06 Linear Regression Test

November 20, 2022

1 Hypothesis Test, Test statistic and P-Value

[83]:
```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

from scipy import stats
from loess.loess_1d import loess_1d
```

1.1 Hypothesis Test

$$H_0 : \beta_1 = 0$$
$$H_A : \beta_1 \neq 0$$

1.2 Test statistic

$$T = \frac{\hat{\beta}_1 - 0}{se(\hat{\beta}_1)}$$

- Test statistic T measures the number of standard deviations that $\hat{\beta}_1$ is away from 0
- If there is really no relationship between X and Y, that is H_0 is true, then we expect that T follows a t-distribution with $n-2$ degrees of freedom.
- p-value: Probability of observing any value of T larger than $|t|$
  - If $p < \alpha$: then we reject H_0 and conclude that there is no relationship between X and Y
  - If $p > \alpha$: then we failed to reject H_0

1.3 t-Test in Linear Regression

- Model

$$Y = \beta_0 + \beta_1 X + \epsilon_i, \quad \epsilon \sim N(0, \sigma^2)$$

- Hypothesis:

$$H_0 : \beta_1 = 0$$
$$H_A : \beta_1 \neq 0 \text{ (two-sided test)}$$

- Test statistic:

$$T = \frac{\hat{\beta}_1 - 0}{se(\hat{\beta}_1)}$$

  Null Distribution assuming H_0 is true: $T \sim t_{n-2}$

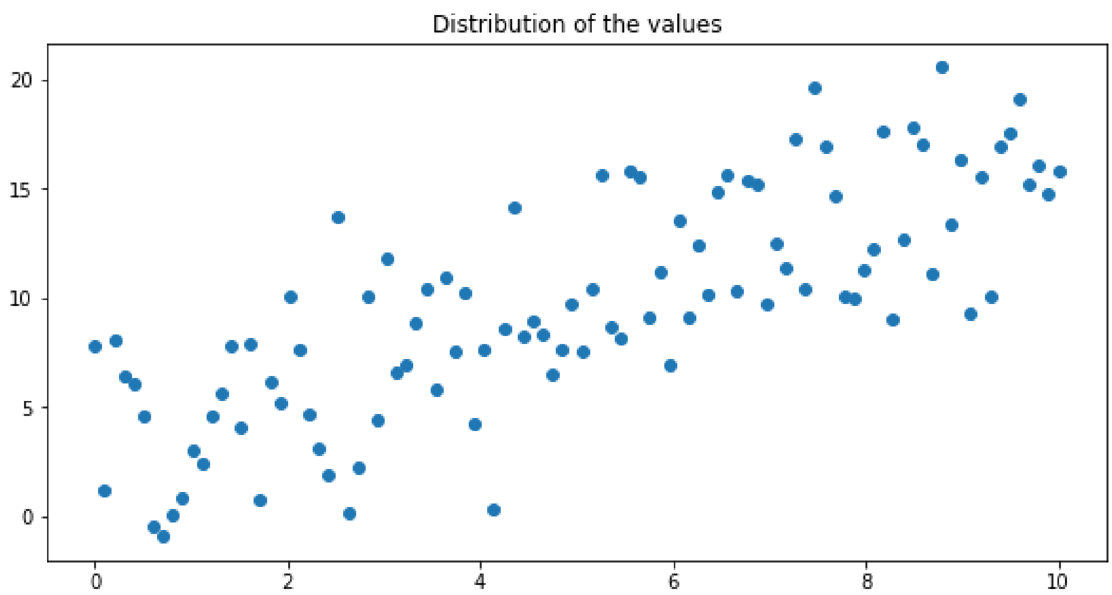
- Significance level: α
- Rejection Region for Test statistic:

$$C = \left(-\inf; t_{n-2;\frac{\alpha}{2}}\right] \cup \left[t_{n-2;1-\frac{\alpha}{2}}; \inf\right)$$

