Physics Cup Problem 5

Satellite Orbit

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Contents

1 Laplace-Runge-Lenz Vector

Firstly, we will define the Laplace-Runge-Lenz Vector, which is constant in magnitude and direction. We will work in a cylindrical coordinate system (r, θ, z) with the origin in the M object, Earth, in our case. Here $k = GMm$.

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - km\hat{\mathbf{r}} \tag{1}$$

Proof that it is constant, using $\dot{\mathbf{L}} = 0$, and $\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\theta}$:

$$\begin{aligned}
\frac{d\mathbf{A}}{dt} &= \dot{\mathbf{p}} \times \mathbf{L} - km\dot{\hat{\mathbf{r}}} \\
&= -\frac{kL}{r^2}(\hat{\mathbf{r}} \times \hat{\mathbf{z}}) - km\dot{\theta}\hat{\theta} \\
&= \frac{L}{mr^2}\hat{\theta} - \dot{\theta}\hat{\theta} \\
&= 0
\end{aligned} \tag{2}$$

Now we will demonstrate a useful identity for the eccentricity: $e = \frac{A}{mk}$.

$$\begin{aligned}
\left(\frac{\mathbf{A}}{mk}\right)^2 &= \frac{p^2L^2}{m^2k^2} + \hat{\mathbf{r}}^2 - \frac{2(\mathbf{p} \times \mathbf{L}) \cdot \hat{\mathbf{r}}}{mk} \\
&= 1 + \frac{p^2L^2}{m^2k^2} - \frac{2Lr\dot{\theta}}{mk} \\
&= 1 + \frac{2EL^2}{m(GmM)^2} = e^2
\end{aligned} \tag{3}$$

It is important to note that in an elliptical motion $\mathbf{A}$ is along the big semi-axis of the ellipse, by invoking Equation 1 at the perihelion, i.e. the minimum distance.

2 Geometrical Interpretation

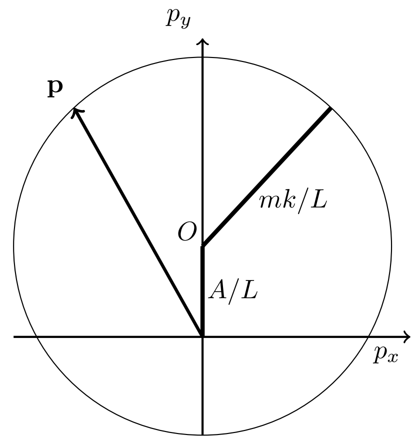
Let the ellipse have the big semi-axis on the x-axis and the small semi-axis on the y-axis. We know that $\mathbf{A} = (A_x, 0, 0)$, and $\mathbf{p} = (p_x, p_y, 0)$. Then $\mathbf{p} \times \mathbf{A} = -\hat{\mathbf{z}}p_yA$.

$$\begin{aligned}
mk\hat{\mathbf{r}} &= \mathbf{p} \times \mathbf{L} - \mathbf{A} \\
(mk)^2 &= p^2L^2 + A^2 + 2\mathbf{L} \cdot (\mathbf{p} \times \mathbf{A}) \\
\left(\frac{mk}{L}\right)^2 &= p^2 + \frac{A^2}{L^2} - 2\frac{p_yA}{L}
\end{aligned} \tag{4}$$

Therefore we can say that $\mathbf{p}$ is constrained to move on a circle with radius mk/L, centered at A/L on the y-axis.

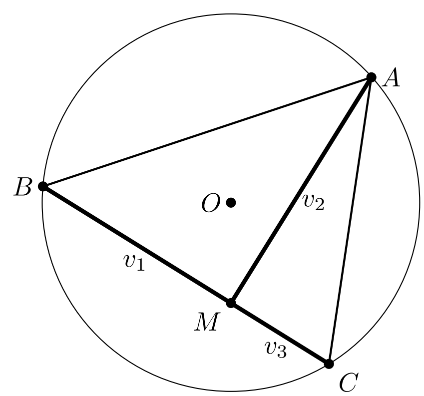
$$p_x^2 + \left(p_y - \frac{A}{L}\right)^2 = \left(\frac{mk}{L}\right)^2 \tag{5}$$

We can define e geometrically as the ratio between the segment A/L and the radius, i.e. mk/L.



3 Solution

Because e is a ratio in the circle, we can scale down the graph by m. Then the problem states: In a triangle ABC a point M is taken on BC such that $AM \perp BC$. Knowing $BM = v_1$, $MC = v_3$ and $AM = v_2$, find $e = \frac{OM}{R}$, where O is the center of the circumcircle and R is the radius.



3.1 Useful formulas

Radius of circumcircle: It is well known that the formula for the radius of the circumcircle of a triangle is:

$$R = \frac{abc}{4S} \tag{6}$$

Where S is the area of the triangle and a, b, c are the sides.
Steward Theorem: If M is a point on BC in triangle BOC then we have the following formula:

$$OM^2 \cdot BC = OB^2 \cdot MC + OC^2 \cdot MB - BC \cdot BM \cdot CM \tag{7}$$

3.2 Final calculations

Knowing that $S = (v_1 + v_3)v_2/2$ we can express the radius R, from Equation 6:

$$\begin{aligned}
R &= \frac{\sqrt{v_1^2 + v_2^2}\sqrt{v_2^2 + v_3^2}(v_1 + v_3)}{2(v_1 + v_3)v_2} \\
R &= \frac{\sqrt{v_1^2 + v_2^2}\sqrt{v_2^2 + v_3^2}}{2v_2}
\end{aligned} \tag{8}$$

Plugging the lengths in Equation 7 we can express e:

$$\begin{aligned}
OM^2(v_1 + v_3) &= R^2 v_3 + R^2 v_1 - (v_1 + v_3)v_1 v_3 \\
OM^2 &= R^2 - v_1 v_3 \\
e &= \sqrt{1 - v_1 v_3 / R^2}
\end{aligned} \tag{9}$$

From the above two equations:

$$\boxed{e = \sqrt{1 - \frac{4 v_1 v_3 v_2^2}{(v_1^2 + v_2^2)(v_2^2 + v_3^2)}}}$$

And for the numerical values $(v_1, v_2, v_3) = (1, 2, 3) km/s$:

$$\boxed{e = 0.5114.}$$