

# Why do Bidders Drop Out from a Sequential Auction\*

Richard Engelbrecht-Wiggans\*\*

Flavio M. Menezes\*\*\*

Summary: 1. Introduction; 2. The basic model; 3. Endogenizing the number of bidders; 4. Expected profits at equilibrium; 5. An example; 6. A necessary condition for equilibrium; 7. Construction of an equilibrium; 8. Why losing bidders drop out; 9. Conclusion.

Key words: sequential auctions; endogenous participation.

JEL code: D44.

In actual sequential auctions, bidders typically incur a cost in continuing from one sale to the next, and they decide whether or not to continue. To investigate the question “why do bidders drop out”, this paper defines a sequential auction model with continuation costs and an endogenously determined number of bidders at each sale, and characterizes the equilibria in this model. Simple examples illustrate the effect of several possible changes to this model.

Este artigo examina um modelo de leilões seqüenciais com custos de participação onde o número de participantes é determinado endogenamente em cada rodada. O artigo caracteriza os lances em equilíbrio em cada rodada do leilão seqüencial e ilustra, através de exemplos, os efeitos de diversas mudanças possíveis no modelo.

## 1. Introduction

Consider the sequential auctioning of several objects one after another. Existing models of such sequential auctions exogenously specify the number of bidders in each auction. But in actual auctions, bidders may choose to drop out. How many bidders continue to the next sale may depend on a variety of auction design factors, including, for example, the reservation prices or the order in which the remaining objects will be sold.

---

\*This paper was received in Jan. and approved in June 2000. The authors are grateful to Charles Kahn for many helpful suggestions. F. Menezes gratefully acknowledges the financial support of CNPq (grant 300065/93-0) and ARC (grant 000000055).

\*\*Department of Business Administration, University of Illinois at Urbana-Champaign.

\*\*\*EPGE/FGV.

This suggests a problem in using existing models to compare the expected outcomes of different types of sequential auctions. In particular, different types of auctions may attract different numbers of bidders. The effect of the difference in the number of bidders needs to be considered in comparing different types of sequential auctions. But the existing models specify the number of bidders exogenously; they do not make the number of bidders adjust appropriately to changes in the auction.

As an example, consider the auctions of used farm machinery. Often, on retiring, a farmer's machinery will be auctioned. The auction occurs on the farm. Between the end of one harvest season and the beginning of the subsequent planting season, several dozen such auctions may be held within a one or two hour drive of each other. And a typical, prospective buyer will want to buy at most one piece of any particular machinery.

At the start of any year's sequence of such auctions, a prospective buyer typically has very little information about what will be offered for sale. A few weeks before an auction occurs, it will be advertised. The advertisement list the equipment to be sold. But obtaining information about the what will be offered before this advertisement is often difficult; indeed, the seller may not have decided to retire his machinery until shortly before the advertisement.

Each prospective buyer is thus faced by a sequence of auctions. There is some cost to attend the next auction. And each prospective buyer must decide whether or not it is worthwhile to continue attending subsequent auctions. But his or her decision may depend on factors such as the type of auction to be used, whether objects will be sold without reserve, and if not, on the reservation price; it seems unreasonable to assume that such factors can be varied without affecting the number of bidders in each auction.

The development of single-object auction theory illustrates the problem. For example, Myerson (1981) and Riley and Samuelson (1981), among others, ask "what reservation price most benefits the seller?" They work with models with a fixed number of bidders. They conclude that introducing a non-trivial reservation price benefits the seller.

But introducing a non-trivial reservation price also makes the auction less attractive to the bidders. So, Engelbrecht-Wiggans (1987 and 1993), McAfee

and McMillan (1987), and Menezes and Monteiro (2000), among others, endogenize the number of bidders. In these models, at equilibrium, as the auction becomes less attractive to any fixed number of bidders, fewer bidders will show up and bid. In such models, introducing a non-trivial reservation price may hurt the seller. This suggests that whether a non-trivial reservation price helps or hurts the seller depends on where the actual auction falls between the two extremes considered by these models.

The practical reasons for endogenizing the number of bidders in sequential auctions go beyond those for doing so in single-object auctions. In both types of auctions, changing the auction design can affect the number of bidders that enter. But in sequential auctions, bidders incur some cost to continue from one sale to another; for example, real bidders typically have non-zero opportunity costs. And in sequential auctions, bidders can, and sometimes do, drop out during the sequence. So, sequential auctions naturally require models with an endogenously determined number of bidders.

Sequential auctions also allow for a much richer collection of models than do single-object auctions. Questions such as “when in the sequence do bidders acquire information about as yet unsold objects”, “how are the objects’ values related”, and of course “what do bidders know about the number of competitors they will face in subsequent sales” simply don’t arise in single-object auctions. Different answers to these questions result in different models, and the number of bidders may vary differently in these different models.

This suggests a very practical question: “Why might bidders choose to drop out?” In what models of sequential auctions, or under what conditions, will bidders never choose to drop out? What characteristics of actual sequential auctions result in bidders choosing to drop out?

