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Shortcomings of the Brazilian Pre-Salt Auction Design*

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Contents: **1. Introduction; 2. The common value model; 3. Share Bidding Auctions; 4. Ranking of expected revenue; 5. Auctions with Sliding Scale Royalty Rates; 6. Conclusions; Appendix.**

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The Brazilian government instituted a new regulatory framework for auctioning pre-salt oil reserves to replace the previous concession regime with a profit-share bidding auction. Motivated by the new rules, we present a model of revenue share bidding auction under affiliation. We prove the existence of monotone pure-strategy equilibrium and characterize its equilibrium bidding function. In addition, we prove that it generates an expected revenue at least as large as the usual bonus bidding auction. Next, we introduce in the model a function representing a royalty rate that is contingent on the value of the object. We suggest instrument improves expected revenue in both models, reducing the gap between them. Analyzing this new regulatory framework from the viewpoint of theoretical and numerical results obtained, we discuss some consequences and argue that the previous regime dominates the new one.

O governo Brasileiro estabeleceu regras de partilha de produção, regulamentando o leilão do pré-sal, em subtituição ao regime de concessão anterior. Motivados por essas mudanças estudamos aqui um modelo de leilão com repartição de receitas. Demonstramos a existência de equilíbrio monótono e determinamos a sua forma analítica. Em seguida introduzimos no modelo uma função representando a taxa sobre os direitos de exploração, contingente no valor da produção. Este instrumento pode aumentar a receita esperada tanto do leilão de bonus quanto do leilão de partilha e reduzir a diferença entre ambos. Analisando o novo regimento do ponto de vista dos resultados teóricos e numéricos obtidos, discutimos algumas consequências e argumentamos que o regime de concessão anterior é superior.

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1. INTRODUCTION

In 2007, Brazil discovered oil reserves in the pre-salt layer, which put the country on track to becoming one of the top oil producers in the world. According to estimates from the *World Energy Outlook 2013* (IEA, 2013), the country's oil production will triple by 2035, turning Brazil into the world's sixth-largest oil producer. The total extent of pre-salt recoverable oil and natural gas is estimated to be around 100 billion barrels of oil equivalent. This discovery tripled the amount of confirmed reserves in the country.

The new discoveries of pre-salt oil fields led to the institution of a new regulatory framework. The Law No. 12351 of December 22, 2010 established a production sharing contract regime to be applied especially for licensing of the pre-salt reserves and other areas considered strategic by the government. In this new regime, government and oil companies (henceforth OC) will share the profit generated by the production of oil and gas, allowing the Union to earn a portion of the wealth yielded by the pre-salt fields. According to the legal rules, the winner will be the one who offers to the Union the highest share of profit oil, defining a profit-share bidding auction.

The article 4 of Law No. 12351 determines that Petrobras must be the sole operator of all blocks contracted under profit-share regime. The operator is entitled to a minimum stake of 30% of the consortium. If Petrobras was not participating in the auction or was part of one of the defeated consortia, the winning bidder must form a new consortium with the Brazilian company. As part of this new consortium, Petrobras adheres to tender protocol rules and winning bid terms. Moreover, the proprietary rights and obligations will be proportional to the stakes of each member of the consortium.

In addition to a share of the profit oil, the winning bidder will have to pay the Union a signing bonus at the moment of signature the contract, and royalties rate of 15% of the value of production. The value of the signing bonus is fixed and predetermined in tender protocol, so it is not part of the bid. Signing bonus and royalties can not be reimbursed under any circumstances. In the event of commercial discovery, royalties will be subtracted from total production in profit calculation, and the winning consortium acquires the right to appropriate the production volume corresponding to the royalties due. However, the same does not happen to the value of the signing bonus, considered a non-recoverable cost.

One important difference between the concession and profit-share models is the ownership of the oil extracted. In the concession regime, all oil extracted is owned by the concessionaire, and all share payments (e.g. royalties) are converted using the oil market price at that specific date. In the profit-share regime the oil is owned by the Union, whether it was extracted or not. The parcel of oil extracted entitled to the winning bidder is the volume corresponding to the costs, royalties, and the portion of profit oil bid. Therefore, the new rule reveals a government preference to keep the oil instead of monetary gains. Apparently, this decision seems to be based on a political rather than economic reasoning. After all the government could purchase the oil in the market after receiving the monetary payment correspondent to it.

In addition to Petrobras and other OC's, representatives of a public enterprise will be part of the consortium. This new public enterprise, called Pré-Sal Petróleo S.A. (PPSA), was created aiming to manage profit-share contracts, and contracts to trade the Union's oil, natural gas and other fluid hydrocarbons. Therefore, PPSA will be in the consortium as representative of Union's interest in the profit-share contract.

The operational committee of the consortium is responsible for its management. The operation committee is in charge of defining the exploration plans, elaborate plan for evaluation of discovery of oil deposits and natural gas, declare commerciality of each deposit discovered and define plan for field production development. However, all of these plans must be submitted to analysis and approval by the National Agency of Petroleum, Natural Gas and Biofuels (ANP).

In our analysis, we highlight three parts of this new framework:

(i) a fixed bonus,
(ii) a profit share bidding auction, and
(iii) Petrobras as obligatory participant and the sole operator of all blocks contracted under profit-share regime.

In assuming this role, Petrobras will be responsible for "running and executing, directly or indirectly, all exploration, evaluation, development, production and decommissioning of exploration and production installation activities."

Libra's oil field auction was the first test of the new regime. According to estimates, Libra holds as much as 12 billion barrels of recoverable oil, which corresponds to almost 2 years of United States' consumption.[1] If this volume of recoverable oil is confirmed, it will be the largest auction of an oil field worldwide.

The tender protocol confirms that Petrobras shall be the sole operator and must join the winning consortium with a minimum interest of 30%. The consortia may be formed by a maximum of five companies, and may include Petrobras. However, a company may not be part of more than one consortium.

In addition to the qualification process, any consortium interested in competing in the bidding round should pays a participation fee of R$ 2,067,400.00.

The signature bonus was set at R$ 15 billion. Thus, the winning consortium will have to pay this amount to be able to sign the production sharing contract. Each member of the consortium will be responsible for paying a portion of the signature bonus corresponding to its stake.

Bids will be classified in order of highest to lowest profit share of oil and the winning consortium will be the one who offers the highest surplus in oil for the Union. The bids should only indicates the percentage of profit oil offered to the Federal Government, respecting the minimum share of 41.65%.

Although, the profit share that will be delivered to the Union throughout the exploration process is adjusted by the oil price and the average daily production. More specifically, the portion of the surplus in oil offered by tenderers should refer to the Brent[2] oil price in a range of US$ 100.01 and US$ 120.00, and to an average daily oil production of active producing wells between 10,001 barrels/day and 12,000 barrels/day. So, the higher is the Brent price or the average daily production, the higher is the government's share. This rule is related to the sliding scale royalty rate in section 5.

The production sharing contract requires a percentage of local content in the exploration and production activities. In the exploration phase, which will last four years, the contracted consortium must meet a minimum of 37% of local content. After that, in the development stage, the minimum percentage will be 55% for modules with first oil up to 2021, and 59% for modules with first oil as of 2022.

On October 21st of 2013 Brazil finally held the first pre-salt field auction. The Libra oil field attracted 11 companies to register for the qualification process, most of them national oil companies. This number was well below the government's expectations of around 40 companies. From these 11 companies, only a five-member consortium was formed and actively participated in the bidding round.

The Libra field was sold to a consortium led by Petrobras (40%), and including the French OC Total (20%), Anglo-Dutch Shell (20%) and China's state-owned CNOOC (10%) and CNPC (10%). The consortium was the only competitor and won the auction with a bid of profit oil into the Union of 41.65%, the floor established in the tender protocol.

It is difficult to argue that the first pre-salt auction was a success. It lacked participation and the winning consortium won with the minimum allowed bid. We show in Monteiro, Araujo, Costellini, &

[1] Based on data from 2012, provided by U.S. Energy Information Administration (EIA).

[2] Brent oil is a specification of crude oil that serves as a reference price for buyers and sellers of oil in the whole world. This benchmark is used mainly in Europe and by OPEC, and it is a mix of crude oil from 15 different oil fields in the North Sea.

