A Selling Mechanism*

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Summary: 1. Introduction; 2. The assignment game; 3. The selling mechanism; 4. The equilibria of the selling mechanism; 5. Some additional examples.

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For the assignment game, we propose the following selling mechanism: sellers, simultaneously, fix their prices first; then buyers, sequentially, decide which object to buy, if any. The first phase of the game determines the potential prices, while the second phase determines the actual matching. We prove that the set of subgame perfect equilibria in pure strategies in the strong sense of the mechanism coincides with the set of sellers' optimal stable outcomes when buyers use maximal strategies.

Propomos o seguinte mecanismo para o assignment game: os vendedores, simultaneamente, fixam seus preços. Então, um após o outro, os compradores decidem que objetos comprar, se houver algum. A primeira fase do jogo determina os preços dos objetos que forem vendidos, enquanto a segunda fase determina o matching real. Provamos que o conjunto dos equilíbrios perfeitos de subjogo em estrategias puras no sentido forte, do jogo induzido pelo mecanismo, coincide com o conjunto dos resultados estáveis ótimos para os vendedores quando os compradores usam estratégias maximais.

1. Introduction

In their seminal contribution, Gale and Shapley (1962) introduce the study of markets where agents from two distinct groups meet. They consider situations where the agents' utilities only depend on their match, monetary transfers are

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not allowed (or they are fixed exogeneously, like in a market where every student who is matched with a college pays a tuition). The analysis of markets where agents take decisions not only on the matching but also on the monetary transfers was started in Shapley and Shubik (1972). They propose the *assignment game*, which is a market where a finite number of heterogeneous sellers and buyers meet, with the particularity that each seller only owns one object and each buyer only wants to buy, at most, one object. In both frameworks, the main solution concept is stability. An outcome is stable if it is individually rational, and no possible partnership has incentives to block the outcome.[1]

Shapley and Shubik (1972) show that the set of stable allocations in the assignment game is non-empty. Moreover, it forms a complete distributive lattice. In particular, the set of stable allocations contains one particular allocation giving an optimal payoff to the buyers (which is the worse for the sellers) and another one giving an optimal payoff to the sellers (the worst for the buyers). The two allocations correspond to the minimum and to the maximum equilibrium prices, respectively.

In this paper, we investigate the result of non-cooperative behavior by sellers and buyers in the assignment game. We propose a very simple non-cooperative mechanism and analyze the equilibrium outcomes. The *selling mechanism* goes as follows: Sellers take their decision first by, simultaneously and non-cooperatively, fixing their prices. Then buyers, sequentially, decide which object to buy, if any. The potential prices for the objects are determined by the choice of the sellers, while the buyers' actions determines the actual matching. The players' strategies are very simple: each seller only proposes a price for his object, each buyer only chooses an object.

We look for the subgame perfect equilibria in pure strategies (SPE) of the previous mechanism. We restrict attention to a certain class of strategies of the buyers, that we call maximal strategies (the matchings induced by these strategies are called maximal matchings). To explain the meaning of a maximal strategy, consider a situation with two buyers, Alph and Bob, and two objects. Suppose that the prices have already been set and that Alph is indifferent between the two objects. However, if she chooses the first object then Bob obtains a high utility by buying the second one, while if Alph chooses the second object then Bob does not want to buy the other. We say that Alph buying the first object (and Bob the

[1] Roth and Sotomayor (1990) provide an extensive overview of the results and extensions of the models proposed by Gale and Shapley (1962) and Shapley and Shubik (1972) until 1990. Recent papers with additional extensions of the models include Dutta and Massó (1997), Sotomayor (1999a,b) and Sotomayor (2003), Martínez et al. (2000) and Martínez et al. (2001), and Camiña (2002).

second) is a maximal strategy, while Alph buying the second object (and Bob not buying) is not. Notice that both strategies are SPE of the game that starts once the prices have been decided. Additionally, we say that a matching is maximal if it is Pareto efficient for the buyers among the matchings that result as SPE of the second phase of the mechanism.

We also concentrate on the analysis of the SPE in the strong sense (Demange and Gale, 1985). To be an equilibrium in the strong sense, the strategies of the participants must be robust to deviations by any optimistic seller. More precisely, we assume that a seller increases his price whenever there is a maximal matching for the buyers where his object is actually sold.

We prove that, when buyers use maximal strategies, the set of SPE in the strong sense of the selling mechanism coincides with the set of sellers' optimal stable outcomes. That is, the mechanism applied to an assignment problem leads to the maximum equilibrium prices and to an optimal matching. In equilibrium, every buyer obtains an object in her demand set and each seller willing to sell can do so.

Several authors have looked for mechanisms that could be applied to matching models. Gale and Shapley (1962) introduce the deferred acceptance algorithm for the marriage problem, where men constitute one side of the market and women are the other side. In this mechanism, each man[2] proposes to his favorite woman, if she is acceptable to him. Each woman accepts the most preferred man among the offers she receives, if he is acceptable to her. Accepted men remain provisionally engaged, while rejected men can make new proposals to their next choice. The algorithm stops at the first step in which no man is rejected. Gale and Shapley (1962) show that when participants declare their true preferences the matching produced by the algorithm is an allocation that all men prefer to any other stable allocation. Even when the participants can act strategically, the outcome is still nice: truthful revelation of preferences is a dominant strategy for men (Dubins and Freedman, 1981) and (Roth, 1982) and the equilibrium of the game when the men state their true preferences and women can choose any preference is still a stable allocation (Roth, 1984). Moreover, Gale and Sotomayor (1985) show that the women's optimal stable allocation is the strong equilibrium of the game, when men play their dominant strategy. Finally, Crawford and Knoer (1981) and Kelso and Crawford (1982) adapt the mechanism for the job matching market by introducing a salary-adjustment process.

Besides the analysis of the deferred acceptance algorithm, several authors have looked for other simple mechanisms that lead to stable allocations for different

[2]The mechanism can also be implemented exchanging the roles of men and women.

matching models. For the marriage problem, Alcalde (1996) presents a mechanism, close to that of Gale and Shapley, which implements the correspondence of stable matchings in undominated equilibria. Alcalde et al. (1998) and Alcalde and Romero-Medina (2000) implement through simple mechanisms the set of stable matchings in the job matching market and in the college-admissions problem, respectively. Moreover, they also implement particular subsets of the stable correspondence.[3]

For the assignment game, Demange and Gale (1985) analyze the properties of a mechanism in which agents announce their demand and supply functions, and then a referee calculates the minimum equilibrium price and allocates the objects accordingly. For the buyers, the mechanism is not coalitionally manipulable in the sense that no coalition of buyers can achieve higher payoffs to all of its members by falsifying demands. However, the mechanism is manipulable for the sellers. It is indeed the case that sellers can lead the payoff to the maximum rather than the minimum equilibrium price by falsifying their supply functions. For the same game, Demange et al. (1986) propose two dynamic auction mechanisms, although they do not analyze the possibility of manipulative behavior.

