Alexander Nes Fredheim WORK SHEET 3

1] a) Vi vil vise at for $f(t) = \begin{cases} t & \text{for } 0 \le t \le a; \\ 0 & \text{for } t > a \end{cases}$

er $\mathcal{L}(f(t)) = F(s) = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}$

$$\int_0^a e^{-st} \cdot t\,dt$$

$$= \frac{1}{-s}\int_0^a -s\,e^{-st}\,dt$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^a \underset{u'}{e^{-st}} \cdot \underset{v}{t}\,dt + \underbrace{\int_a^\infty e^{-st} \cdot 0\,dt}_{=0}$$

$$= \frac{1}{-s}\left(\left[e^{-st} \cdot t\right]_0^a - \int_0^a e^{-st}\,dt\right) = \left(\left[e^{-st} \cdot t\right]_0^a - \frac{1}{(-s)}\left[e^{-st}\right]_0^a\right) \cdot \frac{1}{-s}$$

$$= \frac{1}{-s}\left(\left\{e^{-as} \cdot a - 0\right\} - \frac{1}{-s}\left\{e^{-as} - 1\right\}\right)$$

$$= (-a) \cdot \frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} = \underline{\underline{\frac{1}{s^2} - \frac{e^{-as}}{s^2} - a \cdot \frac{e^{-as}}{s}}}$$

som vi skulle vise

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1 b) ønsker å vise at $\mathcal{L}(f(t)) = F(s) = -\dfrac{e^{-\pi s}}{s^2+1}$

for $f(t) = u(t-\pi)\cdot\sin(t)$

$$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(u(t-\pi)\cdot\sin(t)) = \int_0^\infty e^{-st}\cdot u(t-\pi)\cdot\sin(t)\,dt$$

vi vet at $\sin(t) = (-1)(\sin(t-\pi))$

$g(t) = \sin(t)$

vi får $\int_0^\infty e^{-st}\cdot u(t-\pi)\cdot\sin(t-\pi)\cdot(-1)\,dt$

$= e^{-s\pi}\,G(s)$

$$= (-1)\int_0^\infty e^{-st}\cdot u(t-\pi)\cdot\sin(t-\pi)\,d = (-1)\cdot\mathcal{L}(u(t-\pi)\cdot\sin(t-\pi))$$

$$= (-1)\cdot e^{-s\pi}\cdot\mathcal{L}(\sin(t)) = -e^{-s\pi}\cdot\frac{1}{s^2+1^2} = -\frac{e^{-\pi s}}{s^2+1}$$

som vi skulle vise

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1 c) vil vise at $i(t) = u(t-1)\left(e^{-(t-1)} - e^{-(t-1)}(t-1)\right)$

når $i'(t) + 2\cdot i(t) + \int_0^t i(\tau)\,d\tau = \delta(t-1)$, når $i(0) = 0$

Omgjør alle uttrykk til deres Laplace transformasjon:

$$\mathcal{L}(i') + 2\cdot\mathcal{L}(i) + \mathcal{L}\left(\int_0^t i(\tau)\,d\tau\right) = \mathcal{L}(\delta(t-1))$$

$$s\cdot I(s) - i(0) + 2\cdot I(s) + \frac{1}{s} I(s) = e^{-s}$$

vi vet at

$$\mathcal{L}(u(t-a)\,i(t-a)) = e^{-sa} I(s)$$

$$\Leftrightarrow I(s)\left(s + \frac{1}{s} + 2\right) = e^{-s}$$

$$\Leftrightarrow I(s) = \frac{e^{-s}}{s + \frac{1}{s} + 2} \equiv e^{-s}\cdot\frac{1}{s} + e^{-s}\cdot s + \frac{1}{2} e^{-s}\cdot 1$$

$i(s) =$ $i(s) =$

$$\mathcal{L}(t) = \frac{1}{s^2} = F(s) = \mathcal{L}(f(t))$$

$$\mathcal{L}(e^{-t}\cdot f(t)) = F(s-(-1))$$

$$\mathcal{L}(e^{-t}\cdot t) = F(s+1) = \frac{1}{(s+1)^2}$$

$$I(s) = \frac{e^{-s}}{s+\frac{1}{s}+2} = \frac{e^{-s}\cdot s}{s^2+1+2s} = \frac{e^{-s}\cdot s}{(s+1)^2} = e^{-s}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$= e^{-s}\left(\frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2}\right) = e^{-s}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$\mathcal{L}^{-1} = e^{-t}$ $\mathcal{L}^{-1} = e^{-t}\cdot t$

$(a=1)$

$$= e^{-s}\cdot\frac{1}{s+1} - e^{-s}\frac{1}{(s+1)^2}$$

$$= u(t-1)\cdot e^{-(t-1)} - u(t-1)\,e^{-(t-1)}\cdot(t-1)$$

$$= \underline{\underline{u(t-1)\left(e^{-(t-1)} - e^{-(t-1)}(t-1)\right)}}$$

som vi skulle vise

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$y - y * t = t$

$\Updownarrow$

$Y(s) - Y(s) \cdot F(s) = F(s) \iff \mathcal{L}(y) - \mathcal{L}(y) \cdot \mathcal{L}(t) = \mathcal{L}(t)$