Damé (2016) that the new auction format lacks the existence of nice[3] equilibrium strategies. Of course this is not decisive but one more difficulty for the firms' bidding calculus.

Participation is key to an auction success.[4] For example, Bulow & Klemperer (1996) show, in a private values setting, that the best negotiation with n bidders is not as good as an auction with $n+1$ bidders. But participation could not be lower in the first pre-salt auction. The privileged position of Petrobras as exclusive operator of all pre-salt oil fields is, certainly, a key factor in reducing competition. If one OC has a superior technology why should Petrobras be privy to it? This might also works in the other way around, but Petrobras has no choice here, being an obligatory participant. Another competition reducing factor is Pré-Sal Petróleo S.A. (PPSA). This is a public company that do not enter with any money but might create innumerous inefficiencies.

The need to monitor costs in a profit sharing model is probably the most important reason for the creation of PPSA. This is not needed in a revenue sharing auction. The savings and efficiency gains points to the desirability of a revenue sharing arrangement. Production is clearly easier and less costly to monitor. A revenue sharing concession framework without preferences will have greater participation and competition and is with high probability more profitable for the OC and for the Government.

We discuss, mainly with numerical simulation, alternatives that could lead to a higher expected revenue for the government. Following this line of thought, in section 5 we extend our analysis of bonus bidding and revenue share to auctions with sliding scale royalty rates. We introduce in the models a function representing a royalty rate that is contingent on the value of the object. This instrument increase the expected revenue in both models and reduce the gap between them. With numerical exercises, we show that this result may be obtained by a simple sliding scale royalty rates, easily implemented in practice. This structure is widely used for governments around the world in mineral rights leasing auctions.

2. THE COMMON VALUE MODEL

We study, as briefly as possible, the theory behind the (symmetric) bonus bidding, the revenue share and the profit share bidding auctions. A contract to explore a resource with common value random return, V, is to be awarded through an auction. The contract is disputed by $N > 1$ bidders. To explore the resource the winner incur a cost $c > 0$. We suppose $\mathbb{E}[V] > c$.

Remark 1. It is customary to suppose $V \geq 0$. We however allow for $V < 0$. This gives flexibility and indeed later, we suppose V normally distributed.

Each bidder $n = 1,\ldots,N$ receives a random signal/estimate value S_n. Let $S = (S_1,\ldots,S_N)$. We suppose that (S,V) has a distribution with density $f(s,v) = f(s_1,\ldots,s_N,v)$. The density f is symmetric in $(s_1,\ldots,s_N)$. Let $h(v) = \int f(s,v)\,\mathrm{d}s$ be the density of V. Here $\mathrm{d}s := \mathrm{d}s_1 \ldots \mathrm{d}s_N$. And let

$$g(s_i\,|\,v) = \frac{\int f(s_i,s_{-i},v)\,\mathrm{d}s_{-i}}{\int f(s,v)\,\mathrm{d}s}$$

be the conditional density of s_i for a given v. We suppose $S_1,\ldots,S_N$ conditionally independent for a given V:

Assumption 1. *The density $f(s,v)$ is continuous and can be written as*

$$f(s,v) = h(v)g(s_1\,|\,v)\cdots g(s_N\,|\,v). \tag{DA1}$$

[3] By nice we mean strictly increasing differentiable bidding functions.

[4] From the seller's viewpoint, of course.

If $n \in \mathbb{N}$ and $x,y \in \mathbb{R}^n$ we define

$$x \vee y := \Big(\max\{x_1,y_1\},\ldots,\max\{x_n,y_n\}\Big),$$

and

$$x \wedge y := \Big(\min\{x_1,y_1\},\ldots,\min\{x_n,y_n\}\Big).$$

Definition 1. If $k : \mathbb{R}^n \to \mathbb{R}_+$ then k has multivariate monotone likelihood ratio if

$$k(x)k(y) \leq k(x \vee y)k(x \wedge y),$$

for every $x,y \in \mathbb{R}^n$.

If $n = 2$ we simply say that k has monotone likelihood ratio.

Definition 2. A family of random variables $X_1,X_2,\ldots,X_m$ with joint distribution density f_X is affiliate if f_X has multivariate monotone likelihood ratio.

Remark 2. If the density $f(s,v)$ satisfies (DA1) then it has multivariate monotone likelihood ratio if and only if $g(s_i\,|\,v)$ has monotone likelihood ratio.

We now add our second assumption:

Assumption 2 (Affiliation)**.** *$S_1,\ldots,S_N,V$ are affiliated.*

3. SHARE BIDDING AUCTIONS

We analyze three types of auctions: bonus bidding, revenue share bidding and profit-share bidding auctions.

Notation. Let $G(x\,|\,v) = \int_{-\infty}^{x} g(u\,|\,v)\,\mathrm{d}u$ and $\phi(x,v) = G^{N-1}(x\,|\,v)$. And let the density of S_1 be denoted by $\tilde{g}(s) = \int g(s|v)h(v)\,\mathrm{d}v$. Also define $Y = \max\{S_2,\ldots,S_N\}$.

For convenience and to be in the setup for the numerical simulations we suppose $f(s,v) > 0$ for every $(s,v) \in \mathbb{R}^{N+1}$.

3.1. Bonus bidding auction

The fee or bonus bidding auction is the usual first-price auction. The winner is the bidder who offers the highest bonus. Prospective bidders that have a low estimate S_i may not participate in the bidding due to the fixed cost $c > 0$.

Definition 3. An equilibrium of the bonus auction is a pair $(s_f,B(\cdot))$ such that

(i) Bidder $n = 1,\ldots,N$ enters a bid, $B(s_n)$, if his signal $s_n \geq s_f$. If $s_n < s_f$ he does not participate;
(ii) If each participating bidder bids according to $B(\cdot)$ then the bonus bid $B(s_n)$ maximizes the participating bidder n expected payoff and his optimal payoff is non-negative;
(iii) Any bidder with signal below s_f has a negative payoff if he enters a bid.

To find the equilibrium suppose, let us say, Bidder 1, with signal $S_1 = s$, bids $B(x)$. Then his expected payoff is (see notation above),

$$\begin{aligned}\pi(x) &= \mathbb{E}\left[(V - B(x) - c)I_{Y<x}\,\middle|\,S_1 = s\right] \\ &= \frac{\int (v - c - B(x))g(s\,|\,v)\phi(x,v)h(v)\,\mathrm{d}v}{\tilde{g}(s)}.\end{aligned}$$

Thus s_f is defined by

$$\mathbb{E}\left[(V-c)I_{Y<s_f}\,\middle|\,S_1=s_f\right]=0. \tag{1}$$

We prove in the Appendix the existence of s_f. The first order condition is

$$\pi'(x)\tilde{g}(s)=\int(v-c-B(x))g(s\,|\,v)\phi'(x,v)h(v)\,\mathrm{d}v-B'(x)\int g(s\,|\,v)\phi(x,v)h(v)\,\mathrm{d}v.$$

In equilibrium $\pi'(s)=0$. This gives equation (4) below. Define

$$a_f(x)=\frac{\int(v-c)g(x\,|\,v)\phi'(x,v)h(v)\,\mathrm{d}v}{\int g(x\,|\,v)\phi(x,v)h(v)\,\mathrm{d}v},\qquad c_f(x)=\frac{\int g(x\,|\,v)\phi'(x,v)h(v)\,\mathrm{d}v}{\int g(x\,|\,v)\phi(x,v)h(v)\,\mathrm{d}v}.$$

We omit the proof of the next theorem.

Theorem 1. *The symmetric equilibrium of the fee bidding auction, $(s_f,B(\cdot))$, is given by (i) and (ii):*

(i) The cutoff s_f satisfy

$$\mathbb{E}\left[(v-c)I_{Y<s_f}\,\middle|\,S_1=s_f\right]=0. \tag{2}$$

(ii)

$$B(s)=\int_{s_f}^{s}\exp\left(-\int_{y}^{s}c_f(u)\,\mathrm{d}u\right)a_f(y)\,\mathrm{d}y. \tag{3}$$

Remark 3. The bidding function, $B(\cdot)$, solves the differential equation

$$B'(x)=a_f(x)-c_f(x)B(x) \tag{4}$$

with initial condition $B(s_f)=0$. This equation may be solved using the integrating factor method[5] and gives (3).

