Oblig 8 kalkulus. Alexander. Gilstedt

1 $\int_0^{\pi} \sin x \, dx$

a) Rektangelmetoden der $n = 6$.

$\Delta x = \frac{b-a}{n}$, $\Delta x = \frac{\pi - 0}{6} = \underline{\frac{\pi}{6}}$

$x_0 = a = \underline{0}$

$x_1 = x_0 + \Delta x = \underline{\frac{\pi}{6}}$

kan du beregne $x_1^* = \frac{x_1 + x_0}{2} = \frac{\frac{\pi}{6}}{2} = \frac{\pi}{12}$

$x_2^* = x_1^* + \Delta x = \frac{\pi}{12} + \frac{\pi}{6} = \frac{\pi}{4}$

$x_3^* = x_2^* + \Delta x = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

$x_4^* = x_3^* + \Delta x = \frac{5\pi}{12} + \frac{\pi}{6} = \frac{7\pi}{12}$

$x_5^* = x_4^* + \Delta x = \frac{7\pi}{12} + \frac{\pi}{6} = \frac{\pi}{4}$

$x_6^* + x_5^* + \Delta x = \frac{\pi}{4} + \frac{\pi}{6} = \frac{11\pi}{12}$

$$\int_0^{\pi} \sin x \approx \frac{\pi}{6} \left(\sin\frac{\pi}{12} + \sin\frac{\pi}{4} + \sin\frac{5\pi}{12} + \sin\frac{7\pi}{12} + \sin\frac{\pi}{4} + \sin\frac{11\pi}{12} \right)$$

$\approx \underline{\underline{2.02303}}$

b) $\int_0^{\pi} \sin x \, dx$

Trapesmetoden med $n=6$.

$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{6} = \underline{\frac{\pi}{6}}$

$x_0 = a = \underline{0}$

$x_1 = 0 + \Delta x = \underline{\frac{\pi}{6}}$

$x_2 = \frac{\pi}{6} + \Delta x = \underline{\frac{\pi}{3}}$

$x_3 = \frac{\pi}{3} + \Delta x = \frac{\pi}{2}$

$x_4 = \frac{\pi}{2} + \Delta x = \underline{\frac{2\pi}{3}}$

$x_5 = \frac{2\pi}{3} + \Delta x = \frac{5\pi}{6}$

$x_6 = \frac{5\pi}{6} + \Delta x = \underline{\pi}$

$$\int_0^{\pi} \sin x \, dx \approx \frac{\frac{\pi}{6}}{2}\left(\sin 0 + 2 \cdot \sin\left(\frac{\pi}{6}\right) + 2 \cdot \sin\left(\frac{\pi}{3}\right) + 2 \cdot \sin\left(\frac{\pi}{2}\right) + \right.$$

$$\left. \ldots 2 \cdot \sin\left(\frac{2\pi}{3}\right) + 2\sin\left(\frac{5\pi}{6}\right) + \sin(\pi)\right)$$

$\approx \underline{\underline{1.954097233}}$

c) $\int_0^{\pi} \sin x \, dx$

Simpsons metode med $n = 3$.

$\Delta x = \frac{\pi - 0}{2n} = \frac{\pi}{6}$

$x_0 = 0$

$x_1 = x_0 + \Delta x = \frac{\pi}{6}$

$x_2 = x_1 + \Delta x = \frac{\pi}{3}$

$x_3 = x_2 + \Delta x = \frac{\pi}{2}$

$x_4 = \frac{2\pi}{3}$

$x_5 = \frac{5\pi}{6}$

$x_6 = \pi$

$$\int_0^{\pi} \sin x \, dx = \frac{\frac{\pi}{6}}{3}\left(\sin 0 + 4\sin\left(\frac{\pi}{6}\right) + 2\sin\left(\frac{\pi}{3}\right) + 4\sin\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{2\pi}{3}\right) + 4\sin\left(\frac{5\pi}{6}\right) + \sin\pi\right)$$

$$\approx \underline{\underline{2.00086319}}$$

d) $\int_0^{\pi} \sin x \, dx = \left[-\cos x\right]_0^{\pi}$

$(-\cos \pi) - (-\cos 0)$

$= -1 + 1 = 1 + 1 = \underline{2}$

Vi ser at simpsons metode gir det klart beste svaret.

Rektangel-metoden gir den nest beste verdien.

