Exercise 1.3

Given wave functions

$$\Psi(\vec{R}) \in X = \mathbb{R}^3$$

1.3.1

We first consider

$$\Psi(\vec{r}_1, \vec{r}_2) = e^{-\alpha|\vec{r}_1 - \vec{r}_2|} = e^{-\alpha r_{21}}$$

Permuting the particles in this wave function, we see that we get

$$P_\sigma \Psi = e^{-\alpha|\vec{r}_2 - \vec{r}_1|} = e^{-\alpha r_{12}} = \Psi$$

Since $r_{12} = r_{21}$ is just the distance between the particles, we see that the eigenvalue of P_σ must be 1, and thus Ψ is *totally symmetric*.

We also have that Ψ is square integrable for $\alpha \neq 0$, since

$$\int_{-\infty}^{\infty} \mathrm{d}x\, |f(x)|^2 = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-c^*|x| - c|x|} = \int_{-\infty}^{\infty} \mathrm{d}x\, e^{-2a|x|} = \frac{1}{a} < \infty$$

for $c = a + ib$.

1.3.2

If we now permute

$$\Psi(\vec{r}_1, \vec{r}_2) = \sin(\vec{e}_z \cdot (\vec{r}_1 - \vec{r}_2)) = \sin(z_1 - z_2)$$

instead, we get

$$P_\sigma \Psi = \sin(z_2 - z_1) = -\sin(z_1 - z_2) = (-1)^1 \Psi$$

which means that the wave function is *totally antisymmetric*.

This function is not square integrable, since

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \sin^2(x) = \infty$$

1.3.3

Finally, we look at

$$\Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \sin[\vec{r}_1 \cdot (\vec{r}_2 \times \vec{r}_3)] \prod_{i=1}^{3} e^{-|\vec{r}_i|^2}$$

where we see that the product part can be permuted without consequences to the eigenvalue.

For the *scalar triple product*, we have

$$\begin{aligned}\vec{i}\cdot(\vec{j}\times\vec{k}) &= \vec{k}\cdot(\vec{i}\times\vec{j}) = \vec{j}\cdot(\vec{k}\times\vec{i}) \\ &= -\vec{j}\cdot(\vec{i}\times\vec{k}) = -\vec{k}\cdot(\vec{j}\times\vec{i}) = -\vec{i}\cdot(\vec{k}\times\vec{j})\end{aligned}$$

The upper part here corresponds to *even* number of permutations, and the lower part corresponds to *odd* number of permutations.

Thus

$$P_\sigma\Psi = (-1)^{|\sigma|}\Psi$$

and the function is *totally antisymmetric*.

The

Exercise 1.13

a)

Given the subspace $L^2(X^N)_{AS}$, we want to find a basis for $N = 2, 3, 4$ particles, when we have the orthonormal orbitals $\phi_\mu,\ \mu = 1, \ldots, 6$.

Such a basis can be written as a Slater determinant

$$\begin{aligned}\Phi_{\mu_1,\ldots,\mu_N} &= \langle\vec{x}|\vec{\mu}\rangle = \langle x_1\cdots x_N|\mu_1\cdots\mu_N\rangle \\ &= \frac{1}{\sqrt{N!}}\begin{vmatrix}\phi^1_{\mu_1} & \cdots & \phi^N_{\mu_1} \\ \vdots & \ddots & \vdots \\ \phi^1_{\mu_N} & \cdots & \phi^N_{\mu_N}\end{vmatrix}\end{aligned}$$

N=2 The basis for two particles is then

$$\begin{aligned}\Phi_{\mu_1,\mu_2} &= \langle x_1x_2|\mu_1\mu_2\rangle \\ &= \frac{1}{\sqrt{2}}\left(\phi^1_{\mu_1}\phi^2_{\mu_2} - \phi^1_{\mu_2}\phi^2_{\mu_1}\right)\end{aligned}$$

N=3

$$\begin{aligned}\Phi_{\mu_1,\mu_2,\mu_3} &= \langle x_1x_2x_3|\mu_1\mu_2\mu_3\rangle \\ &= \phi_{\mu_1}\langle x_2x_3|\mu_2\mu_3\rangle - \phi_{\mu_2}\langle x_1x_3|\mu_1\mu_3\rangle + \phi_{\mu_3}\langle x_1x_2|\mu_1\mu_2\rangle\end{aligned}$$

N=4

$$\begin{aligned}\Phi_{\mu_1,\mu_2,\mu_3,\mu_4} &= \langle x_1x_2x_3x_4|\mu_1\mu_2\mu_3\mu_4\rangle \\ &= \phi_{\mu_1}\langle x_2x_3x_4|\mu_2\mu_3\mu_4\rangle - \phi_{\mu_2}\langle x_1x_3x_4|\mu_1\mu_3\mu_4\rangle \\ &+ \phi_{\mu_3}\langle x_1x_2x_4|\mu_1\mu_2\mu_4\rangle - \phi_{\mu_4}\langle x_1x_2x_3|\mu_1\mu_2\mu_3\rangle\end{aligned}$$

b)

We can only have six particles when we have six one-particle functions. If we filled up the determinant with more of the same one-particle functions, we could get equal rows, and the determinant would be zero.

c)

We have

$$|\vec{\mu}\rangle = |\mu_1 \cdots \mu_N\rangle$$

and $\mu = 1, ..., 6$, which means we have six bits that can be on or off

$$|n_1 \cdots n_6\rangle$$

N=2 For only two particles, we get $\{\mu_1, \mu_2\} = \{1, 2\}$, and

$$|\mu_1 \mu_2\rangle = |1, 2\rangle = |110\rangle$$

since

$$110_2 = 2^2 + 2^1$$

N=3

$$|\mu_1 \mu_2 \mu_3\rangle = |1, 2, 3\rangle = |1110\rangle$$

N=4

$$|\mu_1 \mu_2 \mu_3 \mu_4\rangle = |1, 2, 3, 4\rangle = |11110\rangle$$

d)

Since the Fock space is the direct sum of all L^2_N spaces

$$\dim F = \sum_{N=0}^{\infty} \dim L^2(X^N) = \sum_{i=1}^{N} 6^i$$

e)

Since we have L orbitals, we get

$$\dim F = \sum_{i=1}^{N} L^i$$