Entrepreneurial risk and labor's share in output*

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Summary: 1. Introduction; 2. The basic model; 3. The extended model; 4. Conclusion.
Key words: entrepreneurial risk; income distribution.

This paper presents a theoretical model which discusses the role played by the entrepreneurial risk on the distribution of the national income. In a two-period general equilibrium framework with competitive risk-averse entrepreneurs, it is shown that the highest the risk borne by firms, the lower will be real wages, the level of employment and the labor's share in output. The model provides a framework to analyze the impact of the so-called "Brazil risk" in the Brazilian economy. The conclusion is that the reduction of this risk would increase wages, employment and the labor's share in the Brazilian GNP.

Este artigo apresenta um modelo teórico que discute o papel do risco na distribuição da renda nacional. Em um modelo de equilíbrio geral com dois períodos em que empresários avessos ao risco atuam competitivamente, mostra-se que quanto maior for o risco da atividade empresarial, menores serão os salários reais, o nível de emprego e a participação do trabalho na renda nacional. O modelo oferece um arcabouço teórico para se analisar o impacto do chamado "risco Brasil" na economia brasileira. A conclusão é que uma eventual redução desse risco aumentaria os salários reais, o nível de emprego e a participação do trabalho no PIB.

1. Introduction

Since the first contributions to the theory of implicit labor contracts by Baily (1974), Gordon (1974) and Azariadis (1975), economists began to understand some important characteristics of wages. When markets are not complete in the sense of Arrow (1964) or Debreu (1959), firms and workers may find it Pareto improving to assign to wages an insurance role. If this is the case, then gross salaries will differ from the marginal productivity of labor even in a competitive labor market. If workers could obtain insurance against fluctuations of wages or employment, then they would not find any incentive to include an insurance premium in wages.

Incompleteness of insurance markets arises due to many different sources; information asymmetries, moral hazard and adverse selection are sufficient to undermine such kind of markets. The literature on implicit labor contracts puts risk-averse workers as the main source of implicit contracts. Though managers may behave in a risk-averse way, large companies tend to follow a risk-neutral behavior since they can diversify away non-systematic risks for which there are no insurance markets. These issues are discussed in Azariadis (1983), Cooper (1983 and 1985), Fethke & Policano (1984), Grossman & Hart (1981), Grossman (1983), and Hart (1983). Enforcement of firm's fulfillment of labor contracts may be impossible when

bankruptcy is allowed for, as discussed in Farmer (1984), and Kahn & Scheinkman (1985). Rosen (1985) is a good survey on implicit contracts.

The model presented in this paper is based on the assumption that entrepreneurs cannot diversify away the risk they run. In a competitive labor market, each entrepreneur cannot hire workers for a wage lower than the market's. Given the market's wage and the level of risk involved in the production process, the entrepreneur optimally chooses the amount of resources he is willing to allocate to his risky business. It is shown that the higher the wage he has to pay and the higher the risk he has to run, the lower will be his willingness to hire workers.

In partial equilibrium, an individual entrepreneur cannot change the market's wage but he can reduce the number of hours of work he chooses to employ. So, an increase in the risk borne by firms is followed by a reduction of each entrepreneur's demand for labor. When each entrepreneur hires less workers, there is an aggregate reduction in the demand for labor, which leads to a fall in the market's wage. At the new (lower) market wage, entrepreneurs tend to increase their demand for labor, (partially) offsetting the initial reduction brought about by the higher level of risk. The level of employment would remain unchanged only if the supply of labor was infinitely inelastic. The reduction in the market wage is interpreted as an increase in the risk premium that workers implicitly pay to employers who are the agents running the risk of producing goods.

It can be shown that the fall of wages is accompanied by a reduction in the labor's share in the aggregate output. Since the aggregate output decreases due to the reduction in the employment level, one concludes that the fall of the economy's wage payroll is proportionally larger than the fall of the aggregate output. This happens because that share is calculated in an *ex-post* basis. Since the *ex-ante* profits increase due to the fall of wages, and since the number of firms in the economy is large, the Law of Large Numbers apply and the *ex-post* and *ex-ante* worker's share in output are the same.

The model casts light on the distortions brought about by the "Brazil risk" on the determination of workers' income and on the country's GNP. The term "Brazil risk" has been used in an imprecise way to denote many uncertainties that impinge on the entrepreneurial activities in Brazil. Some people use this term to describe aggregate shocks that impact all the Brazilian economic agents, such as sudden currency devaluations, surprise announcements of heterodox plans, etc. The same term, however, has also been used to denote a class of risks that are run by firms operating in Brazil but not by firms operating elsewhere. Finally, the term was sometimes used in a third meaning to describe the fact that some typical entrepreneurial risks are larger in Brazil than in other countries. I shall not go into the semantics of the term "Brazil risk". The reader will conclude that the proxy for the entrepreneurial risk used in the model below is adequate to model the last two of the above interpretations of the term "Brazil risk".

