WORK SHEET 4 Alexander Nes Fredheim

1) $(f, g) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot g(x)\,dx$, $n, m = 1, 2, 3, \dots$

a) $(1, \sin(nx)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\cdot \sin(nx)\,dx$,

$h(x) = \sin(nx)$

$h(-x) = \sin(-nx) = -\sin(nx) = -h(x)$, $\sin(nx)$ er odde

som vil si at $\int_{-\pi}^{\pi} \sin(nx)\,dx = 0$

vi får

$\frac{1}{2\pi}\cdot 0 = 0$, som vi skulle vise

$(\sin(nx), \cos(mx)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\cdot\cos(mx)\,dx$

vi vet at $\cos(mx)$ er like for $m = 1, 2, 3, \dots$

vi vet også at $f_{odde}(x)\cdot g_{like}(x) = h_{odde}(x)$

$\sin(nx)\cdot\cos(mx) =$ odde som gir $\int_{-\pi}^{\pi} \sin(nx)\cdot\cos(mx) = 0$

vi får $\frac{1}{2\pi}\cdot 0 = 0$, som vi skulle vise

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1 b) i) vil vise at $(\sin(mx), \sin(nx)) = \begin{cases} 0 & m \neq n = 0,1,2,\dots \\ \frac{1}{2} & m = n = 1,2,\dots \end{cases}$

For $m \neq n$:

$$(\sin(mx), \sin(nx)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(mx)\cdot\sin(nx)\,dx$$

(*)

vi vet at $\sin(mx)\cdot\sin(nx) = \frac{1}{2}(\cos((m-n)x) - \cos((m+n)x))$

vi får

$$(*) = \frac{1}{2\pi}\cdot\frac{1}{2}\left\{\int_{-\pi}^{\pi}\cos((m-n)x)\,dx - \int_{-\pi}^{\pi}\cos((m+n)x)\,dx\right\}$$

$$\frac{1}{4\pi}\left\{\frac{1}{m-n}\Big[\sin((m-n)x)\Big]_{-\pi}^{\pi} - \frac{1}{m+n}\Big[\sin((m+n)x)\Big]_{-\pi}^{\pi}\right\}$$

vi ser at for alle $m \neq n$ vil vi få enten $\sin(-a\pi)$

$\sin(0\cdot\pi)$ eller $\sin(a\cdot\pi)$ der $a \in \mathbb{N}$, alle uttrykk vil

bli "0"

For $m = n$:

$$(\sin(mx), \sin(nx)) = (\sin(nx), \sin(nx)) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^2(nx)\,dx$$

(α)

vi vet at $\sin^2(nx) = \frac{1}{2}(1 - \cos(2nx))$

$$(\alpha) = \frac{1}{2\pi}\cdot\frac{1}{2}\left\{\int_{-\pi}^{\pi}1\,dx - \underbrace{\int_{-\pi}^{\pi}\cos(2nx)\,dx}_{=0 \text{ pga odde}}\right\} = \frac{1}{4\pi}\Big[x\Big]_{-\pi}^{\pi} = \frac{1}{4\pi}\{\pi - (-\pi)\}$$

$$= \frac{1}{4\pi}\cdot 2\pi = \underline{\frac{1}{2}}$$

vi får $\underline{\underline{(\sin(mx), \sin(nx)) = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}}}$

som vi skulle vise

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1] b) ii)

$$(\cos(nx), \cos(mx)) = \begin{cases} 0 & n \neq m = 0, 1, 2, \ldots \\ \frac{1}{2} & n = m = 1, 2, \ldots \end{cases}$$ skal vises

for $m \neq n$:

$$(\cos(nx), \cos(mx)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$$

vi vet at $\cos(nx) \cdot \cos(mx) = \cos((n+m)x - \sin(nx) \cdot \sin(mx)$

vi får

$$(\cos(nx), \cos(mx)) = \frac{1}{2\pi}\left\{ \underbrace{\int_{-\pi}^{\pi} \cos((n+m)x)\,dx}_{=0 \text{ pga odde}} - \underbrace{\int_{-\pi}^{\pi} \sin(nx) \cdot \sin(mx)\,dx}_{=0 \text{ fra } 1] b) i)} \right\}$$

