Interesting Trajectories of Mobile Robots in Polygons

Alli Nilles
Israel Becerra
Steve LaValle

May 18, 2017

Blind, Bouncing Robots[1]

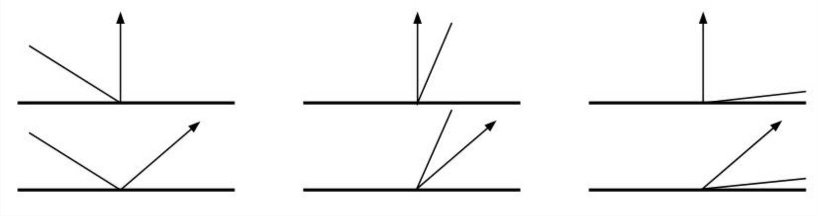
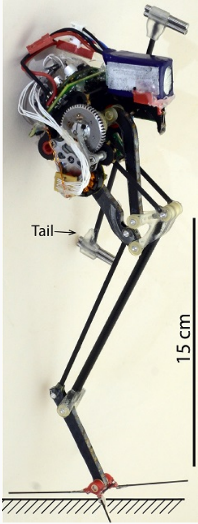


Figure 1: A point robot moving in the plane. The top row shows "bounces" at zero degrees from the normal. The second row shows bounces at 50 degrees clockwise from normal.

[1] (Erickson and LaValle 2013)

Bouncing Robots



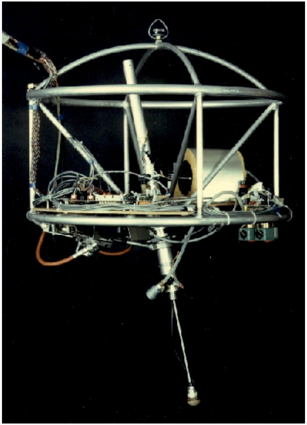


Figure 2: No, not that kind of bouncing robot...

Research Questions

- What kind of tasks are robots with extremely simple control laws capable of performing?
- Will the robot become "trapped" in a certain part of the environment? Or a certain motion pattern?

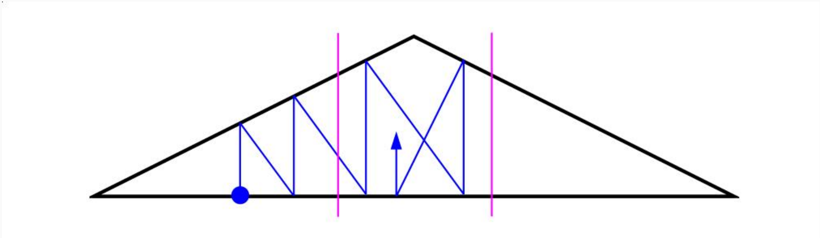


Figure 3: In this environment, bouncing at the normal, the robot will become trapped in the area between the purple lines.

Ok, but why?

- platform invariant: just need two primitives
  - move straight
  - align wrt wall normal
- very simple control schemes
- predictable, reliable motion control in structured spaces
  - warehouses
  - monitor environmental conditions

Ok, but is it physically realizable?

- can implement[2] on a roomba with bump and IR prox detector
- "rotate to parallel"
- maximum error in angle of $\pm 10^\circ$

[2](Lewis and O'Kane 2013)

Related Work

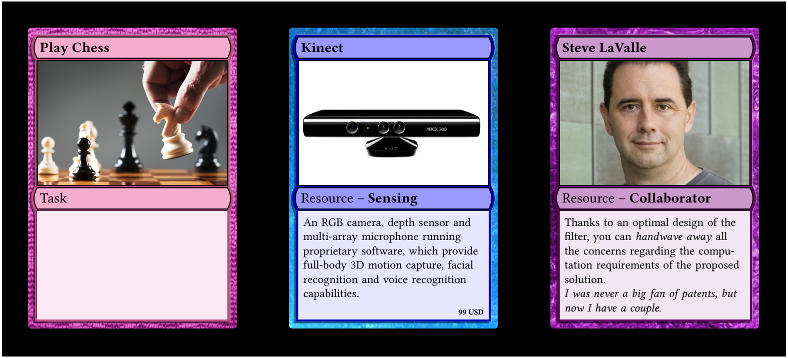
- dynamical billiards: specular bouncing[3]
- pinball billiards[4]
- aspecular billiards, microorganism billiards[5]
- minimal sensing, actuation, computation requirements for point robots in polygons: mapping, navigating, localizing, patrolling, pursuit evasion[6]

