Optimization 1: Location Analysis

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The problems

$$\min_{x\in\mathbb{R}^2} \max_{a\in A} d(a,x) \quad \text{(Center problem)}, \tag{1}$$

$$\min_{x\in\mathbb{R}^2} \sum_{a\in A} d(a,x) \quad \text{(Median problem)}, \tag{2}$$

with objective functions

$$f(x) := \max_{a\in A} d(a,x), \tag{3}$$

$$g(x) := \sum_{a\in A} d(a,x), \tag{4}$$

are considered.

1 Theoretical Work

1. Show that d_1, d_2 and d_∞ are metrics.

Proof. It must be shown that the metric conditions are satisfied for all three metrics. Each vector $x \in \mathbb{R}^2$ will be denoted by $x = (x_1, x_2)^\top$.

First, it is shown that d_1 is a metric. Definiteness of d_1 holds, since

$$0 = d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| \iff x_1 = y_1,\ x_2 = y_2 \iff x = y.$$

Symmetry holds trivially for d_1, since

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d_1(y,x).$$

Finally, the triangle inequality holds, because

$$\begin{aligned} d_1(x,z) &= |x_1 - z_1| + |x_2 - z_2| \\ &= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2| \\ &\leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2| \\ &= d_1(x,y) + d_1(y,z), \end{aligned}$$

since the triangle inequality holds for $|\cdot|$.

Similarly, it is shown that d_2 is a metric. Definiteness of d_2 holds, since

$$0 = d_2(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \iff x_1 = y_1,\ x_2 = y_2 \iff x = y.$$

Symmetry holds for d_2, since

$$d_2(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = \sqrt{(y_1-x_1)^2+(y_2-x_2)^2} = d_2(y,x).$$

Finally, the triangle inequality holds for d_2, because

$$\begin{aligned} d_2(x,z)^2 &= (x_1-y_1)^2+(x_2-y_2)^2 \\ &= ([x_1-z_1]-[y_1-z_1])^2+([x_2-z_2]-[y_2-z_2])^2 \\ &= \sum_{i=1}^{2}(x_i-z_i)^2 - 2\sum_{i=1}^{2}(x_i-z_i)(y_i-z_i) + \sum_{i=1}^{2}(y_i-z_i)^2 \\ &\le d_2(x,z)^2 + 2d_2(x,y)d_2(y,z) + d_2(y,z)^2 \\ &= (d_2(x,y)+d_2(y,z))^2, \end{aligned}$$

which, by taking roots, leads to $d_2(x,z) \le d_2(x,y) + d_2(y,z)$.

Lastly, it is shown that d_∞ is a metric. Definiteness of d_∞ holds, since

$$0 = d_\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\} \iff x_1 = y_1,\ x_2 = y_2 \iff x = y. \tag{5}$$

Equation (5) should be explained in greater detail. The fact that $0 = d_\infty(x,y)$ implies that $|x_1-y_1| = |x_2-y_2| = 0$, since $|\cdot| : \mathbb{R} \to [0,\infty)$. From this, the rest can be deduced. Moreover, symmetry holds for d_∞, because

$$d_\infty(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\} = \max\{|y_1-x_1|, |y_2-x_2|\} = d_\infty(y,x).$$

Lastly, in order to prove that the triangle inequality holds for d_∞, it is first assumed that $|x_1-y_1| > |x_2-y_2|$ for $x, y \in \mathbb{R}^2$. Then

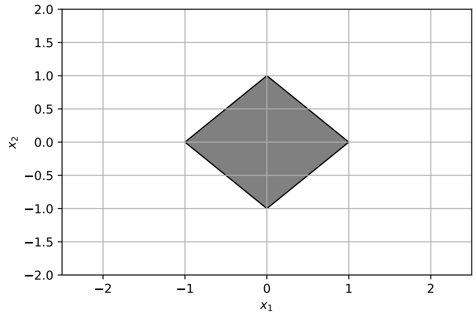
$$\begin{aligned} d_\infty(x,y) &= |x_1-y_1| = |x_1-z_1+z_1-y_1| \\ &\le |x_1-z_1| + |z_1-y_1| \\ &\le \max\{|x_1-z_1|, |x_2-z_2|\} + \max\{|z_1-y_1|, |z_2-y_2|\} \\ &= d_\infty(x,z) + d_\infty(z,y). \end{aligned}$$

The proof is analogous in the case where $|x_1-y_1| < |x_2-y_2|$. Moreover, the triangle inequality holds trivially when $x = y$. Therefore, the triangle inequality holds for d_∞ in all three cases.

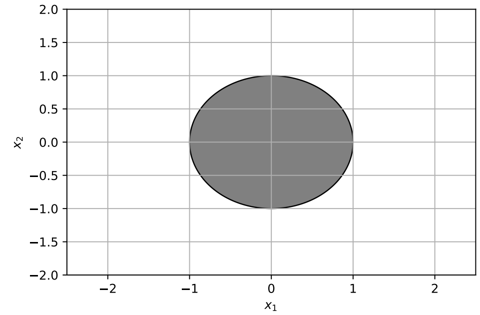
Hence, it has been shown that d_1, d_2 and d_∞ are metrics. $\square$

2. Plot the unit balls $B_i(0,1) := \{x \in \mathbb{R}^2 \mid \|x\|_i \le 1\}$ $i \in (1,2,\infty)$ of the Manhattan norm, the Euclidean norm and of the maximum norm.