$\underline{= 0}$

for $m = n$:

$$(\cos(nx), \cos(mx)) = (\cos(nx), \cos(nx)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2(nx)\,dx$$

vi vet $\cos^2(nx) = \frac{1}{2}(1 + \cos(2nx))$

$$\text{vi får } \frac{1}{2\pi} \cdot \frac{1}{2}\left\{ \int_{-\pi}^{\pi} 1 + \underbrace{\int_{-\pi}^{\pi} \cos(2nx)}_{=0 \text{ pga. odde}} \right\}$$

$$= \frac{1}{4\pi}\Big[x\Big]_{\pi}^{\pi} = \frac{1}{2}, \text{ se } 1] b) i)$$

som vi skulle vise

$$\text{vi får } \underline{\underline{(\cos(nx), \cos(mx)) = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \end{cases}}}$$

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1] c)

vi antar at $f = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$$

$$a_0 = (f, 1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot 1 \, dx = a_0$$

$$a_n = (f, 2\cos(nx))$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot 2 \cdot \cos(nx) \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = a_n$$

$$b_n = (f, 2\sin(nx)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot 2 \cdot \sin(nx) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$$

$$= b_n$$

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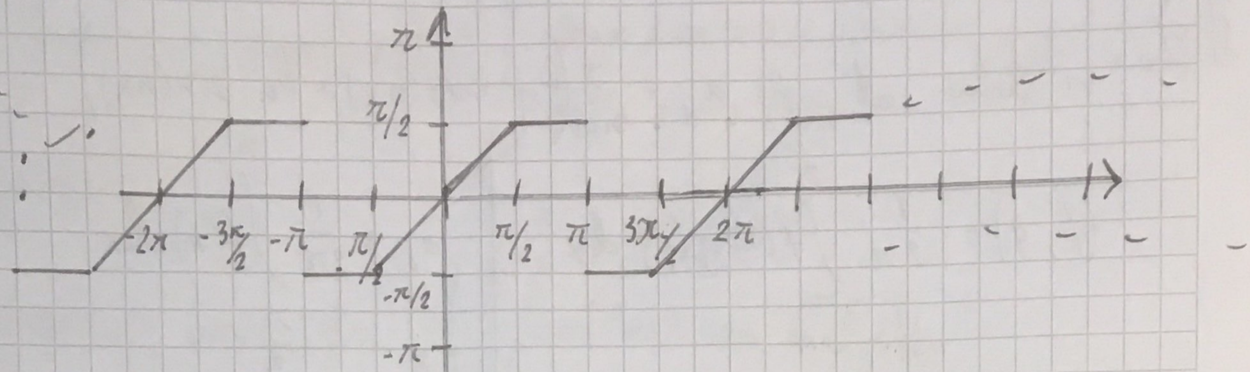
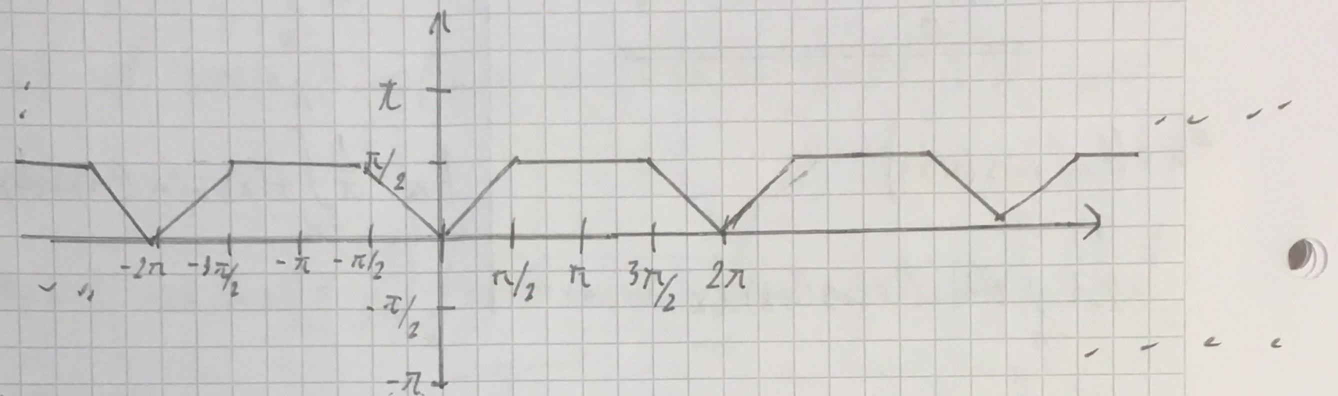
2]

