06 Linear Regression

November 20, 2022

1 Linear Regression

[1]:
```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from scipy import stats
```

1.1 Simple Linear Regression Model

We assume that there is approximately a linear relationship between X and Y:

$$Y \approx \beta_0 + \beta_1 * X$$

1.1.1 Least Squares Method

The least squares approach choosed $\hat{\beta}_0$ and $\hat{\beta}_1$ to minimize the residual sum of squares (RSS)

$$RSS = r_1^2 + r_2^2 + \cdots + r_n^2$$
$$\text{or equivalently } RSS = (y_1 - \hat{\beta}_0 - \hat{\beta}_1 x_1)^2 + \cdots + (y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n)^2$$

1.1.2 Estimation of coefficients

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 * \bar{x}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\text{and } \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

1.1.3 How much do coefficient scatter?

$\hat{\beta}_0$ and $\hat{\beta}_1$ are scattered around the true values β_0 and β_1 with the standard error

$$
\begin{aligned}
se(\hat{\beta}_0)^2 &= \sigma^2\Big(\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\Big) \\
\text{and } se(\hat{\beta}_1)^2 &= \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \\
\text{where } \sigma^2 &= Var(\epsilon)
\end{aligned}
$$

1.1.4 Residuals and Estimation of Variance

In general σ^2 (variance of the error terms) is not known: but can be estimated on the basis of the data. The error term ϵ

- cannot be observed
- cannot be derived from $\epsilon = Y - (\hat{\beta}_0 + \hat{\beta}_1 X)$ since $\hat{\beta}_0$ and $\hat{\beta}_1$ are unknown.

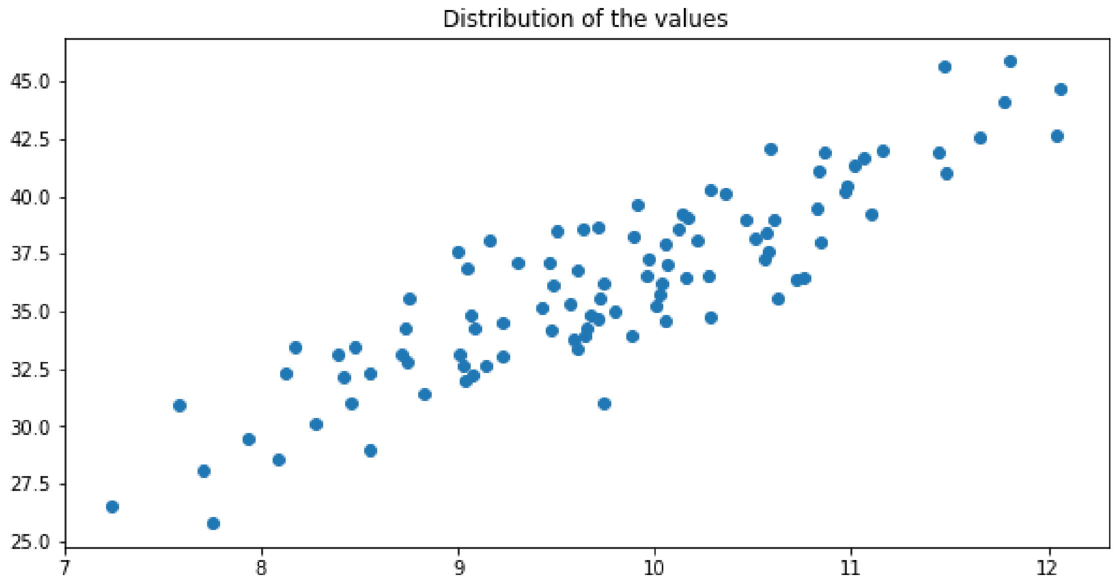
Approximation for ϵ : residuals $r_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)$

$$\hat{\sigma} = RSE = \sqrt{\frac{RSS}{n-2}} = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n-2}}$$