The mechanism that we introduce in this paper shares many features with the one that we propose in Pérez-Castrillo and Sotomayor (2002). The main difference between the two proposals is that in the previous one, the buyers, in addition to deciding which object to by, were asked to report their indifferences (along with the previous buyers' indifferences) to the following buyer in the line. This allowed any buyer to break previous buyers' indifferences in her favor. Therefore, the mechanism that we present in this paper is simpler and closer to the functioning of markets than the previous proposal. However, in order to obtain the implementation result, we need to concentrate in maximal strategies and in equilibria in strong sense.

Finally, our paper is also related to the contribution by Kamecke (1989). This author introduces the following mechanism for the assignment game: First, sellers announce their payoff claims. Second, one buyer after the other chooses a seller and announces a price. Finally, sellers select one of their potential customers. For matched couples, the agents get what they asked for if their claims are feasible. Also, the payoff function assigns to a seller the payoff that was offered to him if it exceeds his claim. Additionally, agents pay a positive cost if they address demands without being successful. This mechanism implements in SPE the seller-optimal stable payoff.

[3]For general cooperative games in characteristic form, Pérez-Castrillo (1994), Perry and Reny (1994), and Serrano (1995) address the question of implementation of the core

The paper is organized as follows. In section 2, we present the assignment game. In section 3, we introduce the selling mechanism, that is analyzed in section 4. In section 5, we present some examples of possible outcomes of the mechanism when we do not restrict attention to maximal strategies or to equilibria in the strong sense. Finally, an appendix contains a technical proof.

2. The Assignment Game

In this section, we describe the assignment game, the solution concepts usually considered, and summarize some of its main properties. In this market, there are two disjoint set of economic agents. On one side, there is a set of buyers $P = \{p_1, ..., p_m\}$. Generic buyers will be denoted by p_i and p_k. Each buyers wants to buy at most one object. On the other side, the set of sellers is $\{q_1, ..., q_{n-1}\}$, where each seller owns only one indivisible object. Generic sellers will be denoted by q_j and q_h. We denote by $Q = \{q_0, q_1, ..., q_{n-1}\}$ the set of objects available in the market, where q_0 is an artificial "null object" that is introduced for technical convenience. This convention allows us to treat a buyer p_i that does not buy any object as if she bought the null object q_0.

The value of any partnership $(p_i, q_j) \in P \times Q$ is $\alpha_{ij} \geq 0$, which can be interpreted as the maximum price that buyer p_i is willing to pay for the object q_j since we normalize the reservation price of each seller to zero. If buyer p_i buys the object q_j at a price v_j then the resulting utilities are $u_i = \alpha_{ij} - v_j$ for the buyer and v_j for the seller. We denote by α the $m \times n$ matrix $(\alpha_{ij})_{i=1,...,m;j=0,1,...,n-1}$, where the value α_{i0} is zero to all buyers. The price of the object q_0 is always zero, $v_0 = 0$. Hence if buyer p_i buys q_0 she obtains a utility $u_i = \alpha_{i0} - v_0 = 0$. The buyer-seller market is denoted by $M \equiv (P, Q, \alpha)$.

Definition 1 *A feasible matching* μ for M is a function from $P \cup Q - \{q_0\}$ onto $P \cup Q$ such that:

- for any $p_i \in P$, $\mu(p_i) \in Q$;
- for any $q_j \in Q - \{q_0\}$, either $\mu(q_j) \in P$ or $\mu(q_j) = q_j$; and
- for any $(p_i, q_j) \in P \times Q - \{q_0\}$, $\mu(p_i) = q_j$ if and only if $\mu(q_j) = p_i$.

If $\mu(p_i) = q_0$, the buyer p_i is unmatched. If $\mu(q_j) = q_j$, the seller q_j is unmatched (or the object q_j is unsold).

Definition 2 *A feasible matching* μ is *optimal* for M if for all feasible matching μ':

$$\sum_{\substack{p_i \in P \\ q_j = \mu(p_i)}} \alpha_{ij} \geq \sum_{\substack{p_i \in P \\ q_j = \mu'(p_i)}} \alpha_{ij}$$

We denote by R_+^n the set of vectors in R^n with non negative coordinates.

Definition 3 A *feasible outcome* $(u, v; \mu)$ for M is a pair of vectors $u \in R_+^m$ and $v \in R_+^n$ and a feasible matching μ such that, for all $(p_i, q_j) \in P \times Q$, $u_i + v_j = \alpha_{ij}$ if $\mu(p_i) = q_j$.

Note that a feasible outcome may have unsold objects with price greater than zero. If $(u, v; \mu)$ is a feasible outcome then (u, v) is called a *feasible payoff.* The matching μ is said to be *compatible* with (u, v) or with the prices v and vice-versa. The vector u is called the *payoff vector of the buyers associated to* (v, μ).

Definition 4 Given the prices $v \in R_+^n$ and a matching μ compatible with v, we say that an object q_j is *μ-expensive under* v if it is unsold under μ, at a price $v_j > 0$.

The next definition introduces the main solution concept for the assignment game.

Definition 5 A feasible outcome $(u, v; \mu)$ is *stable* (or the payoff (u, v) with the matching μ is stable) if $u_i + v_j \geq \alpha_{ij}$ for all $(p_i,\ q_j) \in P \times Q$ and there is no *μ-expensive* object under v.

If $u_i + v_j < \alpha_{ij}$ for some pair (p_i, q_j) we say that (p_i, q_j) *blocks* the outcome $(u, v; \mu)$ or the payoff (u, v). Given this definition of blocking, a feasible outcome is stable if it is not blocked by any buyer-seller pair.

Among the set of stable outcomes, two particularly interesting outcomes can be highlighted.

Definition 6 The payoff $(\underline{u}, \overline{v})$ is called the *seller-optimal stable payoff* if $\overline{v} \geq v$ and $\underline{u} \leq u$ for all stable payoffs (u, v).

If μ is compatible with $(\underline{u}, \overline{v})$ we say that the outcome $(\underline{u}, \overline{v}; \mu)$ is the *seller-optimal stable outcome with the matching* μ. Similarly we can define the *buyer-optimal stable payoff.*

In the assignment model, the concept of stability is equivalent to the concept of the core. Moreover, it is possible to establish a relationship between stable outcomes and competitive equilibria of these markets. In order to do it, let us define the *demand set of a buyer p_i at prices $v \in R^n_+$*, denoted by $D_i(v)$, as the set of all objects which maximize p_i's utility payoffs. That is:

$$D_i(v) = \{q_j \in Q; \alpha_{ij} - v_j \geq \alpha_{ih} - v_h \text{ for all } q_h \text{ in } Q\}$$

The set $D_i(v)$ is always non empty, since buyer p_i has always the option of buying q_0. Also notice that, given v, buyer p_i is indifferent about buying any object in $D_i(v)$.

