# WORK SHEET 8

1 a)

| | $x_0$ | $x_1$ | $x_2$ |
|---|---|---|---|
| $x_i$ | $-2$ | $1$ | $6$ |
| $y_i$ | $1$ | $2$ | $3$ |

$$f(x_0) = 1 = C_0$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{2-1}{1-(-2)} = \frac{1}{3} = C_1$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{3-2}{6-1} = \frac{1}{5}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0} = \frac{\frac{1}{5} - \frac{1}{3}}{6-(-2)} = \frac{\frac{3}{15} - \frac{5}{15}}{8}$$

$$= -\frac{2}{15} \cdot \frac{1}{8} = -\frac{1}{15 \cdot 4} = -\frac{1}{60} = C_2$$

the table of divided differences becomes

$$\begin{array}{c|ccc}
-2 & 1 & & \\
 & & \frac{1}{3} & \\
1 & 2 & & -\frac{1}{60} \\
 & & \frac{1}{5} & \\
6 & 3 & &
\end{array}$$

the polynomial becomes

$$p_2(x) = C_0 + C_1(x - x_0) + C_2(x - x_0)(x - x_1)$$

$$= 1 + \frac{1}{3}(x-(-2)) - \frac{1}{60}(x-(-2))(x-1)$$

$$= 1 + \frac{2}{3} + \frac{1}{3}x - \frac{1}{60}(x+2)(x-1)$$

$$= 1 + \frac{2}{3} + \frac{1}{3}x - \frac{1}{60}(x^2 - x + 2x - 2) = 1 + \frac{1}{2} + \frac{1}{3}x - \frac{1}{60}x^2 - \frac{1}{60}x + \frac{1}{30}$$

$$= -\frac{1}{60}x^2 + \frac{19}{60}x + \frac{46}{30}$$

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1] b)

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|
| $x_i$ | $-2$ | $1$ | $6$ | $-3/4$ |
| $y_i$ | $1$ | $2$ | $3$ | $3/2$ |

we reuse a lot from 1] a), these are the necessary new calculations:

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{3/2 - 3}{-3/4 - 6} = \frac{6/4 - 12/4}{-3/4 - 24/4} = \frac{-6/4}{-27/4} = \frac{6}{27} = \underline{\frac{2}{9}}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 \quad x_1} = \frac{\frac{2}{9} - \frac{1}{5}}{-3/4 - 1} = \frac{\frac{10}{45} - \frac{9}{45}}{-7/4} = \frac{\frac{1}{45}}{-7/4}$$

$$= \underline{-\frac{4}{315}}$$

$$f[x_1, x_2, x_3, x_4] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-\frac{4}{315} - \left(-\frac{1}{60}\right)}{-3/4 - (-2)}$$

$$= \frac{-\frac{240}{18900} + \frac{315}{18900}}{5/4} = \frac{\frac{75}{18900}}{\frac{5}{4}} = \frac{15}{4725} = \frac{1}{315} = C_3$$

Table of divided differences:

$$\begin{array}{c|ccccccc}
-2 & 1 & & & & & & \\
 & & \frac{1}{3} & & & & & \\
1 & 2 & & -\frac{1}{60} & & & & \\
 & & \frac{1}{5} & & \frac{1}{315} & & & \\
6 & 3 & & -\frac{4}{315} & & & & \\
 & & \frac{2}{9} & & & & & \\
-\frac{3}{4} & \frac{3}{2} & & & & & &
\end{array}$$

pol: $p_3(x) =$

$$\underbrace{-\frac{1}{60}x^2 + \frac{19}{60}x + \frac{46}{30}}_{\alpha} + \frac{1}{315}(x+2)(x-1)(x-6)$$

$$= \alpha + \frac{1}{315}(x^2 + x - 2)(x - 6)$$

$$= \alpha + \frac{1}{315}(x^3 + x^2 - 2x - 6x^2 + 6x + 12)$$

$$= \alpha + \frac{1}{315}(x^3 - 5x^2 + 4x + 12)$$

$$\frac{1}{315}x^3 - \frac{1}{63}x^2 - \frac{1}{60}x^2 + \frac{4}{315}x + \frac{19}{60}x + \frac{4}{105} + \frac{46}{30}$$

$$= \underline{\frac{1}{315}x^3 - \frac{41}{1260}x^2 + \frac{83}{252}x + \frac{11}{7}}$$