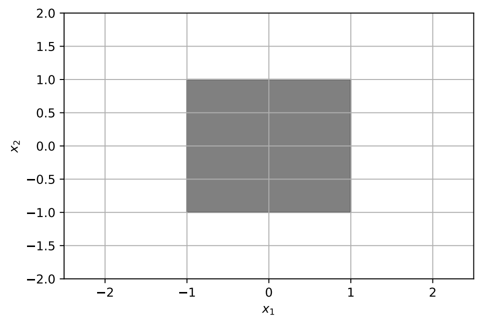
The unit balls are shown in figure 1.



(a) $B_1(0,1) := \{x \in \mathbb{R}^2 \mid \|x\|_1 \leq 1\}$



(b) $B_2(0,1) := \{x \in \mathbb{R}^2 \mid \|x\|_2 \leq 1 \}$



(c) $B_\infty(0,1) := \{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1 \}$

Figure 1: Unit balls plotted for different norms. (a) Manhattan norm, (b) Euclidean norm and (c) Maximum norm.

3. Show that every norm is a convex function.

A function $f : \mathbb{R}^2 \to \mathbb{R}$ is convex if it satisfies

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2), \ \forall \lambda \in [0,1], \ \forall x_1, x_2 \in \mathbb{R}^2. \tag{6}$$

Proof. In order to show that every norm $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ is a convex function, it needs to be shown that all norms satisfy (6). Let $x_1, x_2 \in \mathbb{R}^2$ be two arbitrary vectors and $\lambda \in [0,1]$ arbitrary. Then, the triangle inequality and the positive homogeneity of norms gives

$$\begin{aligned} \|\lambda x_1 + (1-\lambda)x_2\| &\leq \|\lambda x_1\| + \|(1-\lambda)x_2\| \\ &= \lambda\|x_1\| + (1-\lambda)\|x_2\|, \end{aligned}$$

which shows that (6) is satisfied for all norms. □

Note that this proof can be easily generalized to functions and norms on $\mathbb{R}^n$, even though it is proved only for scalar functions on $\mathbb{R}^2$, since we are working in the plane.

4. Show that the objective function of problems (1) **and** (2) **is convex, when defined with respect to the metrics** $d_1(x,y), d_2(x,y)$ **and** $d_\infty(x,y)$**.**

In order to show that both functions (3) and (4) are convex, three subsidiary results will be used.

Result 1. *If f and g are convex functions, then the function* $\max\{f(x), g(x)\}$ *is also convex.*

Proof. A function is convex if and only if the epigraph of the function is a convex set. For convex functions f and g, the epigraph of $\max\{f(x), g(x)\}$ is the intersection of their respective epigraphs. Since the intersection of convex sets is convex, the epigraph of $\max\{f(x), g(x)\}$ is a convex set. Hence, the function itself is convex. □

Result 2. *If f and g are convex functions, then the function $f+g$ is also convex.*

Proof. Select arbitrary $\lambda \in [0,1],\ x_1, x_2 \in \mathbb{R}^n$. By definition of addition of functions

$$\begin{aligned}
(f+g)(\lambda x_1 + (1-\lambda)x_2) &= f(\lambda x_1 + (1-\lambda)x_2) + g(\lambda x_1 + (1-\lambda)x_2) \\
&\le \lambda f(x_1) + (1-\lambda)f(x_2) + \lambda g(x_1) + (1-\lambda)g(x_2), \quad (f, g \text{ convex}) \\
&= \lambda(f(x_1) + g(x_1)) + (1-\lambda)(f(x_2) + g(x_2)) \\
&= \lambda(f+g)(x_1) + (1-\lambda)(f+g)(x_2).
\end{aligned}$$

Hence, the sum $f+g$ is also convex. □

Result 3. *Given a convex function f in x, the function $g(x) := f(ax+b)$ is convex.*

Proof. Select arbitrary $\lambda \in [0,1],\ x_1, x_2 \in \mathbb{R}^n$. Then

$$\begin{aligned}
g(\lambda x_1 + (1-\lambda)x_2) &= f(a(\lambda x_1 + (1-\lambda)x_2) + b + \lambda b - \lambda b) \\
&= f(\lambda(ax_1 + b) + (1-\lambda)(ax_2 + b)) \\
&\le \lambda f(ax_1 + b) + (1-\lambda)f(ax_2 + b), \quad (f \text{ convex}) \\
&= \lambda g(x_1) + (1-\lambda)g(x_2).
\end{aligned}$$

Hence, g is convex. □

The proof of convexity of functions (3) and (4), when defined with respect to the metrics $d_1(x,y), d_2(x,y)$ and $d_\infty(x,y)$, follows.

