9 Appendix

A Proof of Proposition 1

Proposition 1. *The bounce visibility graph for a simple polygon with n vertices has $O(n^2)$ vertices and $O(n^4)$ edges.*

Proof. Consider a polygon P with n vertices, r of which are reflex vertices. To construct the bounce visibility graph, we insert the vertices of the partial local sequence for each vertex in P. For a convex vertex, its partial local sequence will not add any new vertices to P. However, a reflex vertex can add $O(n)$ new vertices.

Up to half of the vertices in the polygon can be reflex, so the complexity of r is $O(n)$. Therefore, the number of vertices in P', returned by Algorithm 1 is $O(n^2)$. Each vertex indexes an edge in P', and thus a node in the edge visibility graph of P'. At most, a given vertex in P' may see all other vertices, so in the worst case, the bounce visibility graph will have $O(n^4)$ edges. See Figure 3 for an example of such a polygon. □

A.1 Worst Case Example for Algorithm 1

We might hope that if r is large, then not all of the reflex vertices will produce a large number of new vertices, and we may bound the size of the edge set in the visibility graph. Unfortunately, the number of reflex vertices, the new vertices produced in their partial local sequence, and the new vertices' visibility can be large at the same time. We will present a family of input polygons with bounce visibility graph edge-set size of $O(n^4)$.

Let $n = 4t+2$, in which t is a positive integer. We design a polygon with $r = 2t$ reflex vertices. The polygon is symmetric with respect to its medium horizontal line. In the top half, the reflex vertices are uniformly located on a circle and thus they are visible to each other; the convex vertices are chosen so that they are outside the circle and the line through an edge will not intersect other edges. Each reflex vertex will have at least $t-1$ new vertices in its partial local sequence. There will be $2t(t-1)+n$ vertices in the polygon after we insert all new vertices in the partial local sequence for all reflex vertices. Each of them can see at least $t(t-1) + n/2$ other vertices. Thus the number of edges in the transition graph for the polygon with inserted vertices is $O((2t(t-1) + n)(t(t-1) + n/2)) = O(t^4) = O(n^4)$. Fig 3 shows the polygon for $t = 4$ with all the vertices in the partial local sequences.

B Proof of Proposition 2

Proposition 2. *Given two entirely visible line segments $e_i = (v_i, v_{i+1})$ and $e_j = (v_j, v_{j+1})$, if a safe action exists from e_i to e_j, the maximum range of safe actions is $\boldsymbol{\tilde{\theta}_{max}} = [\boldsymbol{\theta_r}, \boldsymbol{\theta_l}]$ such that $\theta_r = \pi - \angle v_j v_{i+1} v_i$ and $\theta_l = \angle v_{j+1} v_i v_{i+1}$.*

Proof. Let edge $e_i = (v_i, v_{i+1})$ be aligned with the positive x axis with the clockwise endpoint at the origin, without loss of generality. Due to the edges being entirely visible, $e_j = (v_j, v_{j+1})$ must be in the top half of the plane, above e_i.

Take the quadrilateral formed by the convex hull of the edge endpoints. Let the edges between e_i and e_j be $e_l = (v_i, v_{j+1})$ and the right-hand edge $e_r = (v_{i+1}, v_j)$. Let θ_l be the angle between e_l and the positive x axis ($0 < \theta_l < \pi$); similarly for e_r and θ_r. See Figure 10 for an illustration of the setup.

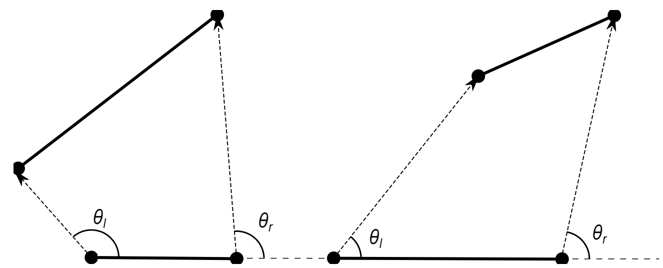

Fig. 10: Angle range such that a transition exists for all points on originating edge (left: such a range exists, right: such a range does not exist)

There are three cases to consider: if e_l and e_r are extended to infinity, they cross either above or below edge e_i, or they are parallel.

Case 1: e_l and e_r meet below edge e_i. In this case, $\theta_l > \theta_r$ and if a ray is cast from any point on e_i at angle $\theta \in [\theta_r, \theta_l]$, the ray is guaranteed to intersect e_j in its interior.

Case 2: e_l and e_r meet above edge e_i. In this case, $\theta_l < \theta_r$, and there is no angle θ such that a ray shot from *any* point on e_i will intersect e_j. To see this, imagine sliding a ray at angle θ_l across the quadrilateral - at some point before reaching v_{i+1}, the ray must stop intersecting e_j, else we would have $\theta_l > \theta_r$.

Case 3: e_l and e_r are parallel. This implies that $\theta_l = \theta_r$, which is the only angle for which a transition from any point on e_i is guaranteed to intersect e_j, and $\tilde{\theta}_{max}$ is a singleton set.