2

a) $\int_0^3 \frac{t}{\sqrt{9-t^2}} \, dt$ $\qquad x^4$

Integranden gir mot uendelig når $x \to 3$.

$$\int_0^3 \frac{t}{\sqrt{9-t^2}} \, dt = \lim_{k \to 3} \int_0^k \frac{t}{\sqrt{9-t^2}} \, dt$$

$u = 9 - t^2 \qquad \frac{du}{dt} = -2t \qquad dt = \underline{-\frac{1}{2t} \, du}$

$$= \int_0^k \frac{\cancel{t}}{\sqrt{u}} \cdot -\frac{1}{2\cancel{t}} \, du = -\frac{1}{2} \int_0^k \frac{1}{\sqrt{u}} \, du$$

$$= -\frac{1}{2} \int_0^t u^{-\frac{1}{2}} = -\frac{1}{2} \left[2 u^{\frac{1}{2}}\right]_{t=0}^{t=k}$$

$$= -\left(\sqrt{9-k^2} - \sqrt{9-0^2}\right) = \underline{-\sqrt{9-k^2} + 3}$$

$$\lim_{k \to 3} -\sqrt{9-3^2} + 3 = 0 + 3 = \underline{\underline{3}}$$

c) $\int_0^1 x^2 \cdot \ln x \, dx$

Integranden går mot uendelig når $x \to 0$.

$$\int_0^1 x^2 \cdot \ln x \, dx = \lim_{t \to 0} \int_t^1 x^2 \cdot \ln x \, dx$$

$V = \ln x \quad V' = \frac{1}{x}$

$U' = x^2 \quad U = \frac{1}{3}x^3$

$$\int x^2 \cdot \ln x = \ln x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx$$

$$= \ln x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^2$$

$$= \ln x \cdot \frac{1}{3}x^3 - \frac{1}{9}x^3 = \frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right) (+C)$$

$$\int_0^1 x^2 \ln x \, dx = \lim_{t \to 0} \left[\frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right)\right]_t^1$$

$$\lim_{t \to 0} \left(\frac{1}{3} \cdot 1^3\left(\overset{0}{\cancel{\ln 1}} - \frac{1}{3}\right) - \left(\frac{1}{3} \cdot t^2\left(\ln t - \frac{1}{3}\right)\right)\right)$$

$$= \frac{1}{3}\left(-\frac{1}{3}\right) - 0 = -\frac{1}{9}$$

$$\int_0^1 x^2 \ln x \, dx = \underline{\underline{-\frac{1}{9}}}$$

c) $\int_0^2 \frac{1}{x^2-1}\,dx$

Her "svikter" divergenten for $x=1$ som jo ligger i intervallet. Deler om i to integraler

$$\int_0^2 \frac{1}{x^2-1}\,dx = \underset{1}{\underset{\uparrow}{\int_0^1 \frac{1}{x^2-1}\,dx}} + \underset{2}{\underset{\uparrow}{\int_1^2 \frac{1}{x^2-1}\,dx}}$$

Undersøke første integral:

$$\int_0^1 \frac{1}{x^2-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x^2-1}\,dx$$

$$\int \frac{1}{(x-1)(x+1)}\,dx = \int \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{(x-1)}\,dx + \frac{1}{2}\int \frac{1}{(x+1)}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| \; (+C)$$

finner grense verdien:

$$\lim_{t\to 1}\left[\frac{\ln|x-1| - \ln|x+1|}{2}\right]_0^t$$

$$= \left(\frac{\ln|t-1| - \ln|t+1|}{2}\right) - \left(\frac{\ln|0-1| - \ln|0+1|}{2}\right)$$

Ser umiddelbart at grenseverdien ikke eksisterer. DVS integralet divergerer.

Vi trenger da ikke å sjekke det andre og kan konkludere med at det opprinnelige divergerer

d) $\int_0^{\infty} 5^{-x} \, dx$

$$\int_a^{\infty} f(x)\, dx = \lim_{t \to \infty} \int_a^{t} f(x)\, dx$$

$$\int_0^{\infty} 5^{-x}\, dx = \lim_{t \to \infty} \int_0^{t} 5^{-x}\, dx$$

$$u = -x \qquad \frac{du}{dx} = -1 \qquad dx = -du$$

$$\int_{x=0}^{t=0} 5^{u} \, -du = -\int_{x=0}^{t=0} 5^{u}\, du = -\left[\frac{5^{u}}{\ln 5}\right]_{t=0}^{x=t}$$

$$= -\left[\frac{5^{-x}}{\ln 5}\right]_0^t = -\left[\frac{1}{\ln 5 \cdot 5^{x}}\right]_0^t$$