What are those risks? Anyone who has run a firm in Brazil, specially during the last decade, has faced umpteen unpredictable setbacks in his everyday managerial work. Discretionary price controls were sometimes adopted. Frequent changes of regulation made full compliance with the law almost impossible, leaving firms at the mercy of officials who were charged of enforcing them. Pernickety labor laws were made ever more complicated to apply, leaving almost all firms liable to high fines from the Labor Ministry. The slowness of the judiciary system makes it preferable to settle disputes by agreements even when there is a clear case for a successful legal process. These examples of distortions increased the uncertainty faced by firms.

Some of the aggregate shocks, on the other hand, impacted firms in idiosyncratic ways. There were spells of very dear money, when the government borrowed massively internally, thus guzzling most of the private savings. There were also spells of very negative real interest rates when the government relented to financing its deficits with monetary expansion. Basic constitutional principles as the one which assures that income (a flow), but never financial assets (a stock), could be source of taxation, were flouted.

It is important to realize that most of these aggregate shocks do not impact all companies in the same way. A company flush with cash during a period of hefty real interest rates could rake in huge gains, while another firm facing a transitory negative cash flow could go bust. Price controls could reach suppliers of raw material but not producers of final goods or just the other way around. This is why I assume in the theoretical model presented below that the risk faced by entrepreneurs is a stochastic production shock which is non-correlated across firms. In the aggregate, the shocks cancel out, but for each firm it varies significantly.

The paper is presented in three sections. Section 2 describes the basic model in which there are only two markets: the market of goods and the market of labor. In this abridged model there is only one market clearing variable, namely the wage. Section 3 extends the basic model, introducing a third market, the market of loans, and consequently a second market clearing variable: the interest rate. Section 4 concludes our findings.

2. The basic model

Consider a competitive two-period economy with one good and two kinds of agents, entrepreneurs and workers. The former own the technology of production and hire the latter for a real wage W during the first period.

Workers' behavior

There are H identical workers who work and consume at period 1. They supply labor competitively according to a consumption-leisure decision: L^s hours of work during period 1 allows consumption of WL^s units of the good. It is assumed that both leisure and consumption are normal goods so that there is a real-valued function $L^s: \Re \to \Re$ with positive derivative which ascribes to each wage W the number of hours of work each worker supplies at that wage.

Entrepreneurs' behavior

There are I identical entrepreneurs with an amount of real resources of A units of the good at period 1. This amount can be either lent at a real riskless rate r in, say, the international bonds market, or used to hire workers who will produce the good according to the risky technology described below.

The production technology is described by a real valued twice differentiable production function $f: \Re^+ \to \Re^+$, and a random variable ε. An entrepreneur who contracts L hours of work during period 1 will have an output of $\tilde{\varepsilon} f(L)$ units of the good at period 2, where the tilde over ε stands for the realization of the random variable. The function f is assumed to be

increasing ($f' > 0$), to exhibit decreasing marginal productivity, i.e., to be strictly concave ($f'' < 0$) and such that $f(0) = 0$. The random variable ε is normally distributed with mean μ and variance σ^2, where μ is sufficiently larger than σ so that a negative realization of ε can be ruled out. The random variables ε_i, $i = 1,2,...,I$ are identical and independently distributed across entrepreneurs.

Each entrepreneur is non-satiable, risk-averse and becomes less risk-averse the richer he gets. His preferences are described by a real valued twice differentiable utility function $U: \Re \to \Re$ with positive first derivative, negative second derivative and decreasing absolute risk aversion.

Given the market real wage W, entrepreneur i chooses how many hours of work L_i^D he will demand during period 1 in order to maximize the expected utility of his period 2 consumption. Since at period 1 he spends WL_i^D with his payroll, he will have $A - WL_i^D$ left to invest at the riskless rate r. At period 2 the amount of goods he will possess will be the sum of his firm's production $\tilde{\varepsilon}_i f(L_i^D)$ and the accrued value of his riskless investment $(A - WL_i^D)(1+r)$. His choice L_i^D is, therefore, the solution of the maximization below:

$$\max_{L_i^D} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{\varepsilon_i - \mu}{\sigma}\right)^2} U\left[\varepsilon_i f(L_i^D) + (A - WL_i^D)(1+r)\right] d\varepsilon_i \qquad (1)$$

The properties of the choice L_i^D are presented in proposition 1.