Proof. It has already been shown that every norm is a convex function. Moreover, each norm on a normed space X induces a metric on the same space. In this case, the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, where $\|\cdot\|_i : \mathbb{R}^2 \to \mathbb{R},\ i \in \{1, 2, \infty\}$, induce the metrics d_1, d_2 and d_∞. More precisely, for $a, x \in \mathbb{R}^2$

$$\begin{aligned}
d_1(a,x) &:= \|a-x\|_1 = |a_1 - x_1| + |a_2 - x_2|, \\
d_2(a,x) &:= \|a-x\|_2 = \sqrt{(a_1-x_1)^2 + (a_2-x_2)^2}, \\
d_\infty(a,x) &:= \|a-x\|_\infty = \max\{|a_1-x_1|, |a_2-x_2|\}.
\end{aligned}$$

Since all these norms are convex functions, their respective induced metrics are convex as well. This is because they are defined as convex functions of an affine function in x, which, according to Result 3, preserves convexity. Thus, Result 1 and Result 2 imply that the functions (3) and (4) are convex, because these results are generalizable to maximums and sums over functions of $a \in A$, assuming that A is a finite set of locations. $\square$

5. Give a geometric interpretation of solving problem (1) with the Euclidean distance function.

The center problem (1) with the Euclidean distance function $d_2(a, x) = \sqrt{(a_1 - x_1)^2 + (a_2 - x_2)^2}$ is given by

$$\min_{x \in \mathbb{R}^2} \max_{a \in A} d_2(a, x). \tag{7}$$

Solving this problem can be interpreted as locating the point x in the entire $\mathbb{R}^2$-plane that gives the shortest distance to the point $a \in A$, which is the furthest away from x. By distance, the straight-line (Euclidean) distance is meant. Hence, one seeks to minimize the largest distance from $a \in A$. Effectively, this means that the minimizer $x \in \mathbb{R}^2$ is located at the center of the smallest circle which encloses all $a \in A$, i.e. the smallest enclosing circle. The distance to each point $a \in A$ from x is as small as possible, since all points a must be reachable from x.

6. Provide a solution approach for problem (2) with Manhattan distance function.

The median problem (2) with the Manhattan distance function $d_1(a, x) = |a_1 - x_1| + |a_2 - x_2|$ is given by

$$\min_{x \in \mathbb{R}^2} \sum_{a \in A} d_1(a, x). \tag{8}$$

It has already been proved that the objective function $\sum_{a \in A} d_1(a, x)$ is convex. As stated in the hint, the objective function $g(x) := \sum_{a \in A} d_1(a, x) = \sum_{a \in A}(|a_1 - x_1| + |a_2 - x_2|)$ can be separated. This means that the problem may be rewritten as

$$\min_{x \in \mathbb{R}^2} \sum_{a \in A} d_1(a, x) = \min_{x_1 \in \mathbb{R}} \sum_{i=1}^{m} |a_1^i - x_1| + \min_{x_2 \in \mathbb{R}} \sum_{i=1}^{m} |a_2^i - x_2|.$$

Hence, minimizers of each of the two subproblems

$$\min_{x_k \in \mathbb{R}} \sum_{i=1}^{m} |a_k^i - x_k|, \; k = 1, 2, \tag{9}$$

can be found separately. A solution approach for each of these problems is to sort the points $a \in A$ in increasing order in each coordinate. After this, the problem (9) may be restated in the reordered fashion

$$\min_{x_k \in \mathbb{R}} \sum_{j=1}^{p} |x_k - a_k^{(j)}|, \; k = 1, 2, \tag{10}$$

where $a_k^{(1)} < a_k^{(2)} < \cdots < a_k^{(p)}$. Note that $p \leq m$, since some of the a_k^i's may be combined (if they are identical) to create a strictly increasing sequence. Next, the derivative of these reordered problems can be calculated. It can be shown that problem (10) is a continuous and piecewise linear function with discontinuities in the first derivative at the points $a_k^{(j)}$, $k = 1, 2$. Moreover, because of the ordered problem, the derivatives increase with the ordering of the points. Hence, the minimizer of the problem is found in two possible occasions. Firstly, if the derivative changes from negative to positive, the minimizer is located at the discontinuity where the change happens. Secondly, if the slope changes from negative to zero, there is an interval of minimizers where the derivative vanishes. Since the objective function is convex, it is known that every local minimizer of the function is a global minimizer. Moreover, since the problem is differentiable, a point x^* is a global minimizer iff the derivative at x^* vanishes, i.e. if it is a stationary point. Hence, the convexity of the objective function is key [4].

7. Consider the Median problem (2) **with squared Euclidean distance function. Show that the objective function is convex, and analytically compute the uniquely determined minimizer of this problem.**

The median problem with squared Euclidean distance function takes the form

$$\min_{x \in \mathbb{R}^2} \sum_{a \in A} (d_2(a, x))^2. \tag{11}$$

In order to show that $f(x) := \sum_{a \in A} (d_2(a, x))^2$ is convex, it is sufficient to show that the Hessian is positive (semi-)definite. First of all, the gradient of f is

$$\begin{aligned}
\nabla_x f(x) &= \nabla_x \sum_{a \in A} (d_2(a, x))^2 = \nabla_x \left(\sum_{a \in A} (a_1 - x_1)^2 + (a_2 - x_2)^2 \right) \\
&= \left(-2 \sum_{a \in A} (a_1 - x_1), -2 \sum_{a \in A} (a_2 - x_2) \right)^\top \\
&= \left(2x_1 \sum_{i \in \mathcal{M}} -2 \sum_{i \in \mathcal{M}} a_1^i, 2x_2 \sum_{i \in \mathcal{M}} -2 \sum_{i \in \mathcal{M}} a_2^i \right)^\top \\
&= (2m(x_1 - \bar{a}_1), 2m(x_2 - \bar{a}_2))^\top,
\end{aligned} \tag{12}$$

where $\bar{a}_j = \frac{1}{m} \sum_{i \in \mathcal{M}} a_j^i$, $j = 1, 2$. This means that the Hessian of f, $\nabla_{xx}^2 f(x)$, is given by

$$\nabla_{xx}^2 f(x) = \begin{pmatrix} 2m & 0 \\ 0 & 2m \end{pmatrix},$$

which is symmetric and positive definite, since it is a strictly diagonally dominant matrix with positive diagonal entries. This can also be shown via the definition $v^\top \nabla_{xx}^2 f(x) v > 0$, $\forall v \in \mathbb{R}^2 \setminus \{0\}$.