f_0 :



f_e :



i)

$f_e(x)$ er en like funksjon, $f_e(x) \sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(nx) + b_n \cdot \sin(nx)\right)$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f_e(x)}_{\text{like}} \cdot \underbrace{\sin(nx)}_{\text{odde}}\, dx \Rightarrow \frac{1}{\pi}\cdot 0 = 0$$

like · odde = odde

pga like mye "over" og "under" integralet

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_e(x)\,dx = \frac{1}{2\pi}\, 2 \cdot \int_0^{\pi} f_e(x)\,dx$$

pga. like

$$= \frac{1}{\pi}\left\{\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} \pi/2\,dx\right\} = \frac{1}{\pi}\left\{\frac{1}{2}\left[x^2\right]_0^{\pi/2} + \left[(\pi/2)\cdot x\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\cancel{\pi}}\left\{\frac{1}{2}\cdot\frac{\pi^{\cancel{2}}}{2^2} + \frac{\pi^{\cancel{2}}}{2} - \frac{\pi^{\cancel{2}}}{2^2}\right\} = \frac{\pi}{8} + \frac{4\pi}{8} - \frac{2\pi}{8} = \underline{\frac{3\pi}{8}}$$

i)

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2] forts.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos(nx)\,dx = \frac{1}{\pi}\cdot 2\int_0^{\pi} f_e(x)\cos(x)\,dx$$

$$= \frac{2}{\pi}\left\{ \int_0^{\pi/2} \underset{u}{x}\cdot \underset{v'}{\cos(nx)}\,dx + \int_{\pi/2}^{\pi} \pi/2\cdot\cos(nx)\,dx \right\}$$

$$= \frac{2}{\pi}\left\{ \left[x\cdot\frac{\sin(nx)}{n}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\sin(nx)}{n} + \frac{\pi}{2n}\left[\sin(nx)\right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi}\left\{ \pi/2\cdot\frac{\sin(n\cdot\frac{\pi}{2})}{n} - \frac{1}{n^2}\left[-\cos(nx)\right]_0^{\pi/2} + \frac{\pi}{2n}\cdot(-1)\sin(n\cdot\frac{\pi}{2}) \right\}$$

NB: $0\cdot\frac{\sin(n\cdot 0)}{n} = 0$ og $\sin(n\pi) = 0$ for alle n

$$= \cancel{\frac{\sin(n\cdot\frac{\pi}{2})}{n}} + \frac{2}{\pi n^2}\cdot\left(\cos(n\tfrac{\pi}{2}) - 1\right) - \cancel{\frac{\sin(n\cdot\frac{\pi}{2})}{n}}$$