# WORK SHEET 8

1] c) $f(x) = x^2 - 3$, $x \in [1, 3]$ $x_0 = 1$, $x_1 = 2$, $x_2 = 3$

$f(x) = 0$ normal : $x^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3} \Rightarrow x = \sqrt{3}, x \in [1, 3]$

$\approx 1.732050808$

"actual"

Inverse interpolation

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $y = f(x)$ | -2 | 1 | 6 |

in respect to $y$ :

| | $y_0$ | $y_1$ | $y_2$ |
|---|---|---|---|
| $y$ | -2 | 1 | 6 |
| $x = f(y)$ | 1 | 2 | 3 |

finding $p_n(y_i) = x_i$ :

$f(y_0) = x_0 = 1 = d_0$

$f[y_0, y_1] = \frac{f(y_1) - f(y_0)}{y_1 - y_0} = \frac{2-1}{1-(-2)} = \frac{1}{3} = d_1$ ...

we see that this is the same that we calculated in 1] a)

we can simply insert 0 into the expression found in 1] a)

from 1] a): $p_2(x) = -\frac{1}{60}x^2 + \frac{19}{60}x + \frac{46}{30}$

$p_2(0) = \frac{46}{30} = 1.5333333$ ...

we test 1] b), where we also have $x_3 = 3/2$ :

$p_3(x) = \frac{1}{315}x^3 - \frac{41}{1260}x^2 + \frac{83}{252}x + \frac{11}{7}$

$p_3(0) = 1.571428571$

we see that $p_3(0)$ gives a closer approximation than $p_2(0)$

```python
    newton_poly = F[0,0]*ones(len(x))  # The Newton polynomial
    for j in range(n-1):
        xpoly = xpoly*(x-xdata[j])
        newton_poly = newton_poly + F[0,j+1]*xpoly
    return newton_poly
```

executed in 10ms, finished 01:34:11 2019-10-28

In [4]:

```python
def f(x):
    return x**2 - 3

n_values = [2, 4, 8, 16]
print('The "actual" value is: 1.73205080757')

for n in n_values:
    xdata = linspace(1, 3, n+1)                      # The x-values in which the polynomial is evaluated
    ydata = f(xdata)                                 # The corresponing y-vaules
    F = divdiff(ydata, xdata)                        # Table of differences, flipped to simulate inversion
    p = newton_interpolation(F, ydata, [0])          # newton interpolation evaluated for f^-1(0)
    print("\nThe newton interpolation for {} uniformly distributed nodes:".format(n))
    print(p)
```

executed in 23ms, finished 01:34:12 2019-10-28

```
The "actual" value is: 1.73205080757

The newton interpolation for 2 uniformly distributed nodes:
[ 1.7]

The newton interpolation for 4 uniformly distributed nodes:
[ 1.73463203]

The newton interpolation for 8 uniformly distributed nodes:
[ 1.7320459]

The newton interpolation for 16 uniformly distributed nodes:
[ 1.73205081]
```

# WORK SHEET 8

3] $b=3$, $a=1$: $\int_1^3 e^{-x}\,dx = [-e^{-x}]_1^3 = [e^{-1} - e^{-3}] \approx 0.3180923728$ $\quad$ $x_i = a + i \cdot \Delta x$

a) i) Simpsons method with $n=1$ intervals:

$$\Delta x = \frac{3-1}{n} = \frac{2}{n} = \frac{2}{1} = 2, \quad x_0 = 1, \quad x_1 = 1 + 1 \cdot 2 = 3$$

$$\int_1^3 e^{-x}\,dx \approx \frac{2}{3}\left[e^{-1} + e^{-3}\right] = \underline{0.2784443397}$$

ii) $n=2$ intervals:

$$\Delta x = 1, \quad x_0 = 1, \quad x_1 = \quad, \quad x_3 = 3$$

$$\int_1^3 e^{-x}\,dx \approx \frac{1}{3}\left[e^{-1} + 4 \cdot e^{-2} + e^{-3}\right] \approx \underline{0.3196692142}$$

iii) Error estimate: $\quad$ we know $|E| \leq \dfrac{K4(b-a)^5}{180 n^4}$, $n=2$

$$f(x) = e^{-x}, \quad f'(x) = -e^{-x}, \quad f''(x) = e^{-x}, \quad f'''(x) = -e^{-x} \quad f^4(x) = e^{-x}$$

