WORK SHEET 2

1] a) $\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = ?$, $\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C\cdot s + D}{s^2+1}$

$\Leftrightarrow s\cdot(s^2+1)\cdot A + (s^2+1)B + s^2(C\cdot s + D) = 1$

$\Leftrightarrow$ $\cancel{s^3 A} + \cancel{s\cdot A} + \cancel{s^2 B} + \cancel{B} + \cancel{s^3 C} + \cancel{s^2 D} = 1$

$\Leftrightarrow \underbrace{(A+C)}_{=0}s^3 + \underbrace{(B+D)}_{=0}\cdot s^2 + \underbrace{A}_{=0}\cdot s + \underbrace{B}_{=1} = 1$, $B = 1$, $B + D = 0 \Leftrightarrow D = -1$

$A = 0$, $A + C = 0 \Leftrightarrow C = 0$

gir oss

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \underline{\underline{t - \sin(t)}}$$

1] b)

$\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+1}\right) = ?$, $s^2 + 2s + 1 = (s+1)^2$

$$\frac{s}{(s+1)^2} = \frac{s+1-1}{(s+1)^2} = \frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+2s+1}\right) = \underbrace{\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)}_{=e^{-t}} - \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$\mathcal{L}(t) = \frac{1}{s^2} = \mathcal{L}(f(t)) = F(s)$, $F(s-a) = \mathcal{L}(e^{at} f(t))$

ønsker $F(s+1) = \frac{1}{(s+1)^2}$, $F(s+1) = F(s-(-1)) = \mathcal{L}(e^{-t}\cdot t)$

$= e^{-t} - e^{-t}\cdot t = \underline{\underline{-e^{-t}(t-1)}}$

WORK SHEET 2

1] c) $\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = ?$ $\quad \mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$, $\mathcal{L}(\cos(t)) = \frac{s}{s^2+1}$

setter $F(s) = \frac{1}{s^2+1}$ og $G(s) = \frac{s}{s^2+1}$ svar: $t \cdot \sin(t)$

$$\frac{2s}{(s^2+1)^2} = 2 \cdot F(s) \cdot G(s)$$, vi har da at

$$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = 2 \cdot \mathcal{L}^{-1}(F \cdot G) = 2 \cdot \int_0^t f(\tau) \cdot g(t-\tau)\, d\tau$$

$$= 2 \cdot \int_0^t \sin(\tau) \cdot \cos(t-\tau)\, d\tau = 2 \cdot \int_0^t \sin(\tau) \cdot (\cos(t) \cdot \cos(\tau) + \sin(t) \cdot \sin(\tau))\, dt$$

$$= \text{- - - - - - - -} = \underline{\underline{t \cdot \sin(t)}}$$

WORK SHEET 2

1] d) $\mathcal{L}^{-1}((s-3)^{-5}) = ?$ $\quad (s-3)^{-5} = \dfrac{1}{(s-3)^5} \quad g(t) = t^4$

$$\mathcal{L}(t^n) = \mathcal{L}(f(t)) = \frac{n!}{s^{n+1}} \quad , \quad \mathcal{L}(t^4) = \mathcal{L}(g(t)) = \frac{4!}{s^5} = G(s)$$

$$\mathcal{L}(e^{3t} g(t)) = G(s-3) = \frac{4!}{(s-3)^5}$$

$$\Leftrightarrow \quad \mathcal{L}(e^{3t} g(t)) = \mathcal{L}(e^{3t} t^4) = \frac{4!}{(s-3)^5}$$

$$\underline{\underline{\mathcal{L}^{-1}\left(\frac{1}{(s-3)^5}\right) = \frac{1}{4!} e^{3t} \cdot t^4}} \qquad u(t) = g(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$$

2] a) $\mathcal{L}(f(t)) = \mathcal{L}((u(t) - u(t-\pi)) \cos(t)) = ?$, $u(t) = 0,\ t<0$

$t<0 \quad : \mathcal{L}((0-0)\cdot\cos(t)) \underline{= 0}$ $\qquad u(t) = 1,\ t \geq 0$

$0 \leq t < \pi : \mathcal{L}((1-0)\cdot\cos(t)) = \underline{\dfrac{s}{s^2+1}}$ $\qquad u(t-\pi) = 0,\ t<\pi$

$\qquad u(t-\pi) = 1,\ t \geq \pi$

$t \geq \pi : \mathcal{L}((1-1)\cdot\cos(t)) = 0$

$$\underline{\underline{\mathcal{L}(f(t)) = \begin{cases} \dfrac{s}{s^2+1} & \text{for } 0 \leq t < \pi \\ 0 & \text{ellers} \end{cases}}}$$

b)

WORK SHEET 1

2] $\mathcal{L}(f(t)) = \mathcal{L}(u(t-3)\, t^4) = ?$ $\qquad u(t-3) = 0$ for $t < 3$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u(t-3) = 1$ for $t > 3$

