TMA4135 Matematikk 4D Fall 2018

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Exercise set 9

1 The error expression for the composite Simpson's formula is given by

$$E_m(a,b) = \int_a^b f(x)\,dx - S_m(a,b) = -\frac{(b-a)h^4}{180} f^{(4)}(\xi), \qquad h = \frac{b-a}{2m}.$$

Thus, the method is of order 4, and the error can be expected to behave like

$$E_m \approx C h^4.$$

This can be verified numerically by using the technique and code described in the Jupyter notebook *Preliminaries*, the section on "Convergence of h-dependent approximations".

a) (J) Given the integral

$$\int_{-1}^{1} x e^x\,dx = \frac{2}{e}$$

Use this to verify numerically the order of the composite Simpson's formula. Include a convergence plot.

b) Repeat the experiment in point **a)** on the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, e^x\,dx = 1.7754996892121809469\ldots.$$

Compare the result with the ones from **a)**, and comment on and try to explain the discrepancies.

Hint: Plot the integrand, and see if that will give you some ideas.

2 (Optional) Sometimes there is a certain factor of the integrands that cause trouble. This can be solved by including this factor in the weights of the quadrature.

Given an integral

$$\int_a^b v(x)\, f(x)\,dx$$

where we expect that the factor $v(x)$ to be the problematic one.

Choose $n+1$ distinct nodes on the interval $[a,b]$ and let $p_n(x)$ be the interpolation polynomial of f through these nodes, written in Lagrange form. Then

$$\int_a^b v(x) f(x) dx \approx \int_a^b v(x) p_n(x) dx = \sum_{i=0}^{n} w_i f(x_i),$$

whith

$$w_i = \int_a^b v(x)\ell_i(x)dx.$$

The functions $\ell_i(x)$ are the cardinal functions.

In this exercise, we consider integrals of the form

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)dx$$

and we will use

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} e^x dx = 3.9774632605064226373$$

as a test problem.

a) (J) Plot the integrand $e^x/\sqrt{1-x^2}$ over the interval $(-1,1)$ (avoid the boundaries, where the function is singular). It should be clear why the standard quadrature rules like the trapezoidal rule or Simpson's formula will fail when applied to this integral.

b) (H) Choose the nodes $(-1,0,1)$ and construct a quadrature formula for such integrals as explained above. Test your rule on the problem above. Write down the numerical approximation and the error. Write down the numerical solution as well as the error.

c) (H) Repeat **b)**, but now with three Chebyshev nodes x_i^{Cheb}, $i = 0,1,2$. (see *Polynomial Interpolation*).

NB! The computations of the the weights w_i in these cases are a bit cumbersome. Feel free use whatever tools you have for symbolic computations to find them.

$\boxed{3}$ Given the equation $f(x) = e^x + x^2 - x - 4 = 0$.

a) (H, J) Prove that f has one and only one zero r in the interval; $[1,2]$.

When this is done, make a plot of $f(x)$, $x \in [1,2]$, and find a rough approximation to r from the plot.

Write down Newton's method for this problem, choose the starting values from the plot above, and do a few iterations by hand, using the approximate root you found from the plot as starting value.

Find a numerical solution by the use of the function `newton` in the Jupyter note.

b) (J) The equation $f(x) = 0$ can be rewritten as $x = g(x)$ by e.g.:

$$\begin{array}{ll} i) & g(x) = \ln(4 + x - x^2) \\ ii) & g(x) = \sqrt{-e^x + x + 4} \\ iii) & g(x) = e^x + x^2 - 4 \end{array}$$

For each of them, it is possible to write a fixed point iteration scheme, $x_{n+1} = g(x_n)$.

Test the three schemes by using the function `fixpoint` from the Jupyter note. Use e.g. $x_0 = 1.5$ as starting value, but you may very well experiment with others. Which of the iteration schemes converges?

c) (H) Verify the numerical results above by the use of the fixed point theorem.

4 **a)** (H) Let $g(x)$ be a continuous function with continuous derivatives on (a, b) and suppose that it has an inverse $g^{-1}(x)$. Show that if $r \in (a, b)$ is a fixed-point of $g(x)$, then r is also a fixed-point of $g^{-1}(x)$.

b) (H) Let $r \in (a, b)$ be a fixed-point of $g(x)$. Show that if $|g'(r)| > 1$, then $|(g^{-1})'(r)| < 1$.

Using the convergence theorem of fixed-point iteration, we now know that if the algorithm does not converge for $g(x)$, then it will converge for $g^{-1}(x)$, if one takes x_0 sufficiently closed to s.

c) (J) With the previous observations in mind, use fixed point iterations to find an approximation to the solution r of the equation $x = \arccos x$.

5 **a)** (H) Write down Newton's method applied to the system of equations

$$\begin{aligned} x^2 + y^2 &= 4 \\ xy &= 1, \end{aligned}$$

Do 2 iterations,, starting with

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

b) (J) Find an approximation to the solution by using the function `newton_sys`. You can also use this to confirm that your hand calculations were correct.

c) (J) Use `newton_sys` to solve the slightly perturbed problem:

$$\begin{aligned} x^2 + y^2 &= 2 \\ xy &= 1, \end{aligned}$$

using the same starting values as in point **a)**. How will the method behave now? Explain!