In fact, since the Hessian is positive definite, the objective function is *strictly* convex. This implies that the uniquely determined global minimizer of this problem can be found by locating the point where the gradient vanishes. Hence, as is apparent from equation (12), the uniquely determined minimizer of this problem is $x^* = (\bar{a}_1, \bar{a}_2)$.

8. Consider the Median problem (2) **with Euclidean distance function. Derive necessary optimality conditions for minimizers of this problem.**

The median problem with Euclidean distance is given by

$$\min_{x\in\mathbb{R}^2} \sum_{a\in A} d_2(a,x) = \min_{x\in\mathbb{R}^2} \sum_{i=1}^{m} \sqrt{(x_1-a_1^i)^2+(x_2-a_2^i)^2} =: \min_{x\in\mathbb{R}^2} f(x). \tag{13}$$

Compute the gradient of (13),

$$\nabla f(x) = \begin{pmatrix} \sum_{i=1}^m (x_1-a_1^i)/d_2(x,a^i) \\ \sum_{i=1}^m (x_2-a_2^i)/d_2(x,a^i) \end{pmatrix}, \tag{14}$$

and notice that it is not defined for $x \in A$. Therefore, assume that $x \notin A$. Since f is both convex and differentiable, $\nabla f(x^*) = 0 \iff x^*$ is a global minimizer of f [3]. Hence, the necessary (and sufficient) optimality conditions for minimizers x^* of (13) are

$$\begin{cases} \sum_{i=1}^m (x_1^*-a_1^i)/d_2(x^*,a^i) & \overset{!}{=} 0 \\ \sum_{i=1}^m (x_2^*-a_2^i)/d_2(x^*,a^i) & \overset{!}{=} 0 \\ x^* = (x_1^*,x_2^*)^\top & \notin A. \end{cases}$$

As noted, it is assumed that $x^* \notin A$. It should always first be checked whether or not this assumption is valid, i.e. if the minimizer of the problem is an existing location in A or not, using Property 2.2 in [4], where all weights are set to 1.

9. Give an example of a set of existing locations, where the set of global minimizers for (13) **and** (11) **do not coincide.**

Let $A = \{a^1, a^2\} = \{(0,0)^\top, (0,1)^\top\}$.

First of all, the minimizer of problem (11) is simple to find, based on the calculations from problem 7: The function $\sum_{a\in A}(d_2(a,x))^2$ is minimized when $x_1 = 0$ and $x_2 = \frac{1}{2}$. Next, it will be shown that this is different from the minimizers of problem (13).

To begin with, one should check, with Property 2.2, if any of the existing locations already are minimizers of $\sum_{a\in A} d_2(a,x)$, by

$$\mathrm{Test}_1 = \left[\left(\frac{a_1^1-a_1^2}{d_2(a^1,a^2)}\right)^2 + \left(\frac{a_2^1-a_2^2}{d_2(a^1,a^2)}\right)^2\right]^{\frac{1}{2}} = 1 \le 1,$$

$$\mathrm{Test}_2 = \left[\left(\frac{a_1^2-a_1^1}{d_2(a^2,a^1)}\right)^2 + \left(\frac{a_2^2-a_2^1}{d_2(a^2,a^1)}\right)^2\right]^{\frac{1}{2}} = 1 \le 1.$$

Since $\mathrm{Test}_k \le 1,\ k = 1,2$, both points $a^1 = (0,0)$ and $a^2 = (0,1)$ are minimizers of problem (13).

Next, the gradient from (14) is

$$\begin{pmatrix} x_1/\sqrt{x_1^2+x_2^2} + x_1/\sqrt{x_1^2+(x_2-1)^2} \\ x_2/\sqrt{x_1^2+x_2^2} + (x_2-1)/\sqrt{x_1^2+(x_2-1)^2} \end{pmatrix}.$$

Setting this to zero gives $x_1 = 0$, from the terms in the first component. The second component thus yields

$$\frac{x_2}{|x_2|} + \frac{x_2 - 1}{|x_2 - 1|} = 0.$$

Two cases need to be investigated: $x_2 > 0$ and $x_2 < 0$. Note that $x_2 \neq 0$ and $x_2 \neq 1$, because then $x = (x_1, x_2)^\top \in A$, and the denominators in the components of the gradient evaluate to zero. The first case, $x_2 > 0$, leads to

$$\begin{aligned} \frac{x_2 - 1}{|x_2 - 1|} &= -1, \\ \implies x_2 - 1 &< 0, \\ x_2 &< 1. \end{aligned}$$

The second case, $x_2 < 0$, yields

$$\begin{aligned} \frac{x_2 - 1}{|x_2 - 1|} &= 1, \\ \implies x_2 - 1 &> 0, \\ x_2 &> 1, \end{aligned}$$

which contradicts the fact that $x_2 < 0$. Thus, the conclusion is that $\sum_{a \in A} d_2(a, x)$ is minimized when $x_1 = 0$ and $\forall x_2 \in [0, 1]$, which does not coincide uniquely with the minimizer of $\sum_{a \in A}(d_2(a, x))^2$, which is contained in the interval of minimizers of the former problem.