[3] (Tabachnikov 2005)
[4] (Markarian, Pujals, and Sambarino 2010)
[5] (Spagnolie et al. 2017)
[6] (Tovar, Guilamo, and LaValle 2005) (Bilò et al. 2012) (J. M. O'Kane and LaValle 2007) (Disser 2011)

Can Imagine Variations on this Model

- **sensors:** gap visibility, detect corners, detect wall normals, pebbles, linear and angular odometers, + noise
- **actuators:** move straight forward, bounce specularly, wall following, rotate in place, + noise
- can we explore this design space in a more systematic and automated way?
    - what is the role of formal methods?

RSS Workshop: *Minimality and Trade-offs in Automated Robot Design*



- July 16
- http://minimality.mit.edu/
- Your chance to become immortalized in a card game!

Limit Cycles In Regular Polygons

Question

If we can only move in straight lines and align relative to wall normal:

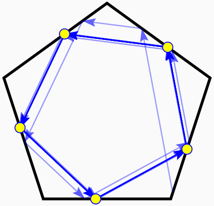
> *Can we guarantee that a robot **patrols** a space on a periodic path?*

Can also phrase as:

> *What are the conditions on limit cycles in this dynamical system?*

Discovery Through Simulation

- Haskell with *Diagrams* library (Yorgey 2012)
- fixed-angle bouncing, specular bouncing, add noise
- render diagrams from simulations automatically[7]



[7] https://github.com/alexandroid000/bounce

In Regular Polygons

Defines a **discrete dynamical system**

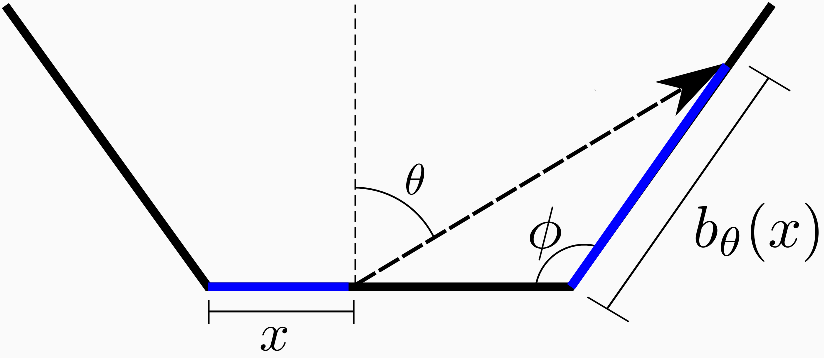
$$x_{n+1} = f(x_n)$$

Given regular polygon, with edge length l and internal angle ϕ, we can define the mapping function

$$b_\theta : (0, l) \to (0, l)$$

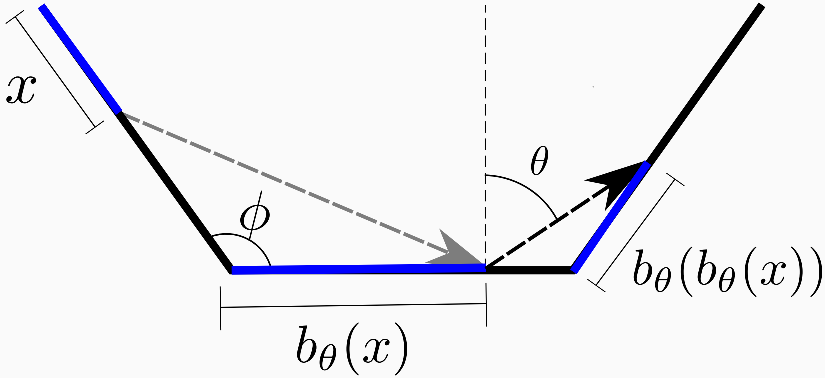
Sequential-Edge Bouncing

$$b_\theta : (0, l) \to (0, l)$$



Sequential-Edge Bouncing

$$b_\theta : (0, l) \to (0, l)$$

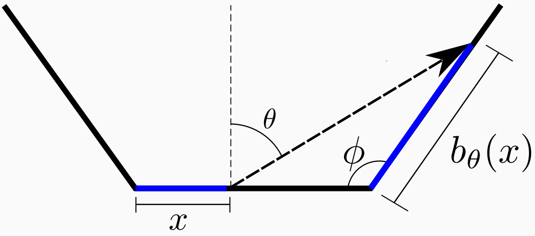


Do the geometry and...

