Physics Cup Problem 4

Axis of a lens

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1 First constraint: Circle

We draw the tangents from O to the ellipse, and take their bisector. We know that the circle is also tangent to the two lines, and consequently the bisector passes through the circle's center. The tangents formed by the intersection of the bisector with the circle are parallel, and their images are the intersections of the bisector with the ellipse. We know that the images of two parallel lines intersect in the focal plane, call the point of intersection F_1. Since the angle $\angle OFF_1 = \pi/2$, F sits on the circle with diameter OF_1, which we can easily draw.

2 Second constraint: Hyperbola

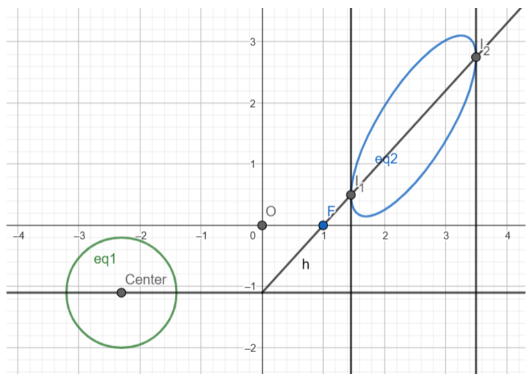

Figure 1: Representation of a perpendicular ray to the lens Oy, passing though the center of the circle.

As seen in Figure 1, the refraction of the perpendicular ray passes through F, I_1 and I_2. I_1 and I_2 are images of points on the circle that have tangents parallel to Oy, therefore their tangents on the ellipse are also parallel to the lens. In conclusion, F sits on the intersection between line I_1I_2(which passes through the center of the ellipse!), and the perpendicular from O to the tangents.

We will now define a function $F'(I)$(varying the point tangent to the ellipse) to determine the possible positions of F, and will show that $F'(I)$ is a conic.

2.1 Mathematical proof that $F'(I)$ is a conic

We will define an axis of coordinates with the origin at the center of the ellipse, with semi-axis a on $O'x$ and b on $O'y$. Let the coordinates of O be (p, q), and the point of tangency I be at (x_0, y_0) on the ellipse. Our goal is to show that the position of the mentioned intersection, F', depends in a conical manner on (x_0, y_0).

The equation of the ellipse is $x^2/a^2 + y^2/b^2 = 1$. The tangent in I has the equation $xx_0/a^2 + yy_0/b^2 = 1$, therefore it has the slope $-\frac{b^2x_0}{a^2y_0}$. The perpendicular from O to the tangent is $y = \frac{a^2y_0}{b^2x_0}(x-p)+q$. F' is the intersection of the previous line with $y = \frac{y_0}{x_0}x$.

$$\frac{y_0}{x_0}x = \frac{a^2y_0}{b^2x_0}(x-p)+q$$

$$x = \frac{a^2}{b^2}x - \frac{a^2}{b^2}p + \frac{x_0}{y_0}q$$

$$x = C_1 \frac{x_0}{y_0} + C_2$$

$$y = \frac{y_0}{x_0} x = C_1 + \frac{y_0}{x_0} C_2$$

So, the coordinates of F' are $(C_1 \frac{x_0}{y_0} + C_2, C_1 + \frac{y_0}{x_0} C_2)$. The x position depends linearly on $\frac{x_0}{y_0}$, while y depends linearly on $1/(\frac{x_0}{y_0})$, so F' is clearly a hyperbola of $\frac{x_0}{y_0}$(which defines a unique position of I), thus a conic.

3 Final determination of F

Now, in a barbaric way, we will determine 5 points on $F'(I)$ by choosing random I's on the ellipse and setting the intersections. Having these 5 points, we will draw the conic through them. Only the construction of the point $F'(B^{''}) = F^{''}$ is left visible in the .ggb file.

The circle from the first section and the conic intersect in only one point except O, thus we have found F. Drawing the line OF and extracting its slope we obtain:

$$a \approx 0.54627393.$$