Periodic Trajectories of Mobile Robots

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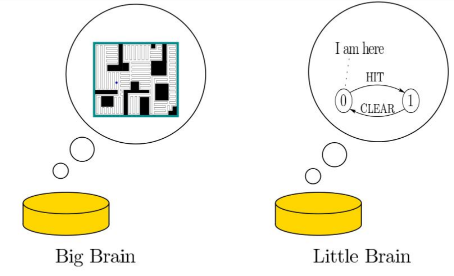
Simple Mobile Robots

- Mobile robots can vacuum floors, transport goods in warehouses, act as security robots (patrol), etc
- We want to **minimize** sensing and computation
  - make robots less expensive, more energy efficient
- Often, robots can bump into things and be ok!
- How can we use **contact with the environment** as a strategy or source of information?



Blind, Bouncing Robots[1]

Abstract the robot as a point moving **in straight lines** in the plane, "bouncing" off the boundary at a **fixed angle** θ from the normal:

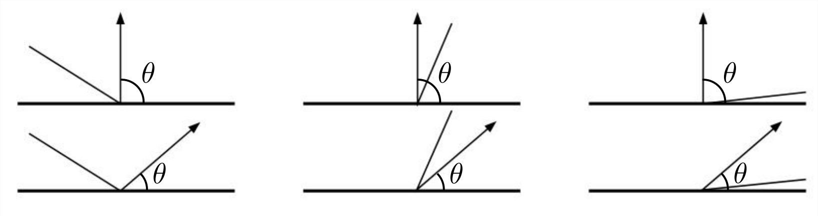


Figure 1: A point robot moving in the plane. The top row shows bounces at zero degrees from the normal. The second row shows bounces at 50 degrees clockwise from normal.

[1](Erickson and LaValle 2013), ICRA

Research Questions

Given a constant control strategy, will the robot become “trapped” in part of the environment? Or in a certain motion pattern? We focus on **patrolling**: periodically orbiting the workspace.

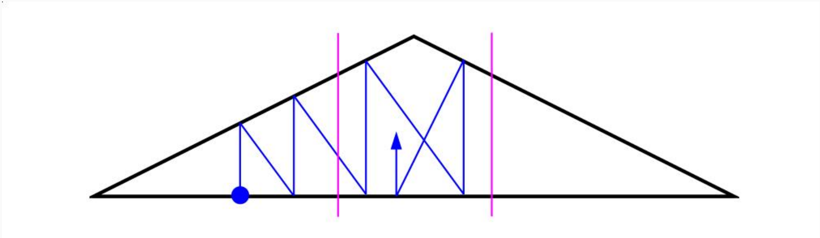


Figure 2: In this environment, bouncing at the normal, the robot will become trapped in the area between the purple lines

Implementation

- In scenarios where we need to patrol, we often know something about the geometry of the space

But is it physically realizable?

- Can implement on a roomba with bump sensor and IR prox detector[2]
- "Collisions" can be virtual - for example, robot stops when it is collinear with two landmarks, and rotates until one landmark is at a certain heading

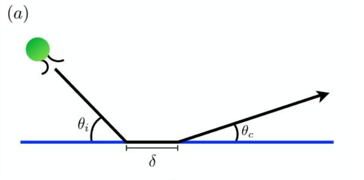
[2] (Lewis and O'Kane 2013), IJRR

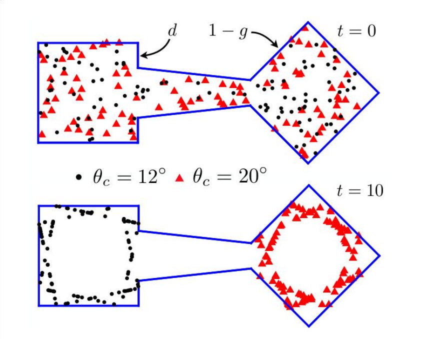
Related Work in Robotics

- Minimal sensing, actuation, computation requirements for mapping, navigating, localizing, patrolling, pursuit evasion *(Tovar, Guilamo, and LaValle (2005), Bilò et al. (2012), J. M. O'Kane and LaValle (2007), Disser (2011))*
- Overlaps with design automation: formalize tradeoffs between sensor and actuator power, computational complexity, energy use, etc
  - ICRA 1996 workshop, RSS '08, '16, '17