3.2. Revenue share auction

In a revenue share auction (also called royalties share auction) each participating Bidder bids a share, $0\le b\le 1$, of the contract. This share is the fraction of the positive returns which is given to the seller. Losses (that is, if $V<0$) are not shared. The winner is the bidder that offers the highest share. A symmetric equilibrium of the revenue share bidding auction $(s_*,b(\cdot))$ is defined analogously as in the bonus bidding auction. The expected revenue of (say) Bidder 1 with signal $S_1=s$ bidding a share $b(x)$ is

$$\psi(x):=\mathbb{E}\left[(V-c-b(x)V^+)I_{Y<x}\,\middle|\,S_1=s\right]. \tag{5}$$

Let s_* be the participation cutoff and $b(\cdot)$ be the equilibrium bidding sharing function. As before, since in equilibrium $b(s_*)=0$,

$$\mathbb{E}\left[(V-c)1_{Y<s_*}\,\middle|\,S_1=s_*\right]=0. \tag{6}$$

We see that $s_*=s_f$. We show in the Appendix that $b(\cdot)$ satisfy the differential equation (ii) below.

Define

$$a(s)=\frac{\int(v-c)g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}{\int v^+g(s|v)\phi(s,v)h(v)\,\mathrm{d}v},\qquad c(s)=\frac{\int v^+g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}{\int v^+g(s|v)\phi(s,v)h(v)\,\mathrm{d}v}.$$

Proposition 1. *A symmetric equilibrium of the revenue share bidding auction, $(s_*,b(\cdot))$, is such that:*

[5] See Menezes & Monteiro (2008), Appendix B

(i) The participation cutoff is given by

$$\int (v-c)g(s_*|v)\phi(s_*,v)h(v)\,\mathrm{d}v = 0. \tag{7}$$

(ii) The bidding function solves the differential equation

$$b'(s) = a(s) - c(s)b(s),$$

with initial condition $b(s_*) = 0$.

Solving the differential equation we get

$$b(s) = \int_{s_*}^{s} \exp\left(-\int_{y}^{s} c(u)\,\mathrm{d}u\right)a(y)\,\mathrm{d}y. \tag{8}$$

The next theorem is proved in the appendix.

Theorem 2. $(s_*,b(\cdot))$ *is an equilibrium.*

3.3. Profit share auction

In the profit share auction the winner pays the seller a fraction of the profit $(v-c)^+$. If we define $\tilde{V} = V - c$ we may apply the previous revenue share equilibrium existence theorem.

4. RANKING OF EXPECTED REVENUE

We will show that the revenue share bidding auction has a higher revenue than the bonus bidding auction. Suppose the density f is continuously differentiable.

Theorem 3. *The revenue share bidding auction expected revenue is higher than the bonus bidding auction expected revenue.*

Proof. We use Proposition 7.1, page 104 in Krishna's book (Krishna, 2002). Let $\psi_{sh}(x) := \psi(x)$ from (5). Note that

$$\begin{aligned}
\psi_f(x) = \pi(x) &= \mathbb{E}\left[(V-c)I_{Y<x}\middle|S_1 = s\right] - \mathbb{E}\left[I_{Y<x}\middle|S_1 = s\right]B(x),\\
\psi_{sh}(x) &= \mathbb{E}\left[(v-c)I_{Y<x}\middle|S_1 = s\right] - b(x)\mathbb{E}\left[V^+I_{Y<x}\middle|S_1 = s\right]\\
&= \mathbb{E}\left[(v-c)I_{Y<x}\middle|S_1 = s\right] - \mathbb{E}\left[I_{Y<x}\middle|S_1 = s\right]\frac{b(x)\mathbb{E}\left[V^+I_{Y<x}|S_1 = s\right]}{\mathbb{E}\left[I_{Y<x}|S_1 = s\right]}.
\end{aligned}$$

If we define $Q^f(x,s) = B(x)$ and

$$Q^{sh}(x,s) = \frac{b(x)\mathbb{E}\left[V^+I_{Y<x}|S_1 = s\right]}{\mathbb{E}\left[I_{Y<x}|S_1 = s\right]},$$

we see that $Q^f(s_*,s_*) = 0 = Q^{sh}(s_*,s_*)$. To apply Proposition 7.1 in Krishna's book we need to check that $\frac{\partial(Q^{sh}(x,s)-Q^f(x,s))}{\partial s} \geq 0$. But this follows from

$$\frac{\mathbb{E}\left[V^+I_{Y<x}|S_1 = s\right]}{\mathbb{E}\left[I_{Y<x}|S_1 = s\right]} = \mathbb{E}\left[V^+\middle|Y < x, S_1 = s\right]$$

being increasing in s. □

Reece (1979) supposes each $S_1,\ldots,S_N,V$ to be log-normally distributed. He shows, numerically, that the profit share auction expected revenue dominates the revenue share auction expected revenue. Our numerical simulations (not included in the figures below) confirms this.

5. AUCTIONS WITH SLIDING SCALE ROYALTY RATES

In the following numerical simulations we suppose returns to be normally distributed. So we introduce the assumption:

Assumption 3. *The return V has a normal distribution, $\mathcal{N}(\mu,\sigma_V^2)$. The signal S_i, $i=1,\ldots,N$ conditional on V is normally distributed with mean V and variance σ^2. Thus*

$$\begin{aligned} f(s_1,\ldots,s_N,v) &= h(v)g(s_1|v)\cdots g(s_N|v),\\ h(v) &= \frac{1}{\sigma_V\sqrt{2\pi}}\exp\left(-\frac{(v-\mu)^2}{2\sigma_V^2}\right),\\ g(s_i|v) &= \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(s_i-v)^2}{2\sigma^2}\right). \end{aligned}$$

Lemma 1. *The density f has multivariate monotone likelihood ratio.*

Proof. It suffices to show that $g(s|v)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(s-v)^2}{2\sigma^2}\right)$ has monotone likelihood ratio. This is true, since if $s''>s'$,

$$\frac{g(s''|v)}{g(s'|v)}=\frac{\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(s''-v)^2}{2\sigma^2}\right)}{\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(s'-v)^2}{2\sigma^2}\right)}=\exp\left(\frac{(s'-v)^2}{2\sigma^2}-\frac{(s''-v)^2}{2\sigma^2}\right)=\exp\left((s''-s')(2v-s'-s'')\right)$$

is increasing in v. □

We now consider an auction where the winner will be required to pay a prefixed sliding scale royalty rate (henceforth SSRR) based on the value of the good in addition to the bonus bid. We show that this rule may increase the expected revenue of both models studied in the previous section and the gap between them shrink.

This type of rule is frequently used on oil lease auctions around the world. Governments set up some ranges of revenue or production volume and determine extra royalty rates to be paid when these ranges are reached. The SSRR is frequently based on the average daily production per well per month on a given area. In most cases the international price of oil is also considered in the arrangement of the SSRRs.

In model developed in this section, we depict the SSRR by a function depending on v. Let $\xi:\mathbb{R}\to\mathbb{R}_+$ be the function representing the SSRR. We suppose $\xi(v)=0$ if $v\leqq 0$.