$x=1: e^{-1} = \frac{1}{e} \approx 0.3679$

$x=2: e^{-2} = \frac{1}{e^2} \approx 0.1353$

$x=3: e^{-3} = \frac{1}{e^3} \approx 0.04979$

the highest value is $e^{-1} = K4$,

we get

$$|E| \leq \frac{e^{-1}(3-1)^5}{180 \cdot 2^4} = \frac{e^{-1} \cdot 2^{\cancel{5}}}{180 \cdot \cancel{2^4}} = \frac{e^{-1}}{90} = \frac{1}{90e}$$

$$\approx \underline{\underline{0.00408755}}$$

"real": 0.31809 ....

estimate: 0.31966 ....

Error $\approx$ 0.00157 ....

which is smaller than 0.00409 ....

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"actual" value

3] $\int_1^3 e^{-x} = \left[-e^{-x}\right]_1^3 = -e^{-3} - (-e^{-1}) = e^{-1} - e^{-3} \approx \underline{0.31809237}$

a) i) $S_1(1,3)$ gives $h = \frac{3-1}{2\cdot 1} = \frac{2}{2} = 1$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$

$$\int_1^3 e^{-x} \approx \frac{1}{3}\left[e^{-1} + 4e^{-2} + e^{-3}\right] \approx \underline{0.31966921}$$

ii) $S_2(1,3)$ gives $h = \frac{3-1}{2\cdot 2} = \frac{2}{4} = \frac{1}{2}$, $x_0 = 1$, $x_1 = \frac{3}{2}$, $x_2 = 2$, $x_3 = \frac{5}{2}$, $x_4 = 3$

$$\int_1^3 e^{-x} \approx \frac{\frac{1}{2}}{3}\left[e^{-1} + 4e^{-\frac{3}{2}} + 2e^{-2} + 4e^{-\frac{5}{2}} + e^{-3}\right] \approx \underline{0.31819962}$$

iii) $\int_1^3 e^{-x}dx - S_2(1,3) = -\frac{(3-1)\cdot\left(\frac{1}{2}\right)^4}{180} f^4(\xi)$

looking at $f^4(\xi)$, $\xi \in [1,3]$ $f^4(x) = f(x)$, $f(1) = e^{-1}$, $f(1.01) = e^{-1.01} < e^{-1}$

choosing $\xi = 1$ gives the highest possible $f(\xi)$ for the values in the interval

we get $|E| \leq \frac{(3-1)\left(\frac{1}{2}\right)^4}{180} e^{-1} \approx 0.00025547$

comparing "actual value and estimate:

theoretical max error

actual: 0.31809237
estimate: 0.31819962

Error $= 0.00010725 < 0.00025547$

# WORK SHEET 8

3] b) the error is given as

$$|E| \leq \frac{(b-a)h^4}{180} f^4(\xi)$$ where we know $a=1$, $b=3$, $f^4(\xi)=e^{-1}$

$$h = \frac{b-a}{2m}$$

we want to know:

$$\frac{(b-a)h^4}{180} f^4(\xi) \leq 10^{-8} \quad \Leftrightarrow \frac{(3-1)\left(\frac{3-1}{2m}\right)^4}{180} e^{-1} \leq 10^{-8}$$

$$\Leftrightarrow \left(\frac{1}{m}\right)^4 \leq 90 \times 10^{-8} \quad \Leftrightarrow \frac{1}{m} \leq \sqrt[4]{9 \times 10^{-7}}$$

$$\Leftrightarrow \frac{1}{\sqrt[4]{9 \times 10^{-7}}} \leq m \quad , \quad \frac{1}{\sqrt[4]{9 \times 10^{-7}}} \simeq 32.47$$

this means m, the number of intervals, must be larger than 32.47 to ensure that the error is smaller than $10^{-8}$

m must be at least 33

```
    # Calculate S1=S_1(a,b), S2=S_2(a,b)
    H = b-a
    S1 = H*(f(a)+4*f(c)+f(b))/6
    S2 = 0.5*H*(f(a)+4*f(d)+2*f(c)+4*f(e)+f(b))/6

    error_estimate = (S2-S1)/15    # Error estimate for S2
    return S2, error_estimate
```