Related Work in Dynamical Systems

- Specular billiards *(Tabachnikov (2005))*
- Pinball billiards *(Markarian, Pujals, and Sambarino (2010))*
- **Aspecular billiards**, microorganism billiards *(Spagnolie et al. (2017))*





Limit Cycles In Regular Polygons

Question

If we can only move in straight lines and align relative to wall normal:

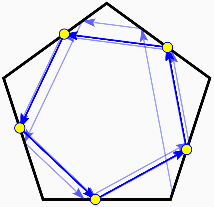
> *Can we guarantee that a robot* ***patrols*** *a space on a periodic path?*

Can also phrase as:

> *What are the conditions on limit cycles in this dynamical system?*

Discovery Through Simulation

- Haskell with *Diagrams* library (Yorgey 2012)
- Fixed-angle bouncing, relative bouncing (rotate θ from previous heading), specular bouncing, add noise
- Render diagrams from simulations automatically[3]



[3] https://github.com/alexandroid000/bounce

In Regular Polygons

Ignore movement in interior, only track position on boundary when robot collides. Defines a **discrete dynamical system**

$$x_{n+1} = f(x_n)$$

Given regular polygon, with edge length ℓ and internal angle ϕ, we can define the mapping function

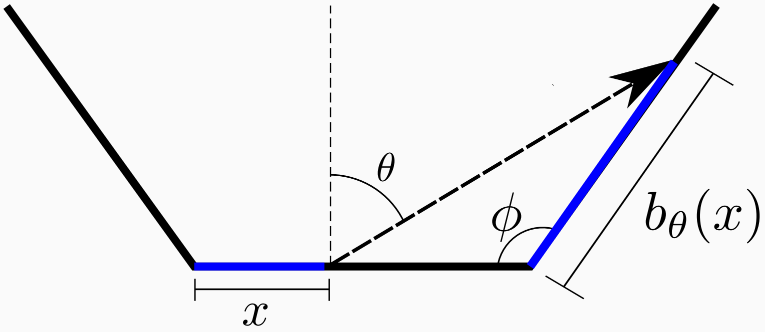
$$b_\theta : (0, \ell) \to (0, \ell)$$

Sequential-Edge Bouncing

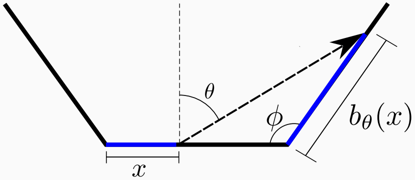
In regular polygons with side length ℓ and internal angle ϕ:

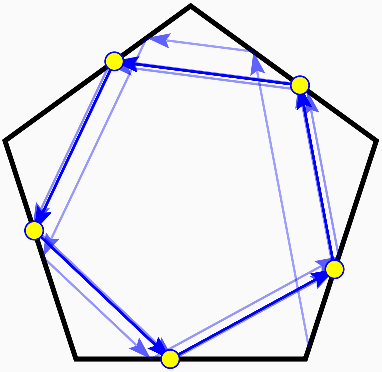
$$b_\theta : (0, \ell) \to (0, \ell)$$

$$b_\theta(x) = c(\theta)(\ell - x) \qquad c(\theta) = \frac{\cos(\theta)}{\cos(\theta - \phi)}$$



Fixed Point of Mapping Function $\iff$ Periodic Orbit

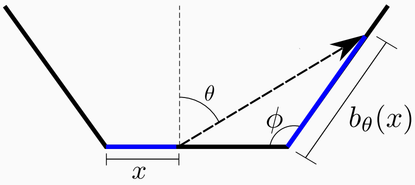




What is the Fixed Point?