- Test decision: Verify whether observed t falls into rejection area

[108]:
```
model = stats.norm(0, 3)
func = lambda x: 2.74 + 1.42 * x

x = np.linspace(0, 10, 100)
y = func(x) + model.rvs(x.shape[0])
```

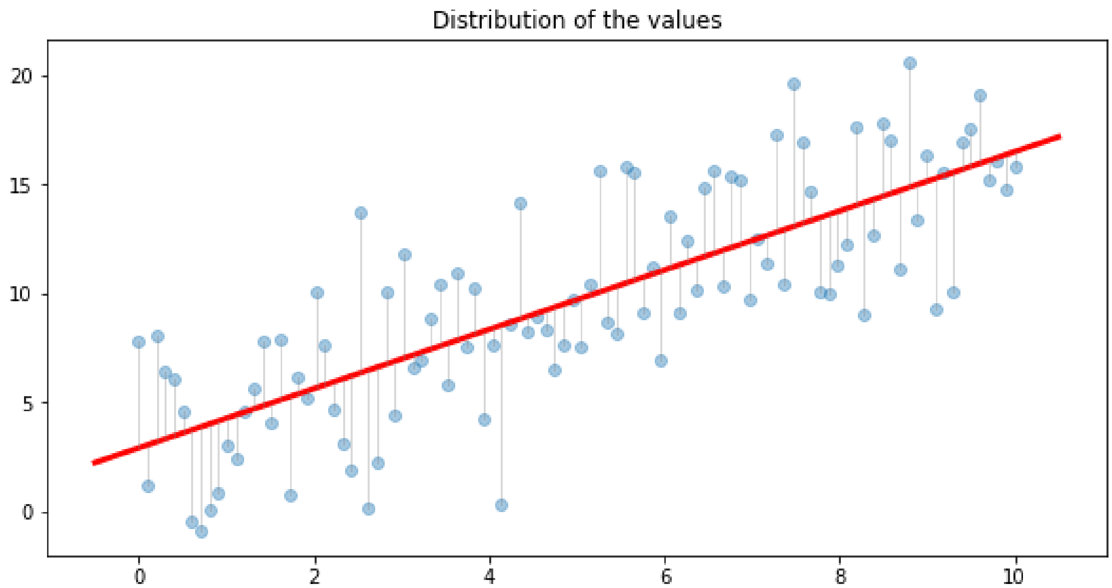
[109]:
```
plt.figure(figsize=(10, 5))
plt.scatter(x, y)
plt.title("Distribution of the values")
plt.show()
```



[110]:
```
x_mean = np.mean(x)
y_mean = np.mean(y)

b1 = np.sum((x - x_mean) * (y - y_mean)) / np.sum((x - x_mean) ** 2)
b0 = y_mean - b1 * x_mean
b1, b0
```

[110]: (1.3572877612681389, 2.9294112666918686)

[111]:
```
plt.figure(figsize=(10, 5))
plt.scatter(x, y, alpha=0.4)
plt.plot(np.array(plt.xlim()), np.array(plt.xlim()) * b1 + b0, c='r', lw=3)

for _x, _y in zip(x, y):
    y_hat = _x * b1 + b0
    plt.plot([_x, _x], [_y, y_hat], alpha=.3, c='grey', lw=1)

plt.title("Distribution of the values")
plt.show()
```



[112]:
```
rss = np.sum((y - (x * b1 + b0)) ** 2)
rss
```

[112]: 1034.4160446891703

[113]:
```
rse = np.sqrt(rss / (x.shape[0] - 2))
rse
```

[113]: 3.2488868496970453

[114]:
```
se_0 = rse ** 2 * ((1 / x.shape[0]) + (x_mean ** 2) / np.sum((x - x_mean) ** 2))
se_1 = (rse ** 2) / np.sum((x - x_mean) ** 2)

se_0, se_1
```

[114]: (0.415940175577177, 0.012415500718233325)

[115]:
```
t = b1 / np.sqrt(se_1)
t
```

[115]: 12.181192694650209

$\hat{\beta}_1$ is approximately 12 standard errors $se(\hat{\beta}_1)$ away from 0

