WORK SHEET 10 ANF

1] a) i) we have $Ax = b$ where $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$

$x^{(0)} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ we rewrite A as a

system of 3 equations:

$$\begin{aligned}
&(1) \quad 3x_1 + x_2 + x_3 = 5 \\
&(2) \quad x_1 + 3x_2 - x_3 = 3 \\
&(3) \quad 3x_1 + x_2 - 5x_3 = 1
\end{aligned}
\Rightarrow
\begin{aligned}
x_1^{(k+1)} &= \frac{1}{3}\left(5 - x_2^{(k)} - x_3^{(k)}\right) \\
x_2^{(k+1)} &= \frac{1}{3}\left(3 - x_1^{(k)} + x_3^{(k)}\right) \\
x_3^{(k+1)} &= \frac{1}{5}\left(1 + 3x_1^{(k)} + x_2^{(k)}\right)
\end{aligned}
, \quad x^{(k+1)} = \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{bmatrix}$$

Jacobi method:

$k = 0$: $x_1^{(1)} = \frac{1}{3}(5 - 0 - 0) = 5/3$

$x_2^{(1)} = \frac{1}{3}(3 - 0 + 0) = 1$

$x_3^{(1)} = \frac{1}{5}(1 + 3 \cdot 0 + 0) = \frac{1}{5}$

$$\Rightarrow x^{(1)} = \begin{bmatrix} 5/3 \\ 1 \\ 1/5 \end{bmatrix} \approx \begin{bmatrix} 1.666667 \\ 1.000000 \\ 0.200000 \end{bmatrix}$$

$k = 1$: $x_1^{(2)} = \frac{1}{3}(5 - 1 - 1/5) = \frac{1}{3} \cdot \frac{1}{5}(25 - 5 - 1) = \frac{19}{15} \approx \underline{1.266667}$

$x_2^{(2)} = \frac{1}{3}(3 - 5/3 + 1/5) = \frac{1}{3} \cdot \frac{1}{15}(45 - 25 + 3) = \frac{23}{45} \approx \underline{0.511111}$

$x_3^{(2)} = \frac{1}{5}(1 + 3 \cdot 5/3 + 1) = \frac{1}{5}(1 + 5 + 1) = \frac{7}{5} = \underline{1.400000}$

$$x^{(2)} = \begin{bmatrix} 19/15 \\ 23/45 \\ 7/5 \end{bmatrix}$$ after two iterations

WORK SHEET 10 ANF

1] a) ii) (1)

$$x_1^{(k+1)} = \frac{1}{3}\left(5 - x_2^{(k)} - x_3^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{3}\left(3 - x_1^{(k+1)} + x_3^{(k)}\right), \quad x^{(k+1)} = \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{bmatrix}, \quad x^{(0)} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3^{(k+1)} = \frac{1}{5}\left(1 + 3x_1^{(k+1)} + x_2^{(k+1)}\right)$$

difference

Gauss - Seidel method:

$$k=0: \quad x_1^{(1)} = \frac{1}{3}(5-0-0) = \frac{5}{3} \approx 1.666667$$

$$x_2^{(1)} = \frac{1}{3}\left(3 - \frac{5}{3} + 0\right) = \frac{1}{3}\cdot\frac{1}{3}(9-5) = \frac{4}{9} \approx 0.444444$$

$$x_3^{(1)} = \frac{1}{5}\left(1 + 3\cdot\frac{5}{3} + \frac{4}{9}\right) = \frac{1}{5}\cdot\frac{1}{9}(9+45+4) = \frac{58}{45} \approx 1.288889$$

$$x^{(1)} = \begin{bmatrix} 5/3 \\ 4/9 \\ 58/45 \end{bmatrix}$$

$$k=1: \quad x_1^{(2)} = \frac{1}{3}\left(5 - 4/9 - 58/45\right) = \frac{1}{3}\cdot\frac{1}{45}(225-20-58) = \frac{147}{135} \approx 1.088889$$

$$x_2^{(2)} = \frac{1}{3}\left(3 - \frac{147}{135} + \frac{58}{45}\right) = \frac{1}{3}\cdot\frac{1}{135}(405-147+174) = \frac{432}{405} = \frac{16}{15} \approx 1.066667$$

$$x_3^{(2)} = \frac{1}{5}\left(1 + 3\cdot\frac{147}{135} + \frac{16}{15}\right) = \frac{1}{5}\cdot\frac{1}{45}(45+147+48) = \frac{240}{225} = \frac{16}{15}$$

$$\approx 1.066667$$

$$x^{(2)} = \begin{bmatrix} 147/135 \\ 16/15 \\ 16/15 \end{bmatrix}$$ after two iterations