Definition 7 The price vector $v \in R^n_+$ is called *competitive* if there exists a matching μ such that $\mu(p_i) \in D_i(v)$, for all p_i in P.

Therefore, at competitive prices v, each buyer can be matched to an object in her demand set. A matching μ such that $\mu(p_i) \in D_i(v)$ for all p_i in P is said to be *competitive* for the prices v. There may be more than one competitive matching for the same price vector v.

Definition 8 The pair (v, μ) is a *competitive equilibrium* if v is competitive, μ is competitive for v, and if $v_j = 0$ for any unsold object q_j.

Thus, at a competitive equilibrium (v, μ), not only does every buyer get an object in her demand set, but there is no *μ-expensive* object under v. If (v, μ) is a competitive equilibrium, v will be called an *equilibrium price vector*.

It is easy to check that, to each competitive equilibrium (v, μ) we can associate a stable outcome $(u, v; \mu)$ and vice-versa, by setting $u_i = \alpha_{ij} - v_j$ if $\mu(p_i) = q_j$, and $u_i = 0$ if $\mu(p_i) = q_0$.

The following well-known results from Shapley and Shubik (1972) will be stated here without proof.[4]

Proposition 1 Every buyer-seller market M has at least one stable outcome.

Consequently, the core and the set of competitive equilibria are non-empty sets.

Proposition 2 If μ is an optimal matching, then it is compatible with any stable payoff.

Thus, if μ is an optimal matching, then it is competitive for any competitive equilibrium.

Proposition 3 If $(u, v; \mu)$ is a stable outcome, then μ is an optimal matching.

[4]See also Roth and Sotomayor (1990) for an exposition of these results.

According to propositions 2 and 3, the set of stable outcomes is the Cartesian product of the set of stable payoffs and the set of optimal matchings.

Proposition 4 Every buyer-seller market M has a seller-optimal stable payoff and a buyer-optimal stable payoff.

The existence of a seller-optimal stable payoff is equivalent to the statement that there is a unique vector of equilibrium prices, $\overline{v}$, that is optimal for the sellers, in the sense that $\overline{v}_j \geq v_j$ for all q_j in Q and for all equilibrium price vector v. Similar statement applies to the buyer-optimal stable payoff. The equilibrium price vector $\overline{v}$ is called the *maximum equilibrium prices* and a competitive equilibrium $(\overline{v}; \mu)$ is called a *maximum competitive equilibrium.*

3. The Selling Mechanism

We attempt to propose and analyze a mechanism for the assignment model as simple as possible, at the same time have it produce both a matching between sellers and buyers and prices for the objects sold. We suggest a mechanism with two phases, we call it the *selling mechanism.* In the first phase, each seller fixes a price at which she is ready to sell her object. Prizes are proposed simultaneously. In the second phase buyers, sequentially, decide which object to buy, if any. Any buyer pays to the owner of the object he has chosen (if any) the price that was fixed at the first phase. Sellers of unsold objects keep their objects and receive nothing.

Formally, the buyers are ordered according to some exogenous protocol. We rename the buyers so that the order is $p_1, ..., p_m$. The *selling mechanism* works as follows:

- First, sellers play simultaneously. A strategy for seller q_j consists of choosing a price $v_j \in R_+$ for his object. We consider that the null object q_0 is always available to every buyer at the price $v_0 = 0$.
- Second, buyers play sequentially. A strategy for buyer p_i is a function that selects an element of Q_i for each vector of offers v, where Q_i is the set of all objects which are still available for p_i after players $p_1, \ldots, p_{i-1}$ have already chosen (notice that $q_0 \in Q_i$).

Since two different buyers cannot choose the same object (except q_0), the buyers' actions produce a feasible matching μ, where $\mu(p_i)$ is the object of Q_i chosen by p_i. Any non-selected object will be unmatched under μ.

Given the matching μ and the proposed prices v, the mechanism allocates the objects according to μ, every buyer p_i pays v_j if $\mu(p_i) = q_j$; every seller q_j receives v_j if q_j is sold and receives nothing if q_j is unsold. That is, denoting by $S(v, \mu)$ the outcome of the selling mechanism when sellers' strategy profile is v and the buyers choose the matching $\mu, S(v, \mu) = (u, v^*; \mu)$, where $v_j^* = v_j$ if $\mu(q_j) \in P$ and $v_j^* = 0$ if q_j is unsold and $u_i = \alpha_{ij} - v_j^* = \alpha_{ij} - v_j$ where $\mu(p_i) = q_j$. Sometimes we will use the notation $S_j(v, \mu)$ for seller q_j's payoff, v_j^*.

We are going to consider the subgame perfect equilibria in pure strategies (SPE) of the selling mechanism. The set of best responses for the buyers to the sellers' joint strategies, say v, is the set of SPE of the game that starts once v has been decided. (Notice that the elements of this set are matchings). For these equilibria, it is always the case that each buyer chooses, once the prices v have been selected, one among the best objects available for her. In other words, each buyer p_i chooses an object in $D_i(v|Q_i)$, which is the set of objects in Q_i which maximize p_i's utility payoff, that is,

$$D_i(v|Q_i) \equiv \{q_j \in Q_i; \alpha_{ij} - v_j \geq \alpha_{ih} - v_h, \text{ for all } q_h \text{ in } Q_i\}$$

A matching μ obtained in this way is called *equilibrium matching for the prices v*. Formally:

Definition 9 Given the feasible price vector v, the matching μ is an *equilibrium matching for v* if every buyer p_i chooses $\mu(p_i) \in D_i(v|Q_i)$.

For some price vectors, there are several equilibrium matchings. To illustrate this situation, consider the following example:

Example 1 Consider a set of objects $Q = \{q_0, q_1, q_2\}$ and the set of buyers $P = \{p_1, p_2\}$. Let the matrix α be such that $\alpha_{11} = \alpha_{12} = \alpha_{22} = 2$ and $\alpha_{21} = 0$. Suppose the vector of prices is $v = (0, 1, 1)$ and consider the matching $\mu : \mu(p_1) = q_2$ and $\mu(p_2) = q_0$. Each buyer is selecting a best response given the prices of the objects and, for p_2, the action of p_1. Hence, μ is an equilibrium matching. Also, the matching μ' where p_1 chooses q_1 and then buyer p_2 can select q_2, is an equilibrium matching. Moreover, under μ', buyer p_1 has the same utility payoff as before but p_2 is strictly better off.