2 Weiszfeld Algorithm

The Median problem with weighted Euclidean distance

$$\min_{x \in \mathbb{R}^2} \sum_{i \in \mathcal{M}} v^i d_2(a^i, x), \tag{15}$$

is considered, which has the objective function

$$f_{d_2}(x) := \sum_{i \in \mathcal{M}} v^i d_2(a^i, x). \tag{16}$$

The main iteration scheme that is used in the Weiszfeld algorithm is

$$\begin{aligned} x_1^{new} &:= \frac{\sum_{i=1}^m v^i \frac{a_1^i}{d_2(a^i, x)}}{\sum_{i=1}^m v^i \frac{1}{d_2(a^i, x)}} \\ x_2^{new} &:= \frac{\sum_{i=1}^m v^i \frac{a_2^i}{d_2(a^i, x)}}{\sum_{i=1}^m v^i \frac{1}{d_2(a^i, x)}}, \end{aligned} \tag{17}$$

1. Explain why the main iteration scheme is chosen according to (17)**.**

Under the assumption that the minimizer $x^* = (x_1^*, x_2^*) \notin A$, the necessary and sufficient optimality condition for the problem (15) is

$$\nabla f_{d_2}(x^*) \overset{!}{=} 0. \tag{18}$$

The idea of the Weiszfeld algorithm is to convert (18) into a fixed-point iteration. Equation (18) can be written component-wise as

$$\frac{\partial}{\partial x_k^*}\left[\sum_{i=1}^{m} v^i d_2(a^i, x^*)\right] = \sum_{i=1}^{m} \frac{v^i(x_k^* - a_k^i)}{d_2(a^i, x^*)} \overset{!}{=} 0, \quad k = 1, 2. \tag{19}$$

Next, equation (19) is rearranged as

$$\begin{aligned} x_k^* \sum_{i=1}^{m} \frac{v^i}{d_2(a^i, x^*)} - \sum_{i=1}^{m} \frac{v^i a_k^i}{d_2(a^i, x^*)} &= 0, \quad k = 1, 2, \\ x_k^* = \frac{\sum_{i=1}^{m} v^i \frac{a_k^i}{d_2(a^i, x^*)}}{\sum_{i=1}^{m} v^i \frac{1}{d_2(a^i, x^*)}} &=: T(x_k^*), \quad k = 1, 2. \end{aligned}$$

Note that these reformulations are not explicitly expressed in terms of x_k^*, even though this looks to be the case, because of the $d_2(a, x^*)$-dependence. Consequently, it is apparent that the iteration scheme (17) essentially is a fixed-point iteration to the minimizer x^*, where

$$x_k^{new} = T(x_k), \quad k = 1, 2.$$

Assuming that x^* and none of the iterates are in A, it can be shown that the iteration scheme (17) converges to x^* (Theorem 5.1 in [1]). The first problem, i.e. that the minimizer is in A, will not be an obstacle for algorithm 2.1, which is given later, because it checks if any of the points $a \in A$ are optimal [1].

2. Prove Theorem 4 using Theorems 2 and 3.

Theorem 4 states that if for the lower bound, $\mathrm{LB}(x) := f_{d_2}(x) - ||\nabla f_{d_2}(x)||\sigma(x)$, we have $\mathrm{LB}(x) > 0$, and for a given $\epsilon > 0$, the property

$$\frac{\mathrm{UB}(x)}{\mathrm{LB}(x)} = \frac{||\nabla f_{d_2}(x)||\sigma(x)}{f_{d_2}(x) - ||\nabla f_{d_2}(x)||\sigma(x)} < \epsilon$$

holds true, then the current iterate has at least a relative accuracy of ϵ. In this expression, $\mathrm{UB}(x) := ||\nabla f_{d_2}(x)||\sigma(x)$ and $\sigma(x) := \max\{d_2(x, y) : y \in \mathrm{conv}\{a^1, \dots, a^m\}\}$, where $\mathrm{conv}\{a^1, \dots, a^m\} := \{x \in \mathbb{R}^2 : \exists \lambda^1 \geq 0, \dots, \lambda^m \geq 0, \sum_{i=1}^{m} \lambda^i = 1, x = \sum_{i=1}^{m} \lambda^i a^i\}$ is the convex hull.