Assumption 4. *The function $\xi(\cdot)$ is increasing and $x-\xi(x)$ is strictly increasing if $x>0$.*

In particular $x>\xi(x)$ if $x>0$. Now, we define a bonus bidding auction with SSRR. If Bidders $i=2,\ldots,N$ bids according to the strictly increasing differentiable function, $b_f(\cdot)$, Bidder 1, with signal $S_1=s$, problem is to choose x maximizing

$$\psi_f(x)=\mathbb{E}\left[(V-b_f(x)-\xi(v))I_{Y<x}\middle|S_1=s\right].$$

Next we define a share bidding auction with SSRR. Bidder's 1 problem is to choose x to maximize

$$\psi_s(x)=\mathbb{E}\left[\left((1-b_s(x))(V^+-\xi(v))-V^-\right)I_{Y<x}\middle|S_1=s\right].$$

Next corollaries present the solution for the fee bidding auction with SSRR and for the share bidding auction with SSRR, respectively:

Corollary 1. *The bidding function $b^*_{\bar{f}}$ is an equilibrium of the bonus bidding auction with sliding scale, such that*

$$b^*_{\bar{f}}(x) = \tau(x,x) - \int_{s_0}^{x} \exp\left(-\int_t^x \frac{\hat{f}(z,z)}{\int_{-\infty}^{z}\hat{f}(z,y)\,\mathrm{d}y}\,\mathrm{d}z\right)\left[\frac{\mathrm{d}\tau(t,t)}{\mathrm{d}t}\right]\mathrm{d}t, \tag{9}$$

where $\tau(s,y) = \mathbb{E}\left[V - \xi(V)\,|\,S_1 = s, Y = y\right]$.

Moreover, $b^*_{\bar{f}}$ solves

$$\begin{aligned} b^{*\prime}_{\bar{f}}(x) &= \frac{\left(\tau(x,x) - b^*_{\bar{f}}(x)\right)\hat{f}(x,x)}{\int_{-\infty}^{x}\hat{f}(x,y)\,\mathrm{d}y}; \\ b^*_{\bar{f}}(s_0) &= \tau(0,0) = 0. \end{aligned} \tag{10}$$

Corollary 2. *The bidding function $b^*_{\bar{s}}$ is an equilibrium of the revenue share bidding auction with sliding scale, such that:*

$$\begin{aligned} b^*_{\bar{s}}(x) = 1 - \mathbb{E}&\left[-\int_{s_0}^{x} \frac{\tau^+(z,z)\hat{f}(z,z)}{\int_{-\infty}^{z}\tau^+(z,y)\hat{f}(z,y)\,\mathrm{d}y}\,\mathrm{d}z\right] \\ &- \int_{s_0}^{x}\left(\frac{[u^-(t,t)]\hat{f}(t,t)}{\int_{-\infty}^{t}\tau^+(t,y)\hat{f}(t,y)\,\mathrm{d}y}\right)\mathbb{E}\left[-\int_t^x \frac{\tau^+(z,z)\hat{f}(z,z)}{\int_{-\infty}^{z}\tau^+(z,y)\hat{f}(z,y)\,\mathrm{d}y}\,\mathrm{d}z\right]\mathrm{d}t, \end{aligned} \tag{11}$$

where $\tau^+(s,y) = \mathbb{E}\left[V^+ - \xi(V^+)\,|\,S_1 = s, Y = y\right]$.

Moreover, $b^*_{\bar{s}}$ solves

$$\begin{aligned} b^{*\prime}_{\bar{s}}(x) &= \frac{\left[(1 - b^*_{\bar{s}}(x))\tau^+(x,x) - u^-(x,x)\right]\hat{f}(x,x)}{\int_{-\infty}^{x}\tau^+(x,y)\hat{f}(x,y)\,\mathrm{d}y}; \\ b^*_{\bar{s}}(s_0) &= 0. \end{aligned} \tag{12}$$

Notice that $\tau(s,y)$ and $\tau^+(s,y)$ are increasing in both variables due to Assumption 3. So, the proof of corollaries 2 and 1 follow from the theorems 2 and 1, respectively.

From this equilibrium result, we would like to show that it is possible to improve expected revenue by introducing a SSRR in the model. It is not easy to reach this result analytically, since a very aggressive royalty rate may result in the opposite effect. The main difficulty is to determine a shape of the function ξ that would lead to an increase in the expected revenue. Thus, the problem will be solved by numerical methods, and we select some examples of SSRRs to verify how the expected revenue changes. Since numerical methods will be used, it is necessary to choose specific values to all model parameters. We assume[6] that distribution of true resource value V is normal with standard deviation $\sigma_v^2 = (10)^2$, and the mean μ_v is set such that approximately 25% of all oil fields are economically viable ($\mu_v \cong -2.133$). We assume that the signals are unbiased, or putting differently, the distribution of the signal S_i was set in such a way that $\mathbb{E}\left[S_i\,|\,v\right] = v$. In addition, the random variable $s|v$ is assumed to be normally distributed with mean v and variance $\sigma_s^2 = 49$. We are considering a symmetric model, so all agents observe signals from the same distribution.

[6]Our main goals with this exercise is to suggest that a revenue improvement is possible if a correct SSRR is chosen. The results seems robust to other values for σ_v^2, μ_v and σ_s^2.

Based on the parameters values, we look into some examples of SSRRs that permit us to identify the results we want to show. We divide support of V^+ into k intervals and define $\xi(\cdot)$ as

$$\xi(v) = \begin{cases} 0 & \text{if } v < v_1 \\ \delta_1(v - v_1) & \text{if } v \in [v_1, v_2) \\ \delta_1(v_2 - v_1) + \delta_2(v - v_2) & \text{if } v \in [v_2, v_3) \\ \quad\vdots & \\ \sum_{i=1}^{k-2} \delta_i(v_{i+1} - v_i) + \delta_{k-1}(v - v_{k-1}) & \text{if } v \geq v_{k-1}. \end{cases} \tag{13}$$

Let $\mathcal{V} = \begin{bmatrix} v_1 & v_2 & v_3 & \dots & v_{k-1} \end{bmatrix}$ and $\Delta = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & \dots & \delta_{k-1} \end{bmatrix}$. We assign different values to these vectors making SSRR more or less aggressive depending on the level of royalty rates (Δ) and the thresholds ($\mathcal{V}$). So, the higher are levels of rates and the lower are the thresholds, the more aggressive is the SSRR.

According to the function ξ described above, the royalty rates are calculated separately in each interval. For example, assume that $\mathcal{V} = \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}$. If the realization of the common value is $v = 4.5$, the amount of royalty due to the auctioneer will be $\xi(4.5) = 0.2(3-2) + 0.3(4-3) + 0.4(4.5-4) = 0.7$. Assuming that $\tilde{b}_{\mathfrak{f}}$ and $\tilde{b}_{\mathfrak{s}}$ are the winning bids of fee and share bidding models respectively, the amount of the final payoffs of the winner will be

$$PO_{\mathfrak{f}} = v - \tilde{b}_{\mathfrak{f}} - \xi v = 4.5 - \tilde{b}_{\mathfrak{f}} - 0.7 = 3.8 - \tilde{b}_{\mathfrak{f}};$$
$$PO_{\mathfrak{s}} = \left(1 - \tilde{b}_{\mathfrak{s}}\right)\left(v - \xi(v)\right) = \left(1 - \tilde{b}_{\mathfrak{s}}\right)(3.8).$$

Next, we present five parametrizations that go from less to more aggressive SSRR:

- Parametrization 1: $\mathcal{V} = \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}$;
- Parametrization 2: $\mathcal{V} = \begin{bmatrix} 1 & 1.5 & 1.8 & 2.2 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 0.2 & 0.3 & 0.4 & 0.5 \end{bmatrix}$;
- Parametrization 3: $\mathcal{V} = \begin{bmatrix} 1 & 1.5 & 1.8 & 2.2 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 0.4 & 0.6 & 0.7 & 0.9 \end{bmatrix}$;
- Parametrization 4: $\mathcal{V} = \begin{bmatrix} 1 & 1.5 & 1.8 & 2.2 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 0.96 & 0.97 & 0.98 & 0.99 \end{bmatrix}$;
- Parametrization 5: $\mathcal{V} = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 0.96 & 0.97 & 0.98 & 0.99 \end{bmatrix}$.