Sometimes it is reasonable to assume that if a buyer is completely indifferent among several actions, but one of them leads to a more efficient outcome for the buyers coming after her, then she will choose this action. When we restrict attention to such strategies we say that buyers are selecting maximal matchings.

Formally we have:[5]

Definition 10 Given the price vector $v \in R^n_+$, we say that μ is *maximal for* v if μ is equilibrium matching for v and it is Pareto-efficient for the buyers among all equilibrium matchings for v. That is, μ is maximal for v if and only if μ is equilibrium matching for v and if, for any other equilibrium matching $\mu^{'}$ such that $\alpha_{ij} - v_j < \alpha_{ij'} - v_{j'}$ for some p_i in P, with $q_j = \mu(p_i)$ and $q_{j'} = \mu^{'}(p_i)$, there exists some p_k in P such that $\alpha_{kh} - v_h > \alpha_{kh'} - v_{h'}$, with $q_h = \mu(p_k)$ and $q_{h'} = \mu^{'}(p_k)$.

Remark 1 From definitions 2 and 10 it follows that if v is an equilibrium price, μ is an optimal matching and u is the payoff vector of the buyers associated to (v, μ), then $u_i \geq \alpha_{ij'} - v_{j'}$ for all $p_i \in P$ and all equilibrium matching $\mu^{'}$ (where $q_j = \mu(p_i)$ and $q_{j'} = \mu^{'}(p_i)$, so μ is maximal for v. Moreover, if μ^* is a maximal matching for v, making use of the fact that v is an equilibrium price, we get that $u^*_i = \alpha_{ij^*} - v_{j^*} = \alpha_{ij} - v_j = u_i$ for all p_i in P and all optimal matchings μ (where $q_{j^*} = \mu^*(p_i)$ and $q_j = \mu(p_i)$). That is, if v is an equilibrium price vector, every maximal matching leads to the same vector of buyers' payoffs.

Let us now look at the possible strategic actions by the sellers. To analyze whether a vector of prices is part of an equilibrium, we must check that no seller is interested in deviating from his proposed price. Sometimes, a deviation by a seller is profitable or not depending on the expected reaction by the buyers. The following example illustrates this situation:

Example 2 Consider a set of objects $Q = \{q_0, q_1, q_2, q_3\}$ and a set of buyers $P = \{p_1, p_2, p_3\}$. Let the matrix α be such that $\alpha_{21} = 1$, $\alpha_{12} = \alpha_{22} = \alpha_{13} = 2, \alpha_{33} = 1$ and the other entries are zero. Suppose that the sellers choose the vector of prices $v = (0, 0, 0)$. If q_1 increases his price from $v_1 = 0$ to $v^{'}_1 = 0.5$, this deviation can be profitable for q_1 if p_1 buys q_2, for then p_2 will buy q_1. However, the deviation is not profitable if p_1 buys q_3. In this case the best response for p_2 is to buy q_2 and p_3 will be unmatched. Observe that the set of best responses for the buyers has only these two matchings and both of them are maximal (under the first matching the payoff vector of the buyers is $(2, 0.5, 1)$ and under the second one is $(2, 2, 0)$).

What buyers' behavior can seller q_1 expect in example 2? We will solve this problem by assuming that a seller who analyses the possibility of deviating takes an *optimistic view*. That is, a seller changes his strategy whenever he has a chance

[5]An equilibrium matching corresponds to a σ-competitive matching in Pérez-Castrillo and Sotomayor (2002). The set of maximal matchings is however larger than the set of σ-maximal matchings in that paper.

to be better off (always taking into account that buyers will play SPE strategies). Therefore we are looking for seller strategies under which no seller has a chance to be better off. This means that we are interested in equilibria in the strong version. (A similar concept has been defined by Demange and Gale (1985).) Also, we continue to require that the buyers use maximal strategies. The formal definition of a SPE in the strong sense is then the following:

Definition 11 Let $v \in R^n_+$ be a price vector and μ some maximal matching for v. We say that (v, μ) is a *SPE in the strong sense* if for no q_j there is a v', with $v'_h = v_h$ for $q_h \neq q_j$, and a maximal matching μ' for v', such that $S_j(v', \mu') > S_j(v, \mu)$.

Notice that by considering only the equilibria in the strong sense we restrict the set of SPE. We denote by SPESS the set of subgame perfect equilibria in the strong sense.

4. The Equilibria of the Selling Mechanism

We suppose that the buyers always choose maximal matchings and we show that, under this condition, equilibria in the strong sense always exist, and that they correspond to the maximum competitive equilibria. That is, the payoff for the sellers under any SPESS is the same, namely the maximum equilibrium price, and the utility for the buyers is also the same.

To characterize the set of SPESS outcomes of the selling mechanism, we will use proposition 5 below, which is an immediate consequence of Hall's theorem (Gale, 1960). To state it, we need the following definition:

Definition 12 Let $v \in R^n_+$, and $P' \subseteq P$ be such that $q_0 \notin D_i(v)$ for all $p_i \in P'$. We say that $D \equiv \cup_{p_i \in P'} D_i(v)$ is an *overdemanded set* under v if $|D| < |P'|$.[6]

That is, a set D is overdemanded if the number of buyers demanding only objects in D is greater than the number of objects in this set.

Proposition 5 (Corollary of Hall's theorem) Let $v \in R^n_+$. A competitive matching for v exists if and only if there is no overdemanded set under v.

We start our analysis of the SPESS of the mechanism by stating two helpful properties. Lemma 1 establishes the first. Consider a price vector which is part of such an equilibrium, and a group of buyers that obtain a strictly positive payoff and that buy objects in their demand sets. Then, there is some object which is

[6] We denote the cardinality of a set A by $|A|$.

not bought by any of the buyers in this group but which belongs to some of their demand sets. That is, the set of objects that the buyers actually buy is strictly included in the union of their demand sets. After this result, proposition 6 shows that, at an SPESS, there exists no overdemanded set.

Lemma 1 Let (v, μ) be an SPESS, and let u be the payoff vector of the buyers associated with (v, μ). Suppose that $\mu(p_i) \in D_i(v)$ and $u_i > 0$ for all $p_i \in P^{'} \subseteq P$. Let $Q^{'} \equiv \mu(P^{'}) \equiv \cup_{p_i \in P'} \mu(p_i)$. Then there exists some $p_i \in P^{'}$ and $q_j \notin Q^{'}$ such that $q_j \in D_i(v)$.