Proof. The proof of this theorem uses Theorem 2 and Theorem 3, from the project description. The relative accuracy at the current iterate x is

$$\text{Relative accuracy} = \frac{f_{d_2}(x) - f_{d_2}(x^*)}{f_{d_2}(x^*)} = \frac{f_{d_2}(x)}{f_{d_2}(x^*)} - 1,$$

where x^* denotes the global minimizer of problem (15). Observe that the relative accuracy is large for large deviations of $f_{d_2}(x)$ from $f_{d_2}(x^*)$. By Theorem 2,

$$f_{d_2}(x) - f_{d_2}(x^*) \leq \text{UB}(x)$$

and by Theorem 3,

$$f_{d_2}(x^*) \geq \text{LB}(x).$$

Assuming $\text{LB}(x) > 0$ yields

$$\frac{f_{d_2}(x) - f_{d_2}(x^*)}{f_{d_2}(x^*)} \leq \frac{f_{d_2}(x) - f_{d_2}(x^*)}{\text{LB}(x)} \leq \frac{\text{UB}(x)}{\text{LB}(x)}.$$

Then, for a given $\epsilon > 0$ such that $\frac{\text{UB}(x)}{\text{LB}(x)} < \epsilon$ holds true, the relative accuracy at the current iterate is bounded above by ϵ. This means that the current iterate has at least a relative accuracy of ϵ. □

3. Derive a termination criterion for the Weiszfeld algorithm based on Theorem 4.

From the previous problem, it is known that the current iterate has at least a relative accuracy of a given $\epsilon > 0$ if $\text{LB}(x) > 0$ and $\frac{\text{UB}(x)}{\text{LB}(x)} < \epsilon$. Hence, a termination criterion for the Weiszfeld algorithm could be based on this. The algorithm should terminate in the case where the lower bound at the current iterate, $\text{LB}(x)$, is greater than zero and the fraction $\frac{\text{UB}(x)}{\text{LB}(x)}$ is smaller than a given $\epsilon > 0$. This ϵ should be supplied to the algorithm, which enables some control over the termination when commencing the computations. Upon termination, the current iterate x should be returned as the approximation to the global minimizer x^* of (15). Moreover, in order to prevent the algorithm from iterating for far too long, an upper bound on the amount of allowed iterations is used as a safeguard in practice.

4. Implement the Weiszfeld algorithm using Python. Give the pseudo-code and a test example.

The pseudo-code is given in Algorithm 2.1.

First, we consider a simple example where $A = \{(1,0)^\top, (-1,0)^\top, (0,1)^\top, (0,-1)^\top\}$ and all weights are 1. The result is depicted in figure 2. As is apparent from the figure, the Weiszfeld algorithm returns origo, $x_W^* = (0,0)^\top$, with a function value $f_{d_2}(x_W^*) = 4$, as expected. This result is returned immediately before the first iteration, because the initial point is chosen to be the solution of the Median problem with unweighted squared Euclidean distance.

Furthermore, we consider example 2.2 presented in Love and Morris [4]. The locations are $A = \{(1,1)^\top, (1,4)^\top, (2,2)^\top, (4,5)^\top\}$, with weights $1, 2, 2$, and 4, respectively. The result is depicted in figure 3. Our implementation of the Weiszfeld algorithm yields the same result as in [4], i.e. $x_W^* \approx (2.577, 3.820)^T$ with a function value $f_{d_2}(x_W^*) \approx 17.618$ after 48 iterations, when $\epsilon = 1.580 \times 10^{-6}$, which is reassuring.

Algorithm 2.1 Weiszfeld Algorithm for the Median problem with weighted Euclidian distances.

Input: Existing unique locations $a^i = \left(a_1^i, a_2^i\right)^T \in \mathbb{R}^2$,
weights $v^i > 0, i \in \{1, \dots, m\} := \mathcal{M}$,
error bound $\epsilon > 0$, maximum allowed iterations μ.

Output: Approximation of the minimizer $x^* = (x_1^*, x_2^*)$

let $\text{Test}_{\,k} := \left[\left(\sum_{i \in \mathcal{M} \setminus \{k\}} v^i \frac{a_1^k - a_1^i}{d_2\left(a^k, a^i\right)}\right)^2 + \left(\sum_{i \in \mathcal{M} \setminus \{k\}} v^i \frac{a_2^k - a_2^i}{d_2\left(a^k, a^i\right)}\right)^2\right]^{\frac{1}{2}}$

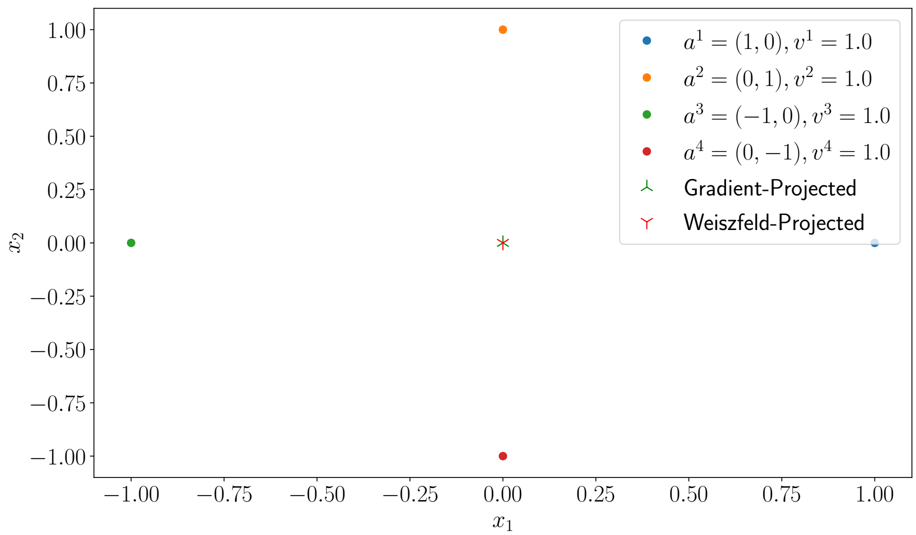
if $\text{Test}_k \leq v^k$ for some $k \in \mathcal{M}$ **then**
  $x^* = a^k$
  return x^*
end if

let $x_j := \frac{1}{m} \sum_{i \in \mathcal{M}} a_j^i, \quad j = 1, 2$
let $z = 0$