The factor $\frac{1}{n-2}$ is chosen so that the stimate of σ turns out to be unbiased. This estimate is known as the residual standard error (RSE)

[98]:
```
model = stats.norm(0, 2)
func = lambda x: 2.74 + 3.42 * x

x = stats.norm(10, 1).rvs(100)
y = func(x) + model.rvs(x.shape[0])
```

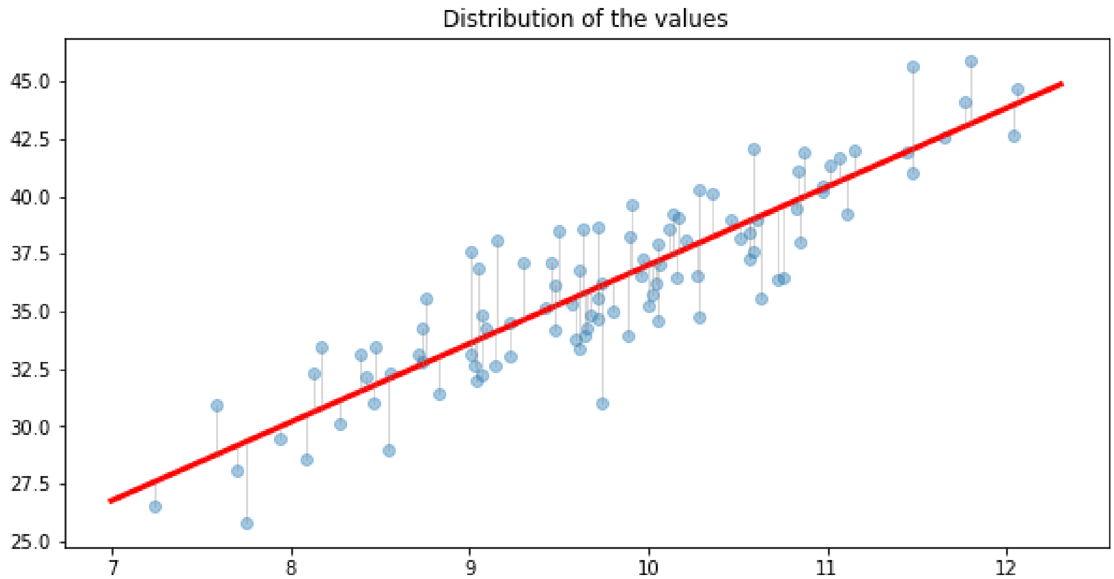
[106]:
```
plt.figure(figsize=(10, 5))
plt.scatter(x, y)
plt.title("Distribution of the values")
plt.show()
```



[100]:
```
x_mean = np.mean(x)
y_mean = np.mean(y)

b1 = np.sum((x - x_mean) * (y - y_mean)) / np.sum((x - x_mean) ** 2)
b0 = y_mean - b1 * x_mean
b1, b0
```

[100]: (3.4136985890489684, 2.862161280507756)

[101]:
```
plt.figure(figsize=(10, 5))
plt.scatter(x, y, alpha=0.4)
plt.plot(np.array(plt.xlim()), np.array(plt.xlim()) * b1 + b0, c='r', lw=3)

for _x, _y in zip(x, y):
    y_hat = _x * b1 + b0
    plt.plot([_x, _x], [_y, y_hat], alpha=.3, c='grey', lw=1)

plt.title("Distribution of the values")
plt.show()
```



[102]:
```
rss = np.sum((y - (x * b1 + b0)) ** 2)
rss
```

[102]: 347.577167145676

[103]:
```
rse = np.sqrt(rss / (x.shape[0] - 2))
rse
```

[103]: 1.8832699719373152

[105]:
```
se_0 = rse ** 2 * ((1 / x.shape[0]) + (x_mean ** 2) / np.sum((x - x_mean) ** 2))
se_1 = (rse ** 2) / np.sum((x - x_mean) ** 2)

se_0, se_1
```

[105]: (3.1349680075129256, 0.03241257143945019)