Mapping Function

$$b_\theta(x) = c(\theta)(\ell - x) \qquad c(\theta) = \frac{\cos(\theta)}{\cos(\theta-\phi)}$$



$$b_\theta(x_{FP}) = x_{FP}$$
$$c(\theta)(\ell - x_{FP}) = x_{FP}$$
$$x_{FP} = \frac{\ell c(\theta)}{1 + c(\theta)}$$

For clockwise bouncing, reflect across the midpoint of the edge:
$\rightarrow \ell - x_{FP}$

Is the Fixed Point Unique?

Mapping Function

$$b_\theta(x) = c(\theta)(\ell - x) \qquad c(\theta) = \frac{\cos(\theta)}{\cos(\theta - \phi)}$$

When we check how distances change under the map, we see that

$$\begin{aligned} d(b_\theta(x), b_\theta(y)) &= |c(\theta)(\ell - y) - c(\theta)(\ell - x)| \\ &= |c(\theta)(x - y)| \\ &= |c(\theta)| d(x, y). \end{aligned}$$

Thus if $|c(\theta)| < 1$, then b_θ is a contraction mapping, and by the Banach fixed-point theorem, it has a unique fixed point (Granas and Dugundji 2003).

Bounds on θ

To get bounds on θ for guaranteeing periodic trajectories:

- Solve $|\frac{\cos(\theta)}{\cos(\theta-\phi)}| < 1$
- Take geometric feasibility into account (for non-regular polygons)

So now we have a statement for the existence and stability of the fixed points, for counterclockwise bouncing striking each edge of a regular polygon:

Proposition

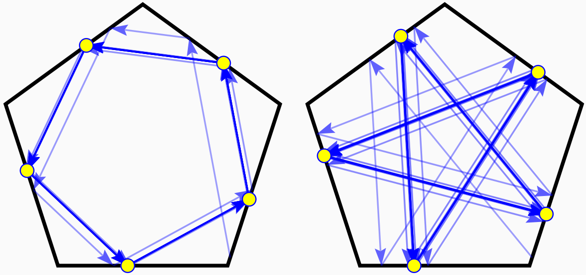
$$x_{FP} = \begin{cases} \frac{\ell c(\theta)}{1+c(\theta)} & \phi/2 < \theta < \pi/2 \\ \frac{\ell}{1+c(\theta)} & -\pi/2 < \theta < -\phi/2 \end{cases}$$

in which $c(\theta) = cos(\theta)/cos(\theta - \phi)$.

Confirmation and Generalization

We have shown the case for counterclockwise bounces, on every sequential edge.

We can also imagine going clockwise, and/or skipping edges:



Mapping function, convergence conditions go through very similarly (see paper)

Generalization

In every regular n-sided polygon with side length l and interior angle $(n-2)\pi/n$, there exists a range for θ such that iterating $B_\theta(x)$ on any $x \in \delta P$, results in a stable limit cycle that strikes the boundary skipping $m-1$ edges, and strikes at points that are distance x_{FP} from the nearest clockwise vertex, with x_{FP} given by

$$x_{FP} = \begin{cases} \frac{l - A(1-c(\theta))}{1+c(\theta)}, & \frac{\phi_m}{2} < \theta < \frac{\phi_{m-1}}{2} \\ \frac{lc(-\theta)+A(1-c(-\theta))}{1+c(-\theta)}, & \frac{-\phi_{m-1}}{2} < \theta < \frac{-\phi_m}{2} \end{cases} \tag{1}$$