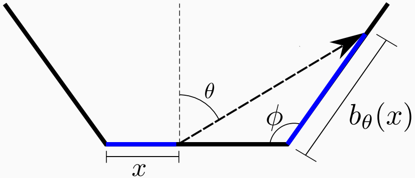
$$\frac{b_\theta(x)}{\sin(\pi/2-\theta)} = \frac{l-x}{\sin(\pi-(\pi/2-\theta)-\phi)}$$

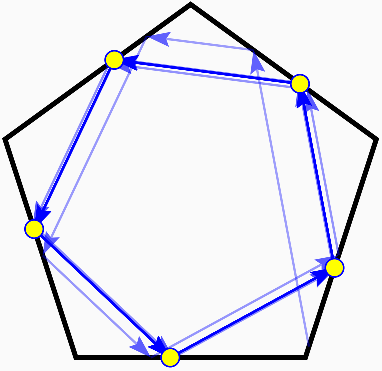
which we can rewrite as

$$b_\theta(x) = \frac{\cos(\theta)}{\cos(\theta-\phi)}(l-x) = c(\theta)(l-x)$$



Fixed Point of Mapping Function $\iff$ Periodic Orbit





Is There a Unique Fixed Point?

Lemma

If $|c(\theta)| < 1$, then $b_\theta(x)$ is a contraction mapping and has a unique fixed point.

Proof:

We take $(0, l)$ to be a metric space with metric $d(x, y) = |y - x|$. To be a contraction mapping, b_θ must satisfy

$$d(b_\theta(x), b_\theta(y)) \leq kd(x, y)$$

for all $x, y \in (0, l)$ and some nonnegative real number $0 \leq k < 1$ (Granas and Dugundji 2003).

Is There a Unique Fixed Point?

$$b_\theta(x) = c(\theta)(l - x)$$

$$c(\theta) = \frac{\cos(\theta)}{\cos(\theta - \phi)}$$

When we check how distances change under the map, we see that

$$\begin{aligned} d(b_\theta(x), b_\theta(y)) &= |c(\theta)(l - y) - c(\theta)(l - x)| \\ &= |c(\theta)(x - y)| \\ &= |c(\theta)| d(x, y). \end{aligned}$$

Thus if $|c(\theta)| < 1$, then b_θ is a contraction mapping, and by the Banach fixed-point theorem, it has a unique fixed point.

Iterate to find fixed point

Proposition

$$x_{FP} = \begin{cases} \frac{lc(\theta)}{1+c(\theta)} & \phi/2 < \theta < \pi/2 \\ \frac{l}{1+c(\theta)} & -\pi/2 < \theta < -\phi/2 \end{cases}$$

in which $c(\theta) = cos(\theta)/cos(\theta - \phi)$*.*

Sketch of Proof:

Bouncing counterclockwise, k sequential edges:

$$\begin{aligned} b_\theta^k(x) &= c(\theta)(l - c(\theta)(l - \ldots c(\theta)(l - x)\ldots)) \\ &= \sum_{i=1}^{k}(-l)(-c(\theta))^i + (-c(\theta))^k x \end{aligned}$$

Iterate for fixed point

taking the limit as $k \to \infty$ and shifting the index gives

$$b_\theta^\infty(x) = l + \sum_{i=0}^{\infty} (-l)(-c(\theta))^i$$

And given same condition

$$|c(\theta)| = |\frac{\cos(\theta)}{\cos(\theta - \phi)}| < 1$$

this geometric sum converges:

$$b_\theta^\infty(x) = \frac{lc(\theta)}{1 + c(\theta)}$$

General Fixed Point and Bounds on θ

Solve $|\frac{\cos(\theta)}{\cos(\theta-\phi)}| < 1$ to get bounds on θ for guaranteeing periodic trajectories.

