Kalkulus Oblig 2. Alexander. Gilstedt

1

a) $f(x) = 5x^6 + 2x^3 - 3x$

$f'(x) = 5 \cdot 6\, x^{6-1} + 2 \cdot 3\, x^{3-2} - 3\, x^{1-1}$

$\underline{f'(x) = 30x^5 + 6x^2 - 3}$

b) $f(x) = \frac{x^7}{7} + \frac{x^4}{2} - \frac{x}{3} + \frac{7}{5}$

$f'(x) = 7 \cdot \frac{x^6}{7} + 4\, \frac{x^3}{2} - 1 \cdot \frac{1}{3} + 0$

$f'(x) = \underline{\underline{x^6 + 2x^3 - \frac{1}{3}}}$

c) $f(x) = (x^2 - 1) \cdot (x + 6)$

$f'(x) = (x^2-1)' \cdot (x+6) + (x^2-1) \cdot (x+6)'$

$f'(x) = 2x \cdot (x+6) + (x^2-1) \cdot 1$

$f'(x) = 2x^2 + 12x + x^2 - 1$

$f'(x) = \underline{3x^2 + 12x - 1}$

d) $f(x) = \frac{2}{x^3} + \frac{3}{x^2} - \frac{6}{x}$

$f(x) = 2x^{-3} + 3x^{-2} - 6x^{-1}$

$f'(x) = \underline{\underline{-6x^{-4} - 6x^{-3} + 6x^{-2}}}$

e) $f(x) = (-4x^3 + 5x^2 + 3)^7$

$f(x) = u^7 \quad , \quad u = -4x^3 + 5x^2 + 3$

$f'(x) = f'(u) \cdot u'$

$f'(x) = 7u^6 \cdot -12x^2 + 10x$

$f'(x) = \underline{\underline{7 \cdot (-4x^3 + 5x^2 + 3)^6 \cdot (-12x^2 + 10x)}}$

2 $y = f(x) = 4x^2 + 5x^3 - 2$

$f(-1) = 4 \cdot (-1)^2 + 5 \cdot (-1)^3 - 2$

$f(-1) = 4 \cdot 1 + 5 \cdot -1 - 2$

$f(-1) = 4 - 5 - 2 = \underline{-3}$

$f(-1) = -3 \Leftrightarrow P(-1,-3)$ må være et punkt på grafen.

Likningen til tangenten gjennom punktet $P(a, f(a))$ er gitt ved:

$$y = f'(a) \cdot (x-a) + f(a)$$

$$f'(x) = 16x^3 + 15x^2 \quad , \quad P(-1,-3)$$

$$f'(-1) = 16 \cdot (-1)^3 + 15 \cdot (-1)^2 = -16 + 15 = \underline{-1}$$

$$y = -1 \cdot (x - (-1)) + (-3)$$

$$y = -1(x+1) - 3$$

$$y = -x - 1 - 3$$

$$y = \underline{\underline{-x-4}}$$

3

a) $f(x) = (2x^3 - 4x^2)^5$

$$f(x) = u^5 \quad , \quad u = 2x^3 - 4x^2$$

$$f'(x) = (u^5)' \cdot u' \quad , \quad u' = 6x^2 - 8x$$

$$f'(x) = 5u^4 \cdot u'$$

$$f'(x) = 5(2x^3 - 4x^2)^4 \cdot (6x^2 - 8x)$$

$$f'(x) = 5 \cdot 2(2x^3 - 4x^2)^4 \cdot (3x^2 - 4x)$$

$$f'(x) = \underline{\underline{10(2x^3 - 4x^2)^4 \cdot (3x^2 - 4x)}}$$

b) $f(x) = \left(\frac{2x-3}{2x-1}\right)^3$

$$f(x) = \left(\frac{u}{v}\right)^3 \quad , \quad u = 2x-3 \ \wedge \ v = 2x-1$$

$$f'(x) = \left(\frac{u}{v}\right)^{3\,\prime} \cdot \frac{u}{v}'$$

$$\frac{u}{v}' = \left(\frac{2x-3}{2x-1}\right)' = \frac{(2x-3)' \cdot (2x-1) - (2x-3) \cdot (2x-1)'}{(2x-1)^2}$$