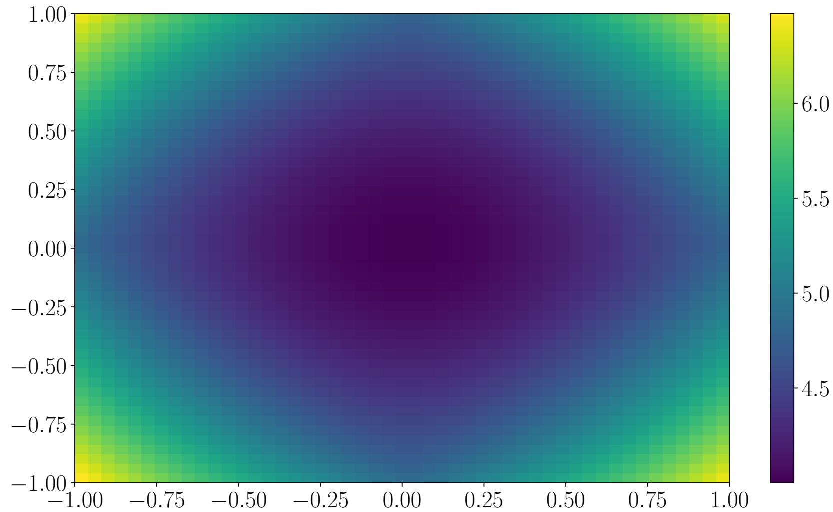
while $\text{LB}(x) > 0$ and $\frac{\text{UB}(x)}{\text{LB}(x)} \geq \epsilon$ and $z \leq \mu$ **do**

  let $x_j^{\text{new}} = \frac{\sum_{i=1}^{m} v^i \frac{a_j^i}{d_2\left(a^i, x\right)}}{\sum_{i=1}^{m} v^i \frac{1}{d_2\left(a^i, x\right)}}, \quad j = 1, 2$

  $x = x^{\text{new}}$
  $z = z + 1$
end while
$x^* = x^{\text{new}}$
return x^*

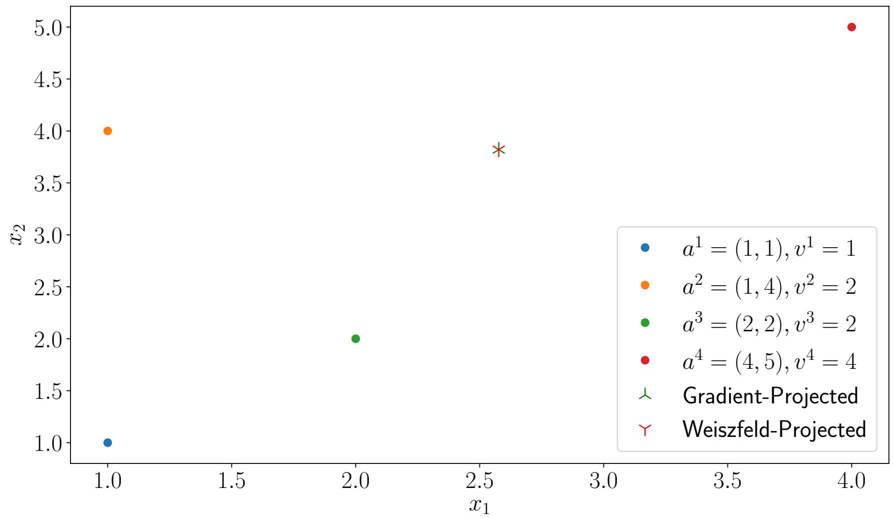


(a) Graphical representation of the numerical solution.

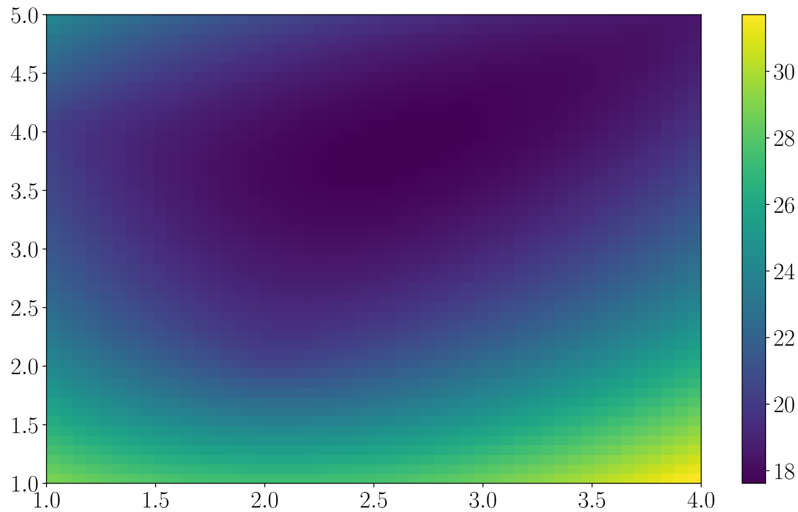


(b) Color map of the objective function.