We focus on this question of why bidders might drop out. To do so, we adapt the model of Engelbrecht-Wiggans (1994) so that:

- a) each bidder who continues to the next stage incurs a continuation cost;
- b) and bidders choose between continuing or dropping out after each sale so as to maximize their net expected profits.

We derive an equilibrium for this model; more precisely, we establish conditions such that the equilibrium of Engelbrecht-Wiggans (1994) remains in equilibrium when we endogenize the number of bidders, and suggest that these conditions can always be satisfied with appropriate adjustments in the number of bidders or objects going into the first sale. We also prove that if the number  $n$  of bidders is allowed to take non-integer values, then at any equilibrium in our model, no bidder ever chooses to drop out.

We also examine our model to see that changes would result in bidders dropping out with positive probability at equilibrium. Our model is symmetric or monotonic with respect to several variables. Destroying any of these might result in bidders choosing to drop out; we provide a couple of examples and suggest that there are many others. But we also consider less obvious changes; in particular, restricting the number of bidders to be integer can result in bidders choosing to drop out.

In this paper we consider the effects of endogenizing the number of bidders in a sequence of second-price (Vickrey) auctions where the winner at each round is the bidder with the highest bid and he pays the amount of the second highest bid. However, with private independent values, as in this paper, Myerson's (1981) revenue equivalence theorem holds so that any auction format that assigns the  $m$  objects to the players with the highest bids and such that the individual with the lowest possible valuation has zero expected prices yields the same expected revenue for the seller.<sup>1</sup> In particular, there will be an equilibrium of first-price sequential auctions (where the winner in each round is the bidder with the highest bid and he or she pays the amount of his bid) that will yield the same expected payment from a bidder and therefore the same ex ante profits as in the case of second-price sequential auctions. We chose to examine the effects of entry on bidding behavior on the latter auction format to simplify computations but the results remain valid for other auction formats satisfying the aforementioned properties.

Finally, there is a growing empirical and theoretical literature on sequential auctions. For example, Weber (1983) considers a sequential auction of identical objects and shows that expected prices follow a martingale, i.e., bid-

---

<sup>1</sup> See Engelbrecht-Wiggans (1988) for a formal proof of the extension of Myerson's result to multi-object auctions.

ders expect prices will remain constant on average throughout the sequence of auctions within a sale. In Weber's model, bidders only purchase one of a fixed number of objects. That is, the winner from each round drops out from the auction.

There is, however, empirical evidence that prices are not constant throughout sequential auction sales. Ashenfelter (1989) reports that identical cases of wine fetch different prices at sequential auctions in three auction houses from 1985 to 1987. Although the most common pattern was for prices to remain constant, prices were at least twice as likely to decline as to increase. Ashenfelter refers to this phenomenon as the "price decline anomaly".

McAfee and Vincent (1993) adopted a similar approach to Ashenfelter and examined data from Christie's wine auctions in Chicago in 1987. In addition to pairwise comparisons, they examined triples of identical wine sold in the same auction sale. Their results are very similar to those of Ashenfelter.

Similar empirical findings were identified in a number of other markets; cable television licenses (Gandal, 1997); condominiums (Ashenfelter & Genesove, 1992; and Vanderporten, 1992a and 1992b); dairy cattle (Engelbrecht-Wiggans & Kahn, 1992); stamps (Taylor, 1991; Thiel & Petry, 1995); and wool (Jones, Menezes & Vella, 1998). Gandal provides evidence that prices increased in the sale of cable-TV licences in Israel. The author attributes the price increases to the interdependencies among licenses that may increase competition in the later rounds of the sale. Jones, Menezes and Vella indicate that prices may increase or decrease in sequential auctions of wool (adjusting prices to estimate wool of homogeneous quality).

Most theoretical explanations for the price decline anomaly have concentrated on models where the winner of each round drops out from the auction. For example, McAfee and Vincent explain the anomaly by considering the effects of risk aversion on bidding strategies. For identical objects they show how bids in the first round are equal to the expected prices in the second round plus a risk premium associated with the risky future price. They assume buyers have nondecreasing risk aversion and can only buy one object.

Von der Fehr (1994) uses participation costs to obtain different net valuations for identical objects. When bidders face a cost of participating in each

auction of two identical objects sold sequentially, price is lower in the second auction than it is in the first. This follows because the number of buyers who stay for the second auction falls by more than the successful bidder in the first auction. Once again, buyers only buy one object. This is also the case of Engelbrecht-Wiggans (1994) and Bernhardt and Scoones (1994), who show how expected prices decline when the objects are statistically identical (i.e., where bidders' valuations for the objects are independent draws from a fixed distribution) and the distribution of values is bounded. An exception is Menezes (1993a), who allows losing bidders to drop out as well.

In this paper we provide a sufficient condition such that losing bidders do not drop out. That is, we provide a sufficient condition under which Weber type models, where the winner drops out at the end of each round, are appropriate even when we allow losing bidders to drop out at the end of each round.