$$\lim_{t \to \infty} -\left(\frac{1}{\ln 5 \cdot 5^{t}} - \left(\frac{1}{\ln 5 \cdot 5^{0}}\right)\right)$$

$$\left(\frac{1}{\infty} \approx \underline{0}\right) \quad = -\left(0 - \frac{1}{\ln 5}\right) = \underline{\frac{1}{\ln 5}}$$

$$\int_0^{\infty} 5^{-x}\, dx = \underline{\underline{\frac{1}{\ln 5}}}$$

e) $\int_{-\infty}^{0} \frac{1}{1-x}\, dx$

$\int_{-\infty}^{b} f(x)\, dx = \lim_{t \to -\infty} \int_{t}^{b} f(x)\, dx$

$\int_{-\infty}^{0} \frac{1}{1-x}\, dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1-x}\, dx$

$u = 1-x \qquad \frac{du}{dx} = -1 \qquad dx = \underline{-du}$

$\int_{x=t}^{x=0} \frac{1}{u} \cdot -du = -\int_{x=t}^{x=0} \frac{1}{u}\, du = -\left[\ln|1-x|\right]_{t}^{0}$

$\lim_{t \to -\infty} \left[-\ln|1-x|\right]_{t}^{0} = \lim_{t \to -\infty} -\left(\ln 1 - \ln|1-t|\right)$

$\lim_{t \to -\infty} 0 - \ln|1+\infty|$ - Ser at denne grenseverdien ikke eksisterer.

Grenseverdien eksisterer ikke, og det uegentlige integralet $\underline{\underline{\text{divergerer}}}$.

f) $\int_{-\infty}^{\infty} x \cdot e^{-x^2} \, dx$

$$\int_{-\infty}^{\infty} x \cdot e^{-x^2} dx = \underbrace{\int_{-\infty}^{0} x \cdot e^{-x^2} dx}_{1} + \underbrace{\int_{0}^{\infty} x \cdot e^{-x^2}}_{2}$$

1) $\int_{-\infty}^{0} x \cdot e^{-x^2} = \lim_{t \to -\infty} \int_{t}^{0} x \cdot e^{-x^2} dx$

$u = -x^2 \quad \frac{du}{dx} = -2x \quad dx = \frac{1}{-2x} du$

$$\int_{x=t}^{x=0} \cancel{x} \cdot e^{u} \frac{1}{-2\cancel{x}} du = -\frac{1}{2} \int_{x=t}^{x=0} e^{u} du =$$

$$-\frac{1}{2}\left[e^{-x^2}\right]_{t}^{0} = -\frac{1}{2}\left(e^{0} - e^{-t^2}\right)$$

e

$$\lim_{t \to -\infty} -\frac{1}{2}\left(1 - e^{-t^2}\right) = -\frac{1}{2}(1-0) = \underline{\underline{-\frac{1}{2}}}$$

2) $\lim_{t \to \infty} \int_{0}^{t} x \cdot e^{-x^2} = -\frac{1}{2}\left[e^{-x^2}\right]_{0}^{t}$

$$-\frac{1}{2}\left(e^{-t^2} - e^{-0^2}\right) = -\frac{1}{2}(0-1) = -\frac{1}{2} \cdot -1 = \underline{\underline{\frac{1}{2}}}$$

$$\int_{-\infty}^{\infty} x \cdot e^{-x^2} dx = \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x \cdot e^{-x^2}$$

$$= -\frac{1}{2} + \frac{1}{2} \underline{\underline{= 0}}$$

3 a) $y' - 7y = 0$

kl: $\lambda - 7 = 0$, $\underline{\lambda = 7}$

$y = e^{\lambda x}$

Almen løsning blir da

$y = e^{7x} \cdot C = \underline{\underline{y = Ce^{7x}}}$

b) $y'' + y' - 6y = 0$

kl: $\lambda^2 + \lambda - 6 = 0$

Gjetter nå heltall $(\lambda - 2)(\lambda + 3)$

$\lambda = \underline{2} \vee \lambda = \underline{-3}$

det gir Almen løsning:

$y = e^{2x} \cdot C_1 + e^{-3x} \cdot C_2 = \underline{\underline{C_1 e^{2x} + C_2 e^{-3x}}}$

c) $y'' + 6y' + 9y = 0$

kl: $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2$. $\underline{\lambda = -3}$

Med støtte i teorem 9.3 er almen løsning

$\underline{\underline{y = C_1 e^{-3x} + C_2 x e^{-3x}}}$