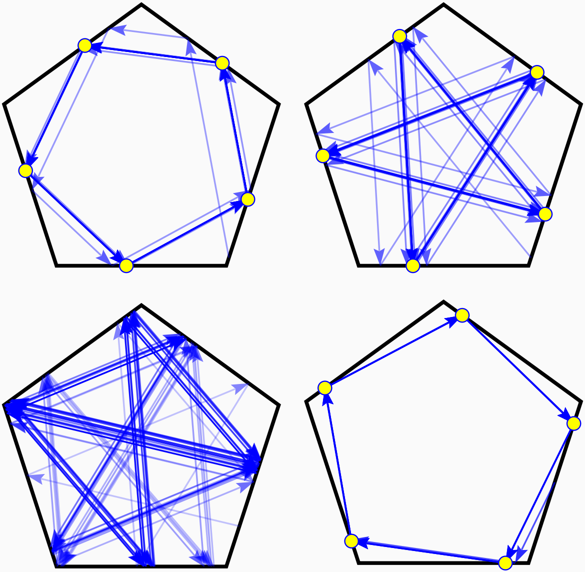
in which

$\phi_m = \frac{\pi(n-2m)}{n}$,

$A = \frac{l \sin(\frac{\pi(m+1)}{n}) \sin(\frac{m\pi}{n})}{\sin(\frac{\pi}{n}) \sin(\frac{\pi(n-2m)}{n})}$, and

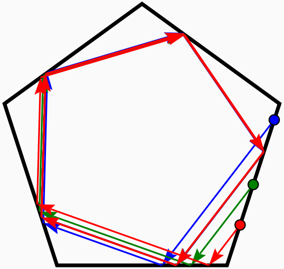
$c(\theta) = \cos(\theta) / \cos\left(\theta - \frac{\pi(n-2m)}{n}\right)$.

Simulation Results



Nice Properties

- Any regular *n*-gon
- Stable orbits are **independent of starting position**
- Exponential convergence



Adding Noise

We got bounds on θ from $|c(\theta)| < 1$: any angle in this range will make similar orbits

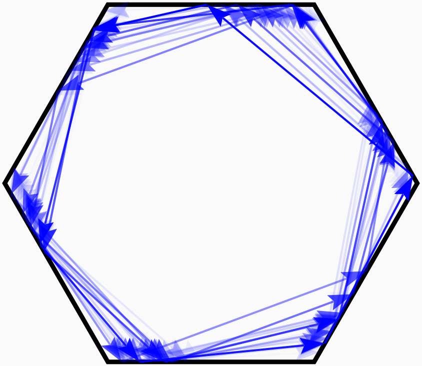
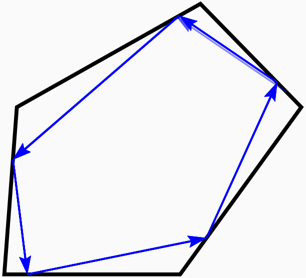
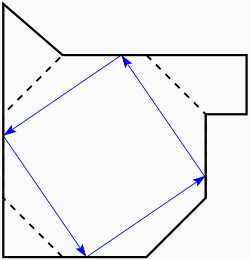


Figure 5: 200 bounces with uniformly distributed error added to θ, -0.1 rad $\leq \epsilon \leq 0.1$ rad

Other Polygons



(a) A stable orbit in a sheared pentagon.

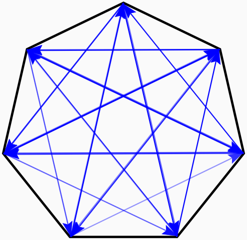


(b) A stable orbit in a nonconvex environment.

Figure 6: Stable orbits also exist in non-regular polygons.

Open Problems & Future Work

- How to characterize and exploit non-periodic dynamics?
  - Lyapunov exponents suggest chaotic dynamics
  - relationship to dispersion: what is the longest unvisited edge interval as the system evolves?
- Extensions to non-regular polygons and smooth environments
- Error bounds: model different disturbances to bounce
- Feedback control: counting number of bounces, make walls distinguishable / colored, allow robot to place and/or detect landmarks



Thank you!

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