## 2. The Basic Model

Imagine that ( $m \geq 2$ ) objects will be sold one after another in a sealed-bid second-price auction as defined by Vickrey (1961). Initially, there are  $n_{m-1}$  expected profits maximizing potential bidders; the  $-1$  simplifies subsequent expressions. Let  $s$  index the sales, with  $s$  equal to 0 for the last sale; think of  $s$  as "sales remaining". For  $s = m - 1, \dots, 0$ , let  $n_s$  denote the number of bidders who bid in sale  $s$ , and define  $\underline{n} = (n_m, n_{m-1}, \dots, n_0)$ . Each bidder may win at most one object; the winner must drop out. In addition, once a bidder skips a sale, he or she may not bid in any subsequent sale. So, after the first sale, the number of bidders drops by at least one per sale. Specifically,  $n_{s-1} \leq n_s - 1$  for  $s = 1, 2, \dots, m$ .

Each bidder who bids in sale  $s$  incurs a continuation cost  $c_s$ . This, especially for  $s = m - 1$ , may be thought of as an entry cost, and might reflect the opportunity cost of time to be spent at a sale, any bid preparation costs, the cost – if the sales are spatially distributed – to travel to this sale, and any other costs incurred by a bidder. Also, for our purposes – since we do not examine the sellers' or auctioneer's expected revenue – any entry fee paid to the sellers and/or the auctioneer may be lumped into this continuation cost. Assume

that  $c_{m-1} \geq c_{m-2} \geq \dots c_0$ . A special case of this arises if bidders incur one cost to enter the sequence of auctions, and then incur another, constant cost on entering each sale.

Let  $X(i, s)$  denote a random variable with outcome  $x(i, s)$ . Bidder  $i$  has a value of  $x(i, s)$  for the object in sale  $s$ . (For notational simplicity, define  $X(i, s)$  for each  $i$  and  $s$  even though some bidders  $i$  may not actually bid in sale  $s$ .) Assume symmetry – but not independence – across bidders; more precisely, the joint distribution of  $X(1, s), X(2, s), \dots, X(n_m, s)$  is symmetric in its arguments. Also assume that for each  $i$ ,  $X(i, m), \dots, X(i, 2), X(i, 1)$  are identically and independently distributed. Next, assume that bidder  $i$  knows  $x(i, s)$  when bidding in sale  $s$ , but does not yet know  $x(i, s-1), x(i, s-2), \dots, x(i, 0)$ . Finally, assume that all bidders know the joint distribution of the  $X(i, j)$ 's.

These assumptions impose quite a bit of structure.<sup>2</sup> In particular, together with a subsequent assumption on the process by which bidders might leave, they imply that ex ante to each auction, the remaining bidders are stochastically equivalent. In addition, they imply that each remaining bidder views each of the remaining objects as being stochastically equivalent.<sup>3</sup>

The model presumes a particular order of events which repeats itself sale after sale. In the beginning, there is a pool of  $n_m-1$  potential bidders, each of whom must decide whether or not to bid in sale  $m-1$ . Of these,  $n_{m-1}$  enter the first sale. (The process that determines how many of the potential bidders actually continue will be defined below.) Then each bidder incurs the continuation cost  $c_{m-1}$ . Next, each bidder  $i$  learns his or her private value  $x(i, m-1)$  for the first object. Now the auction proceeds, and a winner and price are determined. The winner pays up and drops out. This is the end of sale  $m-1$ . Then, going into sale  $s-1$  ( $1 \leq s \leq m-1$ ) there is some new, smaller, number  $n_s-1$  of potential bidders, of whom  $n_{s-1}$  actually enter, each incurring the continuation cost  $c_{s-1}$ . And the process continues as before until the end of sale  $s=0$ .

---

<sup>2</sup> Despite this degree of structure, Engelbrecht-Wiggans and Kahn (1992) show that this model with only the winner dropping out can be parameterised to fit actual sequential auction data.

<sup>3</sup> The stochastic equivalence of the remaining objects makes the analysis tractable, but is not without its limitations. Engelbrecht-Wiggans (1994) and Bernhardt and Scoones (1994) discuss this assumption in more detail. We will illustrate the effects of relaxing it later.

In many ways, the farm auction example illustrates our model. Our model assumes a finite number of sales; sooner or later, a prospective buyer of used farm machinery runs out of chances to buy. There is a cost associated with participating in an auction; a prospective buyer must travel to the auction site. There is only one object per auction; for example, a prospective buyer wants only to buy a combine, each sale offers at most one combine, and the prospective buyer can determine what sales offer a combine from their advertisements. A prospective buyer has no specific information about what will be offered for sale until it is actually offered; a prospective buyer cannot inspect the condition of a particular piece of machinery except at the auction site shortly before the auction commences. Ex ante to the buyer acquiring specific information, all objects are stochastically equivalent; for example, the sales are not ordered so that the better used combines will be sold on earlier dates than the less good used combines. Our model assumes independent private values; a prospective buyer's value for acquiring a particular type of machinery depends on what machinery he or she currently owns and on its condition, the current overall market for used equipment is well known, the average quality of machinery that will be offered is well known, and the cost to resell such equipment is relatively large compared to any allocation inefficiencies arising from the auction.