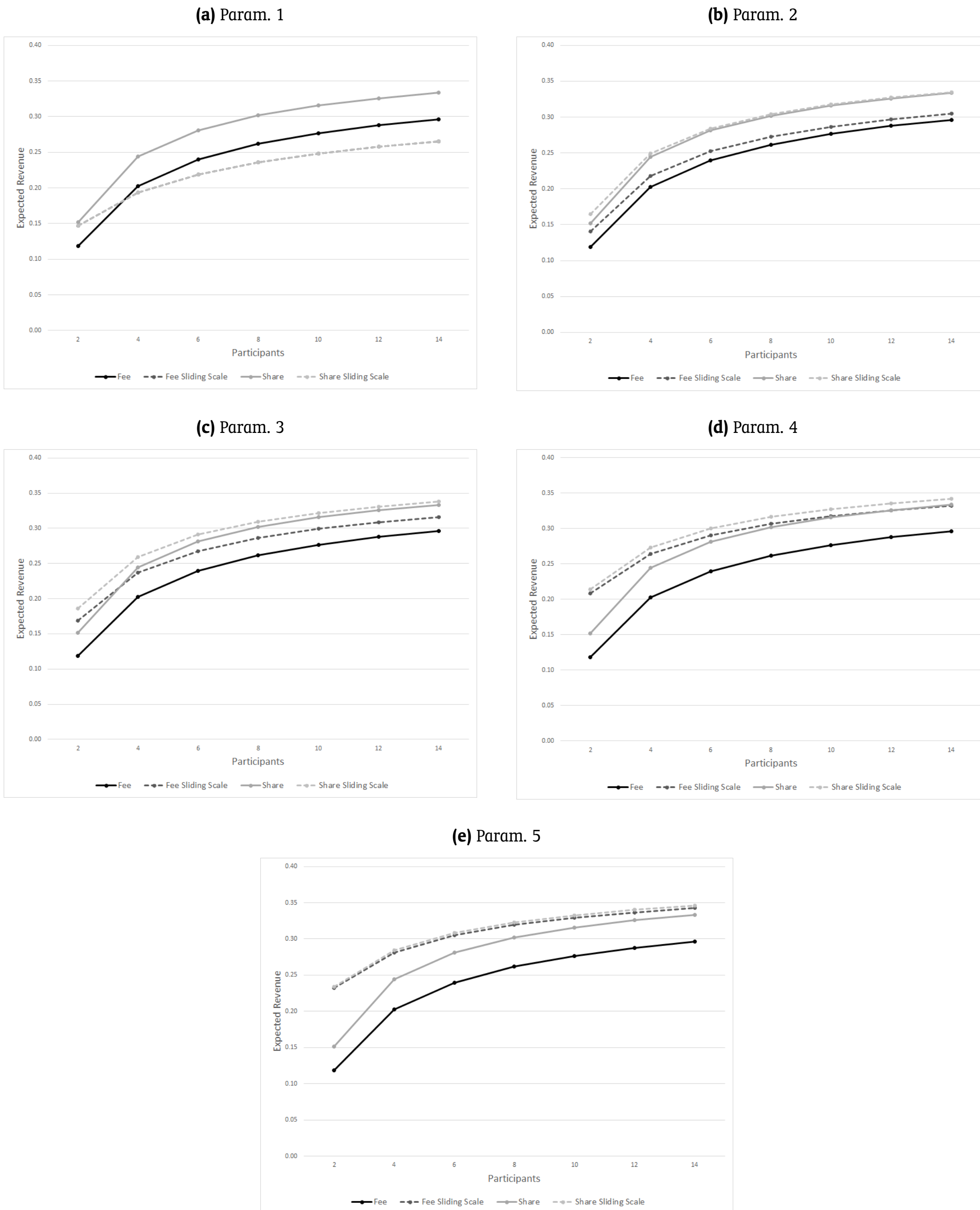
The results of our numerical exercises are presented in Figure 1. The figures illustrate auctioneer's expected revenue resulting from different numbers of participants. Dashed lines represent models with SSRRs, and continuous lines represent regular models.

The expected revenues of both auction schemes is higher when we include the function $\xi(\cdot)$. In figures 1(a) to 1(d) we can observe the evolution of the curves as more aggressive SSRRs are being implemented in the model. Moreover, the gap between the dashed lines were getting smaller and it practically disappears in Figure 1(e).

When a SSRR is present in an auction, the winner's final payment become more connected to the value of the object. In case of overestimation, the punishment of the winning bidder is less severe if compared with the standard auctions. It means that the winner's curse effects will be ameliorated and, consequently, bidders will be more comfortable to bid more aggressively. Thus, the SSRR allows the seller to capture a larger share of bidders' surplus.

However, if the SSRR is too aggressive, bidders will require a very high signal to start bidding and the auctioneer's expected revenue may fall short. On these terms, the bidders' expected payoff for a

Figure 1. Expected revenue and number of participants.



large set of observed signals s will be so small that they will prefer to stay out of the auction. In other words, the value of s_0 will be very high. That is what happened in the exercise presented on Figure 1(e).

This results indicates that there should be an optimal royalty rate function ξ that would lead to maximum and identical expected revenues for both types of auctions. Nevertheless, there seems to be no simple way of establishing this optimal function ξ. The winner's curse is directly related to the number of participants in the auction, thus the effectiveness of a SSRR will be also related with it. The more competition we have in an auction, the more severe will be the effects of winner's curse. The SSRR used in example of Figure 1(e) did not decrease the expected revenue when there were only two participants competing for the object. Hence, the choice of ξ, depends on the level of competition of the auction.

Despite we do not have a result that characterizes an optimal function ξ, the numerical exercises presented here shows that there is no need of a complex function to reach the desired results. With SSRRs that could be easily implemented in a practical case, our models generate higher expected revenue and almost vanish the gap between bonus and revenue share bidding auction. The SSRR used in the exercise exhibited on Figure 1(d) is very similar and even simpler than SSRRs present in oil leasing auctions of many countries.

6. CONCLUSIONS

Motivated by the new pre-salt reserves Brazilian regulatory framework, we analyzed a revenue share bidding auction model. A symmetric common values model with a risk neutral seller and risk neutral bidders was considered. We proved the existence of monotone pure-strategy equilibrium in the revenue share bidding model, and so, we could compare its expected revenue with the bonus bidding model. We conclude that revenue share bidding auction generates expected revenue at least as large as the bonus bidding auction. A numerical example suggests that this expected revenue can be improved if a sliding scale royalty rate is introduced in the model.

We discussed the new Brazilian regulatory framework and its first test. The outcome of this first pre-salt auction was not good for the government. The auction failed to draw a larger number of competitors, and this lack of competition entailed a minimum revenue for the Union. It is possible to identify some reasons for this poor result.

The compulsory participation of Petrobras is a controversial topic of the new regulatory framework that deserves attention. It creates a very unusual situation, where Petrobras is somehow competing against itself, since the competitors will be its partners in the future. It seems that this rule was created to increase government control over the industry via Petrobras; however, it can be detrimental to Petrobras. This rule forces Petrobras to form consortium with (perhaps) unwanted partners, or operate blocks that it considers un-viable. Petrobras will be compelled to join every exploration contract signed and pay for bids that were offered by other companies. Besides risking large losses, Petrobras will have to bear a considerable amount of money as capital expenditure, since oil production is a long-term investment.

The profit-share model is usually chosen by countries[7] with a bad reputation in respecting contracts, which is not the case of Brazil. Countries with unstable political systems or notorious for disrespecting agreements usually fail to draw international OCs to invest in their oil fields when a concession regime is used. In this system, the concessionaire is elected through a bonus bidding auction, thus the winning bidder pays at once a large amount of money and is entitled to explore the oil field for a determined period of time. Thus, this regime is very risky to the international OCs, since there is a high probability of default by the government of these countries. Meanwhile, in the profit-share regime the

[7] See Araujo, Costellini, Damé, & Monteiro (2012) for a review of the main production-sharing agreement worldwide.

payment is made throughout the contract period, so the consequences of breaking an agreement are much severe.

On the other hand, the profit-share regime requires a costly monitoring of costs. The public enterprise PPSA was created with this purpose. However it is unlikely that this monitoring activities do not generate management inefficiencies. This measure and Petrobras privileged position might have scared away several potential competitors. Recently, Mexico has also instituted a profit-share auction to lease oil and gas rights. However, they designated independent audit firms to deal with cost monitoring. The initial tests of their model proved to be much more efficient than Brazilian model, since a large number of international participants were attracted to the auctions.

