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TMA4135
Matematikk 4D
Fall 2018

Exercise set 10

1 We are given the linear system of n equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, 2, \ldots, n, \text{ or in short as } A\mathbf{x} = \mathbf{b}.$$

This can be solved by fixed point iterations as follows. Note that the *system* of equations is a fixed point formulation, not each equation on its own:

1. Solve equation i with respect to x_i for $i = 1, 2, \ldots, n$. The result is the fixed point formulation

$$x_i = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i+1}^{n} a_{ij}x_j\right), \qquad i = 1, 2, \ldots, n. \tag{1}$$

2. The corresponding fixed point iteration scheme (Jacobi-iterations) is given by

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right), \qquad i = 1, 2, \ldots, n. \tag{2}$$

   for $k = 0, 1, 2, 3, \ldots$, together with an appropriate choice of starting values $\mathbf{x}^{(0)}$.

3. This can be improved by using the updated value of x_j for $j < i$, resulting in the scheme (Gauss-Seidel iterations):

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right), \qquad i = 1, 2, \ldots, n.$$

If no starting value is given, start with $\mathbf{x}^{(0)} = \mathbf{0}$.

a) (H) Do two iterations on each of the two iteration schemes on the linear systems of equations

$$A\mathbf{x} = \mathbf{b}$$

with

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

starting with $\mathbf{x}^{(0)} = \mathbf{0}$.

A matrix A is called *strictly diagonally dominant* if

$$|a_{ii}| > \sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|, \qquad \text{for all } i = 1, 2, \ldots, n$$

b) (H) Is the matrix A from **a)** strictly diagonal dominant?

It can be proved that strictly diagonally dominant matrises are always nonsingular, so the system $A\mathbf{x} = \mathbf{b}$ will always have a unique solution.

c) The next task is to prove that the Jacobi-iterations will converge for all choices of starting values if the coefficient matrix A is diagonal dominant. Let $\mathbf{x}$ be the exact solution, then the error after k iterations is

$$\mathbf{e}^{(k)} = \mathbf{x} - \mathbf{x}^{(k)},$$

or in element form:

$$e_i^{(k)} = x_i - x_i^{(k)}, \qquad i = 1, 2, 3, \ldots, n.$$

The max-norm of a vector $\mathbf{y} \in \mathbb{R}^n$ is $\|\mathbf{y}\|_\infty = \max_{i=1,\ldots,n} |y_i|$ (see *Preliminaries*). Your task is to prove that in the case of Jacobi iterations there is an L such that

$$\|\mathbf{e}^{(k+1)}\|_\infty \leq L \|\mathbf{e}^{(k)}\|_\infty$$

and that when A is strictly diagonally dominant, we have $L < 1$. This can be done following the steps below.

- Take the difference between (1) and (2) in order to express $e_i^{(k+1)}$ as some linear combinations of $e_i^{(k)}$, that is

$$e_i^{k+1} = \sum_{j=1, j\neq i}^{n} d_{ij} e_j^{(k)}, \qquad i = 1, 2, \ldots, n.$$

- Take the absolute value on both sides, and use the triangle inequality $|a + b| \leq |a| + |b|$, together with the fact that $|e_i^{(k)}| \leq \|\mathbf{e}^{(k)}\|_\infty$ to find an L such that

$$\|\mathbf{e}^{(k+1)}\|_\infty \leq L \|\mathbf{e}^{(k)}\|_\infty$$

  The constant L will be some combination of the absolute values of the coefficients of A.
- Use the definition of a strictly diagonally matrix to prove that $L < 1$.

Remarks:

- Also Gauss-Seidel iterations converges when A is strictly diagonal dominant.
- Both Jacobi and Gauss-Seidel are examples of a classical iterative methods for linear systems. There are a lot more of them.
- Small systems of linear equations should always be solved by some version of Gauss-elimination (Mathematics 3). Iterative methods are only practical for very large systems of equations where most of the elements in the matrix A are zero.