We do not mean to claim that our model exactly reflects the farm machinery auctions, nor that our model only applies to farm machinery auctions; rather, we simply claim that our model, despite its specialized nature, captures the flavor of certain actual auctions.

### 3. Endogenizing the Number of Bidders

To endogenize the number of bidders, we must model the framework within which losing bidders decide whether or not to continue. Specifically, we assume that bidders make their decisions one by one with full information of the decisions made by others. For example, given appropriate monotonicity of profits in the number of bidders, bidders could leave one by one until those who remain can all profit from continuing. If the losing bidders are stochastically equivalent, the order of the bidders in this process doesn't affect the number



that continue. We assume that the order of bidders is randomly chosen; this preserves the stochastic equivalence of the bidders.

These assumptions have three important, immediate consequences. First, since the losing bidders are stochastically equivalent, we need keep track only of the number of losing bidders who choose to continue. Second, the stochastic equivalence of bidders together with the assumption of full information assures that the number of bidders changes deterministically from sale to sale. In particular, any specific equilibrium uniquely defines the numbers  $n_{m-1}, n_{m-2}, \dots, n_0$  of bidders in each sale; together with the initial number  $n_m - 1$  of potential bidders, this uniquely defines the vector  $\underline{n}$ . In short, each equilibrium has associated with it a unique vector  $\underline{n}$ ; later, in an example, we examine a different set of assumptions, such that an equilibrium defines a non-degenerate stochastic process for the number of bidders.

For the third consequence, look at the process from an individual losing bidder's perspective. Since the order in which the bidders make their decisions is random, each of the losing bidders in sale  $s$  has an equal chance of being one of the  $n_{s-1}$  who continues. So, each of the  $n_s - 1$  losing bidders in sale  $s$  perceives that he or she has an a  $n_{s-1}/(n_s - 1)$  chance of being one of the  $n_{s-1}$  bidders who continue.

Realistically, the number of bidders  $n_s$  must be integer. Unfortunately, this integrality complicates the analysis. Instead, we allow the  $n_s$ 's to be real valued. We will argue that at equilibrium, this implies that if any losing bidders drop out, the  $n_s$  bidders who do continue to sale  $s$  each have exactly zero net expected profit from the remaining sales.<sup>4</sup> Later, in an example, we illustrate the effect of this assumption.

#### 4. Expected Profits at Equilibrium

Consider how the expected profit to bidders at various points in the auction must be related at equilibrium. We consider only perfect – in this case,

---

<sup>4</sup>We need this zero profit condition. If the number of bidders were restricted to be integer, this condition would not hold, and as a subsequent example illustrates, our first proposition may be false.

iterated-dominant-strategy – equilibria. Imagine that such an equilibrium exists; this, together with the number  $n_m - 1$  of potential bidders, defines  $\underline{n}$ .

Start with the last sale. It looks just like a single object auction with  $n_0$  bidders, and each of the bidders has the dominant strategy of bidding equal to his or her known value  $x_{i,s=0}$ . When each bidder follows this strategy, then the  $n_0$  bidders each has an ex ante net expected profit equal to  $n_0(\pi(n_0) - c_0)$ , where  $\pi(n)$  denotes the expected profit per bidder at the dominant strategy equilibrium in a single-object privately-known-values auction without entry costs.<sup>5</sup> Assume that  $\pi(n)$  decreases as  $n$  increases – this is true in most privately-known-values models – and thus  $(\pi(n_s) - c_s)$  decreases as  $s$  increases. Also assume that  $\pi(1) \geq c_0$ ; if not, no bidder would be willing to bid in the last – or any other – sale.

Now, let  $\Pi(\underline{n}, s)$  denote the net expected profit from sales  $s-1, s-2, \dots, 0$  to each losing bidder in sale  $s$ . (For  $s = m$ , interpret  $\Pi(\underline{n}, m)$  as the net expected profit from all the sales to each of the  $n_m - 1$  potential bidders.) At the end of sale  $s = 1$ , each of the  $n_1 - 1$  losing bidders has an equal chance at the combined net expected profit in the last sale. So,  $\Pi(\underline{n}, s = 1) = (n_0/(n_1 - 1))(\pi(n_0) - c_0)$ . Knowing this, each bidder in sale  $s = 1$  has the dominant strategy to bid  $\Pi(\underline{n}, s = 1)$  less than his or her privately known value for the object in this sale, and the combined net expected profit – from this and all remaining sales – to the  $n_1$  bidders who enter this sale equals  $n_1[(\pi(n_1) - c_1 + \Pi(\underline{n}, 1))]$ . Continuing this process yields the following:

$$\Pi(\underline{n}, s) = (n_{s-1}/(n_s - 1)) \left[ (\pi(n_{s-1}) - c_{s-1}) + (n_{s-2}/(n_{s-1} - 1)) \right. \\ \left. [(\pi(n_{s-2}) - c_{s-2}) + \dots] \dots \right]$$

for  $s = 1, \dots, m$ .

