Computational math: Assignment 2

Thanks Ting Gao's support for this HW solutions.

5.2 Using the SVD, prove that any matrix an $\mathbb{C}^{m\times n}$ is the limit of a sequence of matrices of full rank. In other words, prove that the set of full-rank matrices is a dense subset of $\mathbb{C}^{m\times n}$. Use the 2-norm for your proof.

Proof. Let the SVD of an arbitrary matrix $A_{m\times n}$ is

$$A = U\Sigma V^*$$

where $U_{m\times m}$ and $V_{n\times n}$ are unitary matrices.

Suppose that $m \geq n$.

Denote the singular values of A to be $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.
Construct a sequence of matrices $\{A_k\}_{k=1}^{\infty}$ as follows:

$$A_k = U\Sigma_k V^*$$

where

$$\Sigma_k = \begin{bmatrix} \sigma_1 + \frac{1}{k} & & & \\ & \sigma_2 + \frac{1}{k} & & \\ & & \ddots & \\ & & & \sigma_n + \frac{1}{k} \\ & & & \end{bmatrix}_{m\times n}.$$

It's obvious to see that for any $k \in \mathbb{N}$, we have

$$\text{rank}(A_k) = \text{rank}(U\Sigma_k V^*) = \text{rank}(\Sigma_k) = n.$$

Hence, $\{A_k\}_{k=1}^{\infty}$ are a set of full-rank matrices.
Since

$$\|A - A_k\|_2 = \|U(\Sigma - \Sigma_k)V^*\|_2 = \|\Sigma - \Sigma_k\|_2 = \sqrt{\rho[(\Sigma - \Sigma_k)^*(\Sigma - \Sigma_k)]} = \frac{1}{k}.$$

We have, $\|A - A_k\|_2 \to 0$ as $k \to 0$, which implies that the set of full-rank matrices is a dense subset of $\mathbb{C}^{m\times n}$. $\square$

5.3 Consider the matrix

$$A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}.$$

(a) Determine, on paper, a real SVD of A in the form $A = U\Sigma V^T$. The SVD is not unique, so find the one that has the minimal number of minus signs in U and V.
(b) List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.
(c) What are the $1-$, $2-$, $\infty-$ and Frobenius norms of A?
(d) Find A^{-1} not directly, but via the SVD?
(e) Find the eigenvalues of λ_1, λ_2 of A.
(f) Verify that $\det A = \lambda_1\lambda_2$ and $|\det A| = \sigma_1\sigma_2$.
(g) What is the area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$?

Solution:
(a)

$$A^*A = \begin{bmatrix} -2 & -10 \\ 11 & 5 \end{bmatrix} \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix} = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}.$$

Since $A^*A = V(\Sigma^*\Sigma)V^*$ and the eigenvalues of A^*A are 200, 50, the singular values of A are

$$\sigma_1 = 10\sqrt{2},\ \sigma_2 = 5\sqrt{2}.$$

To find U and V, we need to calculate the eigenvectors of AA^* and A^*A. Hence,

$$U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix},\ V = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

Therefore, the real SVD of A with minimal number of minus signs in U and V is

$$A = U\Sigma V^* = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

(b) The singular values of A are $\sigma_1 = 10\sqrt{2},\ \sigma_2 = 5\sqrt{2}$.
The left singular vectors of A are

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix},\ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

The right singular vectors of A are

$$\begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \quad \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}.$$

(c)
$\|A\|_1 = \max\limits_{1\leq j\leq 2} \sum\limits_{i=1}^{2} |a_{ij}| = \max\{12, 16\} = 16.$
$\|A\|_2 = \sqrt{\rho(A^*A)} = 10\sqrt{2}.$
$\|A\|_\infty = \max\limits_{1\leq i\leq 2} \sum\limits_{j=1}^{2} |a_{ij}| = \max\{13, 15\} = 15.$
$\|A\|_F = \sqrt{\mathrm{tr}(A^*A)} = \sqrt{250} = 5\sqrt{10}.$

(d) $A^{-1} = (U\Sigma V^*)^{-1} = V\Sigma^{-1}U^* = \begin{bmatrix} 0.05 & -0.11 \\ 0.1 & -0.02 \end{bmatrix}.$

(e) The eigenvalues of A are $\lambda_1 = \frac{3+\sqrt{391}i}{2}$ and $\lambda_2 = \frac{3-\sqrt{391}i}{2}$.

(f) To find λ_1 and λ_2, let's suppose that $\det(\lambda I - A) = 0$. That is to say,

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det A = 0.$$

Hence, it is easy to see that $\lambda_1\lambda_2 = \det A$.
For arbitrary unitary matrix U, we have $\det(UU^*) = \det U \cdot \det U^* = 1$, hence $\det U = \det U^* = \pm 1$. Therefore, we have

$$(\det A)^2 = \det A^* \cdot \det A = \det(A^*A) = \det(V\Sigma^*\Sigma V^*) = \det V \cdot \det(\Sigma^*\Sigma) \cdot \det V^*.$$

Thus,

$$|\det A| = \sqrt{\det(\Sigma^*\Sigma)} = \sqrt{\sigma_1^2\sigma_2^2} = \sigma_1\sigma_2.$$

(g)The area of the ellipsoid is

$$\pi\sigma_1\sigma_2 = \pi \cdot 10\sqrt{2} \cdot 5\sqrt{2} = 100\pi.$$

6.4 Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

What is the orthogonal projector P onto range(A), and what is the image under P of the vector $(1, 2, 3)^*$?

Solution:

The orthogonal projector P onto range(A) is

$$P = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Hence,

$$P \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

.

6.5 Let $P \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|P\|_2 \geq 1$, with equality if and only if P is an orthogonal projector.

