Title

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Assume that the model assumptions for the Cox proportional hazards model holds for a continuous survival time T with covariates $\mathbf{Z} = (Z_1, \dots, Z_p)'$, i.e., that,

$$\lambda(t; \mathbf{z}) = \lambda_0(t) \exp(\boldsymbol{\beta'} \boldsymbol{z}),$$

but that the survival times are grouped in the intervals $[0 = a_0, a_1), \dots, [a_{g-1}, a_g)$. The corresponding hazards are defined as

$$\lambda_j(\mathbf{z}) = P(T < a_j | T \geq a_{j-1}; \mathbf{z}), \ j = 1, \dots, g.$$

Then we can write

$$\begin{aligned} 1 - \lambda_j(\mathbf{z}) &= P(T \geq a_j | T \geq a_{j-1}; \mathbf{z}) \\ &= \frac{P(T \geq a_j, T \geq a_{j-1}; \mathbf{z})}{P(T \geq a_{j-1}; \mathbf{z})} \\ &= \frac{P(T \geq a_j; \mathbf{z})}{P(T \geq a_{j-1}; \mathbf{z})}. \end{aligned}$$

First we observe that

$$\begin{aligned} P(T \geq a_j; \mathbf{z}) &= S_{\mathbf{z}}(a_j) \\ &= \exp(-\Lambda_{\mathbf{z}}(a_j)) \\ &= \exp(-(\int_{a_0}^{a_1} \lambda_0(t) \exp(\boldsymbol{\beta'} \boldsymbol{z} dt + \dots + \int_{a_{j-1}}^{a_j} \lambda_0(t) \exp(\boldsymbol{\beta'} \boldsymbol{z} dt)))) \\ &= \exp(-(\int_{a_0}^{a_1} \lambda_0(t) dt + \dots + \int_{a_{j-1}}^{a_j} \lambda_0(t) dt) \exp(\boldsymbol{\beta'} \boldsymbol{z})) \\ &= (\exp(-\Lambda_{\mathbf{0}}(a_j)))^{\exp(\boldsymbol{\beta'} \boldsymbol{z})} \\ &= (P(T \geq a_j; \mathbf{0}))^{\exp(\boldsymbol{\beta'} \boldsymbol{z})}. \end{aligned}$$

The exact same argument for $P(T \geq a_{j-1}; \mathbf{z})$ gives us

$$\begin{aligned} 1 - \lambda_j(\mathbf{z}) &= \left(\frac{P(T \geq a_j; \mathbf{0})}{P(T \geq a_{j-1}; \mathbf{0})} \right)^{\exp(\boldsymbol{\beta'} \boldsymbol{z})} \\ &= (1 - \lambda_j(\mathbf{0}))^{\exp(\boldsymbol{\beta'} \boldsymbol{z})} \end{aligned}$$

or equivalently

$$\log(1 - \lambda_j(\mathbf{z})) = (1 - \lambda_j) \exp(\boldsymbol{\beta'} \boldsymbol{z}).$$