$$= \frac{2 \cdot (2x-1) - (2x-3) \cdot 2}{\;} = \frac{4x-2-(4x-6)}{4x^2-4x+1}$$

$$= \frac{4}{(2x-1)^2}$$

$$f'(x) = 3\left(\frac{2x-3}{2x-1}\right)^2 \cdot \frac{4}{(2x-1)^2}$$

$$f'(x) = 3 \cdot \frac{(2x-3)^2 \cdot 4}{(2x-1)^{2+2}} = \underline{\underline{\frac{12(2x-3)^2}{(2x-1)^4}}}$$

c) $f(x) = \frac{1}{\cos x}$

$f(x) = \frac{1}{u}$

$f'(x) = \frac{0 \cdot \cos x - 1 \cdot -\sin x}{(\cos x)^2}$

$f'(x) = \frac{\sin x}{\cos^2 x}$

d) $f(x) = \frac{\cos^2(x)}{\sin(x)}$

$f'(x) = \frac{\cos^2 x' \cdot \sin x - \cos^2 x \cdot \sin x'}{(\sin x)^2}$

$f'(x) = \frac{2\cos x \cdot \overset{\sin x}{\vee} -\sin x - \cos^2 x \cdot \cos x}{\sin^2 x}$

$f'(x) = \frac{-2\cos x \cdot \sin^2 x - \cos^3 x}{\sin^2 x}$

$f'(x) = -\frac{\cos^2 x}{\sin^2 x} - 2\cos x \quad (\sin^2 x + \cos^2 x = 1)$

$f'(x) = -\frac{\sin^2 x + 1}{\sin^2 x} - 2\cos x = -2\cos x - \frac{\cos x}{\sin^2 x} =$

$-\cos x\left(1 + \frac{1}{\sin^2 x}\right)$

c) $f(x) = \frac{\sqrt{\cos(3x^2)} \cdot \sin^3(4x)}{e^x}$

$$f(x) = \frac{\cos(3x^2)^{\frac{1}{2}} \cdot \sin^3(4x)}{e^x}$$

$$\ln|f(x)| = \ln|\cos(3x^2)^{\frac{1}{2}} \cdot \sin^3(4x) \cdot e^{x^{-1}}|$$

$$\ln|f(x)| = \ln(\cos(3x^2)^{\frac{1}{2}} + \ln|\sin^3(4x)| + \ln e^{x^{-1}}|$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{1}{2} \cdot \frac{1}{\cos(3x^2)} \cdot \cos(3x^2)' + \frac{1}{\sin^3(4x)} (\sin^3 4x)' + (-1)|$$

$$\frac{f'(x)}{f(x)} = \frac{1}{2\cos(3x^2)} \cdot -\sin(3x^2) \cdot 6x + \frac{1}{\sin^3(4x)} \cdot 3\sin(4x)^2 \cdot \cos 4x \cdot 4 (-1)$$

$$\frac{f'(x)}{f(x)} = \frac{-6x\sin(3x^2)}{2\cos(3x^2)} + \frac{12\cos(4x) \cdot \sin^2(4x)}{\sin^3(4x)} - 1$$

Forkorter:

$$\underline{\underline{f'(x) = \left(\frac{-3x\sin(3x^2)}{\cos(3x^2)} + \frac{12\cos(4x)}{\sin(4x)} - 1\right) \cdot f(x)}}$$

f) $f(x) = x^4 \cdot \cos^{-1}(x)$

$f'(x) = x^{4'} \cdot \cos^{-1}(x) + x^4 \cdot \cos^{-1}(x)'$

$x^{4'} = 4x^3$

$\cos^{-1}(x)' = \dfrac{-1}{\sqrt{1-x^2}}$

$4x^3 \cdot \cos^{-1}(x) + x^4 \cdot -\dfrac{1}{\sqrt{1-x^2}}$

$$\underline{\underline{f'(x) = 4x^3 \cdot \cos^{-1}(x) + \dfrac{4}{\sqrt{1-x^2}}}}$$

4

$r = (5.0 + 0{,}2)$

$V = \dfrac{4\pi}{3} \cdot r^3 = f(r)$

Ökningen kan uttrykkes som $f(\overset{x=a+h}{5{,}0+0{,}2}) - f(\overset{a}{5{,}0})$