$t < 3 : \mathcal{L}(f(t)) = 0$

$t > 3 : \mathcal{L}(f(t)) = \mathcal{L}(t^4) = \dfrac{4!}{s^5}$

$$\underline{\underline{\mathcal{L}(f(t)) = \begin{cases} 0 & \text{for } t < 3 \\ \dfrac{4!}{s^5} & \text{for } t \geq 3 \end{cases}}}$$

3] $\quad y'' + 2y = \delta(t-1), \quad y(0) = y'(0) = 0$

$\mathcal{L}(y'') + 2 \cdot \mathcal{L}(y) = \mathcal{L}(\delta(t-1))$

$\Leftrightarrow s^2 Y(s) - s \cdot \underbrace{y(0)}_{=0} - \underbrace{y(0)}_{=0} + 2Y(s) = e^{-s \cdot 1}$

$Y(s) = \dfrac{e^{-s}}{s^2+2} = e^{-s} \cdot \dfrac{1}{2} \cdot \dfrac{2}{s^2+2}$

$\mathcal{L}^{-1}(Y(s)) = \dfrac{1}{\sqrt{2}} \mathcal{L}^{-1}\left(e^{-s} \cdot \dfrac{\sqrt{2}}{s^2+2}\right), \qquad F(s) = \mathcal{L}(f(t)) = \mathcal{L}(\sin(\sqrt{2}t))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \dfrac{\sqrt{2}}{s^2+2}$

$\mathcal{L}\left(\frac{1}{\sqrt{2}} u(t-1) f(t-1)\right) = e^{-s} \cdot \dfrac{\sqrt{2}}{s^2+2}$

$\sqrt{2}$

$\Leftrightarrow \mathcal{L}^{-1}(Y(s)) = \dfrac{1}{\sqrt{2}} \cdot u(t-1) \cdot f(t-1) = \underline{\underline{\dfrac{1}{\sqrt{2}} \cdot u(t-1) \cdot \sin(\sqrt{2}t - 1)}}$

WORK SHEET 2

4] a) $f(t+T) = f(t) \Rightarrow$ periode, $T > 0$

vil vise at $\int_{nT}^{(n+1)T} e^{-st} f(t)\,dt = e^{-snT} \int_0^T e^{-st} f(t)\,dt$

$$\int_{nT}^{(n+1)T} e^{-st} f(t)\,dt, \qquad t = \tau - nT$$

$$= \int_0^T e^{-s(\tau + nT)} \underbrace{f(\tau + nT)}_{= f(\tau) \text{ pga periodisk}} d\tau$$

$$= \int_0^T e^{-snT} e^{-s\tau} f(\tau)\,d\tau$$

$$= e^{-snT} \int_0^T e^{-s\tau} f(\tau)\,d\tau = \underline{\underline{e^{-snT} \int_0^T e^{-st} f(t)\,dt}}$$

som vi skulle vise

WORK SHEET 2

4) $f(t+T) = f(t)$, $T > 0$

b) vil vise at $F(s) = \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)\,dt$

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt = \mathcal{L}(f(t))$$

$$\int_0^{\infty} e^{-st} f(t)\,dt = \sum_{n=0}^{\infty} \underbrace{\int_{nT}^{(n+1)T} e^{-st} f(t)\,dt}$$

lar $\tau = t - nT$

$$= \sum_{n=0}^{\infty} \int_0^T e^{-s(\tau + nT)} \underbrace{f(\tau + nT)}_{= f(\tau) \text{ pga. periodisk}}\, d\tau$$

$$= \sum_{n=0}^{\infty} \int_0^T e^{-snT} e^{-s\tau} f(\tau)\,d\tau$$

$$= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-s\tau} f(\tau)\,d\tau = \underline{\underline{\sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)\,dt}}$$

som vi skulle vise

WORK SHEET 2

4] c) $f(t+T) = f(t)$, $T>0$, $s>0$ ← periodisk

skal vise at $F(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$, $s>0$

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} g(t)\,dt$$

La $t = T + \tau$, som vil si at $t - T = \tau$

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(T+\tau)} \underbrace{f(T+\tau)}_{=f(\tau)} d\tau$$

$= f(\tau)$ pga. periodisk

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \underbrace{\int_0^\infty e^{-s\tau} f(\tau)\,d\tau}_{\mathcal{L}(f(t))}$$

, T kun en konstant

$$\mathcal{L}(f(t)) = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\mathcal{L}(f(t))$$

$$\Leftrightarrow \mathcal{L}(f(t)) - e^{-sT}\mathcal{L}(f(t)) = \int_0^T e^{-st} f(t)\,dt$$

$$\Leftrightarrow \mathcal{L}(f(t)) \cdot (1 - e^{-sT}) = \int_0^T e^{-st} f(t)\,dt$$

$$\underset{F(s)}{\underset{\parallel}{\mathcal{L}(f(t))}} = \underline{\underline{\frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt}}$$

som vi skulle vise