In section 4 we demonstrate that the profit-share bidding auction generates a higher expected revenue than the bonus-bidding auction. However, as presented in section 5, by including a sliding scale royalty rate in both models, the expected revenues tend to reach higher levels with a lower gap between them, making the advantage of profit-share model vanish. In other words, the results indicate that, besides achieving a higher level, the expected revenue produced by both revenue share and bonus bidding auctions converge when an appropriate SSRR is introduced. Therefore, we think that a bonus bidding auction with a SSRR might be the most appropriate model for leasing oil and gas rights. Using this approach, the government could avoid the cost monitoring problem, and may attract more participants to the auctions.

The concession model,[8] used in Brazil before the pre-salt discoveries, seems to be much more reasonable than the new production-sharing regime. The Decree No. 2705, of August 3, 1998, establishes the "special participation" for the concession regime, defined as an "extraordinary financial compensation owed by concessionaires of exploration and production of crude oil and natural gas, in the case of a large volume of production or high earnings." Thus, this "special participation" described in this Decree is exactly a SSRR contingent to the production volume. Hence, using the concession regime, the Brazilian government could draw a larger number of foreign competitors and thus would be able to raise more money in each block. It must be emphasized that specifics SSRR could be planned for each block auctioned, considering its individual characteristics. An alternative to make the pre-salt fields more attractive would be replacing PPSA for independent audit firms, following the Mexican profit-share model.

At the time of this writing, a bill was approved that would relieve the mandatory participation in all pre-salt fields. This new proposal lays down that Petrobras may have preference to act as operator in each pre-salt field, and as a consequence to ensure the minimum stake of 30 percent of this field. The National Council for Energy Policy will be responsible to offer Petrobras this right of preference, considering domestic interest. If the right of preference is not offered or not exercised, Petrobras may still participate in the auction on an equal footing with the other competitor companies. We consider the change an improvement but still lacking. If Petrobras is offered the option and exercise it the outcome is probably low competition as before. If Petrobras does not exercise the option and does not participate in the auction low competition is the probable outcome. If Petrobras does not exercise the option and does participate in the auction can the outcome be competitive? It is unclear. It is fair to guess that if Petrobras expected much competition it would have exercised its preference option. This topic needs more investigation.

[8] In the concession regime, the winning consortium was determined by a formula that considers the amount of signature bonus (a fee that should be paid at once in the beginning of the contract), the minimum exploratory program and the percentage of local content used throughout exploration and production period. Despite the auction being decided by these three criteria, as reported by Moura, Canêdo-Pinheiro, & Daitx (2012), in 95% of the blocks successfully auctioned, the winner was the one who offered the higher amount of signature bonus.

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APPENDIX.

A.1.

Definition 4. A function $k:\mathbb{R}^n \to \mathbb{R}$ is increasing if $x,y \in \mathbb{R}^n$ and $x \geq y$ implies $k(x) \geq k(y)$. It is strictly increasing if $x \gneqq y$ implies $k(x) > k(y)$.

For a proof of the Lemma 2 see Karlin & Rinott (1980) page 484.

Lemma 2 (Sarkar (1969)). *If $Z_1,\dots,Z_k$ are affiliated and $H(z_1,\dots,z_k)$ is increasing then the conditional expectation*

$$\mathbb{E}\left[H(Z_1,\dots,Z_k)\,\middle|\,Z_1 = z_1,\dots,Z_k = z_k\right]$$

is increasing.

We omit the (easy) proof of Lemma 3.

Lemma 3. *Suppose $g(u|v)$ has monotone likelihood ratio. Let $G(x|v) = \int_{u<x} g(u|v)\,\mathrm{d}u$. Then the function $v \to \frac{g(x|v)}{G(x|v)}$ is increasing.*

Let $Y \equiv Y_1 := \max\{S_2,\dots,S_N\}$. Let Y_2 be the second highest number in $\{S_2,\dots,S_N\}$, Y_3 the third highest, ..., $Y_{N-1} = \min\{S_2,\dots,S_N\}$ the $(N-1)^{\text{th}}$ highest. Using the conditional independence of $S_1,\dots,S_N$ given V, the density of $(S_1,Y_1,Y_2,\dots,Y_{N-1},V)$ is easily shown to be

$$f_{(S_1,Y,Y_2,\dots,Y_{N-1},V)}(s_1,y_1,y_2,\dots,y_{N-1},v) = \begin{cases}(N-1)!\,f(s_1,y_1,\dots,y_{N-1},v) & \text{if } y_1 \geq y_2 \geq \dots \geq y_{N-1}\\ 0 & \text{otherwise.}\end{cases}$$

It is also immediate to check that $f_{(S_1,Y,Y_2,\dots,Y_{N-1},V)}$ has multivariate monotone likelihood ratio. Hence $S_1,Y,Y_2,\dots,Y_{N-1},V$ are affiliated.

The following proposition is (1.16), page 471 of Karlin & Rinott (1980).

Proposition 2. *Any subfamily of an affiliated family of random variables is affiliated.*

Corollary 3. *S_1,Y,V are affiliated.*

Let $f_{(S_1,Y,V)}$ denote the density of (S_1,Y,V). A routine calculation gives

$$f_{(S_1,Y,V)}(s_1,y,v) = (N-1)g(s_1\,|\,v)g(y|v)G^{N-2}(y|v)h(v). \tag{DSYV}$$

And if $f_{Y,V|S}$ denote the conditional density of (Y,V) for S_1 given we have

$$f_{Y,V|S}(y,v\,|\,s_1) = \frac{(N-1)g(s_1\,|\,v)g(y|v)G^{N-2}(y|v)h(v)}{\int g(s_1\,|\,v)h(v)\,\mathrm{d}v}. \tag{DYV}$$

A.2. Proof of existence of the cutoff s_f

We prove the

Lemma 4. *There is a solution s_f to*

$$\mathbb{E}\left[(v-c)I_{Y<s_f}\,\middle|\,S_1 = s_f\right] = 0.$$

We begin with the following:

Proposition 3. *The function $s \to \mathbb{E}\left[(V-c)1_{Y<s}\,|\,S_1 = s\right]$ is increasing.*

Proof. We have that

$$\begin{aligned}\mathbb{E}\left[(V-c)1_{Y<s}\,\middle|\,S_1 = s\right] &= \frac{\int (v-c)1_{Y<s}h(v)g(s\,|\,v)(N-1)g(y|v)G^{N-2}(y|v)\,\mathrm{d}v\,\mathrm{d}y}{\int h(v)g(s|v)\,\mathrm{d}v}\\ &= \frac{\int (v-c)1_{Y<s}h(v)g(s|v)G^{N-1}(s|v)\,\mathrm{d}v}{\int h(v)g(s|v)\,\mathrm{d}v}.\end{aligned}$$

Suppose $s' < s''$. Let

$$f_1(v) = \frac{h(v)g(s'\,|\,v)G^{N-1}(s'\,|\,v)}{\int h(v)g(s'\,|\,v)\,\mathrm{d}v} \quad \text{and} \quad f_2(v) = \frac{h(v)g(s''\,|\,v)G^{N-1}(s''\,|\,v)}{\int h(v)g(s''\,|\,v)\,\mathrm{d}v}.$$

If we prove that $f_1(v')f_2(v'') \le f_1(v' \wedge v'')f_2(v' \vee v'')$ then from Karlin & Rinott (1980), Theorem 2.2 on page 477, we conclude

$$\int (v-c)f_1(v)\,\mathrm{d}v \le \int (v-c)f_2(v)\,\mathrm{d}v,$$

and this will end the proof. □

We proceed to prove that $f_1(v')f_2(v'') \le f_1(v' \wedge v'')f_2(v' \vee v'')$.