2 (H) Consider the initial value problem

$$y' - xy^2 = 0, \quad y(0) = 1.$$

a) Find the exact solution to the equation.

b) Do 4 steps of Euler's method with $h = 0.1$. Compute the error at the last step.

c) Do 2 steps of Heuns method with $h = 0.2$. Compute the error at the last step

d) Do 1 step of the 4th order Runge-Kutta method (see Problem 4) with $h = 0.4$. Compute the error at the end.

In each case, 4 function evaluations have been done. Which of the methods performed best?

3 Consider two charges (in one dimension), with positions $u_1(x)$ and $u_2(x)$ at time $x > 0$. The positions of the charges is described by a system of two second order differential equations:

$$\begin{aligned} u_1''(x) &= -\frac{1}{(u_1(x) - u_2(x))^2}, \\ u_2''(x) &= \frac{1}{(u_1(x) - u_2(x))^2}, \end{aligned} \tag{3}$$

with initial values $u_1(0) = 0$, $u_1'(0) = 1$, $u_2(0) = 1$, $u_2'(0) = 0$.

a) (H) Rewrite (3) to a system of first order differential equations.

b) (H) Do one step by Heun's method, using $h = 0.1$ to solve this system of equations.
Hint: This is messy, and it is easy to make mistakes. To check your hand calucations, do one step with the function `heun` in the Jupyter notebook, and print out all intermediate vectors. But do remember that this notebook is not available at the exam!

c) (J) Solve the problem numerically over the interval $[0, 1]$, by a method of your own choice.

4 **a)** (J) Implement the classical Runge–Kutta method (RK4):

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(x_n, \mathbf{y}_n) \\ \mathbf{k}_2 &= \mathbf{f}\left(x_n + \frac{h}{2}, \mathbf{y}_n + \frac{h}{2}\mathbf{k}_1\right) \\ \mathbf{k}_3 &= \mathbf{f}\left(x_n + \frac{h}{2}, \mathbf{y}_n + \frac{h}{2}\mathbf{k}_2\right) \\ \mathbf{k}_4 &= \mathbf{f}(x_n + h, \mathbf{y}_n + h\mathbf{k}_3) \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{h}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4). \end{aligned}$$

and verify numerically that the order is 4 (see `oving10_python_files.ipynb` for how to do this).

b) (J) Use RK4 to solve the Lotka–Volterra equation:

$$y_1' = 2.0\, y_1 - y_1 y_2, \qquad y_1(0) = 2.0$$
$$y_2' = 0.5\, y_1 y_2 - y_2, \qquad y_2(0) = 0.5$$

over the interval $[0, 10]$. Try both the stepsize$h = 0.1$ and $h = 0.5$.

What happens if you try to solve the same problem with the same stepsizes with Heuns method?

NB! If you have solved the previous points correctly, you should have, for $h = 0.5$ got a numerical solution corresponding to the •'s in Figure 1. The dotted lines serves as the exact solution, calculated by a numerical method with high accuracy, and are here only for demonstration purpose. We observe that the solution is quite accurate in the points $(x_n, \mathbf{y}_n)$, but we are loosing useful information between.

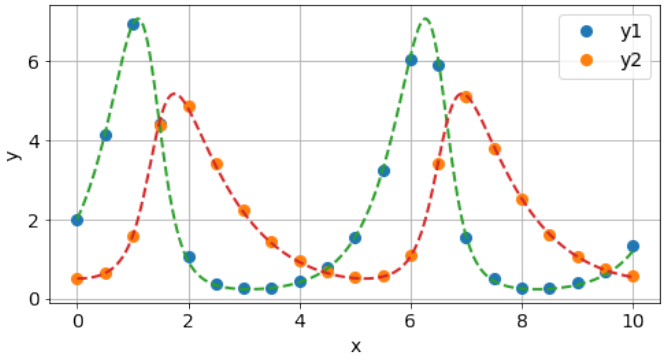

Figure 1: The Lotka–Volterra equation, where • represents the solution from RK4 with $h = 0.5$ and $--$ the exact solution.