WORK SHEET 10 ANF

1b) we know A is strictly diagonally dominant if

$$|a_{ii}| > \sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}| \quad \text{for all } i = 1, 2, \dots, n$$

or more simply put, a matrix is strictly diagonally dominant if every diagonal entry is strictly larger than the sum of the other entries on the corresponding row.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \qquad \begin{aligned} |3| &> |1| + |1| \quad \checkmark \\ |3| &> |1| + |-1| \quad \checkmark \\ |-5| &> |3| + |1| \quad \checkmark \end{aligned}$$

the matrix is strictly diagonally dominant

c) we want to prove that jacobi iterations converges for all starting values if the matrix is diagonal dominant

$$e_i^{(k+1)} = \sum_{j=1, j\neq i}^{n} d_{ij} e_j^{(k)} \quad i = 1, 2, \dots, n.$$

writing out for clarification:

$$\Leftrightarrow e_1^{(k+1)} = d_{12} e_2^{(k)} + d_{13} e_3^{(k)} + \dots + d_{1(n-1)} e_{(n-1)}^{(k)} + d_{1n} e_n^{(k)}$$

$$e_2^{(k+1)} = d_{21} e_1^{(k)} + d_{23} e_3^{(k)} + \dots + d_{2(n-1)} e_{(n-1)}^{(k)} + d_{2n} e_n^{(k)}$$

$$\vdots$$

$$e_{n-1}^{(k+1)} = d_{(n-1)1} e_1^{(k)} + d_{(n-1)2} e_2^{(k)} + \dots + d_{(n-1)(n-2)} e_{(n-2)}^{(k)} + d_{(n-1)n} e_n^{(k)}$$

$$e_n^{(k+1)} = d_{n1} e_1^{(k)} + d_{n2} e_2^{(k)} + \dots + d_{n(n-1)} e_{(n-1)}^{(k)}$$

WORK SHEET 10 ANF

1] c) continued $\quad e^{(k)} = x - x^{(k)}$, x exact solution

we want to prove that there exists an L such that

$$\|e^{(k+1)}\|_\infty \leq L \|e^{(k)}\|_\infty$$

$$|e_i^{(k+1)}| = |x_i - x_i^{(k)}| = \left| \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j\right) - \frac{1}{a_{ii}}\left(b_i - \sum_{j=1, j\neq i}^{n} a_{ij} x_j^{(k)}\right)\right|$$

$$= \left| \frac{1}{a_{ii}} \left(\sum_{j=1, j\neq i}^{n} (x_i^{(k)} - x_i)\right) a_{ij} \right| = \left| \sum_{j=1, j\neq i}^{n} \frac{a_{ij}}{a_{ii}} e_i^{(k)} \right| \quad \left(d_{ij} = \frac{a_{ij}}{a_{ii}}\right)$$

$$= \left| \sum_{j=1, j\neq i}^{n} d_{ij} e_i^{(k)} \right| \leq \sum_{j=1, j\neq i}^{n} |d_{ij}| |e_i^{(k)}| \leq \|e^{(k)}\| \sum_{j=1, j\neq i}^{n} |d_{ij}|$$

defining L as $\max\{L_i \mid i = 1, 2, \dots, n\}$, where $L_i = \sum_{j=1, j\neq i}^{n} |d_{ij}|$

by definition of a strictly diagonally dominant matrix we have that $a_{ii} > \sum_{j=1, j\neq i}^{n} a_{ij} \Rightarrow 1 > \sum_{j=1, j\neq i}^{n} \frac{a_{ij}}{a_{ii}} \Rightarrow \underline{L_i < 1}$

$\|e^{(k)}\| \sum_{j=1, j\neq i}^{n} |d_{ij}| = \|e_i^{(k+1)}\| \leq L \|e^{(k)}\|$, every element in

$e^{(k+1)} \leq L \|e^{(k)}\| \Rightarrow \underline{\underline{\|e^{(k+1)}\| \leq L \|e^{(k)}\|}}$ ✓

