Exercise 1.20. *sdfsadfsadfsadf We want to show that*

$$\overset{\sqcap}{XY} = XY - N(XY)$$

using

$$\overset{\sqcap}{XY} \equiv \langle -|XY|-\rangle$$

where X and Y are arbitrary creation and annihilation operators.

N is the operator that brings a string of creation and annihilation operators $A_1 \cdots A_N$ to a desired order such that

$$\begin{aligned} N(A_1 \cdots A_N) &\equiv (-1)^{|\sigma|}[\textit{creation operators}] \cdot [\textit{annihilation operators}] \\ &= (-1)^{|\sigma|} A_{\sigma(1)} \cdots A_{\sigma(N)} \end{aligned} \tag{1}$$

where σ is the permutation.

Exercise 1.20. // Solution

We consider the four cases possible, where $X \in \{c_\mu, c_\mu^\dagger\}$ and $Y \in \{c_\nu, c_\nu^\dagger\}$

$$\begin{aligned} \overset{\sqcap}{XY} &\equiv \langle -|XY|-\rangle \\ &= \begin{cases} \langle -|c_\mu^\dagger c_\nu^\dagger|-\rangle = 0 \\ \langle -|c_\mu c_\nu|-\rangle = 0 \\ \langle -|c_\mu^\dagger c_\nu|-\rangle = 0 \\ \langle -|c_\mu c_\nu^\dagger|-\rangle = \delta_{\mu,\nu} \end{cases} \\ &= \begin{cases} \{c_\mu^\dagger, c_\nu^\dagger\}ra = c_\mu^\dagger c_\nu^\dagger + c_\nu^\dagger c_\mu^\dagger \\ \{c_\mu, c_\nu\} = c_\mu c_\nu + c_\nu c_\mu \\ \{c_\mu^\dagger, c_\nu\} = c_\mu^\dagger c_\nu + c_\nu c_\mu^\dagger \\ \{c_\mu, c_\nu^\dagger\} = c_\mu c_\nu^\dagger + c_\nu^\dagger c_\mu \end{cases} \\ &= \begin{cases} c_\mu^\dagger c_\nu^\dagger - N(c_\mu^\dagger c_\nu^\dagger) \\ c_\mu c_\nu - N(c_\mu c_\nu) \\ c_\mu^\dagger c_\nu - N(c_\mu^\dagger c_\nu) \\ c_\mu c_\nu^\dagger - N(c_\mu c_\nu^\dagger) \end{cases} \\ &= XY - N(XY) \end{aligned}$$

since we do one permutation and thus contract a minus sign.

Exercise 1.21. *We want to prove that for any permutation $\sigma \in S_N$,*

$$N(A_1 \cdots A_N) = (-1)^{|\sigma|} N(A_{\sigma(1)} \cdots A_{\sigma(N)}) \tag{2}$$

So for any such σ, we can find another normal-ordered product with the same sign.

Exercise 1.21. // Solution

Assume that we have a string of creation and annihilation operators arbitrarily ordered on the form $A_1 \cdots A_N$, and that we perform a permutation σ_1 so that we get the normal-ordered equation

$$\Sigma_1 = N(A_1 \cdots A_N) = (-1)^{|\sigma_1|} A_{\sigma_1(1)} \cdots A_{\sigma_1(N)}$$

Assume another normal-ordering permutation σ_2, such that

$$\Sigma_2 = N(A_1 \cdots A_N) =$$

Exercise 1.24. *We let $\boldsymbol{\mu} = (\mu_1 \cdots \mu_N)$ for $N \geq 2$, and want to compute the matrix elements $\langle \boldsymbol{\mu} | \hat{H}_0 | \boldsymbol{\mu} \rangle$ and $\langle \boldsymbol{\mu} | \hat{W} | \boldsymbol{\mu} \rangle$ using* Wick's theorem *applied to vaccum expectation values.*

Exercise 1.24. // Solution

The operators in terms of creation and annihilation operators are

$$\hat{H}_0 = \sum_{\mu,\nu} \langle \mu | \hat{h} | \nu \rangle c_\mu^\dagger c_\nu$$

$$\hat{W} = \frac{1}{4} \sum_{\substack{\nu_1,\nu_2 \\ \mu_1,\mu_2}}^{N} \langle \mu_1 \mu_2 | \hat{w} | \nu_1 \nu_2 \rangle c_{\mu_1}^\dagger c_{\mu_2}^\dagger c_{\nu_1} c_{\nu_2}$$

and $\boldsymbol{\mu} = c_{\mu_1}^\dagger \cdots c_{\mu_N}^\dagger |-\rangle$ means that

$$\begin{aligned}
\langle \boldsymbol{\mu} | \hat{H}_0 | \boldsymbol{\mu} \rangle &= \sum_{\mu,\nu} \langle \mu | \hat{h} | \nu \rangle \langle - | c_{\nu_N} \cdots c_{\nu_1} c_\mu^\dagger c_\nu c_{\mu_1}^\dagger \cdots c_{\mu_N}^\dagger | - \rangle \\
&= \sum_{\mu,\nu} \langle \mu | \hat{h} | \nu \rangle \langle - | \nu_N \cdots \nu_1 \mu^\dagger \nu \mu_1^\dagger \cdots \mu_N^\dagger | - \rangle \\
\langle \boldsymbol{\mu} | \hat{W} | \boldsymbol{\mu} \rangle &= \sum_{\substack{\alpha_1,\alpha_2 \\ \beta_1,\beta_2}} \langle \alpha_1 \alpha_2 | \hat{w} | \beta_1 \beta_2 \rangle \langle - | c_{\nu_N} \cdots c_{\nu_1} c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger c_{\beta_2} c_{\beta_1} c_{\mu_1}^\dagger \cdots c_{\mu_N}^\dagger | - \rangle
\end{aligned}$$