$f((5{,}0+0{,}2) - f(5{,}0) \approx f'(5{,}0) \cdot 0{,}2$

$f'(r) = 4\pi r^2$

$f'(5{,}0) = 4\pi \cdot 5^2 = 100\pi$

Differansen er da : $100\pi \cdot 0.2 = 20\pi$

<u>Differansen er $20\pi\ cm^3 \approx 62.8\ cm^3 = 63\ cm^3$</u>

5 $f(x)=\sqrt{x}$

- Søker $f(35)$, vet at $f(36)=\underline{6}$.

$$f(x)\approx f(a)+f'(a)(\overset{35}{x}-\overset{36}{a})$$

$$f(35)=f(36)+f'(36)(35-36)$$

$$f'(x)=(x^{\frac{1}{2}})'=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2}\cdot\frac{1}{x^{\frac{1}{2}}}=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}=\underline{\frac{1}{2\sqrt{x}}}$$

$$f'(36)=\frac{1}{2}\cdot\frac{1}{6}=\frac{1}{12}$$

Gir da : $6+\frac{1}{12}\cdot -1=6-\frac{1}{12}\approx\underline{5,92}$

6

$$f(x)=\frac{(x-2)^4(x+3)^{\frac{2}{3}}e^{x^2-x}}{(x-1)^2}$$

$$\ln|f(x)|=\ln\left|\frac{(x-2)^4(x+3)^{\frac{2}{3}}e^{x^2-x}}{(x-1)^2}\right|$$

$$\ln|fx|=\ln|(x-2)^4(x+3)^{\frac{2}{3}}(e^{x^2-x})\cdot(x-1)^{-2}|$$

$$\ln|fx|=\ln(x-2)^4+\ln(x+3)^{\frac{2}{3}}+\ln e^{x^2-x}+\ln(x-1)^{-2}$$

$$\ln|f(x)=4\ln(x-2)+\frac{2}{3}\ln(x+3+\ln e^{x^2-x}-2\ln(x-1)$$

$$(\ln x'=\frac{1}{x})$$

$$\frac{1}{f(x)}\cdot f'(x)=4\cdot\frac{1}{(x-2)}\cdot(x-2)'+\frac{2}{3}\cdot\frac{1}{(x+3)}\cdot(x+3)'+(x^2-x)'-2\frac{1}{x-1}\cdot x-1'$$

$f'(x)$ $\quad 4 \quad\quad 2 \quad\quad\quad\quad 2$

$f(x) = \frac{}{x-2} \cdot 1 + \frac{}{3(x+3)} \cdot 1 + 2x - 1 - \frac{}{(x-1)} \cdot 1$

$$\underline{\underline{f'(x) = \left(\frac{4}{(x-2)} + \frac{2}{3(x+3)} + 2x - 1 - \frac{2}{(x-1)}\right) \cdot f(x)}}$$

7

$f(x) = x^3 \sin x$

finner først den deriverte:

$f'(x) = 3x^2 \cdot \sin x + x^3 \cdot \cos x = x^2(3\sin x + x \cdot \cos x)$

$f''(x) = (6x \cdot \sin x + 3x^2 \cdot \cos x) + (3x^2 \cdot \cos x + x^3 \cdot -\sin x)$

$f''(x) = (6x \cdot \sin x + 3x^2 \cdot \cos x) + (3x^2 \cdot \cos x - x^3 \sin x)$

$f'''(x) = (6 \cdot \sin x + 6x \cdot \cos x) + (6x \cdot \cos x + 3x^2 \cdot -\sin x)$

Trekker sumen $: (6\sin x + 12\cos x - 3x^2 \sin x)$

Parentes 2:

$(6x \cdot \cos x + 3x^2 - \sin x) + (-3x^2 \cdot \sin x + x^3 \cdot \cos x)$

$6x\cos x - 3x^2 \sin x + -3x^2 \sin x - x^3 \cos x$

$\underline{(x(6\cos x - x^2 \cos x) - 6x^2 \sin x)}$

$(6\sin x + 12x\cos x - 3x^2 \sin x) + (6x\cos x - x^3 \cos x - 6x^2 \sin x$

$\underline{6\sin x - x^3 \cos x + 18\cos x - 9x^2 \sin x}$

$= \underline{\underline{3(2 - 3x^2)\sin x + x(18 - x^2)\cos x}}$

8 $x^2y^2 + 4xy^2 - 5x = 15 \quad , \quad P(1,2)$

a) Setter inn $x=1$, og $y=2$.