WORK SHEET 10 ANF

2]

$$y' - xy^2 = 0 \,,\; y(0) = 1 \qquad \text{solution: } -\frac{2}{x^2-2}$$

a) we see that the ODE is separable as

$$dy/dx = F(x,y) = f(x)\cdot g(y) = x\cdot y^2$$

$$y' = x\cdot y^2 \iff \frac{1}{y^2}y' = x$$, integrating for x on both sides

Left side: $\int \frac{1}{y^2}\cdot y' \, dx = \int \frac{1}{y^2}\, dy = -\frac{1}{y} + C_1$

Right side: $\int x \, dx = \frac{1}{2}x^2 + C_2$

we now have $-\frac{1}{y} + C_1 = \frac{1}{2}x^2 + C_2 \implies -\frac{1}{y} = \frac{1}{2}x^2 + C$

solving for C, $y(0) = 1$: $\quad -\frac{1}{1} = C \implies C = \underline{-1}$

we have $-\frac{1}{y} = \frac{1}{2}x^2 - 1 \;\; |\cdot y \implies -1 = \frac{1}{2}x^2 y - y$

$$\iff -1 = y\left(\frac{1}{2}x^2 - 1\right) \iff -1 = y\left(\frac{x^2-2}{2}\right) \;\; \Big|\cdot \frac{1}{\left(\frac{x^2-2}{2}\right)}$$

$$\iff y = -\frac{1}{\frac{x^2-2}{2}} \iff \underline{\underline{y = -\frac{2}{x^2-2}}}$$

WORK SHEET 10 ANF

$y(0) = 1$

2] b) Doing 4 iterations of Eulers method

$y_0 = 1,\ x_0 = 0$

$h = 0.1$

$y_{n+1} = y_n + h \cdot f(x_n, y_n) \qquad y' = x \cdot y^2 = f(x, y)$

$x_1 = 0.1$

$x_2 = 0.2$

$n = 0: \quad y_1 = y_0 + h \cdot f(x_0, y_0) = y_0 + h \cdot x_0 \cdot y_0^2 = y_0 = \underline{1} \qquad x_3 = 0.3$

$n = 1: \ y_2 = y_1 + h \cdot f(x_1, y_1) = 1 + 0.1 \cdot 0.1 \cdot 1^2 = \underline{1.01}$

$n = 2: \ y_3 = y_2 + h \cdot f(x_2, y_2) = 1.01 + 0.1 \cdot 0.2 \times 1.01^2 \approx \underline{1.030402}$

$n = 3: \ y_4 = y_3 + h \cdot f(x_3, y_3) = 1.030402 + 0.1 \cdot 0.3 \cdot 1.030402^2 \approx \underline{1.062254}$

$\underline{\underline{\text{after 4 iterations}}}$

Error: $|y(0.3) - y_4| = |1.047120 - 1.062254| = \underline{\underline{0.015134}}$

$-\dfrac{2}{(0.3)^2 - 2} \approx 1.047120$

WORK SHEET 10 ANF

2] c) Doing 2 iterations of Heuns method $\quad h = 0.2$

$$u_{n+1} = y_n + h f(x_n, y_n) \qquad y_0 = 1, \; x_0 = 0$$

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, u_{n+1})\right), \quad f(x,y) = x \cdot y^2 = y'$$

$n=0$: $u_1 = y_0 + \underbrace{h f(x_0, y_0)}_{=0} = 1$

$$y_1 = y_0 + \frac{h}{2}\left(\underbrace{f(x_0, y_0)}_{=0} + f(\overset{0.2}{x_1}, \overset{1}{u_1})\right) = 1 + \frac{0.2}{2}\left(0.2 \times 1^2\right) = \underline{1.02}$$

$n=1$: $u_2 = y_1 + h f(x_1, y_1) = 1.02 + 0.2\left(0.2 \times 1.02^2\right) \approx \underline{1.061616}$

$$y_2 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(\overset{0.4}{x_2}, \overset{1.061616}{u_2})\right) = 1.02 + \frac{0.2}{2}\left((0.2 \times 1.02^2) + (0.4 \times 1.061616^2)\right)$$