Proof We do the proof by contradiction. Suppose that for all $p_i \in P^{'}$ and all $q_j \notin Q^{'}$ we have that $q_j \notin D_i(v)$. Then $Q^{'} \equiv \mu(P^{'}) = \cup_{p_i \in P'} D_i(v)$. Let p_k be the last buyer in $P^{'}$ and $q_h = \mu(p_k)$. Since $u_k > 0$ we have that there exists some $\lambda > 0$ such that:

$$u_k - \lambda = \alpha_{kh} - (v_h + \lambda) > \alpha_{kj} - v_j \text{ for all } q_j \notin Q^{'}$$

Let v' be such that $v_h^{'} = v_h + \lambda$ and $v_j^{'} = v_j$ for all $q_j \neq q_h$. Since p_k is the last buyer in $P^{'}$, it follows that p_k will buy q_h at $v_h^{'}$ if the previous buyers follow μ. This is immediate from the previous equation and the fact that q_h is the unique available object belonging to $Q^{'}$ when p_k is called to play. Hence μ is an equilibrium matching for $v^{'}$. We now show that μ is maximal for $v^{'}$, which will contradict the fact that (v, μ) is SPESS. Suppose that μ is not maximal for $v^{'}$. Then there is some equilibrium matching $\mu^{'}$ for $v^{'}$ such that $u_i^{'} \geq u_i$ for all $p_i \in P - \{p_k\}$, $u_k^{'} \geq u_k - \lambda$, with at least one strict inequality, where $u^{'}$ is the payoff vector for the buyers associated to $(v^{'}, \mu^{'})$. Since $\mu(p_i) \in D_i(v)$ for all $p_i \in P^{'} - \{p_k\}$ and $q_j \notin D_i(v)$ for all $q_j \notin Q^{'}$ it follows that $u_i^{'} = u_i > \alpha_{ij} - v_j$ for all $p_i \in P^{'} - \{p_k\}$ and $q_j \notin Q^{'}$, so $\mu^{'}(p_i) \in Q^{'}$ for all $p_i \in P^{'} - \{p_k\}$. Also if $p_i \neq p_k$ then $\mu^{'}(p_i) \neq q_h$, for if not $\alpha_{ih} - v_h > \alpha_{ih} - v_h - \lambda = u_i^{'} = u_i \geq \alpha_{ih} - v_h$, which is a contradiction. Therefore all objects in $Q^{'} - \{q_h\}$ are already matched when p_k comes to play under $\mu^{'}$. Due to this fact, to the previous equation, and to $u_k^{'} \geq u_k - \lambda$ it follows that $\mu^{'}(p_k) = q_h$. Since λ can be taken arbitrarily small, it is then easily seen that if $\mu^{'}$ is an equilibrium matching for $v^{'}$ then $\mu^{'}$ is also equilibrium matching for v. Moreover, $u_i^{'} \geq u_i$ for all $p_i \in P$ and $u_i^{'} > u_i$ for some $p_i \in P (p_i \in P - P^{'})$, which contradicts the maximality of μ for v. Hence, μ is maximal for v', so there is a profitable deviation from v, which is a contradiction. Q.E.D.

Proposition 6 Let v be the strategies of the sellers in an SPESS for the market M. Then there is no overdemanded set under v.

Proof Denote $P^r \equiv \{p_1, ..., p_r\}$. To validate the proposition, it is enough to prove that for all $1 \leq r \leq m$, and all $P^{'} \subseteq P^r$ with $q_0 \notin D_i(v)$ for every $p_i \in P'$, it is the case that $| \cup_{p_{i \in P'}} D_i(v)| \geq |P'|$. We prove this property by induction on r. If $r = 1$ it is obvious.

Suppose that for all $P' \subseteq P^{r-1}$, with $q_0 \notin D_i(v)$ for every $p_i \in P'$, we have that $|P'| \leq | \cup_{p_i \in P'} D_i(v)|$. We first show that there exists a maximal matching μ' for v such that $\mu'(p_i) \in D_i(v)$ for all $i = 1, 2, ..., r-1$. Indeed, consider the market $M' = (P^{r-1}, Q, \alpha')$ where α' is the restriction of α to $P^{r-1} \times Q$. By the induction hypothesis there is no overdemanded set of objects in M' under v. Hall's theorem implies that there exists some competitive matching for v in M'. We will show that there is a maximal matching for $M = (P, Q, \alpha)$ whose restriction to M' is a competitive matching for v. In fact, let $S = \{\mu; \mu$ is equilibrium matching for v in M and $\mu(p_i)$ in $D_i(v)$ for all $i = 1, 2, ..., r-1\}$.

That is, S is the set of all equilibrium matchings whose restriction to M' is competitive. The set S is not empty, since the matching given by Hall's theorem can be easily extended to an equilibrium matching for v (we only need to take a best response for the buyers from p_r on, which always exists). Since S is not empty and finite, there is at least a matching $\mu' \in S$ which is Pareto-efficient for the buyers among all matchings in S. We claim that μ' is a maximal matching for v. That is, μ' is Pareto-efficient for the buyers among all equilibrium matchings. In fact, if μ' is not maximal then there is an equilibrium matching μ such that $u_i \geq u_i^{'}$ for all $p_i \in P$, with strict inequality holding for at least one buyer, where u' and u are the payoff vectors of the buyers associated with (v, μ') and (v, μ), respectively. Then $u_i = u_i^{'}$ for all $p_i \in P^{r-1}$, since all these players are maximizing their utility payoff under μ'. Hence $\mu \in S$, which contradicts the assumption that μ' is Pareto-efficient for the buyers.

We now prove the induction property for r by contradiction. Suppose that there is some $P' \subseteq P^r$ with $q_0 \notin D_i(v)$ for every $p_i \in P'$, and such that $|P'| > | \cup_{p_i \in P'} D_i(v)|$. Let $D \equiv \cup_{p_i \in P'} D_i(v)$. It follows by the induction hypothesis that $p_r \in P'$ and $|P' - \{p_r\}| \leq |D| < P'|$. Denoting $P^* \equiv P' - \{p_r\}$, we also have that $|D| = |P^*|$ and $D = \cup_{p_i \in P'} D_i(v)$. Moreover, since $\mu' \in S$ (the matching previously found), it follows that it is competitive for P^{r-1}, so $D = \mu'(P^*)$.

We can now use an argument similar to the one in lemma 1 to prove that it is the case that there is a profitable deviation from v. We only need to consider the last buyer p_k in P^* and $q_h = \mu'(p_k) \in D_k(v)$. Since $u'_k = \alpha_{kh} - v_h > 0$ we have that there exists some $\lambda > 0$ such that $u'_k - \lambda = \alpha_{kh} - (v_h + \lambda) > \alpha_{kj} - v_j$ for all $q_j \notin D$. Let v' be such that $v'_h = v_h + \lambda$, $v'_j = v_j$ for all $q_j \neq q_h$. Following the line of the proof of lemma 1, it is easily shown that μ' is maximal for v'. Therefore, v'_h is a profitable deviation from v for q_h, which contradicts the fact that v is part of an SPESS for M. Q.E.D.