[118]:
```
alpha = 0.05
c_left = stats.t(x.shape[0] - 2).ppf(alpha / 2)
c_right = stats.t(x.shape[0] - 2).ppf(1 - alpha / 2)

print(f"Rejection area: (inf; {c_left}] u [{c_right};inf)")
```

```
Rejection area: (inf; -1.9844674544266925] u [1.984467454426692;inf)
```

[119]:
```
p_value = 2 * (1 - stats.t(x.shape[0] - 2).cdf(t))
print(f"P-Value:             {p_value}")
print(f"Significant result: {p_value < alpha}")
```

```
P-Value:             0.0
Significant result: True
```

1.4 Confidence interval for coefficients

$$\left[\hat{\beta}_1 - t_{n-2;1-\frac{\alpha}{2}} * se(\hat{\beta}_1); \hat{\beta}_1 + t_{n-2;1-\frac{\alpha}{2}} * se(\hat{\beta}_1)\right]$$

[120]:
```
t = stats.t(x.shape[0] - 2).ppf(1 - alpha / 2)
lo = b1 - t * np.sqrt(se_1)
up = b1 + t * np.sqrt(se_1)

print(f"Confidence Interval for coefficient 1: [{lo}; {up}]")
```

```
Confidence Interval for coefficient 1: [1.1361687414769794; 1.5784067810592983]
```

1.5 Model Assumptions fro the Error Terms

The error terms ϵ_i are independent and normally distributed random variables with a constant variance

$$\epsilon_i \sim N(0, \sigma^2)$$

1. For the expected value of all ϵ_i we have

$$E(\epsilon_i) = 0$$

2. The error terms ϵ_i all have the same constant variance

$$Var(\epsilon_i) = \sigma^2$$

3. The error terms ϵ_i are normally distributed
4. The error terms ϵ_i are independent

1.6 R-squared Statistic

The R^2 statistic provides an alternative measure of fit

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} = 1 - \frac{\text{variance left after regression fit}}{\text{total variance}}$$

- R^2 takes the form of a proportion - the proportion of variance explained: R^2 always takes on a value between 0 and 1, and is independent of the scale of Y
- If model fits perfectly the data, then $\hat{y}_i = y_i$ for all $i \Rightarrow R^2 = 1$

[121]:
```
y_hat = func(x)
r_squared = 1 - np.sum((y - y_hat) ** 2) / np.sum((y - y_mean) ** 2)
print(f"R-Squared: {r_squared}")
```

```
R-Squared: 0.6003644390601466
```

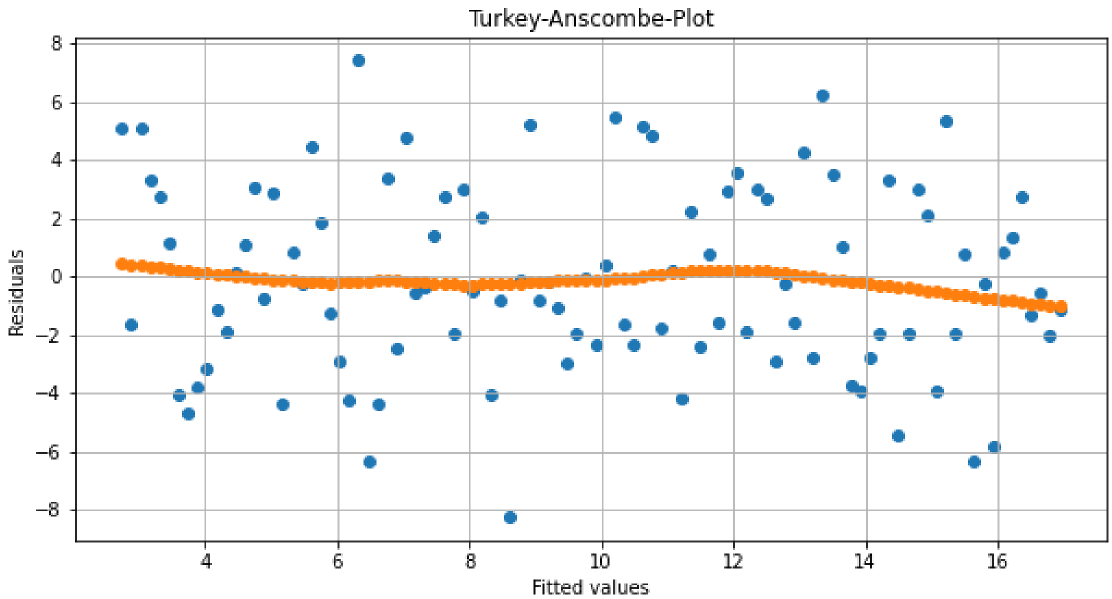
1.7 Diagnostics Tools

1.7.1 Testing Model Assumtion expected value of error terms

We want to identify non-linearity of the regression function f, that is, we want to verify model assumption $E(\epsilon) = 0$. The relevant residual plot is called **Turkey-Anscombe-Plot**

- We plot on the vertical axis the residuals $r_i = y_i - \hat{y}_i$
- We plot on the horizontal axis the fitted or predicted values $\hat{y}_i$
- We thus plot the points $(\hat{y}_i, r_i)$ for $i = 1, \ldots, n$

[122]:
```
n = x.shape[0]
r = y - y_hat
rse = np.sqrt(1 / (n - 2) * np.sum(r ** 2))

# Turkey-Anscombe plot
plt.figure(figsize=(10, 5))
plt.scatter(y_hat, r)

res = loess_1d(y_hat, r)  # smoothing approach
plt.scatter(res[0], res[1])

plt.xlabel("Fitted values")
plt.ylabel("Residuals")
plt.title("Turkey-Anscombe-Plot")
plt.grid()
plt.show()
```



Non-constant variances in the errors ϵ_i: **heteroscedasticity**

1.8 Scale Location Plot

- Measure of scattering amplitude of errors: square root of the absolute value of the standardized residuals, that is

$$\sqrt{|\tilde{r}_i|}$$

- Standardized residuals $\tilde{r}_i$ are defined as follows

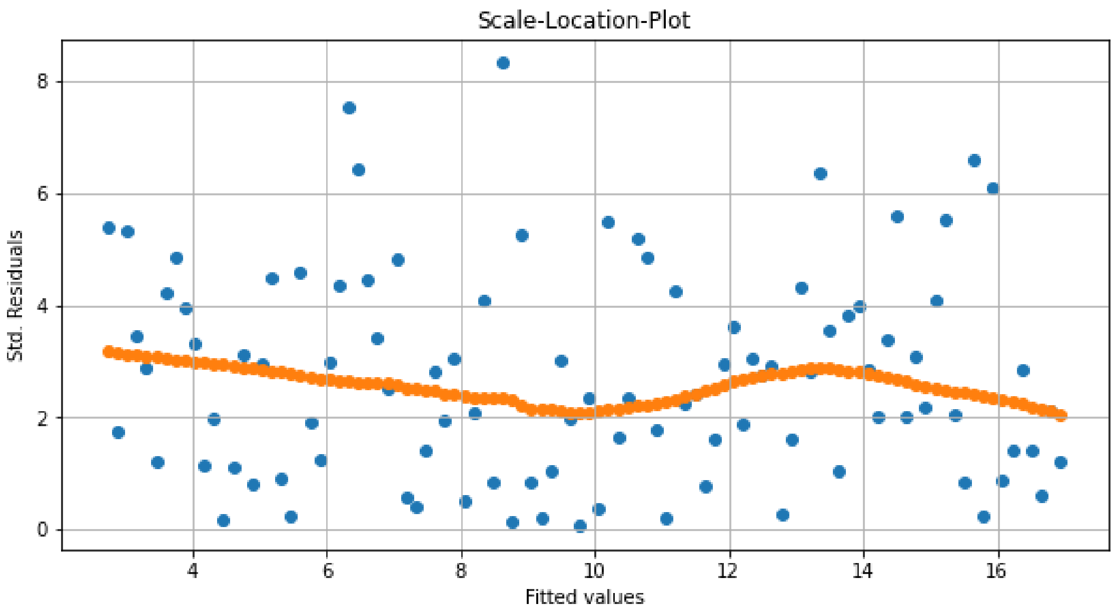
$$\tilde{r}_i = \frac{r_i}{\hat{\sigma}\sqrt{1 - \left(\frac{1}{n} + \frac{(x_i - \bar{x}^2)}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)}}$$

- $\hat{\sigma}$: Estimate of standard deviation of error terms (estimated by RSE)
- If error terms ϵ_i are normally distributed, then

$$\tilde{r}_i \sim N(0, 1)$$

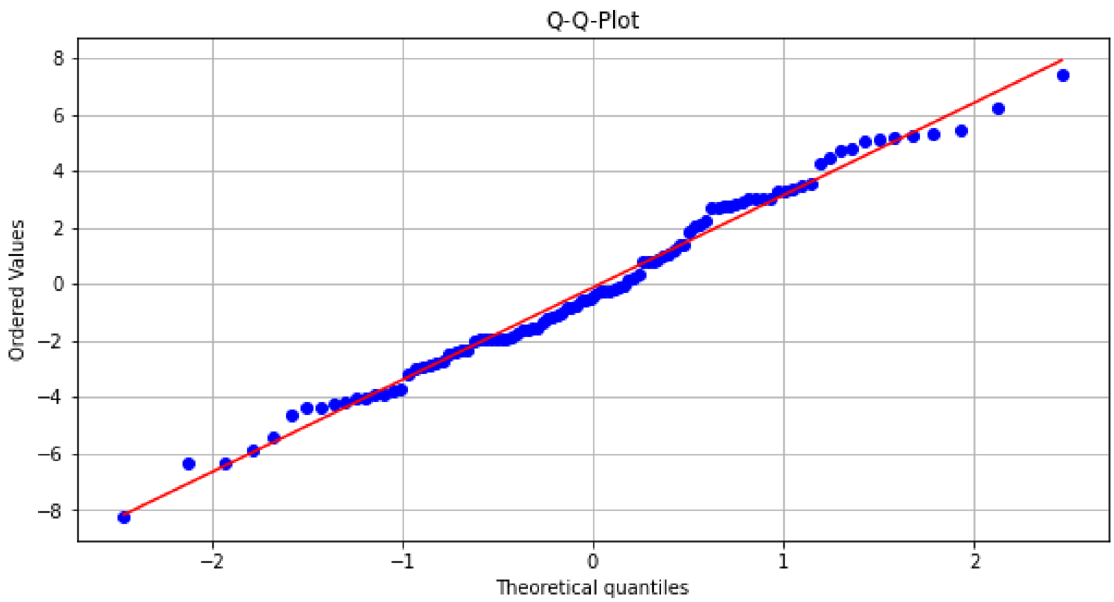
[127]:
```
factor = np.sqrt(1 - (1 / x.shape[0] + (x - x_mean) ** 2 / (np.sum((x - x_mean)
↪** 2)) * rse))
std_r = r / factor
res = loess_1d(y_hat, np.sqrt(std_r ** 2))

plt.figure(figsize=(10, 5))
plt.scatter(y_hat, np.sqrt(std_r ** 2))
plt.scatter(res[0], res[1])
plt.xlabel("Fitted values")
plt.ylabel("Std. Residuals")
plt.title("Scale-Location-Plot")
```

```
plt.grid()
plt.show()
```



1.8.1 Normal Distribution Assumption of the Errors

[128]:
```
plt.figure(figsize=(10, 5))
stats.probplot(r, plot=plt.gca())

plt.title("Q-Q-Plot")
plt.grid()
plt.show()
```


Q-Q-Plot
Ordered Values
Theoretical quantiles
8
6
4
2
0
−2
−4
−6
−8
−2
−1
0
1
2

```
[ ]:
```