$\approx \underline{\underline{1.085889}}$ after two iterations

Error: $|f(0.4) - y_2| \approx |1.086957 - 1.085889| = \underline{\underline{1.068 \times 10^{-3}}}$

$$-\frac{2}{(0.4)^2 - 2} \approx 1.086957$$

WORK SHEET 10 ANF

2] d) Doing 1 iteration of the 4th order Runge-Kutta method

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad , \quad h = 0.4$$

$$x_{n+1} = x_n + h \, , \quad f(x,y) = xy^2 = y' \qquad x_0 = 0 \, , \, y_0 = 1$$

$$k_1 = h\,f(x_n, y_n) \, , \quad n=0 \Rightarrow \underline{k_1 = 0}$$

$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \quad n=0 \Rightarrow 0.4(0.2 \times 1^2) \underline{= 0.08}$$

$$k_3 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \quad n=0 \Rightarrow 0.4(0.2 \times 1.04^2) \underline{\approx 0.086528}$$

$$k_4 = h\,f(x_n + h, y_n + k_3), \quad n=0 \Rightarrow 0.4(0.4 \times 1.086528^2)$$

$$\underline{\approx 0.188887}$$

$n=0$:

$$y_1 = y_0 + \frac{1}{6}(\underbrace{k_1}_{0} + \underbrace{2k_2}_{2\times 0.08} + \underbrace{2k_3}_{2\times 0.086528} + \overbrace{k_4}^{0.188887})$$

$$= 1 + \frac{1}{6}(0 + 2\times 0.08 + 2\times 0.086528 + 0.188887)$$

$$\underline{\underline{\approx 1.086991}} \text{ after 1 iterations}$$

Error: $|f(0.4) - y_1| \approx |1.086957 - 1.086991| \underline{\underline{= 3.4 \times 10^{-5}}}$

2] c) $\rightarrow 1.086957$

Runge-Kutta performed best, as it had the smallest error with "just" one iteration and a equal number of function evaluations

WORK SHEET 10 ANF

3] a) we have

$$u_1'' = -\frac{1}{(u_1(x) - u_2(x))^2} \quad (1) \quad u_1(0) = 0, \quad u_1'(0) = 1$$

$$u_2'' = \frac{1}{(u_1(x) - u_2(x))^2} \quad (2) \quad u_2(0) = 1, \quad u_2'(0) = 0$$

defining new functions:

$$x_1 = u_1(x) \Rightarrow x_1' = u_1'(x) = x_2$$

$$x_2 = u_1'(x) \Rightarrow x_2' = u_1'' = -\frac{1}{(u_1(x) - u_2(x))^2} = -\frac{1}{u_1^2 - 2u_1u_2 - u_2^2}$$

$$= -\frac{1}{u_1^2} + \frac{1}{2u_1u_2} + \frac{1}{u_2^2} = -\frac{1}{x_1^2} + \frac{1}{x_1y_1} - \frac{1}{y_1^2}$$

$$y_1 = u_2(x) \Rightarrow y_1' = u_2'(x) = y_2$$

$$y_2 = u_2'(x) \Rightarrow y_2' = u_2'' = \frac{1}{u_1^2 - 2u_1u_2 + u_2^2} = \frac{1}{x_1^2} - \frac{1}{2x_1y_1} - \frac{1}{y_1^2}$$

$$x_1' = x_2 \quad \underline{x_1(0) = 0}$$

$$x_2' = -\frac{1}{x_1^2} + \frac{1}{2x_1y_1} + \frac{1}{y_1^2} \quad \underline{x_2(0) = 1} \quad \Rightarrow \vec{X}(0) =$$

$$y_1' = y_2 \quad \underline{y_1(0) = 1}$$

$$y_2' = \frac{1}{x_1^2} - \frac{1}{2x_1y_1} - \frac{1}{y_1^2} \quad \underline{y_2(0) = 0}$$

$$y' = \begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{1}{x_1^2} + \frac{1}{2x_1y_1} + \frac{1}{y_1^2} \\ y_2 \\ \frac{1}{x_1^2} - \frac{1}{2x_1y_1} - \frac{1}{y_1^2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{1}{(x_1 - y_1)^2} \\ y_2 \\ \frac{1}{(x_1 - y_1)^2} \end{bmatrix} = f(y)$$