• Proposition 1 — The solution to (1) is a continuously differentiable function $L^D: \Re^5 \to \Re$, $L^D = L^D(W, \mu, \sigma, A, r)$ with the following partial derivatives:

a) $\partial L^D / \partial W < 0$, $\partial L^D / \partial \mu > 0$, $\partial L^D / \partial \sigma < 0$, $\partial L^D / \partial A > 0$;

b) if entrepreneurs borrow resources at the riskless rate, i.e., if $A - WL^D < 0$, then $\partial L^D / \partial r < 0$

Proof : See the appendix.

The partial derivatives above show that a non-satiable risk averse entrepreneur will demand more labor, i.e., allocate a higher share of his available resources A to the risky investment opportunity, the lower the cost of labor (wage), the higher the expected productivity of labor, the lower the underlying risk. Since he becomes less risk-averse the richer he gets, the higher the amount of available resources he has, the higher his demand for labor.

The partial derivative with respect to the riskless rate r is indeterminate. On the one hand the higher this rate, the higher the opportunity cost of investing in the risky business, which tends to reduce the demand for labor. On the other hand, the higher the riskless rate, the higher is the (future) value of the entrepreneurs' available resources in the second period. Hence, a higher interest rate has an income effect of increasing his wealth. Since entrepreneurs are assumed to have decreasing absolute risk aversion, a higher interest rate increases their will-

ingness to run risks, which tends to increase the demand for labor. If the entrepreneur's relative risk aversion were constant, this income effect would be null and an increase in the riskless rate would reduce the demand for labor. If the solution of (1) is such that the entrepreneur finances part of his risky business with borrowed money, then the higher the riskless rate the lowest his demand for labor, since an increase in r would increase his exposure to risk.

The risk aversion hypothesis also implies that given the market wage W, each entrepreneur will hire labor at an amount sufficiently low such that:

a) the expected present value of the marginal productivity of labor be greater than the real wage;

b) the expected rate of return on each unit of resource ploughed into his business be greater than the riskless rate;

These results are presented in proposition 2:

• Proposition 2 — Given the market real wage W, the demand for labor L^D is such that:

$$\frac{\mu\, f'[L^D]}{1+r} > W ;$$ (2.a)

$$\frac{\mu\, f(L^D) - WL^D}{WL^D} > r$$ (2.b)

Proof: See the appendix.

General equilibrium

The market real wage $W*$ is competitively determined in the labor market such that the aggregate supply of labor $HL^s(W*)$ be equal to the aggregate demand $IL^D(W*, \mu, \sigma, A, r)$:

$$\lambda L^s(W*) = L^D(W*, \mu, \sigma, A, r)$$ (3)

where $\lambda = H/I$ is the number of workers hired by each entrepreneur. The properties of the equilibrium real wage $W*$ are presented in proposition 3.

• Proposition 3 — The equilibrium real wage $W*$ is a continuously differentiable function, $W*: \Re^5 \to \Re$, $W* = W*(\mu, \sigma, A, r, \lambda)$ with the following partial derivatives:

a) $\partial W*/\partial \mu > 0$, $\partial W*/\partial \sigma < 0$, $\partial W*/\partial A > 0$, $\partial W*/\partial \lambda < 0$;

b) if entrepreneurs borrow resources at the riskless rate, i.e., if $A - W*L^D < 0$, then $\partial W*/\partial r < 0$.

Proof: See the appendix.

The above partial derivatives are straightforward consequences of the demand side of the market. The equilibrium real wage increases (decreases) when the demand for labor also increases (decreases).

Labor's share in output

At period 2 the i-th entrepreneur will have a level of production of $\tilde{\varepsilon}_i f(L_i^D)$ units of the good. The national product Y is the sum over i of each firm's output:

$$Y = \sum_{i=1}^{I} \tilde{\varepsilon}_i f(L_i^D) = f(L^D) \sum_{i=1}^{I} \tilde{\varepsilon}_i$$

where the second equality follows from the fact that all entrepreneurs employ the same amount of labor. The hypothesis that the random variables are independently distributed across entrepreneurs and that the number of entrepreneurs is sufficiently large gives, by the Law of Large Numbers:

$$\frac{1}{I} \sum_{i=1}^{I} \tilde{\varepsilon}_i = \mu$$

These last two expressions give the national product:

$$Y = I\mu f(L^D)$$

The labor income is the aggregate payroll $IL^D W*$. Thus the labor's share in output θ is given by:

$$\theta = \frac{I L^D W*}{I\mu f(L^D)} = \frac{W*(\mu, \sigma, A, r, \lambda) L^D[W*(\mu, \sigma, A, r, \lambda), \mu, \sigma, A, r]}{\mu f\big(L^D[W*(\mu, \sigma, A, r, \lambda), \mu, \sigma, A, r]\big)}$$

The properties of the labor's share in output are given in proposition 4.