---

<sup>5</sup> Engelbrecht-Wiggans (1993) shows that for  $n > 1$  in sales without reserve, this  $\pi(n)$  equals  $v(n) - v(n-1)$ , where  $v(n)$  denotes the expected social value  $E[\max\{X(1,s), X(2,s), \dots, X(n,s)\}]$ . In particular,  $\pi(n)$  depends on the distribution of the  $X(i,s)$ 's. In the case of a single bidder,  $\pi(n)$  presumably equals the amount by which the bidder's expected value for the object exceeds the reservation price.

## 5. An Example

Consider a simple, specific example that satisfies all the assumptions of the basic model. This example illustrates what the equilibrium looks like; subsequent propositions formalize these characterizations. This example also provides a reference point for our subsequent examples, which illustrate the effect of various possible changes to the basic model.

**Example 1:** Five expected profit maximizing potential bidders face a sequence of two sales. Assume that  $\pi(n) = 1/(n(n+1))$  for all  $n > 0$ . (This would be the case for integer  $n \geq 2$  if the bidders' privately-known values for the objects are independent – across both bidders and sales – samples from a uniform distribution on  $[1, 2]$  and the objects will be sold without reserve; it also holds for  $n = 1$  if that lone bidder faces an appropriate reservation price.) Assume  $c_1 = c_0 = 0.09$ .

To solve this example, start with sale  $s = 0$ . Assume each bidder follows the dominant strategy of bidding equal to his or her value for the object. Then, ex ante, each of the  $n_0$  bidders has a net expected profit of  $\pi(n_0) - 0.09$ . At equilibrium, this must be non-negative, and so  $n_0 \leq 2.795^+$ .

Now look at sale  $s = 1$ . At the iterated-dominant-strategy equilibrium, each bidder bids  $(n_0/(n_1 - 1))(\pi(n_0) - 0.09)$  less than his or her value for the object in sale  $s = 1$ . So each of the  $n_1$  bidders has an ex ante net expected profit of  $(\pi(n_1) - 0.09) + (n_0/(n_1 - 1))(\pi(n_0) - 0.09)$  from the two sales. At equilibrium, this must be non-negative.

Two cases need to be considered. In the first,  $n_0 = 2.795^+$  and, therefore,  $n_1 \geq 3.795^+$ . This forces the ex ante net expected profit for the two sales together to be negative, and thus this case can't occur at equilibrium. In the second case,  $n_0 < 2.795^+$  and this can happen only if  $n_1 = n_0 + 1$ . In this case, the ex ante net expected profit from two sales is non-negative if and only if  $n_1 \leq 3.480^+$ .

Thus, at (the unique perfect) equilibrium to this first example,  $3.480^+$  of the five potential bidders bid on the first object, and  $2.480^+$  continue to the last sale. In particular, only the winner in the first sale drops out. A subsequent proposition establishes that this must be true at any equilibrium to the basic model.

## 6. A Necessary Condition for Equilibrium

At equilibrium, bidders act in their own best interests. So, at any (perfect) equilibrium, each of the bidders entering a sale must have a non-negative expected profit from doing so. This implies that  $\Pi(\underline{n}, s) \geq 0$  for  $s = m, m-1, \dots, 1$ .

In addition, no bidder who drops out at equilibrium can have a positive expected profit from continuing. In particular, if someone chooses to drop out, then there exists an  $s^*$ ,  $1 \leq s^* \leq m$ , and a  $\delta > 0$  such that  $n_{s^*-1} \leq n_{s^*} - 1 - \delta$ . For  $1 \leq s \leq m$ , define  $\underline{n}(s, \delta) = (n_m, \dots, n_s, n_{s-1} + \delta, n_{s-2}, \dots, n_0)$ ; this is the number of bidders that would result if  $\delta$  (or a fraction  $\delta$  of one bidder) were to continue one sale longer than in the equilibrium. At equilibrium, these  $\delta$  bidders could not have a positive expected profit from continuing. This implies that  $\Pi(\underline{n}(s^*, \delta), s^*) \leq 0$ . Now let  $\delta$  decrease to 0. Then the two above conditions together imply that at equilibrium, if there exists an  $s^*$  ( $1 \leq s^* \leq m$ ) such that  $n_{s^*-1} < n_{s^*} - 1$ , then  $\Pi(\underline{n}, s^*) = 0$ . In short, if any losing bidders drop out, then the continuing bidders must have zero expected profit from continuing.<sup>6</sup> This gives us the following proposition:

**Proposition 1:** *At any perfect equilibrium to our model, all losing bidders in each sale continue to the next (if there is one.) Specifically,  $n_{s-1} = n_s - 1$  for  $s = m-1, m-2, \dots, 1$ .*

**Proof:** By contradiction, imagine that for  $m$  objects and a pool  $n_m$  of potential bidders, there exists an equilibrium resulting in  $n_{m-1}, n_{m-2}, \dots, n_0$  bidders bidding in sales  $m-1, m-2, \dots, 0$ , such that for some  $s^*$  ( $1 \leq s^* \leq m-1$ ),  $n_{s^*-1} < n_{s^*} - 1$ . Thus  $\Pi(\underline{n}, s^*) = 0$ . But since  $(\pi(n_s) - c_s)$  decreases as  $s$  increases,  $\Pi(\underline{n}, s^*)$  can be 0 only if  $\pi(n_{s^*-1}) - c_{s^*-1} < 0$ , and so  $\pi(n_{s^*}) - c_{s^*}$  must also be negative. Write  $\Pi(\underline{n}, s^* + 1) = (n_{s^*} / (n_{s^*+1} - 1))[(\pi(n_{s^*}) - c_{s^*}) + \Pi(\underline{n}, s^*)]$ . The first term in the square brackets is negative and the second is 0. Thus, the total expression is negative, which can't happen at equilibrium. So, no such equilibrium can exist.

