Residuals

Schoenfeld Residuals

Assume p covariates and n independent observations of time, covariates, and censoring, which are represented as (t_i, x_i, c_i), where $i = 1, 2, ..., n$, and $c_i = 1$ for uncensored observations and zero otherwise. Schoenfeld residuals "are based on the individual contributions to the derivative of the log partial likelihood" (Hosmer and Lemeshow 1999, 198). To derive the Schoenfeld residuals, one takes the derivative for the kth covariate,

$$\begin{aligned} \frac{\partial L_p(\beta)}{\partial \beta_k} &= \sum_{i=1}^{n} c_i \left\{ x_{ik} - \frac{\sum_{j \in R(t_i)} x_{jk} e^{x'_j \beta}}{\sum_{j \in R(t_i)} e^{x'_j \beta}} \right\} \\ &= \sum_{i=1}^{n} c_i \left\{ x_{ik} - \overline{x}_{w_i k} \right\}, \end{aligned} \tag{1}$$

where

$$\overline{x}_{w_i k} = \frac{\sum_{j \in R(t_i)} x_{jk} e^{x'_j \beta}}{\sum_{j \in R(t_i)} e^{x'_j \beta}}. \tag{2}$$

Hosmer and Lemeshow (1999, 198) show, the estimator of the Schoenfeld residual for the ith subject on the kth covariate are then obtained by substituting the partial likelihood estimator of the coefficient, $\hat{\beta}$:

$$\hat{r}_{S_i k} = c_i (x_{ik} - \hat{\overline{x}}_{w_i k}), \tag{3}$$

where

$$\hat{\overline{x}}_{w_i k} = \frac{\sum_{j \in R(t_i)} x_{jk} e^{x'_j \beta}}{\sum_{j \in R(t_i)} e^{x'_j \beta}}. \tag{4}$$

Highly nonobvious, but there *is* some intuition here!

• Schoenfeld residual can be thought of as the observed minus the expected values of the covariates at each failure time.

• If the residual exhibits a random (i.e. unsystematic) pattern at each failure time, then this gives evidence the covariate effect is not changing with respect to time—precisely the PH assumption. If it is systematic, it suggests that as time passes, the covariate effect is changing.

• Why? Because if the PH property holds, then we would expect the difference between covariate values at failure times versus a weighted average of the covariate values to display no temporal trends. In residual plots, we might expect the slope of the (re-scaled) Schoenfeld residuals with respect to time should be zero. This is the basic idea behind graphical methods.