$\iff \mathcal{L}(y)(1 - \mathcal{L}(t)) = \mathcal{L}(t) \iff \mathcal{L}(y) = \dfrac{\mathcal{L}(t)}{1 - \mathcal{L}(t)}$

vi vet at $\mathcal{L}(t) = \frac{1}{s^2}$, vi får

$$\mathcal{L}(y) = \frac{\frac{1}{s^2}}{1 - \frac{1}{s^2}} \Big/ \cdot \frac{s^2}{s^2} = \frac{1}{s^2 - 1}$$

$\mathcal{L}(\sinh(1 \cdot t))$

$= \dfrac{1}{s^2 - 1}$

vi får da:

$\underline{\underline{y = \sinh(t)}}$

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3] Finner løsningen for $\begin{cases} x' = 2x - y & (1) \\ y' = 3x - 2y & (2) \end{cases}$, $x(0)=0$, $y(0)=1$

(1): Anvender Laplace: $s \cdot X(s) - \underbrace{x(0)}_{0} = 2 \cdot X(s) - Y(s)$

(2): — " —: $s \cdot Y(s) - \underbrace{y(0)}_{1} = 3 \cdot X(s) - 2 \cdot Y(s)$

(1) $Y(s) = X(s)(2-s) \iff \underline{(2-s)X(s) - Y(s) = 0}$

(2) $3 \cdot X(s) = Y(s)(s+2) - 1 \iff \underline{(-3) \cdot X(s) + (s+2)Y(s) = 1}$

setter opp matrise:

$$\left[\begin{array}{cc|c} 2-s & -1 & 0 \\ -3 & s+2 & 1 \end{array}\right]$$

bruker Cramers regel:

$$X(s) = \frac{\begin{vmatrix} 0 & -1 \\ 1 & (s+2) \end{vmatrix}}{\begin{vmatrix} (2-s) & -1 \\ -3 & (s+2) \end{vmatrix}} = \frac{0-(-1)}{2^2 - s^2 - 3}$$

$$= \underline{\frac{1}{1-s^2}} = (-1) \cdot \underbrace{\frac{1}{s^2-1}}_{\mathcal{L}^{-1} = \sinh(t)}$$

Cramers regel

$$Y(s) = \frac{\begin{vmatrix} (2-s) & 0 \\ -3 & 1 \end{vmatrix}}{\begin{vmatrix} (2-s) & -1 \\ -3 & (s+2) \end{vmatrix}} = \frac{2-s}{1-s^2} = \frac{s-2}{s^2-1}$$

$$= \underbrace{\frac{s}{s^2-1}}_{\mathcal{L}^{-1} = \cosh(t)} - 2 \cdot \underbrace{\frac{1}{s^2-1}}_{\mathcal{L}^{-1} = \sinh(t)},$$

vi får:

$\underline{\underline{x = -\sinh(t)}}$

$\underline{\underline{y = \cosh(t) - 2 \cdot \sinh(t)}}$

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4] a) skal vise at $x = \sum_{n \neq 0} \frac{i(-1)^n}{n} e^{inx}$ når $-\pi < x < \pi$

[vi ser at funksjonen er odde]

Den komplekse Fourier serien til f på $(-\pi, \pi)$ er definert som

$$\sum_{n \in \mathbb{Z}} c_n e^{inx} \quad , \quad c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \quad \Big| \quad n \in \mathbb{Z} , \; n = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$$

Fra oppgaven: $f(x) = x$, vi får

$$c_n = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} \underbrace{x}_{u} \cdot \underbrace{e^{-inx}}_{v'} \, dx\right) = \frac{1}{2\pi}\left(\left[- \frac{x \cdot e^{-inx}}{in}\right]_{-\pi}^{\pi} - \left(-\frac{1}{in}\right) \int_{-\pi}^{\pi} e^{-inx}\right)$$

$$= \frac{1}{2\pi}\left(-\frac{1}{in}\left[x \cdot e^{-inx}\right]_{-\pi}^{\pi} - \left(-\frac{1}{in}\right)\left(-\frac{1}{in}\right)\left[e^{-inx}\right]_{-\pi}^{\pi}\right)$$

$$= \frac{1}{2\pi}\left(-\frac{1}{in}\left\{\pi \cdot e^{-in\pi} - (-\pi) \cdot e^{-in(-\pi)}\right\} - \left(\frac{1}{in}\right)^2 \left\{e^{-in\pi} - e^{-in(-\pi)}\right\}\right) \quad n = \text{heltall}$$