This necessary condition that  $n_{s-1} = n_s - 1$  for  $s = m-1, m-2, \dots, 1$  simplifies the expected profit expression  $\Pi(n, s)$ . In particular,  $n_0$  now completely determines  $\underline{n}$  for any fixed number  $n_m - 1$  of potential entrants; to

---

<sup>6</sup>If the number of bidders must be an integer, then  $\delta$  must be a positive integer, and  $\Pi(\underline{n}, s^*)$  may be strictly positive.

emphasize this, we now write  $\Pi(n_0, s)$  in place of  $\Pi(\underline{n}, s)$ . And appropriate use of the necessary condition yields  $\Pi(n_0, s) = \sum_{k=0,1,\dots,s-1} (\pi(n_0 + k) - c_k)$  for  $s = m, m-1, \dots, 1$ .

This function has three noteworthy properties. One, since  $\pi(n)$  decreases as  $n$  increases,  $\Pi(n_0, s)$  decreases as  $n_0$  increases. Two, since  $(\pi(n_k) - c_k)$  decreases as  $k$  increases,  $\Pi(n_0, s)$  is concave in  $s$ . And three, since  $\pi(1) \geq c_0$ ,  $(n_0 = 1, s = 1) \geq 0$ .

## 7. Construction of an Equilibrium

We now construct an equilibrium. More precisely, we establish conditions such that the iterated-dominant-strategy equilibrium of Engelbrecht-Wiggans (1994) for sequential auctions with an exogenously specified number of bidders will still be an equilibrium in our adaptation of that model. In this equilibrium, only the winning bidders drop out.

To establish the desired conditions, we start by defining sales and bidder capacities for sequential auctions. Then we derive the necessary conditions for an equilibrium in terms of these capacities. Finally, we discuss how an equilibrium can be constructed for cases not satisfying these conditions.

In particular, for any sequential auction, define the sales capacity  $m^* = \max\{s \text{ st } \Pi(1, s) \geq 0\}$ . Property 3 above assures that  $m^*$  exists and that  $m^* \geq 1$ . Property 2 assures that  $\Pi(1, s) \geq (<)0$  for  $s \leq (>)m^*$ . In particular,  $\Pi(1, m^*) \geq 0$ . Intuitively,  $m^*$  is the maximum number of objects so that if the auction starts with as many bidders as objects, no loser would ever benefit from dropping out and, thus, in the end, no object would be left without anyone bidding for it.

For any sales capacity  $m^*$  and any actual number of objects  $m$ , define the bidder capacity  $n^* = \max\{n \text{ st } \Pi(n, \min\{m, m^*\}) \geq 0\}$ . Note that  $\Pi(1, m^*) \geq 0$  implies that  $n^*$  exists and that  $n^* \geq 1$ . Property 1 above assures that  $\Pi(n, m^*) \geq (<)0$  for  $n \leq (>)n^*$ . This, together with property 2, yields the following condition:

$$\Pi(n, s) \geq 0 \quad \text{for all } n \leq n^* \text{ and } s \leq m^* \quad (1)$$

Intuitively,  $n^* - 1$  is the maximum number of excess bidders – bidders in excess of the number of objects to be sold – that the auction can accommodate without some losing bidder dropping out at some point (or some potential bidding not entering into the auction). This suggests the following proposition.

**Proposition 2:** *If the number  $m$  of objects to be sold is at most the capacity  $m^*$  and the number  $n_m - 1$  of potential bidders is at most  $n^* + m - 1$ , then there exists an iterated dominant strategy equilibrium in which everyone who may bid chooses to bid.*

**Proof:** Set  $\underline{n} = (n_m, n_m - 1, \dots, n_m - m)$ . Thus  $n_0 = n_m - m \leq n^*$ . Condition (1) above then assures that  $\Pi(\underline{n}, s) \geq 0$  for  $s = m, m - 1, \dots, 0$ , and so no losing bidder ever wants to drop out. In particular, the iterated-dominant-strategy equilibrium defined by Engelbrecht-Wiggans (1994) remains even though bidders may drop out.