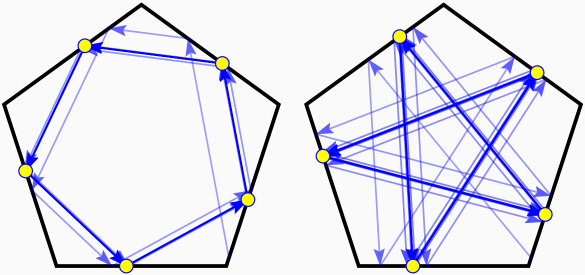
Position of collisions in limit cycle (distance from nearest clockwise vertex):

$$x_{FP} = \begin{cases} \frac{lc(\theta)}{1+c(\theta)} & \phi/2 < \theta < \pi/2 \\ \frac{l}{1+c(\theta)} & -\pi/2 < \theta < -\phi/2 \end{cases}$$

Do the geometry and...

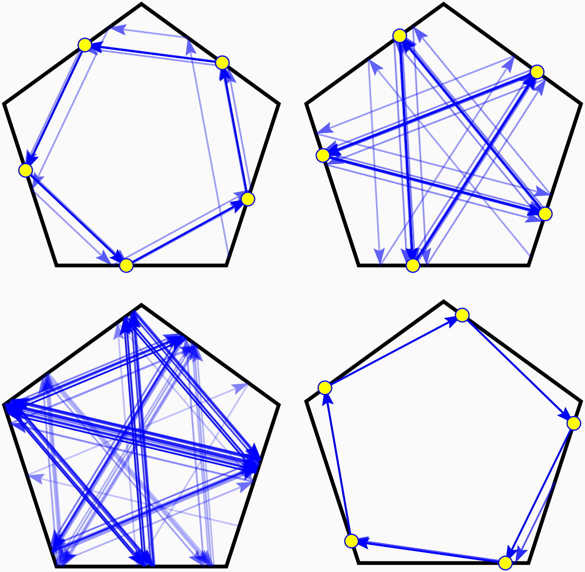
We have shown the case for clockwise bounces, on every sequential edge.

We can also imagine going counterclockwise, and skipping edges:



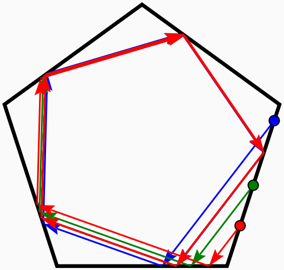
Mapping function, convergence conditions go through very similarly

Simulation Results

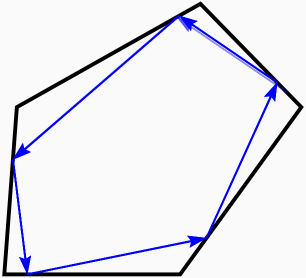


Nice Properties

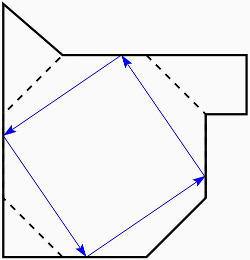
- any regular *n*-gon
- stable orbits are independent of starting position
- exponential convergence
- bounds on θ from $|c(\theta)| < 1$: any angle in this range will make similar orbits



Other Polygons



(a) A stable orbit in a sheared pentagon.



(b) A stable orbit in a nonconvex environment.

Figure 7: Stable orbits also exist in non-regular polygons.

Adding Noise

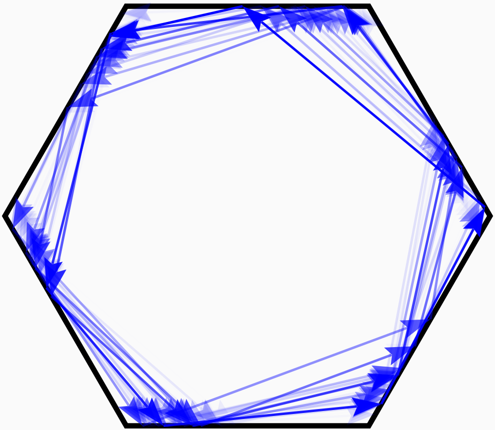
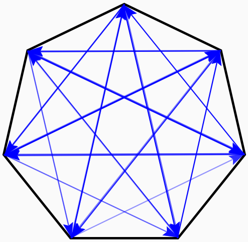


