Complementary Filter

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1 Problem

Suppose we have some measurements about a car:

- Positions: x_0, x_1 .. x_n
- Speeds: v_0, v_1, .. v_n
- Time: Δt

Unfortunately, all of them have some **noise** that we can't get rid of. Let's say we add some Gaussian noise, such that:

- $x_i \sim \mathcal{N}(x_i^R, \sigma_x^2)$
- $v_i \sim \mathcal{N}(v_i^R, \sigma_v^2)$

How can we get the best estimates $\hat{x_i}$ for the position of the vehicle?

2 Solutions

2.1 Use only the positions

We can simply just set the position estimate equal to the measurement (pretty bad, we add all the uncertainty surrounding x_i):

- $\hat{x_i} = x_i$

But we are ignoring the speeds.

2.2 Use only the speeds

If we build upon the past solution, we can also add the speeds:

- $\hat{x_i} = \hat{x_{i-1}} + \Delta t v_{i-1}$
- $\hat{x_0} = x_0$, initial condition

But now we're ignoring the positions. Let's combine them.

2.3 Combine positions and speeds

We can average the estimate obtained from the speed $\bar{x_i}$ with the position measurement from the sensor x_i and get $\hat{x_i}$:

- $\bar{x_i} = \hat{x_{i-1}} + \Delta t v_{i-1}$
- $\hat{x_i} = \frac{\bar{x_i} + x_i}{2}$
- $\hat{x_0} = x_0$

But why use arithmetic mean? We can use any weight we like for a weighted arithmetic mean.

2.4 Use a weighted mean

- $\bar{x_i} = \hat{x_{i-1}} + \Delta t v_{i-1}$
- $\hat{x_i} = \alpha \bar{x_i} + (1 - \alpha) x_i$
- $\hat{x_0} = x_0, \alpha \in [0, 1]$

This is the best estimate we have so far. However, how can we choose alpha? Well, there is still one piece of information we haven't used: **the noise of the sensors**.

3 Optimal solution

We need some equations, given $x \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $x \sim \mathcal{N}(\mu_y, \sigma_y^2)$:

$$x + y \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2) \tag{1}$$

$$\alpha x \sim \mathcal{N}(\alpha \mu_x, (\alpha \sigma_x)^2) \tag{2}$$

$$\mathcal{N}(\mu_1, \sigma_1^2) \cdot \mathcal{N}(\mu_2, \sigma_2^2) = \mathcal{N}(\frac{\mu_1 \sigma_2 + \mu_2 \sigma_1}{\sigma_1 + \sigma_2}, \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2}) \tag{3}$$

Observe that in equation (3), the resulting variance is **smaller than both of the component variances**.

3.1 Single variable

- $\hat{x_0} = x_0$
- $\hat{\sigma_0^2} = \sigma_x^2$
- $\bar{x_i} = \hat{x_{i-1}} + v_{i-1} \Delta t$
- $\bar{\sigma_i^2} = \hat{\sigma_{i-1}^2} + (\Delta t \sigma_v)^2$, from (1) and (2)
- $\hat{x_i} = \frac{\bar{x_i} \sigma_x^2 + x_i \bar{\sigma_i^2}}{\sigma_i^2 + \sigma_x^2}$
- $\hat{\sigma_i} = \frac{\bar{\sigma_i^2} \cdot \sigma_x^2}{\sigma_i^2 + \sigma_x^2}$, from (3)

Now, we are no longes guessing α. However, we can generalize this for the multivariable case.

3.2 Multivariable

Suppose we have $\boldsymbol{X_1} \sim \mathcal{N}(\boldsymbol{\mu_1}, \boldsymbol{\Sigma_1})$ and $\boldsymbol{X_2} \sim \mathcal{N}(\boldsymbol{\mu_2}, \boldsymbol{\Sigma_2})$ and we want to combine them. From the single variable case, we get:

$$\mu = \frac{\sigma_2^2 \mu_1 + \sigma_1^2 \mu_2}{\sigma_1^2 + \sigma_2^2} \tag{4}$$

$$\sigma_2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \tag{5}$$

which can be rewritten as

$$\mu = \mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(\mu_2 - \mu_1) \tag{6}$$

$$\sigma_2 = \sigma_1^2 - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \cdot \sigma_1^2 \tag{7}$$

From eqs. (6) and (7) we can generalize for the multivariable case:

$$\boldsymbol{\mu} = \boldsymbol{\mu_1} + \boldsymbol{\Sigma_1}(\boldsymbol{\Sigma_1} + \boldsymbol{\Sigma_2})^{-1} \cdot (\boldsymbol{\mu_2} - \boldsymbol{\mu_1}) \tag{8}$$

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma_1} - \boldsymbol{\Sigma_1}(\boldsymbol{\Sigma_1} + \boldsymbol{\Sigma_2})^{-1}\boldsymbol{\Sigma_1} \tag{9}$$

Which can be furthered simplified to:

$$\boldsymbol{K} = \boldsymbol{\Sigma_1}(\boldsymbol{\Sigma_1} + \boldsymbol{\Sigma_2})^{-1} \tag{10}$$

$$\boldsymbol{\mu} = \boldsymbol{\mu_1} + \boldsymbol{K} \cdot (\boldsymbol{\mu_2} - \boldsymbol{\mu_1}) \tag{11}$$

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma_1} - \boldsymbol{K}\boldsymbol{\Sigma_1} \tag{12}$$