This proposition also suggests an equilibrium for cases with too many potential bidders and/or too many objects. If  $m > m^*$ , then some objects will see no bidders. Throw away (the last)  $m - m^*$  objects. If  $n_{\min\{m^*, m\}}$  exceeds  $n^* + \min\{m^*, m\}$ , then not all of the potential bidders will enter the auction. These adjustments yield an auction with an equilibrium in which no losing bidder will drop out, and in which all the objects will be bid on. Note that if the numbers of bidders must be integer, then the results of this section still hold, but there might also be equilibria in which losing bidders choose to drop out.

## 8. Why Losing Bidders Drop Out

Now let us examine why losing bidders might drop out. In our basic model, they don't. But, any one of several changes to the basic model introduces factors that can result in losing bidders dropping out. We illustrate the possibilities through appropriate examples.

The following two examples illustrate the effect of changing the assumptions on the process by which losing bidders decide whether or not to continue. Example 2 restricts the number of bidders that continues to be integer, while example 3 alters the assumption of full information about others' decisions. Either change can result in equilibria in which some losing bidders drop out; the first proposition no longer holds.

**Example 2:** Everything is the same as in example 1 except that now the numbers  $n_0$  and  $n_1$  of bidders must be integer.

To solve this example, proceed as before. Rounding down the previous  $2.795^+$  gives that, at equilibrium,  $n_0 \leq 2$ . If  $n_0 < 2$ , then  $n_1 = n_0 + 1 < 3$ , and at least one additional bidder would want to enter into sale  $s = 1$ ; there can be no equilibrium with  $n_0 < 2$ . Then, if  $n_0 = 2$ , each of these bidders in sale  $s = 0$  has an ex ante net expected profit of  $0.07666\dots$  and all five of the potential bidders will choose to enter sale  $s = 1$ .

Thus, at (the unique perfect) equilibrium to this second example two losing bidders chose to drop out after the first sale. The intuition goes as follows: rounding down the number of bidders in sale  $s = 0$  increases the expected profit per bidder from that sale. This increases the bidders' expected profit for the two sales combined, and sale  $s = 1$  can now support more bidders. And in example 2, the increase is large enough so that, after rounding down  $n_1$ ,  $n_1 - 1$  still exceeds  $n_0$ . But if the continuation cost were  $0.11$ , only three of the five potential bidders would enter sale  $s = 1$  and both losing bidders would continue to sale  $s = 0$ . The parameters must be balanced just so to get losers to drop out.

**Example 3:** Everything is the same as in example 1 except for the process by which bidders chose to drop out. In particular, assume that each losing bidder in sale  $s$  chooses a probability  $p_s$  and then continues to sale  $s - 1$  with this probability.

To solve this example, again work backwards from sale  $s = 0$ . We consider only the symmetric equilibrium and thus assume that all the potential bidders choose the same value for  $p_s$ . If there are at most two potential bidders, then they will all continue with probability 1. If there are three or more potential bidders, then, at (any perfect) equilibrium,  $p_s$  must be less than 1, and each potential bidder must be indifferent between continuing and dropping out. In particular, for three potential bidders, each of the three bidders has an expected profit of  $[p_s^2(1/12) + 2p_s(1 - p_s)(1/6) + (1 - p_s)^2(1/2)] - 0.09$  from continuing, and this must be equal to the zero expected profit from dropping out. Solving this quadratic equation yields  $p_s = 0.9621^+$ ; similarly, for four potential bidders,  $p_s$  is the appropriate root of a cubic equation.

Now look at sale  $s = 1$ . If all five potential bidders were to bid with probability 1, there would be four potential bidders for sale  $s = 0$ , each of which would have zero expected profit from sale  $s = 0$ . So, in sale  $s = 1$ , each of the five bidders would have the dominant strategy of bidding equal to

their value, which would net each of them an ex ante expected profit of  $1/30 - 0.09$  from the two sales combined. This is negative. Thus, the five bidders can't all bid with probability 1. In fact, at equilibrium,  $p_{s=1} < 1$ ; the actual calculations are quite involved, and we refer the interested reader to Menezes (1993b) for the details.

So, in example 3, an equilibrium implicitly defines a stochastic process for the vector  $\underline{n}$ . An integer, but possible random, number of bidders continue from one stage to the next. For example, there is a positive probability of five bidders in sale  $s = 1$  and zero bidders in sale  $s = 0$ ; all of the maximum possible number of losing bidders might drop out. In sequences of more than two sales, there may be several sales in which some losing bidders drop out. But all the dropping out of losing bidders can be interpreted as a random adjustment process toward a sufficiently small number of continuing bidders.

In the above three examples,  $\pi(n_s) - c_s$  is monotonically non-increasing in  $s$ . We used this monotonicity in the proof of the first proposition. The next two examples illustrate different changes to the basic model assumptions that underlie this monotonicity; either change can result in equilibria in which losing bidders drop out.

**Example 4:** Everything is as in example 1, except that the objects are no longer stochastically equivalent. In particular, let the object in sale  $s = 1$  be one hundred times as valuable as before; the bidders' values for this object are now independent samples distributed uniformly on the interval  $[100, 200]$ .

**Example 5:** Everything is as in example 1, except that the continuation costs increase as  $s$  decreases. In particular, let  $c_1 = 0$  (and  $c_0 = 0.09$  as before).