Our first theorem asserts that the SPESS outcomes are competitive equilibria. The insight obtained from proposition 6 is very useful for both the understanding and the proof of theorem 1.

Theorem 1 Let (v, μ) be a SPESS. Then $S\ (v, \mu)$ is a competitive equilibrium.

Proof By proposition 6 there is no overdemanded set of objects at the prices v. Therefore, Hall's theorem guarantees that there is a matching μ' which is a competitive matching for v. Observe that μ is also a competitive matching for v. This is immediate from the fact that the buyers maximize their utility payoffs under μ' and μ is maximal. Then v is a competitive price with matching μ. Let $S(v, \mu) = (u, v^*; \mu)$, where $v^*_j = v_j$ if q_j is sold and $v^*_j = 0$ otherwise. To prove that $S(v, \mu)$ is a competitive equilibrium, we have to show that v^* is a competitive price vector with matching μ. Since there are no μ-expensive objects under v^* we only need to show that if q_j is unsold and $v_j > v^*_j = 0$, then $u_i \geq \alpha_{ij} - v^*_j$ for all $p_i \in P$. In fact, if there exists some unsold object, say q_j, such that $v_j > v^*_j = 0$ and $u_i < \alpha_{ij} - v^*_j$ for some $p_i \in P$, we can choose $\lambda > 0$ and $\gamma > 0$ such that $u_i + \lambda = \alpha_{ij} - \gamma$. Let v' be defined by $v'_j = \gamma$ and $v'_h = v_h$ if $q_h \neq q_j$. Since v is part of an SPESS it follows that q_j is unsold at any maximal matching for v'. However, $\max_{h \neq j} \alpha_{ih} - v_h = u_i < \alpha_{ij} - v'_j$, hence buyer p_i is not playing her best response at any matching in which q_j is unsold, which is a contradiction. Therefore, $S(v, \mu)$ is a competitive equilibrium. Q.E.D.

Theorem 1 ensures that only competitive equilibria are candidates for an SPESS of the selling mechanism. Theorem 2 goes a step further: only the maximum equilibrium prices can be part of an SPESS of the mechanism. To prove the theorem, we will use the following lemma:

Lemma 2 Let (v^1, μ^1) be an SPESS and set $S(v^1, \mu^1) \equiv (u^1, v*; \mu^1)$. Let (u^2, v^2, μ^2) be some feasible outcome and $Q^+ = \left\{q_j \in Q; \mu^2(q_j) \in P \text{ and } v^2_j > v*_j\right\}$. If $Q^+ \neq \emptyset$ then there exists some pair $(p_i, q_h) \in P \times Q$ such that $u^2_i + v^2_h < \alpha_{ih}$.

Proof *Case 1.* $\mu^1(Q^+) \neq \mu^2(Q^+)$. Since every seller in Q^+ is matched by μ^2, choose $p_i \in \mu^2(Q^+) - \mu^1(Q^+)$, say $p_i = \mu^2(q_j)$. By theorem 1 $(u^1, v^*; \mu^1)$ is stable. It then follows that $0 \leq u^2_i < u^1_i$, for if not $\alpha_{ij} = u^2_i + v^2_j > u^1_i + v^*_j$, which would contradict the stability of $(u^1, v^*; \mu^1)$. Hence p_i is matched under μ^1, say

$p_i = \mu^1(q_j)$, where $q_j \notin Q^+$. Then $\alpha_{ih} = u_i^1 + v_h^* > u_i^2 + v_h^2$, and the assertion is proved.

Case 2. $\mu^2(Q^+) = \mu^1(Q^+)$. By the definition of Q^+, $v_j^2 > v_j^*$ for all $q_j \in Q^+$. By theorem 1, $(u^1, v^*; \mu^1)$ is stable. These two facts together imply that $0 \leq u_i^2 < u_i^1$, for all $p_i \in \mu^2(Q^+)$. Moreover, stability implies that $\mu^1(p_i) \in D_i(v^1)$ for all $p_i \in P$. By lemma 1, making $Q' \equiv Q^+$ and $P' \equiv \mu^1(Q^+)$, there exists some $p_i \in \mu^1(Q^+)$ and $q_h \notin Q^+$ such that $q_h \in D_i(v^1)$. (Observe that the fact that $(u^1, v^*; \mu^1)$ is stable and so μ^1 is competitive was used here). Hence $\alpha_{ih} = u_i^1 + v_h^* > u_i^2 + v_h^2$, and the result follows. Q.E.D.

Theorem 2 Let (v, μ) be an SPESS. Then $S(v, \mu) = (\underline{u}, \overline{v}; \mu)$, where $\overline{v}$ is the maximum equilibrium price vector.

Proof Let $S(v, \mu) = (u, v^*; \mu)$. We show that $v^* \geq v'$ for all equilibrium prices v'. Let v' be some equilibrium price vector, μ' a competitive matching for v', and u' the payoff vector of the buyers associated with (v', μ'). Denote

$$Q^+ = \left\{ q_j \in Q; \mu^{'}(q_j) \in P \text{ and } v_j^{'} > v_j^* \right\}$$

If $Q^+ \neq \emptyset$, then lemma 2 asserts that there is some pair $(p_i, q_h) \in P \times Q$ such that $u_i^{'} + v_h^{'} < \alpha_{ih}$, which is impossible because of the competitivity of the price vector v'. Therefore $Q^+ = \emptyset$. Since there are no expensive objects at any equilibrium prices it follows that $v* \geq v'$. That is, $v* = \overline{v}$. Q.E.D.

Remark 2 It follows easily from theorem 2 that if (v, μ) is an SPESS and $v_j = 0$ for all unmatched seller q_j, then v is the maximum equilibrium price vector. Of course any prices v', with $v_j^{'} = v_j$ if q_j is matched under μ, and $v_j^{'} \geq v_j$ if q_j is unmatched under μ, is part of an SPESS leading to the maximum competitive equilibrium.

From theorems 1 and 2 we know that the selling mechanism must necessarily lead to a maximum competitive equilibrium. We now show that it is indeed the case that every maximum competitive equilibrium can be the outcome of an SPESS. This theorem is proved with the help of lemma 3 which, in addition, highlights an interesting property of these equilibria.

Lemma 3 Let $(\underline{u}, \overline{v})$ be the seller-optimal stable payoff. Let μ be an optimal matching. Construct a graph whose vertices are $P \cup Q$ and with two types of arcs.

If $\mu(p_i) = q_j$ there is an arc from q_j to p_i; if $q_j \in D_i(\overline{v})$ and $q_j \neq \mu(p_i)$ there is an arc from p_i to q_j. Let $p_k \in P$ with $\underline{u}_k > 0$. Then there exists an oriented path starting at p_k and ending at an unsold object or at a buyer with a payoff zero.