(*)

$$e^{i\omega t} = \cos(\omega t) + i \cdot \sin(\omega t) \quad , \quad e^{in\pi} = \cos(n\pi) + i \cdot \underbrace{\cancel{\sin(n\pi)}}_{= 0 \text{ for alle } n}$$

$$= \pi\left(\cos((-n)\cdot\pi) + \cos(n \cdot \pi)\right)$$

$$= \frac{1}{2\pi}\left(-\frac{1}{in}\left\{\pi \cdot \cos((-n)\pi) + \pi \cdot \cos(n \cdot \pi)\right\} - \left(\frac{1}{in}\right)^2 \cdot \left\{\cos((-n)\pi) - \cos(n \cdot \pi)\right\}\right)$$

ser bort ifra $n = 0$ pga. oppgave teksten

ser på $\cos((-n)\cdot\pi) + \cos(n \cdot \pi)$: n like gir $1 + 1 = 2$

n odde gir $-1 - 1 = -2$

som kan skrives som $2 \cdot (-1)^n$

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4] a) forts.

se på $\underbrace{\cos((-n)\pi) - \cos(n\cdot\pi)} = 0$ for alle n

n like gir $1 - 1 = 0$

n odde gir $-1 - (-1) = 0$

vi får da

$$C_n = \left(-\frac{1}{i\cdot n}\cdot\pi\cdot 2\cdot(-1)^n\right)\cdot\frac{1}{2\pi} = -\frac{(-1)^n}{i\cdot n}\,\Big|\cdot\frac{i}{i} = -\frac{i\cdot(-1)^n}{(-1)\cdot n}$$

$$= \frac{i\cdot(-1)^n}{n}\ , \quad x = \sum_{n\neq 0} C_n\cdot e^{inx} = \underline{\underline{\frac{i\cdot(-1)^n}{n}\cdot e^{inx}}}$$

$n\in\mathbb{Z}$ som vi skulle vise

4] b) skal vise at $x(2\pi - x) = -\frac{\pi^2}{3} + \sum_{n\neq 0}\left(\frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{(n+1)}}{n^2}\right)e^{inx}$

når $-\pi < x < \pi$

$u = x\cdot(2\pi - x) = 2\pi x - x^2$

$u' = 2\pi - 2x$

starte med C_n som i 4] a):

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{x\cdot(2\pi - x)}_{u_1}\cdot \underbrace{e^{-inx}}_{v_1'}\,dx = \frac{1}{2\pi}\left(\left[\frac{x\cdot(2\pi - x)\cdot e^{-inx}}{(-in)}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{(2\pi - 2x)e^{-inx}}{(-in)}dx\right)$$

$$= -\frac{1}{in2\pi}\left(\left[x\cdot(2\pi - x)\cdot e^{-inx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\underbrace{(2\pi - 2x)}_{u_2}\underbrace{e^{-inx}}_{v_2'}\,dx\right)$$

$$= -\frac{1}{in2\pi}\left(\left[x\cdot(2\pi - x)\cdot e^{-inx}\right]_{-\pi}^{\pi} - \left[\frac{(2\pi - 2x)e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{(-2)e^{-inx}}{-in}dx\right)$$

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$$= -\frac{1}{in\,2\pi}\left(\left[x\cdot(2\pi-x)e^{-inx}\right]_{-\pi}^{\pi} - \left(-\frac{1}{in}\right)\left\{\left[(2\pi-2x)e^{-inx}\right]_{-\pi}^{\pi} - \frac{(-2)}{-in}\underbrace{\left[e^{-inx}\right]_{-\pi}^{\pi}}\right\}\right)$$

(*) fra oppg 4] a) gjelder for alle e^{-inx} her $\Big|$ $= 0$ (fra 4] a))

$$-\frac{1}{i\cdot n\cdot 2\cdot\pi}\left(\underbrace{\pi\cdot(2\pi-\pi)\cos((-n)\pi) - ((-\pi)(2\pi-(-\pi))\cos(n\cdot\pi))}\right.$$

$$\left. - \left(-\frac{1}{in}\right)\left\{\underbrace{(2\pi-2\pi)}_{=0}\cdot\cos(\dots) - \underbrace{(2\pi-(-2\pi)\cos(n\cdot\pi)}\right\}\right)$$

For n like får vi: $\pi^2\cdot 1 - 3\pi^2\cdot 1 = -2\pi^2$

For n odde: $2\pi^2$, kan skrives $(-1)^{n+1}\cdot 2\pi^2$

For n like: 4π, odde: -4π, blir $(-1)^n\cdot 4\cdot\pi$

vi får da

$$-\frac{1}{in\,2\pi}\left((-1)^{n+1}\cdot 2\pi - \right.$$

Gir opp, ønsker svar på:

1) kunne jeg funnet fourrier koeffisientene istedet og gjort det lettere?

2) Hvordan henger $a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_m\sin(mx)\right)$

og $\sum_{n\in\mathbb{Z}} C_n\cdot e^{inx}$ sammen?