To solve these examples, start with sale  $s = 0$ . It looks just like sale  $s = 0$  in example 1. But now in examples 4 and 5, all five potential bidders will actually bid in sale  $s = 1$ . So,  $4 - 2.795^+ = 1.205^-$  losing bidders ( $4 - 2 = 2$ , if the numbers must be integer) drop out after sale  $s = 1$ . In both examples, the condition that  $n_{s+1} \geq n_s + 1$  for all  $s$  no longer suffices to make  $\pi(n_s) - c_s$  a monotonically non-increasing function of  $s$ . This illustrates that if, for whatever reason, early sales offer objects with a sufficiently higher value (net of continuation costs) than do later sales, losing bidders may drop out at equilibrium.



These two examples destroy an essential symmetry or monotonicity of our model. Of course, there are many other ways to destroy the symmetry. Each of these, if taken to a sufficient extreme, might result in losing bidders dropping out. To illustrate, consider our assumption that bidders learn their values for an object just before the object is offered for sale. This helped make the remaining objects stochastically equivalent both across objects and across bidders.

Alternatively, the model might assume that bidders know something about their values for an object earlier. Possibly, a potential bidder knows the value for the next object to be sold before he or she must decide whether or not to continue. Or, bidders know something about each object relatively early in the sequence. Either possibility can result in different bidders having different expected values from continuing. Now, with positive probability, a losing bidder might perceive the remaining objects to be of low enough value that he or she decides to drop out.

## 9. Conclusion

Bidders might drop out of a sequential auction for a variety of reasons. In our basic model, there can be no equilibrium at which losing bidders decide to drop out, and equilibria do exist. But any one of several changes to the model can result in bidders dropping out at equilibrium; our examples illustrate several possibilities, and there are many others.

## References

Ashenfelter, O. How auctions work for wine and art. *Journal of Economic Perspectives*, 3:23-26, 1989.

\_\_\_\_\_ & Genesove, D. Testing for price anomalies in real-estate auctions. *American Economic Review*, 82:501-5, 1992.

Bernhardt, D. & Scoones, D. A note on sequential auctions. *American Economic Review*, 84(3):653-7, 1994.

Engelbrecht-Wiggans, R. Optimal reservation prices in auctions. *Management Science*, 33:763-70, 1987.

- \_\_\_\_\_. Revenue equivalence in multi-object auctions. *Economics Letters*, 26(1):15-9, 1988.
- \_\_\_\_\_. Optimal auctions revisited. *Games and Economic Behavior*, 5:227-39, 1993.
- \_\_\_\_\_. Sequential auctions with stochastically equivalent objects. *Economics Letters*, 44(1-2):87-90, 1994.
- \_\_\_\_\_ & Kahn, C. *An empirical analysis of the afternoon effect in dairy cattle auctions*. University of Illinois, 1992. (Working Paper.)
- Gandal, N. Sequential auctions of interdependent objects: Israeli cable television licenses. *Journal of Industrial Economics*, 45(3):227-44, 1997.
- Jones, C.; Menezes, F. M. & Vella, F. *Auction price anomalies: evidence from wool auctions in Australia*. Australian National University, 1998. (Working Paper.)
- McAfee, R. P. & McMillan, J. Auctions with entry. *Economics Letters*, 23:343-7, 1987.
- \_\_\_\_\_ & Vincent, D. The declining price anomaly. *Journal of Economic Theory*, 60:191-212, 1993.
- Menezes, Flavio. Sequential auctions with delay costs: a two-period example. *Economics Letters*, 42:173-8, 1993a.
- \_\_\_\_\_. *Four essays in auction theory*. Department of Economics, University of Illinois, 1993b. (Ph.D. Dissertation.)
- \_\_\_\_\_ & Monteiro, P. K. Auctions with endogenous participation. *Review of Economic Design*, 5:71-89, 2000.
- Myerson, R. B. Optimal auction design. *Mathematics of Operations Research*, 6:58-63, 1981.
- Riley, J. G. & Samuelson, W. Optimal auctions. *American Economic Review*, 7:381-92, 1981.
- Taylor, W. *Declining prices in sequential auctions: an empirical investigation*. Rice University, 1991. (Working Paper.)
- Thiel, S. & Petry, G. Bidding behavior in second-price auctions: rare stamp sales 1923-1937. *Applied Economics*, 27(1):11-6, 1995.

Vanderporten, B. Strategic behavior in pooled condominium auctions. *Journal of Urban Economics*, 31:123-37, 1992a.

\_\_\_\_\_. Timing of bids at pooled real estate auctions. *Journal of Real Estate Finance and Economics*, 5:255-67, 1992b.

Vickrey, W. Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance*, 16:8-37, 1961.

Von der Fehr, N-H M. Predatory bidding in sequential auctions. *Oxford Economic Papers*, 46:345-56, 1994.

Weber, R. Multi-object auctions. In: Engelbrecht-Wiggans, R.; Shubik, M. & Stark, R. M. (eds.). *Auctions, bidding, and contracting: uses and theory*. New York University Press, 1983. p. 165-194.