WORK SHEET 10 ANF

3] b) Doing 1 iteration of heuns method on A from 3] a)
with initial condition $\vec{x}(0)$, $h = 0.1$

$$u_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, u_{n+1})\right)$$

sometimes written as $k_1 = f(x_n, y_n)$

$$k_2 = f(x_n + h, y_n + h k_1)$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

$$n = 0: \quad k_1 = y'(y_0) = \begin{matrix} 1 \\ -\frac{1}{(0-1)^2} \\ \frac{0}{(0-1)^2} \end{matrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$y_0 + h k_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 0.1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 9/10 \\ 1 \\ 1/10 \end{bmatrix}$$ which gives us

$$y'(x) = \begin{bmatrix} 9/10 \\ \frac{-1}{(9/10-1)^2} \\ 1/10 \\ \frac{1}{(9/10-1)^2} \end{bmatrix} = \begin{bmatrix} 9/10 \\ -100/81 \\ 1/10 \\ 100/81 \end{bmatrix} = k_2$$

3] b) cont.

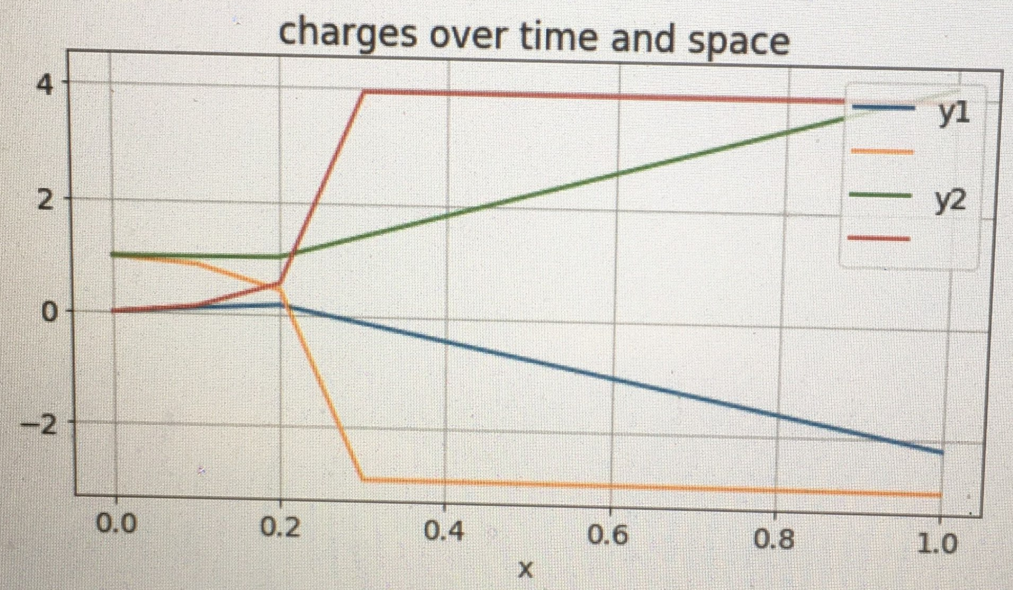
this finally gives us

$$y_1 = y_0 + \frac{h}{2}(k_1 + k_2) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{0.1}{2}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4/10 \\ -100/81 \\ 1/10 \\ 100/81 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 0.05 \begin{bmatrix} 19/10 \\ -181/81 \\ 1/10 \\ 181/81 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 19/200 \\ -181/1620 \\ 1/200 \\ 181/1620 \end{bmatrix}$$

$$= \begin{bmatrix} 19/200 \\ 1439/1620 \\ 201/200 \\ 181/1620 \end{bmatrix} \approx \begin{bmatrix} 0.095000 \\ 0.888272 \\ 1.005000 \\ 0.111728 \end{bmatrix}$$

```
3] b)
y0 og y1:
[[ 0.          1.          1.          0.        ]
 [ 0.095       0.871875    1.005       0.128125]]

3] c)
[[ 0.          1.          1.          0.        ]
 [ 0.095       0.871875    1.005       0.128125  ]
 [ 0.17314859  0.43918885  1.02685141  0.56081115]
 [-0.12095349 -2.94128949  1.42095349  3.94128949]
 [-0.415188   -2.94339924  1.815188    3.94339924]
 [-0.70963334 -2.94550641  2.20963334  3.94550641]
 [-1.00428928 -2.94761101  2.60428928  3.94761101]
 [-1.29915554 -2.94971304  2.99915554  3.94971304]
 [-1.59423189 -2.95181251  3.39423189  3.95181251]
 [-1.88951805 -2.95390943  3.78951805  3.95390943]
 [-2.18501377 -2.9560038   4.18501377  3.9560038 ]]
```



```python
        order = NaN                        # Nothing
    else:
        order = log2(error_old/error)
    print(format('{:.3e}  {:.3e}  {:7
    h = 0.5*h                              # Reduce th
    error_old = error
print("we can see that the order is 4")
```