• Proposition 4 — The labor's share in output θ is a continuously differentiable function $\theta : \mathfrak{R}^5 \to \mathfrak{R}$, $\theta = \theta(\mu, \sigma, A, r, \lambda)$, with the following partial derivatives:

a) $\partial\theta / \partial\sigma < 0$, $\partial\theta / \partial A > 0$, $\partial\theta / \partial\lambda < 0$;

b) if entrepreneurs borrow resources at the riskless rate, i.e., if $A - WL^D < 0$, then $\partial \theta / \partial r < 0$.

Proof: See the appendix.

The partial derivatives above show that the labor's share in output decreases when the risk born by entrepreneurs increases. The impact of μ on the labor's share in output is uncertain. On the one hand it increases the demand for labor and the wage; on the other, it also increases the aggregate output through its direct impact on the average productivity as well as on the larger level of employment.

The results above are illustrated in graph 1. The vertical axis represents the wage and the horizontal axis the labor supply and demand for each firm. If there were no risk in the production technology, i.e., if σ were zero, than the demand for labor would be given by the usual first order condition in which the wage is equal to the (present value of the) marginal productivity of labor:

$$W^* = \mu f'(L^D) / (1 + r)$$

According to proposition 2, the demand for labor for a positive σ, i.e., L^D is a curve below the previous one. The labor supply L^s is an upward sloping curve, and the equilibrium wage W^* is determined at the point where L^s crosses L^D.

Graph 1
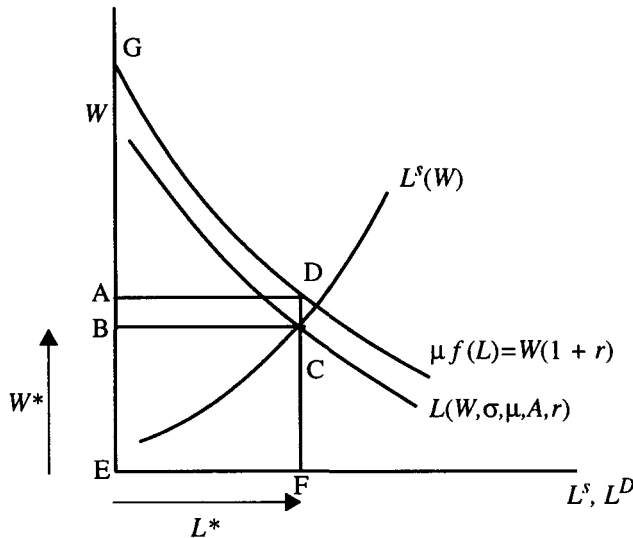
The (average) output per firm y is given by:

$$y = Y\!\!\big/\!\!_I = \mu\, f(L^*) = \mu \int_0^{L^*} f'(L)\, dL$$

This is represented by the area below the curve $\mu f'(L) = W(1+r)$ and the horizontal axis up to the point where $L = L^*$ (points *GEFD*). The payroll per firm is given by the area of the rectangle *BEFC*. The area of the rectangle *ABCD* represents the risk premium implicitly paid by workers to entrepreneurs who bear the risk of producing goods. The area *GAD* represents the pure profits received by entrepreneurs due to the ownership of the technology. An increase of σ shifts the demand $L(W, \sigma, \mu, A, r)$ downwards. This reduces the ratio of the area of rectangle *BEFC* to *GEFD*.

3. The extended model

The basic model of section 2 has a unique market clearing variable, namely the real wage. The risk premium implicitly paid to the risk bearers was given from workers to entrepreneurs solely through this variable. In this section, the riskless rate r is endogenized so that a second variable clearing a savings market is introduced. The goal is to ascertain whether the impact of the entrepreneurial risk on the workers' share in output is changed when a more complete setting is allowed for.

Workers' behavior

In this section each worker works at period 1 and consumes at periods 1 and 2. He supplies labor competitively according to a consumption-savings-leisure decision: L^s hours of work during period 1 gives WL^s units of the good. If he consumes C_1 units at period 1 he will have $(WL^s - C_1)(1+r)$ units for consumption at period 2. Each worker's endowment of time as of period 1 is $\overline{L}$. Thus he has to decide how many hours of leisure $\ell^D = \overline{L} - L^s$ he will demand at period 1, and how many units C_1 and C_2 of the good he will consume at periods 1 and 2, subject to the budget constraint below:

$$\overline{L}(1+r)W = \ell^D W(1+r) + C_1(1+r) + C_2$$

Increases of the real wage have both income and substitution effects. The income effect is the fact that the worker feels richer and, therefore, tends to increase his demand for leisure and consumption in both periods. The substitution effect, on the other hand, makes leisure relatively more expensive *vis-à-vis* consumption, which tends to reduce leisure.

Likewise, increases of the interest rate have both income and substitution effects. The income effect, which tends to increase the worker's demand for leisure and consumption in both periods, is the fact that the higher the interest rate, the richer a worker feels since his consumption possibilities increase. On the other hand, a higher interest rate makes leisure and period 1 consumption more expensive relative to period 2 consumption, which tends to reduce the former.

It is assumed that leisure and periods 1 and 2 consumption are not Giffen goods so that the substitution effects are larger than the income effects. This means that an increase in the wage increases period 1 consumption and reduces the demand for leisure, i.e., increases the

supply of labor. An increase in the interest rate reduces period 1 consumption and leisure, i.e., increases the supply of labor:

$$L^S = L^S(W, r), \quad L^S_w \geq 0 \quad \text{and} \quad L^S_r \geq 0$$

$$C_1 = C_1(W, r), \quad C_{1w} \geq 0, \quad \text{and} \quad C_{1r} \leq 0$$

Entrepreneurs' behavior

Entrepreneurs face the same choices of the basic model. The only difference is that instead of borrowing or lending at the international (exogenous) riskless rate r, they will not have access to foreign markets and r will represent the home interest rate. While making his choice of L^D, each entrepreneur takes the interest rate r as given. But r will be determined endogenously in general equilibrium. Propositions 1 and 2 still hold but for this reinterpretation of the riskless rate r.

General equilibrium

The market real wage W^* and the interest rate r^* are competitively determined in the labor and loan markets. The workers' supply of loans is $H(WL^S - C_1)$ and the entrepreneur's supply is $I(A - WL^D)$:

$$H(WL^D - C_1) + I(A - WL^D) = 0$$

The labor market equilibrium requires $HL^s = IL^D$; the equilibrium real wage W^* and riskless rate r^* in both labor and loans markets are determined by the following two equations:

$$\lambda L^s(W^*, r^*) = L^D(W^*, r^*, \mu, \sigma, A) \tag{4}$$

$$\lambda C_1(W^*, r^*) = A \tag{5}$$

Since workers must save something in order to consume at period 2, it follows that their savings will be lent to entrepreneurs. Therefore condition (b) of proposition 1 will hold and the demand for labor will be lower the higher the interest rate. Workers are now partially financing their own wages. Proposition 5 substitutes for proposition 3 of the basic model.

• Proposition 5 — The equilibrium real wage W^* and interest rate r^* are continuously differentiable functions $W^*: \Re^4 \to \Re$, $r^*: \Re^4 \to \Re$, $W^*(\mu, \sigma, A, \lambda)$, $r^*(\mu, \sigma, A, \lambda)$, with the following partial derivatives:

$$\partial W^*/\partial \mu > 0, \qquad \partial W^*/\partial \sigma < 0, \qquad \partial W^*/\partial A > 0, \qquad \partial W^*/\partial \lambda < 0$$

$$\partial r^*/\partial \mu > 0, \qquad \partial r^*/\partial \sigma < 0, \qquad \partial r^*/\partial A \leq or \geq 0 \qquad \partial r^*/\partial \lambda \leq or \geq 0$$

Proof: See the appendix.

The results above show that the higher the expected productivity of labor, the higher will be the equilibrium values of factor prices. On the other hand, the higher the risk borne by firms, the lower will be those prices, for the risk-averse entrepreneurs will require a premium in order to run the risk of employing workers as well as of borrowing resources. The impact of the entrepreneurs' available resources on the wage is positive because the entrepreneurs become less risk-averse, and hence less loath to borrow, the richer they are. The impact on the interest rate is ambiguous because the higher those resources, the higher the aggregate supply of loans.

Labor's share in output

As in section 2 the labor's share in output is the aggregate payroll divided by the aggregate output:

$$\theta^* = \frac{I L^D W}{I \mu f(L^D)} = \frac{W^*(\mu,\sigma,A,\lambda) L^D [W^*(\mu,\sigma,A,\lambda), r^*(\mu,\sigma,A,\lambda),\mu,\sigma,A]}{\mu f \left(L^D [W^*(\mu,\sigma,A,\lambda), r^*(\mu,\sigma,A,\lambda),\mu,\sigma,A] \right)} \tag{6}$$

The effect of the exogenous parameters on θ are qualitatively the same as those found in the basic model. The results are summarized in proposition 6.

• Proposition 6 — The labor share in output in the extended model is a continuously differentiable function $\theta^*:\Re^4 \to \Re$, $\theta^* = \theta^*(\mu,\sigma,A,\lambda)$, with the following partial derivatives:

$$\partial\theta^*/\partial\sigma < 0 \qquad \partial\theta^*/\partial A \geq 0 \ or \ \leq 0 \qquad \partial\theta^*/\partial\lambda \geq 0 \ or \ \leq 0$$

Proof: See the appendix.

The results above show that the introduction of a second endogenous variable in the general equilibrium, namely the interest rate, does not modify the impact of the risk borne by firms on the workers' share in output. Actually, it just provides the economy with a second means of conveying the risk premium from the workers to the entrepreneurs.

4. Conclusion

The model presented shows that, in an environment where markets are incomplete, firms that run the risks of producing goods charge a premium via low wages. If one interprets the parameter σ as a measure of the risk faced by the Brazilian entrepreneurs, the conclusion is that the adoption of coherent policies designed to reduce the "Brazil risk" would be represented by a reduction of this parameter. The model suggests that this would produce an (once-and-for-all) increase of real wages and of the full-employment output, and reduce income in-

equalities. Hence, the model suggests that the "Brazil risk" is partly to blame for the low wages in the Brazilian economy and the low share of workers in the GNP. Needless to say, there are other effects that contribute to these facts, such as the high taxation on the payroll and the low level of education, that were not addressed by the paper.

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Appendix

Proposition 1

Proof — The maximization (1) can be rewritten with $x = \dfrac{\varepsilon - \mu}{\sigma}$ to yield:

$$\max_{L} \int_{-\infty}^{+\infty} e^{-x^2/2} U[(\sigma x + \mu) f(L) + (A - WL)(1 + r)] dx$$

The first order condition is:

$$\int_{-\infty}^{+\infty} e^{-x^2/2} U'[(x)][(\sigma x + \mu) f'(L) - W(1 + r)] dx = 0$$

where:

$$(x) = (\sigma x + \mu) f(L) + (K - WL)(1 + r)$$

The second order condition is:

$$I(L) = \int_{-\infty}^{+\infty} e^{-x^2/2} \{ U''[(x)][(\sigma x + \mu) f'(L) - W(1 + r)]^2 +$$
$$U'[(x)](\sigma x + \mu) f''(L) \} dx < 0$$

From $U'' < 0, f'' 0, W(1 + r) > 0$ and the first order condition, it follows that $I(L) < 0$. The implicit function theorem applied to the first order condition yields:

$$-I(L) dL = I(\mu) d\mu + I(W) dW + I(\sigma) d\sigma + I(A) dA + I(r) dr$$

where:

$$I(\mu) = \int_{-\infty}^{+\infty} e^{-x^2/2} \{ U''[(x)] f(L)[(\sigma x + \mu) f(L) - W(1 + r)] + U'[(x)] f'(L) \} dx$$

$$I(W) = -\int_{-\infty}^{+\infty} e^{-x^2/2} \{ U''[(x)](1 + r) L[(\sigma x + \mu) f'(L) - W(1 + r)] + U'[(x)](1 + r) \} dx$$

$$I(\sigma) = \int_{-\infty}^{+\infty} e^{-x^2/2} x \{ U''[(x)] f(L)[(\sigma x + \mu) f'(L) - W(1 + r)] + U'[(x)] f'(L) \} dx$$

$$I(A) = (1 + r) \int_{-\infty}^{+\infty} e^{-x^2/2} U''[(x)][(\sigma x + \mu) f'(L) - W(1 + r)] dx$$

$$I(r) = \int_{-\infty}^{+\infty} e^{-x^2/2} \{(A - WL)U''[(x)][(\sigma x + \mu)f'(L) - W(1 + r)] - U'[(x)]W\}dx$$

Lemmas 4, 5, 6 and 7 prove that $I(\mu) > 0$, $I(W) < 0$, $I(\sigma) < 0$, $I(A) > 0$. Lemma 8 proves that if $A - WL < 0$ then $I(r) < 0$. Lemmas 1, 2 and 3 are used to prove the other Lemmas.

• Lemma 1

Define the constant $\bar{x} \in \Re$ such that $(\sigma\bar{x} + \mu)f'(L) - W(1 + r) = 0$
Then $\bar{x} < 0$:

Proof — From $\int_{-\infty}^{+\infty} e^{-x^2/2} x \, dx = 0$, it follows that:

$$-\bar{x}\int_{-\infty}^{+\infty} e^{-x^2/2} dx = \int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})dx$$

It will be shown that the integral on the right hand side of the above equality is positive. In order to avoid cluttering with too many symbols, define:

$$U'[(\bar{x})] = U'[(\sigma\bar{x} + \mu)f'(L) + (A - WL)(1 + r)]$$

Taking any $x < \bar{x}$ and recalling that $U'' < 0$, one finds that: $U'[(x)] > U'[(\bar{x})]$. From $(x - \bar{x}) < 0$ it follows that: $U'[(x)](x - \bar{x}) < U'[(\bar{x})](x - \bar{x})$, for all $x < \bar{x}$. By the same token, taking any $x > \bar{x}$ and recalling that $U'' < 0$, one has that: $U'[(x)] < U'[(\bar{x})]$. From $(x - \bar{x}) > 0$ it follows that: $U'[(x)](x - \bar{x}) < U'[(\bar{x})](x - \bar{x})$ for all $x > \bar{x}$. This proves that for any $x \in \Re$ the inequality below holds:

$$U'[(\bar{x})](x - \bar{x}) \geq U'[(x)](x - \bar{x}), \text{ with equality only when } x = \bar{x}. \qquad (*)$$