$(n \neq 0), \in \mathbb{N}$

ser på $\cos(n\frac{\pi}{2}) - 1 = \alpha_n$

$\alpha_0 = 1 - 1 = 0$ $\quad \alpha_4 = 1 - 1 = 0$

$\alpha_1 = 0 - 1 = -1$ $\quad \alpha_5 = 0 - 1 = -1$

$\alpha_2 = -1 - 1 = -2$ $\quad \alpha_6 = -1 - 1 = -2$

$\alpha_3 = 0 - 1 = -1$ $\quad \alpha_7 = 0 - 1 = -1$

$\vdots$

$$f_e(x) = \underbrace{a_0}_{3\pi/8} + \sum_{i=1}^{\infty}\left(a_n\cdot\cos(nx) + \underbrace{b_n\cdot\sin(nx)}_{=0}\right)$$

gir oss $f_e(x) - \frac{3\pi}{8} + \frac{2}{\pi}\left(-\frac{\cos(1\cdot x)}{1^2} - \frac{2\cdot\cos(2x)}{2^2} - \frac{\cos(3x)}{3^2}\right.$

$\left. + 0 - \frac{\cos(5x)}{5^2} - \ldots \right)$

som vi skulle vise

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2] ii) $f_0(x)$ er en odde funksjon

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_0(x)\,dx = \frac{1}{2\pi}\cdot 0 = 0 \qquad \text{pga. odde}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f_0(x)}_{\text{odde}}\cdot \underbrace{\cos(nx)}_{\text{like}}\,dx = \frac{1}{\pi}\cdot 0 \quad -''-$$

odde · like = odde

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_0(x)\cdot\sin(nx)\,dx = \frac{1}{\pi}\cdot 2\int_{0}^{\pi} f_0(x)\cdot\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \underset{u}{x}\cdot \underset{v'}{\sin(nx)}\,dx + \int_{\pi/2}^{\pi} \frac{\pi}{2}\cdot\sin(nx)\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\underbrace{\frac{1}{n}\Big[x\cdot(-\cos(nx))\Big]_0^{\pi/2}}_{=0} - \frac{1}{n}\int_0^{\pi/2} -\cos(nx) + \frac{\pi}{2n}\Big[-\cos(nx)\Big]_{\pi/2}^{\pi}\right\}$$

→ gir 0 for alle n

$$= \frac{2}{\pi}\left\{\frac{1}{n^2}\cdot\Big[\sin(nx)\Big]_0^{\pi/2} - \frac{\pi}{2n}\cos(n\pi)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n^2}\cdot\sin\left(n\frac{\pi}{2}\right) - \frac{\pi}{2n}(-1)^n\right\} = \underbrace{\frac{2}{\pi n^2}\cdot\sin\left(n\frac{\pi}{2}\right) - \frac{1}{n}(-1)^n}_{\beta_n}$$

$$\beta_1 = \frac{2}{\pi\cdot 1^2}\cdot 1 + \frac{1}{1} \qquad \beta_3 = \frac{2}{\pi\cdot 3^2}\cdot(-1) + \frac{1}{3} \qquad \beta_5 = \frac{2}{\pi\cdot 5^2}\cdot 1 + \frac{1}{5}$$

$$\beta_2 = \frac{2}{\pi 2^2}\cdot 0 - \frac{1}{2} \qquad \beta_4 = \frac{2}{\pi\cdot 4^2}\cdot 0 - \frac{1}{4}$$

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2] i) forts.

$$f_0(x) \sim \underbrace{a_0}_{=0} + \sum_{n=1}^{\infty}\left(\underbrace{a_n}_{=0}\cdot\cos(nx) + b_n\cdot\sin(nx)\right)$$

som gir oss

$$f_0(x) \sim \sum_{n=1}^{\infty} b_n \cdot \sin(nx) = \left(\frac{2}{\pi}+1\right)\cdot\sin(1\cdot x) + \left(0-\frac{1}{2}\right)\cdot\sin(2x)$$

$$+\left(\frac{-2}{3^2\pi}+\frac{1}{3}\right)\sin(3x) + \left(0-\frac{1}{4}\right)\sin(4x)$$

$$+\left(\frac{2}{5^2\pi}+\frac{1}{5}\right)\cdot\sin(5x) + \ldots\ldots$$

som vi skulle vise