$1^2 \cdot 2^2 + 4 \cdot 1 \cdot 2^2 - 5 \cdot 1 = 15$

$1 \cdot 4 + 4 \cdot 4 - 5 = \underline{15}$

$4 + 16 - 5 = 15$

$\underline{15 = 15}$

<u>HS er lik VS, og punktet må ligge på kurven</u>

b) (Implisitt derivasjon vil gi uttrykket for stigningstallet (når vi setter inn (x,y))

$x^2y^2 + 4xy^2 - 5x = 15$

$2x \cdot y^2 + \underline{x^2 \cdot 2y \cdot y'} + 4 \cdot y^2 + 4y^2 + \underline{4x \cdot 2y \cdot y'} - 5 = 0$

$x^2 2y\, y' + 4x\, 2y \cdot y' = -2x \cdot y^2 - 4y^2 + 5$

$y'(x^2 \cdot 2y + 4x \cdot 2y) = y^2(-2x-4) + 5$

$y' = \dfrac{y^2(-2x-4)+5}{x^2 \cdot 2y + 4x \cdot 2y}$

Setter inn $(1,2)$ i uttrykket for y'

$$a = \frac{y^2(-2x-4)+5}{2y(x^2+4x)} = \frac{2^2(-2\cdot1-4)+5}{2\cdot2(1^2+4\cdot1)} = \frac{4\cdot(-6)+5}{4\cdot5}$$

$$= -\frac{19}{20}$$

Bruker deretter ettpunktsformelen

$$y - y_0 = a(x - x_0)$$

$$y - 2 = -\frac{19}{20}(x - 1)$$

$$y = -\frac{19}{20}x + \frac{19}{20} + 2$$

$$y = -\frac{19}{20}x + \frac{19}{20} + \frac{38}{20}$$

$$\underline{y = -\frac{19}{20}x + \frac{57}{20}}$$

<u>Linjen for tangenten er: $y = -\frac{19}{20}x + \frac{57}{20}$</u>

9

$f(x) = \tan x \;/\; y = \tan x$

$x = f^{-1}(y) \;/\; x = \tan^{-1} x \;/\; \tan y = x$

$f(x)^{-1} = g(y)$

Den deriverte av den inverse funksjonen kan skrives som $\frac{1}{f'(x)}$, (en delt på den deriverte av den opprinnelige funksjonen.

$$\frac{d\,\arctan}{dy} = \frac{1}{f'(x)} = \frac{1}{(\tan x)'} = \frac{1}{\tan^2 x + 1} =$$

$\tan^2 x + 1 = \sin^2 x$

$$\frac{1}{\sin^2 x}$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad / : \cos^2 x$$

$\tan^2 x + 1 = \frac{1}{\cos^2 x}$. $\tan^2 x + 1 = \sin^2 x$

Dette gir.

$$\frac{1}{\sin^2(y)} = \frac{1}{1 + (\tan y)} \;,\; \tan y = x$$

$$= \underline{\underline{\frac{1}{1 + x^2}}}$$

1

10

a) $\lim\limits_{x\to 0} \frac{\sin^2 x}{x} = \frac{(\sin x)^2}{x} \quad \frac{(\sin 0)^2}{0} \quad \frac{0^2}{0} = \frac{0}{0}$

Vi har et $\frac{0}{0}$ uttrykk

$\underset{=}{l'h} \lim\limits_{x\to 0} \frac{2\sin x \cdot \cos x}{1} = \frac{2\cdot \sin 0 \cdot \cos 0}{1} = \frac{0}{1} = \underline{\underline{0}}$

b) $\lim\limits_{x\to 10} \frac{\ln(x-9)}{x-10} ; \frac{\ln(10-9)}{10-10} = \frac{\ln 1}{0} = \frac{0}{0}$

Vi har et $\frac{0}{0}$ uttrykk

$\underset{=}{l'h} \lim\limits_{x\to 10} \frac{\frac{1}{x-9}}{1} = \frac{\frac{1}{10-9}}{1} = \frac{\frac{1}{1}}{1} = \frac{1}{1} = \underline{\underline{1}}$

c) $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{(e^x - 1)} \right) = \left(\frac{1}{0} - \frac{1}{e^0 - 1} \right) = (\infty - \infty)$

$\lim\limits_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \frac{0}{0} \overset{l'h}{=} \lim\limits_{x\to 0} \frac{e^x - 1}{1(e^x - 1) + x \cdot e^x}$

