta^+(y)) \leq kd(x, y)$ for all $x, y \in (0, l)$ and some nonnegative real number $0 \leq k < 1$.

$$d(b_\theta^+(x), b_\theta^+(y))$$

$$\begin{aligned}
&= \Bigg| \sum_{j=0}^{m-1} (-1)^{m-1-j} l_{(i+j) \bmod n} \prod_{k=j}^{m-1} \mathrm{C}_{(i+k) \bmod n} + (-1)^m x \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \\
&- \sum_{j=0}^{m-1} (-1)^{m-1-j} l_{(i+j) \bmod n} \prod_{k=j}^{m-1} \mathrm{C}_{(i+k) \bmod n} - (-1)^m y \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \Bigg| \\
&= \left| (-1)^m \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} (x - y) \right| \\
&= \left| \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \right| d(x, y).
\end{aligned}$$

Thus if $\left| \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \right| < 1$, then b_θ^+ is a contraction mapping, and by the Banach fixed-point theorem, it has a unique fixed point. □

Theorem 5. *In every convex n-sided polygon, there exists a range for θ such that iterating $b_\theta^+(x)$ results in a stable limit cycle of period n, which strikes e_i at point distance x_{FP} from vertex v_i. Distance x_{FP} is given by*

$$x_{FP} = \frac{\sum_{j=0}^{m-1} (-1)^{m-1-j} l_{(i+j) \bmod n} \prod_{k=j}^{m-1} \mathrm{C}_{(i+k) \bmod n}}{1 - (-1)^m \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n}} \tag{9}$$

in which $\mathrm{C}_j = cos(\theta)/cos(\theta - \phi_j)$, and $\theta \in (\min\{\phi_i/2\}, \pi/2)$.

Proof. The function b_θ^+ given in Lemma **??** describes a robot that bounces at angle θ between adjacent edges e_i and $e_{(i+1) \bmod n}$ in a convex n-gon, beginning its trajectory at a point $p \in P$ which is at a distance x from v_i. If the robot is constrained to hit adjacent walls, then θ must be greater than $\min\{\phi_i/2\}$ and less than $\pi/2$. Setting $\theta \in (\min\{\phi_i/2\}, \pi/2)$ also yields $\left| \prod_{k=0}^{m-1} \mathrm{C}_{(i+k) \bmod n} \right| < 1$, hence, by Lemma **??**, $b_\theta^+(x)$ has a unique fixed point, which is an orbit of B_θ since the robot keeps contacting e_i at the same distance from v_i every time it returns to it.

To get the fixed point x_{FP} equate Equation (**??**) to x, and solve for x yielding Equation (**??**). The result follows. □