executed in 12ms, finished 01:34:15 2019-10-28

```
[9]: def g(x):                    # Integrand
         return e**-x

     a, b = 1, 3                  # Integration interval

     I_exact = e**-1 - e**-3          # Exact solution for comparision

     # Simpson's method over two intervals, with error estimate
     S, error_estimate = simpson_basic(g, a, b)

     # Print the result and the exact solution
     print('Numerical solution = {:.8f}, exact solution = {:.8f}'.format(S, I_exact))

     # Compare the error and the error estimate
     print('Error in S2 = {:.3e},  error estimate for S2 = {:.3e}'.format(I_exact-S, error_estimate))

     isless = error_estimate <= 10**-8

     print('the statement that {0} is less than 10^-8 is: {1}'.format(error_estimate, isless))
```

executed in 15ms, finished 01:38:47 2019-10-28

```
Numerical solution = 0.31819962, exact solution = 0.31809237
Error in S2 = -1.072e-04,  error estimate for S2 = -9.797e-05
the statement that -9.797304299630956e-05 is less than 10^-8 is: True
```

# WORK SHEET 8

4] a) $L_m(x) = \frac{d^m}{dt^m}(x^2-1)^m$

$L_3(x) = \frac{d^3}{dt^3}(x^2-1)^3 = 24x(5x^2-3)$ which gives us

$x_0 = 0, \; x_1 = -\sqrt{\frac{3}{5}}, \; x_2 = \sqrt{\frac{3}{5}}$

$l_0 = \frac{\left(x+\sqrt{\frac{3}{5}}\right)\left(x-\sqrt{\frac{3}{5}}\right)}{\left(\sqrt{\frac{3}{5}}\right)\left(-\sqrt{\frac{3}{5}}\right)} = -\frac{5}{3}x^2+1 \Rightarrow w_0 = \int_{-1}^{1}\left(-\frac{5}{3}x^2+1\right)dx = \frac{8}{9}$

which gives $l_1 = \frac{5}{9}$ & $l_2 = \frac{5}{9}$. we end up with

$Q = \frac{8}{9}\cdot f(0) + \frac{5}{9}\left(f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right)$

b) From notebook:

A numerical quadrature has degree of precision $d$ if:

$Q[p](a,b) = I[p](a,b)$ for all $p \in \mathbb{P}_d$

due to linearity we get

$I[x^j](a,b) = Q[x^j](a,b), \; j = 0,1,\dots,d$

$I[x^{d+1}](a,b) \neq Q[x^{d+1}](a,b)$ where $d$ is the degree of precision

see table in next page

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b) cont.

| $j$ | $I[x^j](-1,1)$ | $Q[x^j](-1,1)$ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 2/3 | 2/3 |
| 3 | 0 | 0 |
| 4 | 2/5 | 2/5 |
| 5 | 0 | 0 |
| 6 | 2/7 | 6/25 ← not equal |

this proves that the degree of precision is 5

c) Transfering the quadrature to the arbitary interval $[a,b]$

$h = (b-a)/2$ , $c = (b+a)/2$

$$x = \frac{b-a}{2} t + \frac{b+a}{2} = ht - c$$

$$\Rightarrow dx = h \cdot dt \qquad \tilde{f}(t) = f(x(t))$$

$$t_0 = c$$

$$t_1 = \frac{(5+\sqrt{15})a - (\sqrt{15}-5)b}{10}$$

$$t_2 = \frac{(5-\sqrt{15})a - (5+\sqrt{15})b}{10}$$

$$x[t](-1,1) = \frac{8}{9} f(0) + \frac{5}{9}\left(f\left(-\sqrt{\tfrac{3}{5}}\right) + f\left(\sqrt{\tfrac{3}{5}}\right)\right)$$

$$x[t](a,b) = \left(\frac{8}{9} f(0) + \frac{5}{9}\left[f(\alpha a + \beta b) + f(\beta a + \alpha b)\right]\right) \cdot c$$

$$\alpha = \frac{5-\sqrt{15}}{10} \qquad \beta = \frac{5+\sqrt{15}}{10}$$

$$Q[e^{-x}](1,3) = 1 \cdot \frac{8}{9} e^{-2} + \frac{5}{9}\left[e^{\frac{(10+\sqrt{15})}{5}} + e^{\frac{(10-\sqrt{15})}{5}}\right] \approx 0.31808351$$

$$E = I - Q = 8.85886 \cdot 10^{-6}$$