Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4135
Matematikk 4D
Fall 2019

Exercise set 3

Submission deadline: 24th September

$\boxed{1}$ Try to verify the following computations

a) The Laplace transform of

$$f(t) = \begin{cases} t & \text{if } 0 \le t \le a; \\ 0 & \text{if } t > a \end{cases}$$

is

$$F(s) = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}.$$

b) The Laplace transform of $f(t) = u(t-\pi)\sin t$ is $F(s) = -\frac{e^{-\pi s}}{s^2+1}$.

c) The solution $i(t)$ of

$$i'(t) + 2\,i(t) + \int_0^t i(\tau)\,d\tau = \delta(t-1),\ i(0) = 0$$

is

$$i(t) = u(t-1)(e^{-(t-1)} - e^{-(t-1)}(t-1)).$$

$\boxed{2}$ Use Laplace transform to solve this convolution equation: $y - y \star t = t$.

Remark: One may also use Laplace transform method to solve some *boundary value problems*. For example: consider the following ODE

$$f'' = f,$$

with boundary restrictions

$$f(0) = 0, \quad f(1) = 1.$$

If we apply the Laplace transform to the equation, we would get

$$s^2F - sf(0) - f'(0) = F.$$

Since $f(0) = 0$, it reduces to

$$s^2F - f'(0) = F.$$

Thus
$$F = \frac{f'(0)}{s^2 - 1}.$$
Taking the inverse transform we get
$$f(t) = f'(0)\sinh t.$$
Thus $f(1) = 1$ is equivalent to
$$f'(0)\sinh 1 = 1,$$
which implies $f'(0) = 1/\sinh 1$ and
$$f(t) = \frac{\sinh t}{\sinh 1}.$$
You might also try to find other examples. Another application of Laplace transform is to solve *system of ODEs*. Please try to do the following exercise:

$\boxed{3}$ Solve the following system of equations:
$$\begin{cases} x' = 2x - y \\ y' = 3x - 2y \end{cases}$$
with initial conditions $x(0) = 0,\ y(0) = 1$.

Hint: apply Laplace transform to each equation and then solve the linear equation for X and Y.

Now let us move to the Fourier series part, recall that the complex Fourier series of a function f on $(-\pi, \pi)$ is defined by (you might compare it with the finite Fourier transform in the week 2 Exercise)
$$\sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$
Try to use the above formula to verify the followings:

$\boxed{4}$ Prove the following formulas for complex Fourier series expansion:

a) $x = \sum_{n\neq 0} \frac{i(-1)^n}{n} e^{inx}$ when $-\pi < x < \pi$.

b) $x(2\pi - x) = -\frac{\pi^2}{3} + \sum_{n\neq 0}\left(\frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{(n+1)}}{n^2}\right) e^{inx}$ when $-\pi < x < \pi$.