Norwegian University of Science
and Technology
Department of Mathematical
Sciences

Solution to Exercise set 4

Submission deadline: 1th October

1. The aim of this exercise is to show you the fundamental fact behind the *real* Fourier series: $\{1, \cos nx, \sin nx\}_{n\geq 1}$ is an orthogonal system with respect to the following inner product

$$(f, g) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

   a) Show that

   $$(1, \sin nx) = 0, \quad (\sin nx, \cos mx) = 0, \quad \forall\, n, m = 1, 2, \cdots.$$

   b) Show that

   $$(\sin mx, \sin nx) = (\cos nx, \cos mx) = \begin{cases} 0 & n \neq m = 0, 1, 2, \cdots \\ \frac{1}{2} & n = m = 1, 2, \cdots. \end{cases}$$

   c) Assume that

   $$f = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

   use a) and b) to prove

   $$a_0 = (f, 1), \quad a_n = (f, 2\cos nx), \quad b_n = (f, 2\sin nx)$$

   and the following identity

   $$||f||^2 := (f, f) = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2).$$

2. Draw the graph of the odd extension f_o and the even extension f_e of

   $$f(x) = \begin{cases} x, & 0 < x \leq \frac{\pi}{2}; \\ \frac{\pi}{2}, & \frac{\pi}{2} < x < \pi. \end{cases}$$

   Verify the following Fourier cosine series expansion of f_e

   $$f_e(x) = \frac{3\pi}{8} + \frac{2}{\pi}\left(-\cos x - \frac{2\cos 2x}{2^2} - \frac{\cos 3x}{3^2} - \frac{\cos 5x}{5^2} - \cdots\right)$$

and the Fourier sine series expansion of f_o

$$
\begin{aligned}
f_o(x) &= \left(\frac{2}{\pi} + 1\right)\sin x + \left(0 - \frac{1}{2}\right)\sin 2x + \left(\frac{-2}{3^2\pi} + \frac{1}{3}\right)\sin 3x \\
&\quad + \left(0 - \frac{1}{4}\right)\sin 4x + \left(\frac{2}{5^2\pi} + \frac{1}{5}\right)\sin 5x + \cdots.
\end{aligned}
$$

Further notes: We know the Fourier series comes from the finite Fourier transform. Now we will show that *how to use the Fourier series to get the usual Fourier transform* (thus the usual Fourier transform also comes from the finite Fourier transform).

The *main idea* is to consider the following T-periodic function

$$
f_T(x) := \sum_{k \in \mathbb{Z}} f(x + kT).
$$

Similar as $T = 2\pi$ case, we can write the above T-periodic function f_T as

$$
f_T(x) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n \frac{x}{T}}, \quad c_n := \frac{1}{T}\int_{-T/2}^{T/2} f_T(y) e^{-2\pi i n \frac{y}{T}}\, dy.
$$

Notice that

$$
\frac{1}{T}\int_{-T/2}^{T/2} f_T(y) e^{-2\pi i n \frac{y}{T}}\, dy = \sum_{k \in \mathbb{Z}} \frac{1}{T}\int_{-T/2}^{T/2} f(y + kT) e^{-2\pi i n \frac{y}{T}}\, dy = \int_{\mathbb{R}} f(y) e^{-2\pi i n \frac{y}{T}}\, dy.
$$

We call

$$
\hat{f}(\omega) := \int_{\mathbb{R}} f(y) e^{-2\pi i \omega y}\, dy
$$

the *Fourier transform* of f (the definition here is slightly different from the notes!). Then

$$
c_n = \hat{f}(n/T),
$$

thus the Fourier series of f_T gives the following *Poisson summation formula*

$$
f_T(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n/T) e^{2\pi i n \frac{x}{T}}. \tag{1}
$$

Assume that f goes to zero very fact when $|x|$ goes to infinity, then we have

$$
f(x) = \lim_{T \to \infty} f_T(x) = \lim_{T \to \infty} \sum_{n \in \mathbb{Z}} \hat{f}(n/T) e^{2\pi i n \frac{x}{T}} = \int_{\mathbb{R}} \hat{f}(\omega) e^{2\pi i x \omega}\, d\omega,
$$

which is precisely the famous *Fourier inversion formula*! Similarly the $T \to \infty$ limit of the Parseval identity associated to (1) gives the following *Plancherel identity*

$$
\int_{\mathbb{R}} |f(x)|^2\, dx = \int_{\mathbb{R}} |\hat{f}(\omega)|^2\, d\omega.
$$