Multiplying (*) by $\sigma f'(L)e^{-x^2/2}$ and integrating, one can write:

$$= \sigma f'(L)\int_{-\infty}^{+\infty} e^{-x^2/2}U'[(\bar{x})](x - \bar{x})dx > \sigma f'(L)\int_{-\infty}^{+\infty} e^{-x^2/2}U'[(x)](x - \bar{x})dx =$$

$$\int_{-\infty}^{+\infty} e^{-x^2/2}U'[(x)][(\sigma x + \mu)f'(L) - W(1 + r)]dx = 0$$

where the quality to zero is given by the first order condition. Dividing through by $\sigma f'(L)U'[(\bar{x})]$ gives:

$$\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})dx > 0$$

which completes the proof.

• Corolary

The definition of $\bar{x}$ implies: $\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})U'\big[(x)\big]dx = 0 \cdot$

• Lemma 2

Define $Z = \lim\limits_{x \to +\infty} e^{-x^2/2}(x - \bar{x})U'[(x)] - \lim\limits_{x \to -\infty} e^{-x^2/2}(x - \bar{x})U'[(x)]$. Then $Z \geq 0$

Proof — Since $\lim\limits_{x \to +\infty} e^{-x^2/2} = \lim\limits_{x \to -\infty} e^{-x^2/2}$, Z can be rewritten as:

$$Z = \lim\limits_{x \to +\infty} e^{-x^2/2}\left\{ \lim\limits_{x \to +\infty}(x - \bar{x})U'[(x)] - \lim\limits_{x \to -\infty}(x - \bar{x})U'[(x)] \right\}$$

The same argument used in the proof of lemma 1 can be repeated in order to find that for any x_1 and x_2 such that $x_1 < \bar{x} < x_2$, the following inequality holds:

$$(x_1 - \bar{x})U'[(x_1)] < 0 < (x_2 - \bar{x})U'[(x_2)]$$

letting $x_1 \to -\infty$ and $x_2 \to +\infty$ the proof is completed.

• Lemma 3

The following inequality holds: $\int_{-\infty}^{+\infty} e^{-x^2/2}x(x - \bar{x})U'[(x)]dx > 0$

Proof — Since $x = (x - \bar{x}) + \bar{x}$, the integral above is equal to:

$$\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})^2 U'[(x)]dx + \bar{x}\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})U'[(x)]dx$$

The first integral above is positive and the second one is zero according to the corolary. This completes the proof.

• Lemma 4:

$I(\mu) > 0$

Proof — Noting that:

$$\frac{d}{dx}[U'[(x)][(\sigma x + \mu)f'(L) - W(1+r)]] =$$

$$= \sigma\{U''[(x)]f(L)[(\sigma x + \mu)f'(L) - W(1+r)] + U'[(x)]f'(L)\}$$

one can write:

$$I(\mu) = \int_{-\infty}^{+\infty} e^{-x^2/2}\frac{1}{\sigma}\frac{d}{dx}\{U'[(x)][(\sigma x + \mu)f'(L) - W(1+r)]\}dx$$

Integration by parts gives:

$$I(\mu) = \frac{1}{\sigma}[e^{-x^2/2}U'[(x)][..]\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-x^2/2}x\,U'[(x)][..]dx]$$

where $[..] = (\sigma x + \mu)f'(L) - W(1+r) = \sigma f'(L)(x - \bar{x})$.

From lemmas 2 and 3, it follows that $I(\mu) > 0$.

• Lemma 5

$I(W) < 0$

Proof — $I(W) = -(1+r)\int_{-\infty}^{+\infty} e^{-x^2/2}\{U''[(x)]L\sigma f'(L)(x - \bar{x}) + U'[(x)]\}dx$

Integrating by parts the first integral, one finds:

$$\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})U''[(x)]dx = \frac{1}{\sigma f(L)}\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})\frac{d}{dx}U'[(x)]dx =$$

$$\frac{1}{\sigma f(L)}[e^{-x^2/2}(x - \bar{x})U'[(x)]\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-x^2/2}[1 - x(x - \bar{x})]U'[(x)]dx]$$

therefore:

$$I(W) = -(1+r)\frac{Lf'(L)}{f(L)}\left\{e^{-x^2/2}(x - \bar{x})U'[(x)]\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-x^2/2}x(x - \bar{x})U'[(x)]dx + \right.$$

$$\left.(\frac{f(L)}{Lf'(L)} - 1)\int_{-\infty}^{+\infty} e^{-x^2/2}U'[(x)]dx\right\}$$

The strict concavity of f, $f(0) = 0$ and the mean value theorem assures the existence of $\tilde{L}$, $0 \leq \tilde{L} \leq L$ such that

$$\frac{f(L)}{L} = \frac{f(L) - f(0)}{L - 0} = f'(\tilde{L}) \geq f'(L)$$

therefore $f(L) / Lf'(L) \geq 1$. Thus, from lemmas 2 and 3 it follows that $I(W) < 0$.