Proof Suppose that there is no such a path and denote by S and T the sets of objects and buyers, respectively, that can be reached from p_k. Then $\underline{u}_i > 0$ for all $p_i \in T$ and each object in S is sold to some buyer in T. Furthermore, if $q \notin S$, then there is no buyer in T who demands q_j at prices $\overline{v}$. Then we can decrease $\underline{u}_i$ for all p_i in T by some $\lambda > 0$ and we can increase $\overline{v}_j$ for all $q_j \in S \cup \{\mu(p_k)\}$ by the same $\lambda > 0$ and still have a stable outcome which contradicts the maximality of $\overline{v}$.

Theorem 3 Let $(\underline{u}, \overline{v}; \mu)$ be a seller-optimal stable outcome. Then $(\overline{v}, \mu)$ is an SPESS.

Proof The outcome of the strategies $(\overline{v}, \mu)$ is $S(\overline{v}, \mu) = (\underline{u}, \overline{v}, \mu)$. Since μ is competitive for $\overline{v}$ it is the case that μ is maximal. Hence, we only need to prove that $\overline{v}$ is an equilibrium for the sellers. Let $q_j \in Q - \{q_0\}$. We are going to show that q_j will be unsold at any maximal matching for v', where $v_j^{'} > \overline{v}_j$ and $v_h^{'} = \overline{v}_h$ if $q_h \neq q_j$. We first show that there is some competitive matching for $\overline{v}$, say μ', which leaves q_j unsold. The cases to be considered are the following:

Case 1. q_j is unmatched at μ. Then take $\mu' = \mu$.

Case 2. $\mu(q_j) = p_i$ and $\underline{u}_i = 0$. In this case take μ' so that μ' agrees with μ on the choices of the buyers other than p_i and gives p_i the null object.

Case 3. $\mu(q_j) = p_i$ and $\underline{u}_i > 0$. By lemma 3, there exists an oriented path c starting at p_i and ending at an unsold object q_s or at a buyer p_s with payoff zero. Since c does not cycle then q_j is not in c. Set $c \equiv (p = p_1, q_1, p_2, q_2, ..., p_s, q_s)$ or $c \equiv (p = p_1, q_1, p_2, q_2, ..., p_s, q_s, p_s)$. Now consider the matching μ' that matches p_t to q_t, for all $t = 1, 2, ..., s$, that leaves p_s unmatched if p_s is on the path, that otherwise agrees with μ with regard to every object in $Q - \{q_j\}$ and every buyer in P which are not on the path, and that sets $\mu^{'}(q_j) = q_j$. Every buyer obtains the same utility under μ' as under μ, since $\mu'(p_t) = q_t \in D_t(v)$, for all $t = 1, 2, ..., s$, and μ' agrees with μ for the other buyers. Therefore, μ' is a competitive matching for $\overline{v}$.

In all of the three cases, we have found a matching μ' for $\overline{v}$ such that every buyer maximizes her utility payoff under $\overline{v}$ and q is not sold. Therefore, under μ' every buyer p_k will be maximizing her utility payoff also for $v^{'}$, and she obtains a utility of $\underline{u}_k$. Then, q_j will be unsold at any maximal matching for v': if q_j was sold at the price $v_j^{'}$ to some p_k under some matching, we would have that $\alpha_{kj} - v_j^{'} < \alpha_{kj} - \overline{v}_j \leq \underline{u}_k$, while the utility of the other buyers can not be higher than $\underline{u}$, so this matching could not be maximal. Q.E.D.

Theorems 1 and 2 say that the only outcomes that can be reached through the selling mechanism are maximum competitive equilibria if we use equilibria in the strong sense. Theorem 3 asserts that any maximum equilibrium price vector is part of an SPESS. As a consequence, the mechanism implements in SPESS the set of maximum competitive equilibria.

Corollary 1 The selling mechanism implements in SPESS the set of maximum competitive equilibria.

Proof Immediate from theorems 2 and 3.

5. Some Additional Examples

In order to obtain our results, we have restricted the analysis to what we have called maximal strategies by the buyers, and to equilibria in the strong sense by the sellers. We show here, through examples, that without such restrictions the implementation result (Corollary 1) no longer holds.

The first question is what happens if we still restrict attention to SPE in the strong sense, while allowing the buyers to use any equilibrium strategies. Note that this change diminishes the set of equilibria since the optimistic seller looking for a deviation considers as possible a larger set of buyers' strategies. Hence, a seller's deviation that was not profitable before may become now worthwhile. Example 3 shows that the set of equilibria may be empty. In fact, the set may be empty for any possible ordering of the players. In the example, we will use the following result, whose proof is relegated to an Appendix.

Proposition 7 If (v, μ) is an SPE in the strong sense then $S(v, \mu)$ is a competitive equilibrium. Consequently μ is a maximal matching for v.

Example 3 (The set of SPE may be empty) Consider a set of objects $Q = \{q_0, q_1, q_2, q_3\}$ and a set of buyers $P = \{p_1, p_2, p_3\}$. Let α be such that $\alpha_{11} =$

$\alpha_{33} = 5, \alpha_{12} = \alpha_{32} = 1, \alpha_{13} = \alpha_{21} = \alpha_{23} = \alpha_{31} = 4$ and $\alpha_{22} = 0$. There is only one stable payoff in this market: $u = (1,0,1)$ and $v = (4,0,4)$. There are three optimal matchings, denoted $\mu_k, k = 1,2,3$: $\mu_1(p_1) = q_1, \mu_1(p_2) = q_2$, $\mu_1(p_3) = q_3$; $\mu_2(p_1) = q_1$, $\mu_2(p_2) = q_3$, $\mu_2(p_3) = q_2$; and $\mu_3(p_1) = q_2$, $\mu_3(p_2) = q_1$, $\mu_3(p_3) = q_3$. Under any of them all objects are sold at v. Then, by proposition 7, $(v, \mu_k), k = 1,2,3$, are the only candidates for an SPE in the strong sense. However, the strategies (v, μ_i) do not constitute an SPE in the strong sense because the deviation $v_2^{'} = 0.5$ followed by the equilibrium matching μ_2 is profitable to seller q_2. Moreover, it is easy to check that the strategies are also not SPE for any other order of the players. (Note that μ_2 or μ_3 are not maximal matchings for v' in none of the cases, confirming theorem 3.)

The second question is whether restricting attention to equilibria in the strong sense is actually a restriction (note that the set of equilibria is a superset of the set of equilibria in the strong sense). When buyers only use maximal strategies, example 4 shows that it is indeed the case that there exist equilibria different from the maximum competitive equilibria.