Thus, for each case, we can either compute the maximum angle range or determine that no such angle range exists. □

C Proof of Lemma 1

Lemma 1. *If two segments are entirely visible to each other, there will be at least one safe action between them.*

Proof. From the proof of Proposition 2, we can see that if case one holds in one direction, case two will hold in the other direction, so a safe action must exist from one edge to the other in one direction. If case three holds, there is a safe action both directions but $\tilde{\theta}_{max}$ is a singleton set.

D Proof of Lemma 2

Lemma 2. *If the transition from segment e_i to segment e_j is a left transition, then the transition function $f(x, \theta)$ between segments e_i and e_j is a contraction mapping if and only if $\theta > \frac{\pi}{2} + \frac{\phi_{i,j}}{2}$; otherwise, the transition function $f(x, \theta)$ is a contraction mapping if and only if $\theta < \frac{\pi}{2} - \frac{\phi_{i,j}}{2}$.*

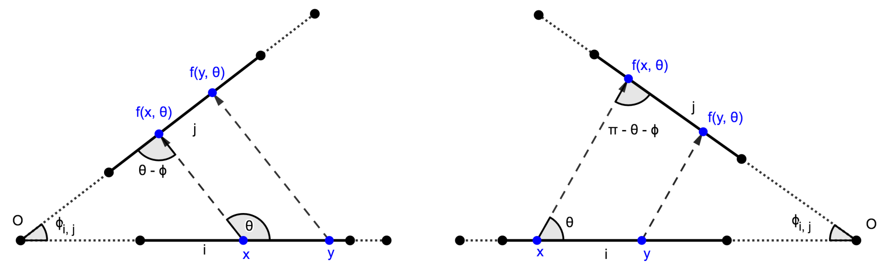

Fig. 11: The two cases for computing contraction mapping conditions.

Proof. We consider the two cases of transition separately:

1. For the transition shown in the left hand side of Figure 11, $\overline{xf(x,\theta)} \parallel \overline{yf(y,\theta)} \Rightarrow \frac{|f(x,\theta)-f(y,\theta)|}{|x-y|} = \frac{|f(x,\theta)|}{|x|} = \frac{\sin(\pi-\theta)}{\sin(\theta-\phi_{i,j})} = \frac{\sin(\theta)}{\sin(\theta-\phi_{i,j})}$. The transition will be contraction if and only if $\frac{|f(x,\theta)-f(y,\theta)|}{|x-y|} < 1 \iff \sin(\theta) < \sin(\theta - \phi_{i,j})$. If $\theta < \frac{\pi}{2}$, then $\sin(\theta) > \sin(\theta - \phi_{i,j})$. Thus we need $\theta > \frac{\pi}{2}$. If $\theta - \phi_{i,j} > \frac{\pi}{2}$, then $\sin(\theta) < \sin(\theta - \phi_{i,j})$ and we are done; otherwise we need $\pi - \theta < \theta - \phi_{i,j} \Rightarrow \theta - \frac{\phi_{i,j}}{2} > \frac{\pi}{2}$. Combining all conditions, we have the transition will be contraction if and only if $\theta > \frac{\pi}{2} + \frac{\phi_{i,j}}{2}$.
2. Similarly, for a right transiton shown in the right diagram of figure 11, $\overline{xf(x,\theta)} \parallel \overline{yf(y,\theta)} \Rightarrow \frac{|f(x,\theta)-f(y,\theta)|}{|x-y|} = \frac{|f(x,\theta)|}{|x|} = \frac{\sin(\pi-\theta)}{\sin(\pi-\theta-\phi_{i,j})} = \frac{\sin(\theta)}{\sin(\theta+\phi_{i,j})}$. The transition will be contraction if and only if $\frac{|f(x,\theta)-f(y,\theta)|}{|x-y|} < 1 \iff \sin(\theta) < \sin(\theta + \phi_{i,j})$. If $\theta > \frac{\pi}{2}$, then $\sin(\theta) > \sin(\theta + \phi_{i,j})$. Thus we need $\theta < \frac{\pi}{2}$. If $\theta + \phi_{i,j} < \frac{\pi}{2}$, then $\sin(\theta) < \sin(\theta + \phi_{i,j})$ and we are done; otherwise we need $\theta < \pi - \theta - \phi_{i,j} \Rightarrow \theta < \frac{\pi}{2} - \frac{\phi_{i,j}}{2}$. Combining all conditions, we have the transition will be contraction if and only if $\theta < \frac{\pi}{2} - \frac{\phi_{i,j}}{2}$.

□

E Supplementary Figure for Theorem 1

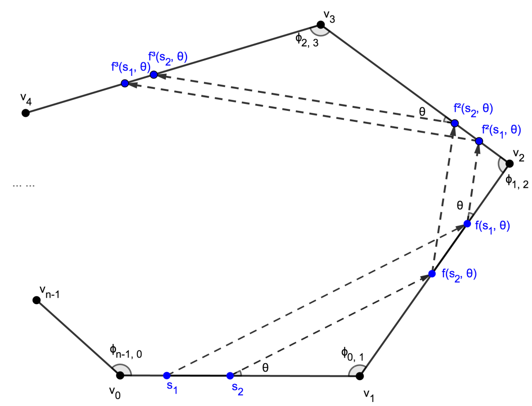

Fig. 12: The notation setup for the proof of contracting cycle in a convex polygon.