• Lemma 6

$I(\sigma) < 0$

Proof — $I(\sigma) = f'(L) \displaystyle\int_{-\infty}^{+\infty} e^{-x^2/2} x\{U''[(x)]f(L)\sigma(x - \bar{x}) + U'[(x)]\}dx =$

$f'(L)\left\{ \sigma f(L)\displaystyle\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})^2 U''[(x)]dx + \sigma f(L)\bar{x}\displaystyle\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})U''[(x)]dx + \right.$

$$\left. \displaystyle\int_{-\infty}^{+\infty} e^{-x^2/2} x U'[(x)]dx \right\}$$

Integrating by parts the second integral:

$$I(\sigma) = f'(L)[\sigma f(L)\displaystyle\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})^2 U''[(x)]dx + \bar{x}e^{-x^2/2}(x - \bar{x})U'[(x)]\Big|_{-\infty}^{+\infty} +$$

$$\bar{x}\displaystyle\int_{-\infty}^{+\infty} e^{-x^2/2} x(x - \bar{x})U'[(x)]dx + \displaystyle\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})U'[(x)]dx] < 0$$

where the inequality is assured by lemmas 1, 2, and 3 and the corolary.

• Lemma 7

$I(A) > 0$

Proof — Define $g: \Re \to \Re$, such that $g(x) = -\dfrac{U''[(x)]}{U'[(x)]}$. The hypothesis about U assures that: $g(x) > 0$, $g'(x) < 0$. Then, the mathematical expression of $I(A)$ is given by:

$$-I(A) = (1 + r)\sigma f'(L)\displaystyle\int_{-\infty}^{+\infty} e^{-x^2/2} U'[(x)](x - \bar{x})g(x)dx$$

Since $g' < 0$, the same argument used in lemma 1 can be used to conclude that for any $x \in \Re$, $(x - \bar{x})g(x) \leq (x - \bar{x})g(\bar{x})$, where the equality holds only if $x = \bar{x}$.

Therefore:

$$-I(A) < (1+r)\sigma f'(L)g(\bar{x})\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})U'[(x)]dx = 0$$

where the equality to zero is assured by the corolary.

• Lemma 8

If $A - WL < 0$, then $I(r) < 0$.

$$\text{Proof} \;—\; I(r) = \frac{A - WL}{1+r}I(A) - W\int_{-\infty}^{+\infty} e^{-x^2/2}U'[(x)]dx < 0$$

where the inequality is assured by lemma 7.

Proposition 2

Proof of (a) — From lemma 1:

$$\mu f'(L) - W(1+r) = -\sigma\bar{x}f'(L) > 0$$

Proof of (b) — In order for maximization (1) to have a positive solution $L^D > 0$, it is necessary that the expected period 2 amount of goods be higher with $L^D > 0$ than with $L^D = 0$. That is:

$$\mu f(L^D) + (1+r)(A - WL) > (1+r)A$$

which implies inequality (b).

Proposition 3

Proof — Apply the implicit function theorem to the labor market equilibrium equation (3) and use the partial derivatives given by proposition 1.

Proposition 4

Proof — The proof given below for σ can easily be repeated for μ, λ and r:

$$\frac{d\theta}{d\sigma} = \frac{1}{\mu f(L^D)^2}[L^D f(L^D)\frac{dW}{d\sigma} + W(f(L^D) - L^D f'(L^D))\frac{dL^D}{d\sigma}]$$

From proposition 3, $dL^D / d\sigma$ is given by:

$$\frac{dL^D}{d\sigma} = L_\sigma + L_W \frac{dW}{d\sigma} = L_\sigma^D + L_W^D \frac{L_\sigma^D}{\lambda L_W^S - L_W^D} = \frac{\lambda L_W^S L_\sigma^D}{\lambda L_W^S - L_W^S} < 0$$

Since $dW / d\sigma < 0$ and from lemma 5 $f(L^D) > L^D f'(L^D)$, it follows that $d\theta / d\sigma < 0$.

Proposition 5

Proof — Apply the implicit function theorem to the system of equations (4) and (5).

Proposition 6

Proof — Apply the implicit function theorem to equations (4), (5) and (6).