Figure 8: 200 bounces with uniformly distributed error added to θ, -0.1 rad $\leq \epsilon \leq 0.1$ rad

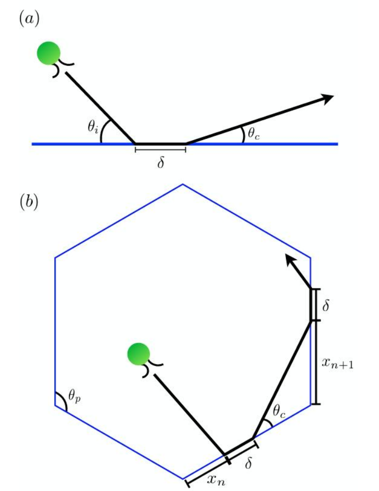
Open Problems & Future Work

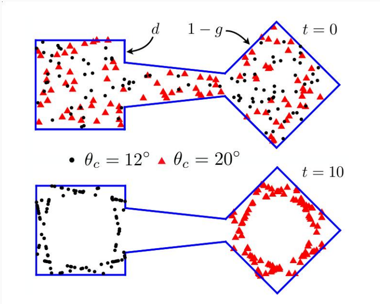
- How to characterize non-periodic dynamics?
  - frequency doubling at transition angles
  - Lyapunov exponents
- Analytic extension to non-regular polygons?
- Smooth environments?
- Error bounds?



Microorganism Billiards (Spagnolie et al. 2017)

- Applications: sorting, driving movement of objects in environment (Di Leonardo et al. 2010)
- switch between modes; let passive dynamics evolve





References

Bilò, Davide, Yann Disser, Matús Mihalák, Subhash Suri, Elias Vicari, and Peter Widmayer. 2012. "Reconstructing Visibility Graphs with Simple Robots." *Theoretical Computer Science* 444. Elsevier: 52–59.

Di Leonardo, R, L Angelani, D Dell'Arciprete, Giancarlo Ruocco, V Iebba, S Schippa, MP Conte, F Mecarini, F De Angelis, and E Di Fabrizio. 2010. "Bacterial Ratchet Motors." *Proceedings of the National Academy of Sciences* 107 (21). National Acad Sciences: 9541–5.

Disser, Yann. 2011. *Mapping Polygons*. Logos Verlag Berlin GmbH.

Erickson, L. H., and S. M. LaValle. 2013. "Toward the Design and Analysis of Blind, Bouncing Robots." In *IEEE International Conference on Robotics and Automation*.

Granas, Andrzej, and James Dugundji. 2003. "Elementary Fixed Point Theorems." In *Fixed Point Theory*, 9–84. New York, NY: Springer New York.

Haldane, Duncan W., M. M. Plecnik, J. K. Yim, and R. S. Fearing. 2016. "Robotic Vertical Jumping Agility via Series-Elastic Power Modulation." Science Robotics.

Lewis, Jeremy S., and Jason M. O'Kane. 2013. "Planning for Provably Reliable Navigation Using an Unreliable, Nearly Sensorless Robot." *International Journal of Robotics Research* 32 (11): 1339–54.

Markarian, Roberto, Enrique Pujals, and Martin Sambarino. 2010. "Pinball Billiards with Dominated Splitting." *Ergodic Theory and Dynamical Systems* 30: 1757–86.

O'Kane, J. M., and S. M. LaValle. 2007. "Localization with Limited Sensing." *IEEE Transactions on Robotics* 23 (4): 704–16.

Raibert, Marc H, H Benjamin Brown Jr, and Michael Chepponis. 1984. "Experiments in Balance with a 3d One-Legged Hopping Machine." *The International Journal of Robotics Research* 3 (2). Sage Publications Sage CA: Thousand Oaks, CA: 75–92.

Spagnolie, Saverio E, Colin Wahl, Joseph Lukasik, and Jean-Luc Thiffeault. 2017. "Microorganism Billiards." *Physica D: Nonlinear Phenomena* 341. Elsevier: 33–44.