executed in 147ms, finished 21:58:48 2019-11-05

```
4] a)
h            error          order
-----------------------------------
1.000e-01    1.625e-06        nan
5.000e-02    1.025e-07       3.99
2.500e-02    6.407e-09       4.00
1.250e-02    3.999e-10       4.00
6.250e-03    2.497e-11       4.00
3.125e-03    1.559e-12       4.00
1.563e-03    9.520e-14       4.03
7.813e-04    8.549e-15       3.48
we can see that the order is 4
```

[28]:

```python
def lotka_volterra(x, y):
    alpha, beta, delta, gamma = 2, 1, 0.5,
    dy = array([alpha*y[0]-beta*y[0]*y[1],
                delta*y[0]*y[1]-gamma*y[1]
    return dy
```

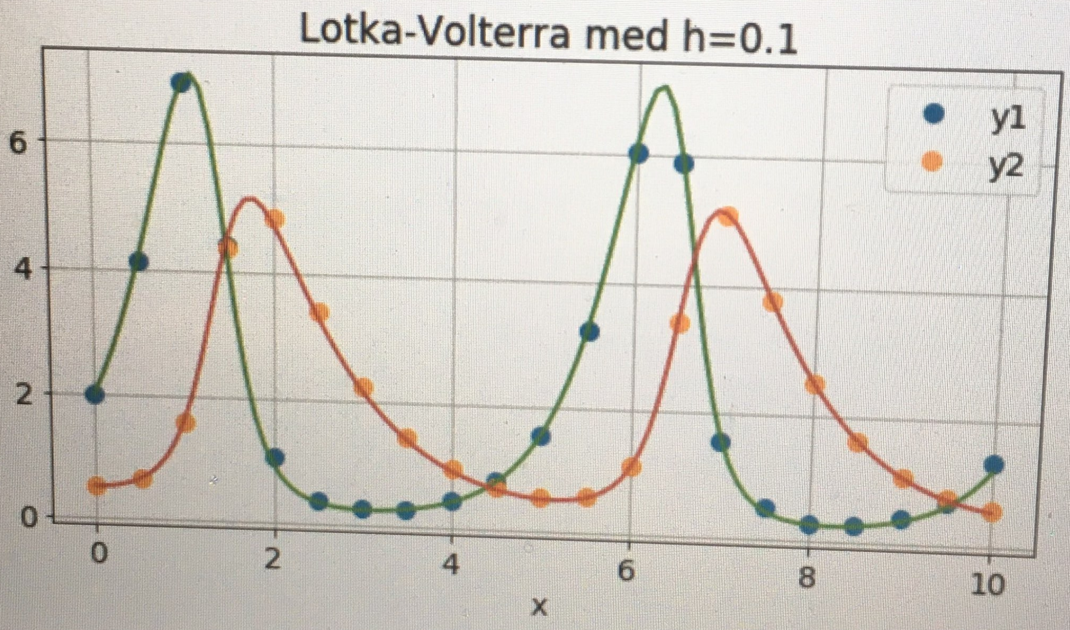
executed in 7ms, finished 22:00:08 2019-11-05

[30]:

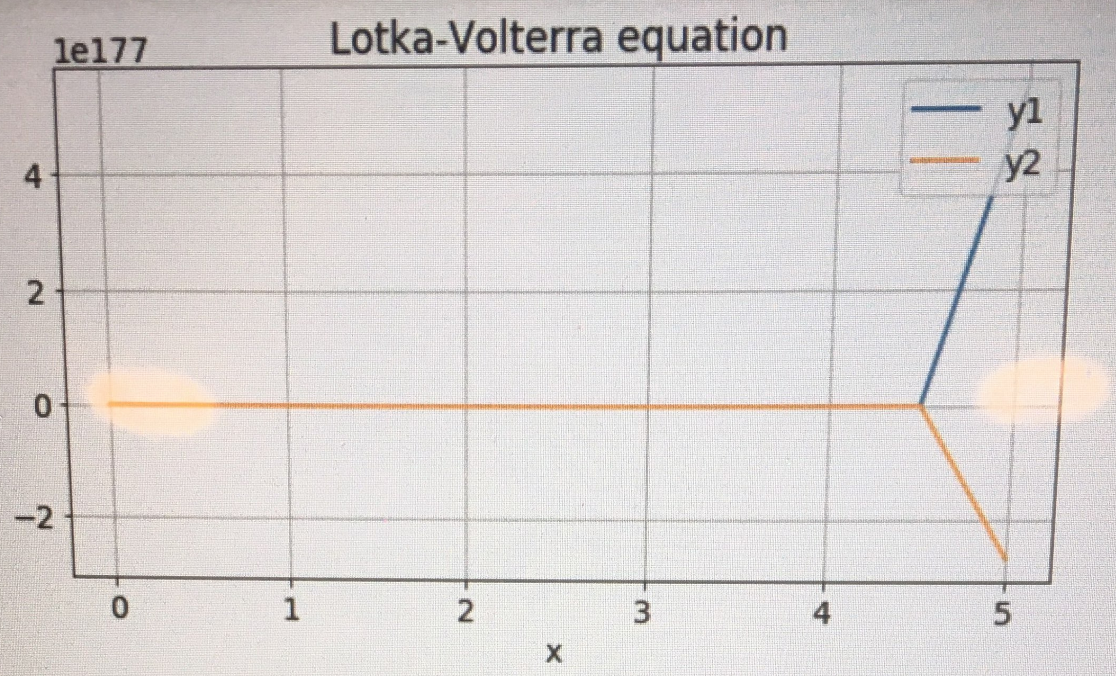
```python
f = lotka_volterra
```

```
x0, xend = 0, 10
y0 = array([2, 0.5])
h = 0.5

x_lv, y_lv = ode_solver(f, x0, xend, y0, h, RK4)
plot(x_lv,y_lv, 'o');
xlabel('x')
title('Lotka-Volterra med h=0.5')
legend(['y1','y2'],loc=1);

h = 0.1

x_lv, y_lv = ode_solver(f, x0, xend, y0, h, RK4)
plot(x_lv,y_lv);
xlabel('x')
title('Lotka-Volterra with h=0.1')
legend(['y1','y2'],loc=1);
```

executed in 1.03s, finished 22:00:22 2019-11-05



Run Code nbdiff



In [32]:

```
h = 0.1

x_lv, y_lv = ode_solver(f, x0, xend, y0, h, heun)
plot(x_lv, y_lv);
xlabel('x')
title('Lotka-Volterra ligningen')
legend(['y1','y2'], loc=1);
```

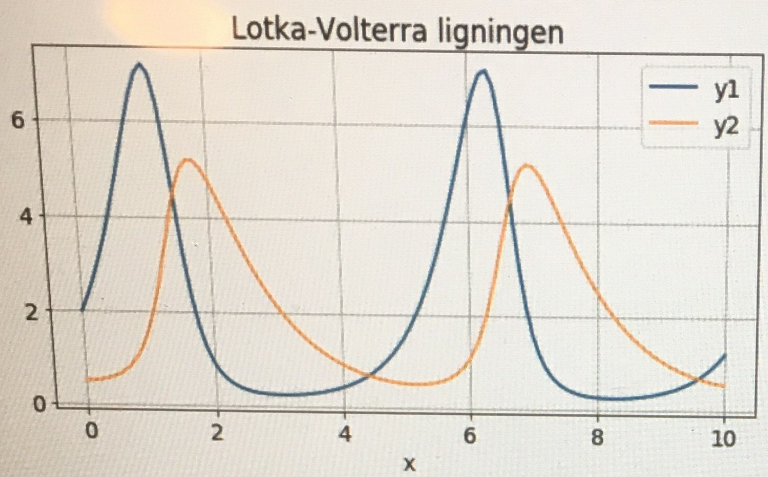
executed in 952ms, finished 22:01:01 2019-11-05

søke

[32]:

```
h = 0.1

x_lv, y_lv = ode_solver(f, x0, xend, y0, h, heun)
plot(x_lv, y_lv);
xlabel('x')
title('Lotka-Volterra ligningen')
legend(['y1','y2'], loc=1);
```

executed in 952ms, finished 22:01:01 2019-11-05



we see that Heuns method does not find a solution for h=0.5, but it does for h=0.1. We can see that Heuns demands shorter intervals than RK4.

In [ ]:

In [ ]: