1 Contraction Ratio with Uncertainty in Regular Polygons

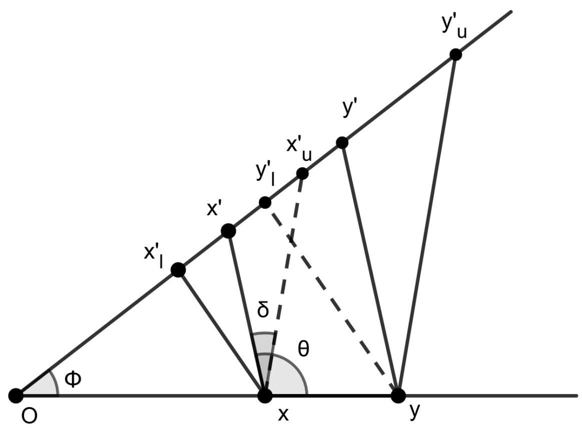

Figure 1: Bounce with uncertainty

Suppose our environment is a regular polygon with edge length 1. To compute fix angle bouncing for an interval on the boundary, we can consider the individual bounce of the end points of the interval and the length of the range at each bounce will be the difference between the end points positions. When we add a fix δ uncertainty to such bounce, i.e., instead of bouncing at θ, the robot chooses an arbitrary angle in $[\theta - \delta, \theta + \delta]$, the worse case results is when the end point on the right bounces at $\theta - \delta$ and the end point on the left bounces at $\theta + \delta$, as shown by y'_u and x'_l in figure **??**. In a sequence of bounce for a fix interval, the end points of the interval alternates between being

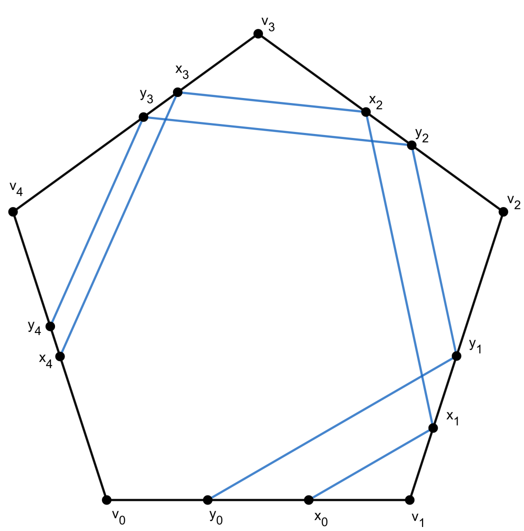

Figure 2: The end points of an interval alternate between being the right end point and the left end point.

the right end point and the left end point, as shown by the sequence of x_i and y_i in figure 2.

In a regular polygon with n sides, assume $\theta + \delta < \frac{2\pi}{2n}$. When the bounce angle is $\theta + \delta$, we have $x_t/(1-x_{t-1}) = \frac{\sin(\theta+\delta)}{\sin(\phi-\theta-\delta)} =: r_1$; similarly, when the bounce angle is $\theta - \delta$, we have $x_t/(1-x_{t-1}) = \frac{\sin(\theta-\delta)}{\sin(\phi-\theta+\delta)} =: r_2$. So $x_t = r_1(1-x_{t-1})$ or $r_2(1-x_{t-1})$ depending on whether x_{t-1} is a left end point or right end point. Let x^+ denotes the sequence of positions that started from

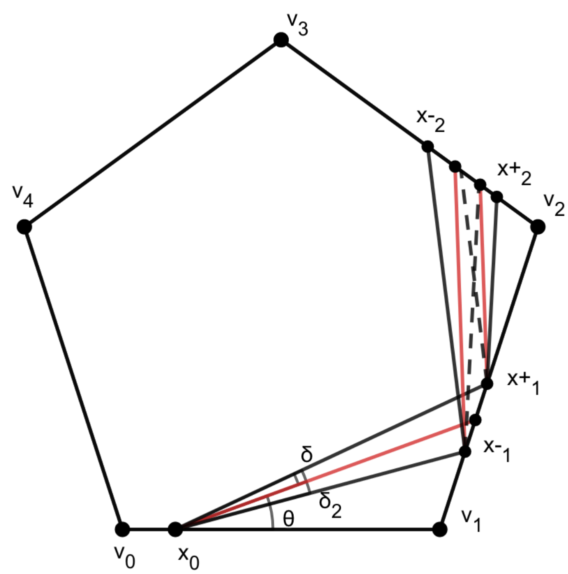

Figure 3: The x^+ sequence starts with $\theta+\delta$ bounce and the x^- sequence starts with $\theta-\delta$ bounce.

x_0 with a $\theta+\delta$ bounce, and x^- denotes the sequence of positions that started from x_0 with a $\theta-\delta$ bounce, as shown in figure 3.

We can show by induction (or unrolling) that

$$x_t^+ = \frac{r_1(1-r_2)(1-(r_1r_2)^n)}{1-r_1r_2} + (-1)^t r_1^{\lfloor t/2 \rfloor} r_2^{\lceil t/2 \rceil} x_0$$

$$x_t^- = \frac{r_2(1-r_1)(1-(r_1r_2)^n)}{1-r_1r_2} + (-1)^t r_1^{\lceil t/2 \rceil} r_2^{\lfloor t/2 \rfloor} x_0$$

If $r_1r_2 < 1$, then $x_\infty^+ = \frac{r_1(1-r_2)}{1-r_1r_2}, x_\infty^- = \frac{r_2(1-r_1)}{1-r_1r_2}, |x_\infty^+ - x_\infty^-| = |\frac{r_1-r_2}{1-r_1r_2}|$.

Remark: since the term containing the start position x_0 vanishes, the results for $t=\infty$ does not depends on the initial condition. Even if the start position for the x^+, x^- sequence are different, we will still have the same convergence.

For an $n-$gon,