$\lim\limits_{x\to 0} \frac{e^0 - 1}{1 \cdot 0 + 0} = \frac{0}{0}$, Benytter regelen igjen:

$\lim\limits_{x\to 0} \frac{e^x}{e^x + 1 \cdot e^x + x \cdot e^x} = \frac{e^0}{e^0 + e^0 + 0} = \frac{1}{1+1} = \underline{\underline{\frac{1}{2}}}$

d) $\lim_{x\to 0} (1+3x)^{\frac{1}{x}} = (1+3\cdot 0)^{\frac{1}{0}} = 1^{\infty}$

Har et ubestemt uttrykk på formen 1^{∞}

$\lim_{x\to 0} \ln(1+3x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{1}{x} \ln(1+3x) = \infty\cdot 0$

e) $\lim_{x\to 0} (\ln x)^{\frac{1}{x}}$ $\quad \lim_{x\to 0} \frac{1}{x}\cdot \ln x = \frac{1}{0}\cdot \ln 0 = \underline{\infty\cdot\infty}$

Gjør om: $\frac{\frac{1}{x}}{(\ln x)^{-1}}$ $\quad \frac{\infty}{\infty}$ $\quad \underline{l'h}$

$\frac{-x^{-2}}{\left(\frac{1}{x}\right)^{-1}} = \frac{0}{\frac{1}{0}}$ — $\underline{\text{Ingen grenseverdi}}$

f) $\lim_{x\to 0} \frac{e^x - e^{-x} - 2\sin x}{x^3} \left(\frac{e^0 - (e^{-0}) - 2\sin 0}{0^3} = \frac{1-1-0}{0} = \frac{0}{0} \right)$

$\overset{l'h}{=} \lim_{x\to 0} \frac{e^x - (-e^{-x}) - 2\cdot\cos x}{3x^2} = \frac{1-(-1)-2\cdot 1}{0} = \frac{2-2}{0}$

Bruker l'hopitals regel på nytt:

$\overset{l'h}{=} \lim_{x\to 0} \frac{e^x - (e^{-x}) - 2\cdot -\sin x}{3x} = \frac{1-1\cdot 0}{0} = \frac{0}{0}$

Bruker l. hopitals på nytt:

$\overset{l'h}{=} \lim_{x\to 0} \frac{e^x - (-e^{-x}) - 2\cdot -(\cos x)}{6} = \frac{1+1+2\cdot 1}{6} = \frac{4}{6} = \underline{\underline{\frac{2}{3}}}$

11

$f(x) = \frac{x\ln(x^2) - 1}{\ln x}$ $\quad D_f = \langle 0, \to \rangle = \mathbb{R}^+$

Sjekker grense verdien når $x \to \pm\infty$

Skråasymptote: $y = kx + C$.

finner k

$\lim_{x\to\infty} \frac{f(x)}{x}, \lim_{x\to\infty} \frac{x\ln(x^2) - 1}{x\cdot\ln x} = \left(\frac{\infty \cdot \ln(\infty^2) \cdot 1}{\infty \cdot \ln\infty} \right) = \frac{\infty}{\infty}$

$\overset{l'h}{=} = \frac{1\cdot\ln(x^2) + x\cdot\frac{1}{x^2}\cdot 2x}{1\cdot\ln x + x\cdot\frac{1}{x}} = \frac{\ln(x^2) + 2}{\ln x + 1}$

$\overset{l'h}{} \lim_{x\to\infty} \frac{\infty + 3}{\infty + 1} = \frac{2}{1} = \underline{2}$ $\quad k = \underline{2}$

b finnes ved

$$\lim_{x \to \infty} (f(x) - hx) = \lim_{x \to \infty} \frac{x\ln(x^2) - 1}{\ln x} - 2x$$

$$\left(\frac{\infty}{\infty}\right) \overset{l'h}{=} \lim_{x \to \infty} \frac{1 \cdot \ln(x^2) + x \cdot \frac{1}{x^2} \cdot 2x}{\frac{1}{x}} - 2 = \frac{\ln(x^2) + 2 - 2}{\frac{1}{x}}$$

$$\lim_{x \to \infty} \frac{\ln(x^2)}{\frac{1}{x}} = \frac{\infty}{0}$$ - Ingen grenseverdi

b eksisterer ikke / er lik null.

<u>Vi får ikke en asymptote i $y = 2x$</u>