Proof. Since P is a nonzero projector, we have $P = P^2$ and $\|P\|_2 \neq 0$. Then, based on Cauchy-Schwarz inequality, we have

$$\|P\|_2 = \|P^2\|_2 \leq \|P\|_2^2.$$

Hence, $\|P\|_2 \geq 1$.

If P is an orthogonal projector, then $P^* = P$. Suppose P has the SVD of the form $P = U\Sigma V^*$, where $UU^* = VV^* = I$.
Hence,

$$\| P \|_2 = \| P^2 \|_2 = \| PP^* \|_2 = \| \Sigma\Sigma^* \|_2 = \sigma_1^2,$$

where σ_1 is the largest singular value of Σ.
Since $\| P \|_2 = \| \Sigma \|_2 = \sigma_1 > 0$. We have $\sigma_1^2 = \sigma_1$. Therefore, $\sigma_1 = 1$. i.e., $\| P \|_2 = 1$.

Assume that the projector P is not orthogonal. i.e., $range(P)$ is not perpendicular to $range(I - P)$. Then, we can find a vector a such that $Pa \neq a$ and $a \perp range(I - P)$. Hence,

$$\|Pa\|_2 = \|a + (P - I)a\|_2 > \|a\|_2.$$

Therefore,

$$\|P\|_2 = \sup_{\|a\|_2=1} \|Pa\|_2 > \sup_{\|a\|_2=1} \|a\|_2 = 1.$$

□

7.1 Consider matrix

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Using any method you like, determine reduced and full QR factorizations $B = \hat{Q}\hat{R}$ and $B = QR$.

Solution: Rewrite B as $B = [b_1\ b_2]$, where

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\quad b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$

which are linearly independent. Hence,

$$q_1 = \frac{b_1}{\|b_1\|} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix},$$

$$q_2 = \frac{b_2 - (q_1^* b_2)q_1}{\|b_2 - (q_1^* b_2)q_1\|} = \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2}\begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}}{\|b_2 - (q_1^* b_2)q_1\|} = \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{3}} = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ -\sqrt{3}/3 \end{bmatrix}.$$

Hence,

$$B = \hat{Q}\hat{R} = [q_1\ q_2]\begin{bmatrix} \|b_1\| & q_1^* b_2 \\ 0 & \|b_2 - (q_1^* b_2)q_1\| \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{3}/3 \\ 0 & \sqrt{3}/3 \\ \sqrt{2}/2 & -\sqrt{3}/3 \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

When finding Q, we need to find another vector q_3 satisfying $q_1^* q_3 = 0$, $q_2^* q_3 = 0$ and $\|q_3\| = 1$. Hence, we have $q_3 = [\frac{\sqrt{6}}{6}\ \ -\frac{\sqrt{6}}{3}\ \ -\frac{\sqrt{6}}{6}]^T$. Therefore, the full QR factorization is

$$B = QR = \begin{bmatrix} \sqrt{2}/2 & \sqrt{3}/3 & \sqrt{6}/6 \\ 0 & \sqrt{3}/3 & -\sqrt{6}/3 \\ \sqrt{2}/2 & -\sqrt{3}/3 & -\sqrt{6}/6 \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}.$$

7.5 Let A be an $m \times n$ matrix ($m \geq n$), and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization.

(a) Show that A has rank n if and only if all the diagonal entries of $\hat{R}$ are nonzero.

(b) Suppose $\hat{R}$ has k nonzero diagonal entries for some k with $0 \leq k \leq n$. What does this imply about the rank of A?

Proof. Let's first denote that

$$A = \left[\begin{array}{ccc} a_1 & \cdots & a_n \end{array} \right], \quad \hat{Q} = \left[\begin{array}{ccc} q_1 & \cdots & q_n \end{array} \right], \quad \hat{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}.$$

Using the induction method to show the proof of (a) and (b), we only need to show that if $r_{kk} \neq 0$ then

$$rank(A_k) \geq rank(A_{k-1}) + 1,$$

where

$$A_k := (a_1 | \cdots | a_k).$$

Combing the formula

$$a_k = \sum_{j=1}^{k-1} r_{jk} q_j + r_{kk} q_k, \quad k = 1, \cdots, n,$$

we have

$$a_k \in span \{q_1, \cdots, q_{k-1}, q_k\},$$

but

$$a_1, \cdots, a_{k-1} \in span \{q_1, \cdots, q_{k-1}\}.$$

It implies that

$$a_k \notin span \{a_1, \cdots, a_{k-1}\}.$$

Thus

$$rank(A_k) \geq rank(A_{k-1}) + 1.$$

Therefore

$$rank(A) = n,$$

when all the diagonal entries of $\hat{R}$ are nonzero, and

$$rank(A) \geq k,$$

when $\hat{R}$ has k nonzero diagonal entries.

□

8 Prove $P_j = P_{\perp q_{j-1}} \cdots P_{\perp q_2} P_{\perp q_1}, \ \ j = 2, 3, \cdots, n.$

Proof. Since

$$P_j = I - \begin{bmatrix} q_1 & \cdots & q_{j-1} \end{bmatrix} \begin{bmatrix} q_1^* \\ \vdots \\ q_{j-1}^* \end{bmatrix} = I - \sum_{i=1}^{j-1} q_i q_i^*.$$

and

$$\prod_{i=1}^{j-1} P_{\perp q_i} = \prod_{i=1}^{j-1} (I - q_i q_i^*) = I - q_{j-1}q_{j-1}^* - \cdots - q_1 q_1^*$$

for $q_i \perp q_j (i \neq j)$, i.e., $q_i q_i^* q_j q_j^* = 0 (i \neq j)$(This is a zero matrix). Therefore,

$$P_j = P_{\perp q_{j-1}} \cdots P_{\perp q_2} P_{\perp q_1}.$$

□