Example 4 Consider a set of objects $Q = \{q_0, q_1, q_2, q_3, q_4\}$, a set of buyers $P = \{p_1, p_2, p_3, p_4\}$, let $\alpha_{14} = \alpha_{23} = \alpha_{31} = \alpha_{34} = \alpha_{42} = \alpha_{43} = 0$ and let the other entries be equal to 2. The maximum price vector is $\overline{v} = (2,2,2,2)$ and an optimal matching is $\mu : \mu(p_1) = q_1$, $\mu(p_2) = q_2$, $\mu(p_3) = q_3$, $\mu(p_4) = q_4$. However, we claim that $v = (1,1,1,1)$ followed by the maximal matching μ is part of an SPE (not in the strong sense) in which the out-of-equilibrium maximal strategies for the buyers are the following:

(a) If $v_1^{'} > v_1$ and $v_i^{'} = v_i$, for $i = 2,3,4$, then $\mu_1(p_1) = q_3$, $\mu_1(p_2) = q_2$, $\mu_1(p_3) = q_0$, $\mu_1(p_4) = q_4$.

(b) If $v_2^{'} > v_2$ and $v_i^{'} = v_i$, for $i = 1,3,4$, then $\mu_2(p_1) = q_1$, $\mu_2(p_2) = q_4$, $\mu_2(p_3) = q_3$, $\mu_2(p_4) = q_0$.

(c) If $v_3^{'} > v_3$ and $v_i^{'} = v_i$, for $i = 1,2,4$, then $\mu_3(p_1) = q_1$, $\mu_3(p_2) = q_4$, $\mu_3(p_3) = q_2$, $\mu_3(p_4) = q_0$.

(d) If $v_4^{'} > v_4$ and $v_i^{'} = v_i$, for $i = 1,2,3$, then $\mu_4(p_1) = q_3$, $\mu_4(p_2) = q_2$, $\mu_4(p_3) = q_0$, $\mu_4(p_4) = q_1$.

(e) If v' is different from the previous (a)–(d), then take any maximal strategy.

To check that the strategies are maximal, take for example case (a). If μ_1 is not maximal, then it is necessarily the case that $\mu^{'}(p_3) \in \{q_1, q_2, q_3\}$, for any μ' Pareto-superior for the buyers to μ_1. But in this case either p_1, or p_2, or p_4 are strictly worse-off with μ' than with μ_1, so μ' is not Pareto-superior to μ_1.

Our final question is what happens if we look for SPE without restricting attention either to maximal strategies or to equilibria in the strong sense. The following example shows that the SPE are not necessarily stable.

Example 5 Consider $Q = \{q_0, q_1, q_2, q_3, q_4\}$, $P = \{p_1, p_2, p_3\}$, let α be such that $\alpha_{11} = \alpha_{22} = 5$, $\alpha_{13} = \alpha_{24} = 3$, $\alpha_{31} = \alpha_{32} = 7$, and the other entries are equal to zero. The price vector $v = (0, 2, 2, 0, 0)$ with the matching $\mu(p_1) = q_1$, $\mu(p_2) = q_2$, $\mu(p_3) = 0$, is not stable. However, we claim that they constitute an SPE with the following out-of-equilibrium continuation:

(a) If $v_1^{'} > v_1$ and $v_i^{'} = v_i$, for $i = 2, 3, 4$, then $\mu_1(p_1) = q_3$, $\mu_1(p_2) = q_4$, $\mu_1(p_3) = q_2$.

(b) If $v_2^{'} > v_2$ and $v_i^{'} = v_i$, for $i = 1, 3, 4$, then $\mu_2(p_1) = q_3$, $\mu_2(p_2) = q_4$, $\mu_2(p_3) = q_1$.

(e) If v' different from the previous (a)–(b), then take any equilibrium strategy.

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Appendix

Definition 13 Let (v, μ) be some strategy. We say that (v, μ) is an SPE in the *strong sense* if for no q_j there is a v', with $v_h^{'} = v_h$ for $q_h \neq q_j$, and an equilibrium matching μ' for v', such that $S_j(v', \mu') > S_j(v, \mu)$.

Proposition 7 If (v, μ) is an SPE in the strong sense then $S(v, \mu)$ is a competitive equilibrium. Consequently μ is a maximal matching for v.

Proof Let $S(v, \mu) = (u, v^*; \mu)$. Suppose by way of contradiction that (v^*, μ) is not a competitive equilibrium. Then there is a pair (p_i, q_j) such that $u_i + v_j^* < \alpha_{ij}$. Either (a) q_j is a μ-expensive object for $v(q_j$ is unsold and $v_j > v_j^* = 0)$, or (b) q_j was sold at price $v_j^* = v_j$ to p_k, with $i' < i$. In this case, choose q_j so that $\mu(q_j) = p_{i'}$ is the last buyer of an object such that $u_i + v_j^* < \alpha_{ij}$ for some pair (p_i, q_j).

Let $\lambda > 0$ and $\gamma > 0$ be such that $(u_i + \lambda + (v_j^* + \gamma) = \alpha_{ij}$ and let v' be such that $v_j' = v_j^* + \gamma$ and $v_h^{'} = v_h$ for all $q_h \neq q_j$. Then:

$$u_i + \lambda = \alpha_{ij} - v_j^{'} > u_i \geq \alpha_{ih} - v_h^* \text{ for all } q_h \in Q_i$$

where Q_i is the set of available objects for p_i under μ. In case (a) let every buyer p_k, $k < i$, choose $\mu(p_k)$, which is still a best response for p_k to v'. Thus the set of available objects for p_i under v' is still Q_i. Hence, by the previous equation, q_j will be the only object in the demand set of p_i at prices v'. Therefore q_j will be sold to p_i. In this case seller q_j wins by deviating, which contradicts the fact that (v, μ) is a SPE.

Consider now case (b). Let every buyer p_k with $k < i'$ play $\mu(p_k)$, which is still a best response for p_k to v'. Then at the time p_i is called to play, if q_j is still available, no matter which were the choices of i' and of the buyers who came after i', q_j will be the only object in the demand set of p_i at prices v'. In fact, if $\alpha_{ih} - v_h^{'} > \alpha_{ij} - v_j^{'}$ then $\alpha_{ih} - v_h^* > \alpha_{ij} - (v_j^* + \gamma)$. Thus if we have chosen a γ small enough, we have that $\alpha_{ih} - v_h \geq \alpha_{ij} - v_j^* > u_i$, so (p_i, q_h) also blocks $(u, v^*; \mu)$. Then $\mu(q_h)$ is prior to $i^{'} < i$. By hypothesis this implies that $\mu(q_h)$ buys q_h at v', so q_h is not available to p_i when she comes to play. Now use the previous equation to get that p_i will buy q_j at $v_j^{'}$. Therefore, in any case, $v_j^{'}$ is a profitable deviation. Hence (v^*, μ) is a competitive equilibrium. Moreover, as a consequence of this result, μ is an optimal matching so it is a maximal matching for v and the proof is complete. Q.E.D.