Proof. This will be true if and only if

$$\frac{h(v')g(s'|v')G^{N-1}(s'|v')}{\int h(v)g(s'|v)\,\mathrm{d}v}\,\frac{h(v'')g(s''|v'')G^{N-1}(s''|v'')}{\int h(v)g(s''|v)\,\mathrm{d}v}$$
$$\leq \frac{h(v'\wedge v'')g(s'|v'\wedge v'')G^{N-1}(s'|v'\wedge v'')}{\int h(v)g(s'|v)\,\mathrm{d}v}\,\frac{h(v'\vee v'')g(s''|v'\vee v'')G^{N-1}(s''|v'\vee v'')}{\int h(v)g(s''|v)\,\mathrm{d}v}.$$

Canceling the denominators:

$$h(v')g(s'|v')G^{N-1}(s'|v')h(v'')g(s''|v'')G^{N-1}(s''|v'')$$
$$\leq h(v'\wedge v'')g(s'|v'\wedge v'')G^{N-1}(s'|v'\wedge v'')h(v'\vee v'')g(s''|v'\vee v'')G^{N-1}(s''|v'\vee v'').$$

Noting that $s''=s'\vee s''$ and $s'=s'\wedge s''$ this inequality is equivalent to say that $\phi:=g(s|v)G^{N-1}(s|v)h(v)$ has multivariate monotone likelihood ratio. But from (DYV) we see that $N\phi$ is the density of (Y,V) if there are $N+1$ bidders. Since (N,Y) is affiliated the proof is finished. □

Thus if $\mathbb{E}\left[(V-c)1_{Y<s}|S_1=s\right]>0$ for every s, every bidder participates (that is $s_*:=-\infty$). If $\mathbb{E}\left[(V-c)1_{Y<s}|S_1=s\right]<0$ for every s there will be no participating bidders. Otherwise there will be a s_f satisfying (6).

Remark 4. We easily show that the case of no-participating bidders does not occur since

$$\begin{aligned}\mathbb{E}\left[(v-c)1_{Y<S_1}\right]&=\int_{y<s_1}(v-c)g(s_1|v)(N-1)g(y|v)G^{N-2}(y|v)h(v)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}s_1\\&=\int(v-c)g(s_1|v)G^{N-1}(s_1|v)h(v)\,\mathrm{d}v\,\mathrm{d}s_1\\&=\frac{1}{N}\int(v-c)h(v)\,\mathrm{d}v>0\end{aligned}$$

implies the existence of s such that $\mathbb{E}\left[(V-c)1_{Y<s}|S_1=s\right]>0$.

A.3. The revenue share bidding equilibrium

A.3.1. Differential equation for $b(\cdot)$

Let $x^+:=\max\{x,0\}$. Suppose bidders $n=2,\ldots,N$ bids according to $b(\cdot)$. If Bidder 1 has signal s and bids as if his signal is x his expected profit is

$$\psi(x):=\mathbb{E}\left[(V-c-b(x)V^+)I_{Y<x}\middle|S_1=s\right]. \tag{A-1}$$

Note that we used the assumption that losses are not shared. Thus the seller payment is $b(x)V^+$. And we also used that $I_{b(Y)<b(x)}=I_{Y<x}$. We may write

$$\begin{aligned}\psi(x)\tilde{g}(s)&=\int_v\int_{y<x}\left(v-c-b(x)v^+\right)g(s|v)\phi'(y,v)h(v)\,\mathrm{d}v\\&=\int\left(v-c-b(x)v^+\right)g(s|v)\phi(x,v)h(v)\,\mathrm{d}v.\end{aligned}$$

Differentiating:

$$\psi'(x)\tilde{g}(s)=-b'(x)\int v^+g(s|v)\phi(x,v)h(v)\,\mathrm{d}v+\int\left(v-c-b(x)v^+\right)g(s|v)\phi'(x,v)h(v)\,\mathrm{d}v. \tag{A-2}$$

If $b(\cdot)$ is an equilibrium then $\psi'(s) = 0$. Thus

$$b'(s)\int v^+ g(s|v)\phi(s,v)h(v)\,\mathrm{d}v = \int \Big(v - c - b(s)v^+\Big)g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v.$$

Define

$$a(s) = \frac{\int (v-c)g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}{\int v^+ g(s|v)\phi(s,v)h(v)\,\mathrm{d}v}, \qquad c(s) = \frac{\int v^+ g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}{\int v^+ g(s|v)\phi(s,v)h(v)\,\mathrm{d}v}.$$

Then

$$\begin{aligned} b'(s) &= a(s) - b(s)c(s);\\ b(s_*) &= 0. \end{aligned}$$

A.3.2. Proof that $b(\cdot)$ is an equilibrium

We need the

Lemma 5. *The ratio* $\frac{a(s)}{c(s)} < 1$ *is increasing.*

Proof. Note first that since $v - c = v^+ - (v^- + c) \le v^+$ it is immediate $\frac{a(s)}{c(s)} < 1$. We may write

$$\frac{a(s)}{c(s)} = \frac{\int (v-c)g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}{\int v^+ g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v} = 1 - \frac{\int (v^- + c)g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}{\int v^+ g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}.$$

Thus it suffices to prove that $\frac{\int (v^- + c)g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}{\int v^+ g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}$ is decreasing. Note that

$$\begin{aligned} \frac{\int (v^- + c)g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}{\int v^+ g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v} &= \frac{\mathbb{E}\left[V^- + c \,\middle|\, S_1 = s, Y = s\right]\int g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}{\mathbb{E}\left[V^+ \,\middle|\, S_1 = s, Y = s\right]\int g(s|v)\phi'(s,v)h(v)\,\mathrm{d}v}\\ &= \frac{\mathbb{E}\left[V^- + c \,\middle|\, S_1 = s, Y = s\right]}{\mathbb{E}\left[V^+ \,\middle|\, S_1 = s, Y = s\right]}. \end{aligned}$$

Lemma 2 implies that the denominator is increasing and the numerator is decreasing. □

Lemma 6. *The function* $b(\cdot)$ *is strictly increasing and smaller than* 1.

Proof. We will prove that $b'(s) > 0$. For this it suffices to prove that $b(s) < \frac{a(s)}{c(s)}$. Thus

$$\begin{aligned} b(s) &= \int_{s_*}^{s} \exp\left(-\int_y^s c(u)\,\mathrm{d}u\right)a(y)\,\mathrm{d}y = \int_{s_*}^{s} \exp\left(-\int_y^s c(u)\,\mathrm{d}u\right)\frac{a(y)}{c(y)}c(y)\,\mathrm{d}y\\ &\le \frac{a(s)}{c(s)}\int_{s_*}^{s} \exp\left(-\int_y^s c(u)\,\mathrm{d}u\right)c(y)\,\mathrm{d}y = \frac{a(s)}{c(s)}\left[\exp\left(-\int_y^s c(u)\,\mathrm{d}u\right)\Big|_{s_*}^{s}\right]\\ &= \frac{a(s)}{c(s)}\left[1 - \exp\left(-\int_{s_*}^{s} c(u)\,\mathrm{d}u\right)\right] < \frac{a(s)}{c(s)} < 1. \end{aligned}$$

□

Theorem 4. $(s_*, b(\cdot))$ *is an equilibrium of the revenue share auction.*

Proof. Let $\widehat{f}(s,y) = \int g(s|v)\phi'(y,v)h(v)\,\mathrm{d}v$ and $\widehat{g}(s,y,v) = g(s|v)\phi'(y,v)h(v)$. From equation (A-2):

$$\psi'(x)\tilde{g}(s) = -b'(x)\int v^{+}g(s|v)\phi(x,v)h(v)\,\mathrm{d}v + \int \left(v - c - b(x)v^{+}\right)\widehat{g}(s,x,v)\,\mathrm{d}v. \tag{A-3}$$

Suppose $x < s$. We separate the proof in two cases. In the first case we suppose

$$\frac{\widehat{f}(s,x)}{\widehat{f}(x,x)} \geq \frac{\int v^{+}g(s|v)\phi(x,v)h(v)\,\mathrm{d}v}{\int v^{+}g(x|v)\phi(x,v)h(v)\,\mathrm{d}v}.$$

Since $v \to v - c - b(x)v^{+}$ is increasing, Lemma 2 implies

$$\mathbb{E}\left[V - c - b(x)V^{+}\,|\,S = s, Y = x\right] \geq \mathbb{E}\left[V - c - b(x)V^{+}\,|\,S = x = Y\right].$$

Thus

$$\begin{aligned}\frac{\int (v - c - b(x)v^{+})\widehat{g}(s,x,v)\,\mathrm{d}v}{\widehat{f}(s,x)} &= \mathbb{E}\left[V - c - b(x)V^{+}\,\middle|\,S = s, Y = x\right]\\ &\geq \mathbb{E}\left[V - c - b(x)V^{+}\,\middle|\,S = x = Y\right] = \frac{\int (v - c - b(x)v^{+})\widehat{g}(x,x,v)\,\mathrm{d}v}{\widehat{f}(x,x)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int (v - c - b(x)v^{+})\widehat{g}(s,x,v)\,\mathrm{d}v &\geq \frac{\widehat{f}(s,x)}{\widehat{f}(x,x)}\int (v - c - b(x)v^{+})\widehat{g}(x,x,v)\,\mathrm{d}v\\ &= b'(x)\frac{\widehat{f}(s,x)}{\widehat{f}(x,x)}\int v^{+}g(x|v)\phi(x,v)h(v)\,\mathrm{d}v\\ &\geq b'(x)\int v^{+}g(s|v)\phi(x,v)h(v)\,\mathrm{d}v.\end{aligned}$$

Hence from (A-3), $\psi'(x) \geq 0$. In the second case, we suppose

$$\frac{\widehat{f}(s,x)}{\widehat{f}(x,x)} < \frac{\int v^{+}g(s|v)\phi(x,v)h(v)\,\mathrm{d}v}{\int v^{+}g(x|v)\phi(x,v)h(v)\,\mathrm{d}v}.$$

Let us write

$$\int (v - c - b(x)v^{+})\widehat{g}(s,x,v)\,\mathrm{d}v = (1 - b(x))\alpha - \beta,$$

where $\alpha = \int v^{+}\widehat{g}(s,x,v)\,\mathrm{d}v$, and $\beta = \int (v^{-} + c)\widehat{g}(s,x,v)\,\mathrm{d}v$. The function $v \to v^{-} + c$ is decreasing thus using Lemma 2:

$$\begin{aligned}\beta &= \int (v^{-} + c)\widehat{g}(s,x,v)\,\mathrm{d}v = \mathbb{E}\left[V^{-} + c\,|\,S = s, Y = x\right]\widehat{f}(s,x)\\ &\leq \mathbb{E}\left[V^{-} + c\,|\,S = x = Y\right]\widehat{f}(s,x) = \frac{\widehat{f}(s,x)}{\widehat{f}(x,x)}\int (v^{-} + c)\widehat{g}(x,x,v)\,\mathrm{d}v\\ &< \frac{\int v^{+}g(s|v)\phi(x,v)h(v)\,\mathrm{d}v}{\int v^{+}g(x|v)\phi(x,v)h(v)\,\mathrm{d}v}\int (v^{-} + c)\widehat{g}(x,x,v)\,\mathrm{d}v = \left(c(x) - a(x)\right)\int v^{+}g(s|v)\phi(x,v)h(v)\,\mathrm{d}v.\end{aligned}$$

Now from the Claim below,

$$\alpha \geq \frac{\int v^+ g(s|v)\phi(x,v)h(v)\,\mathrm{d}v}{\int v^+ g(x|v)\phi(x,v)h(v)\,\mathrm{d}v} \int v^+ \widehat{g}(x,x,v)\,\mathrm{d}v = c(x) \int v^+ g(s|v)\phi(x,v)h(v)\,\mathrm{d}v.$$

Thus

$$\begin{aligned}
\int (v - c - b(x)v^+)\widehat{g}(s,x,v)\,\mathrm{d}v &= (1-b(x))\alpha - \beta \\
&\geq (1-b(x))c(x) \int v^+ g(s|v)\phi(x,v)h(v)\,\mathrm{d}v - (c(x)-a(x)) \int v^+ g(s|v)\phi(x,v)h(v)\,\mathrm{d}v \\
&= \big(a(x) - b(x)c(x)\big) \int v^+ g(s|v)\phi(x,v)h(v)\,\mathrm{d}v = b'(x) \int v^+ g(s|v)\phi(x,v)h(v)\,\mathrm{d}v.
\end{aligned}$$

Hence $\psi'(x) \geq 0$ in any case. Finally the case $x > s$ is analogous. □

Claim 1. *If $s > x$,*

$$\frac{\int v^+ \widehat{g}(s,x,v)\,\mathrm{d}v}{\int v^+ g(s|v)\phi(x,v)h(v)\,\mathrm{d}v} \geq \frac{\int v^+ \widehat{g}(x,x,v)\,\mathrm{d}v}{\int v^+ g(x|v)\phi(x,v)h(v)\,\mathrm{d}v}.$$

To prove the claim we need the following

Proposition 4. *Let f_1, f_2, f_3, f_4 be non-negative functions on $\mathbb{R}^m$ such that*

$$f_1(x) f_2(y) \leq f_3(x \vee y) f_4(x \wedge y), \forall x,y \in \mathbb{R}^m.$$

Then

$$\int f_1(x)\,\mathrm{d}x \cdot \int f_2(x)\,\mathrm{d}x \leq \int f_3(x)\,\mathrm{d}x \cdot \int f_4(x)\,\mathrm{d}x.$$

For a proof see Theorem 2.1 page 475 of Karlin & Rinott (1980). The following inequality implies the Claim: If $s'' > s'$ then

$$\frac{\int v^+ g(s''|v)\phi'(x,v)h(v)\,\mathrm{d}v}{\int v^+ g(s''|v)\phi(x,v)h(v)\,\mathrm{d}v} \geq \frac{\int v^+ g(s'|v)\phi'(x,v)h(v)\,\mathrm{d}v}{\int v^+ g(s'|v)\phi(x,v)h(v)\,\mathrm{d}v}.$$

Thus let us suppose $s'' > s'$. Let

$$f_1(v) = v^+ g(s'|v)\phi'(x,v)h(v) \tag{A-4}$$

$$f_2(v) = v^+ g(s''|v)\phi(x,v)h(v) \tag{A-5}$$

$$f_3(v) = v^+ g(s''|v)\phi'(x,v)h(v) \tag{A-6}$$

$$f_4(v) = v^+ g(s'|v)\phi(x,v)h(v). \tag{A-7}$$

The claim is true[9] if we show that

$$\int f_3(x)\,\mathrm{d}x \cdot \int f_4(x)\,\mathrm{d}x \geq \int f_1(x)\,\mathrm{d}x \cdot \int f_2(x)\,\mathrm{d}x.$$

Thus from Proposition 4 it suffices to check that

$$\begin{aligned}
(v' \vee v'')^+ g(s''|v' \vee v'')\phi'(x,v' \vee v'')h(v' \vee v'')\,(v' \wedge v'')^+ g(s'|v' \wedge v'')\phi(x,v' \wedge v'')h(v' \wedge v'') \\
\geq (v')^+ g(s'|v')\phi'(x,v')h(v')\,(v'')^+ g(s''|v'')\phi(x,v'')h(v'').
\end{aligned}$$

[9]Just choose $s' = x$.

If $v' \wedge v'' \leq 0$ both lines are null. If $v', v'' > 0$ we may cancel $(v')^+(v'')^+$ on both sides. We may also cancel $h(v')h(v'')$ on both sides of the inequality. Thus it suffices to prove

$$g(s''|v' \vee v'')g(s'|v' \wedge v'')\phi'(x|v' \vee v'')\phi(x, v' \wedge v'') \geq g(s'|v')g(s''|v'')\phi'(x, v')\phi(x, v'').$$

The monotone likelihood ratio property of $g(s|v)$ implies that it is enough to prove the inequality without the g's:

$$\phi'(x|v' \vee v'')\phi(x, v' \wedge v'') \geq \phi'(x, v')\phi(x, v'').$$

Thus we need to prove that

$$g(x|v' \vee v'')G^{N-2}(x|v' \vee v'')G^{N-1}(x|v' \wedge v'') \geq g(x|v')G^{N-2}(x|v')G^{N-1}(x|v'').$$

Dividing both sides by $G^{N-2}(x|v')G^{N-2}(x|v'')$ we get

$$g(x|v' \vee v'')G(x|v' \wedge v'') \geq g(x|v')G(x|v'').$$

If $v' \geq v''$, both sides are equal. If $v' < v''$ we need to prove that

$$g(x|v'')G(x|v') \geq g(x|v')G(x|v'').$$

But this is Lemma 3.