Figure 2: Simple test example of use of the Weiszfeld algorithm:
$A = \{(1, 0)^\top, (-1, 0)^\top, (0, 1)^\top, (0, -1)^\top\}$ and all weights are 1. (a) Shows the numerical result graphically. (b) Shows a color map of the objective function, which displays the vicinity of the minimum.



(a) Graphical representation of the numerical solution.



(b) Color map of the objective function.

Figure 3: Example 2.2 from Love and Morris [4]: $A = \{(1,1)^\top, (1,4)^\top, (2,2)^\top, (4,5)^\top\}$, with weights $1, 2, 2$, and 4, respectively. (a) Shows the numerical result graphically. (b) Shows a color map of the objective function, which displays the vicinity of the minimum.

5. Implement the gradient descent method with backtracking and replace steps 2.-4. in the algorithm with it. Compare with the Weiszfeld algorithm and discuss the difference between the two algorithms and their performance. Which algorithm should be suggested to use, and why?

A termination criterion for gradient descent with backtracking needs to be found. This is given in the following discussion. In order to compare the performance of both algorithms, and our implementations of them, as objectively as possible, one possibility is to use the same stopping criterion as in the Weiszfeld Algorithm. Another possible solution is to assert that the algorithm improves in each step, i.e. to assert that $\left|f\left(x^{k+1}\right)-f\left(x^{k}\right)\right|<\epsilon$. We chose the former, because of the mentioned benefits, which in the implementation was combined with a maximum amount of allowed iterations, which was not reached in our test examples.

Consider the simple example where $A=\{(1,0)^\top,(-1,0)^\top,(0,1)^\top,(0,-1)^\top\}$ and all weights are 1. The result is depicted in figure 2. As shown in the figure, gradient descent with backtracking yields the same result as the Weiszfeld algorithm, i.e $x^*_{GD}=(0,0)^T$ and $f_{d_2}(x^*_{GD})=4$. This solution is found before the first iteration when using the same stopping criteria as in the Weiszfeld algorithm. This is because the initial point is chosen to be the solution of the Median problem with unweighted squared Euclidean distance.

The methods are compared by first computing a reference solution, $\tilde{x}^*$, with $\epsilon=10^{-14}$, with the Weiszfeld algorithm. Then, how the quantity $\mathcal{X}:=|f_{d_2}(x)-f_{d_2}(\tilde{x}^*)|$ changes is studied for both methods, both as a function of time and number of iterations.

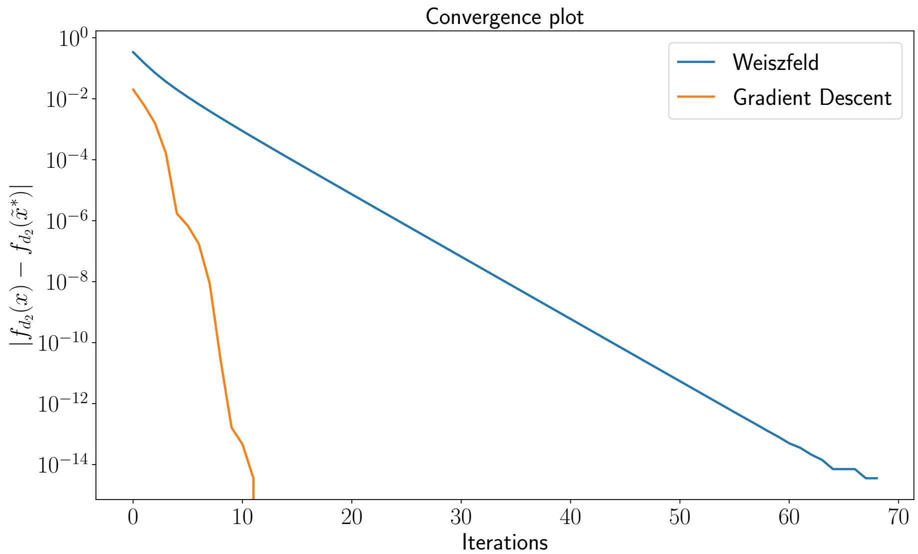
We consider two more test examples. In the **first test example**, the problem from Love and Morris [4] is again considered, i.e with locations $A=\{(1,1)^\top,(1,4)^\top,(2,2)^\top,(4,5)^\top\}$ and weights $1,2,2,$ and 4, respectively. The result is depicted in figure 3. Both gradient descent with backtracking and the Weiszfeld algorithm return $x^*\approx(2.577,3.820)^\top$ and a function value $f_{d_2}(x^*)\approx 17.62$. This was observed when $\epsilon=10^{-8}$. Moreover, how the quantity $\mathcal{X}$ changes as a function of number of iterations and time, is shown in figure 4. It can be observed that the gradient descent method is both faster and reaches the reference solution in less iterations than Weiszfeld's algorithm for this problem.

In the **second test example**, a problem with $A=\{(2,0)^\top,(0,1)^\top,(-1,0)^\top,(0,-1)^\top\}$ and respective weights 0.5, 1, 1 and 1, is considered. The graphical plot of the solution in the plane is omitted. Both methods return $x^*\approx(-0.258,0)^\top$ and a function value $f_{d_2}(x^*)\approx 3.94$. This was observed when $\epsilon=10^{-8}$. How the quantity $\mathcal{X}$ changes as a function of number of iterations and time, is shown in figure 5. Once again, gradient descent with backtracking is superior with respect to the number of iterations and time usage.

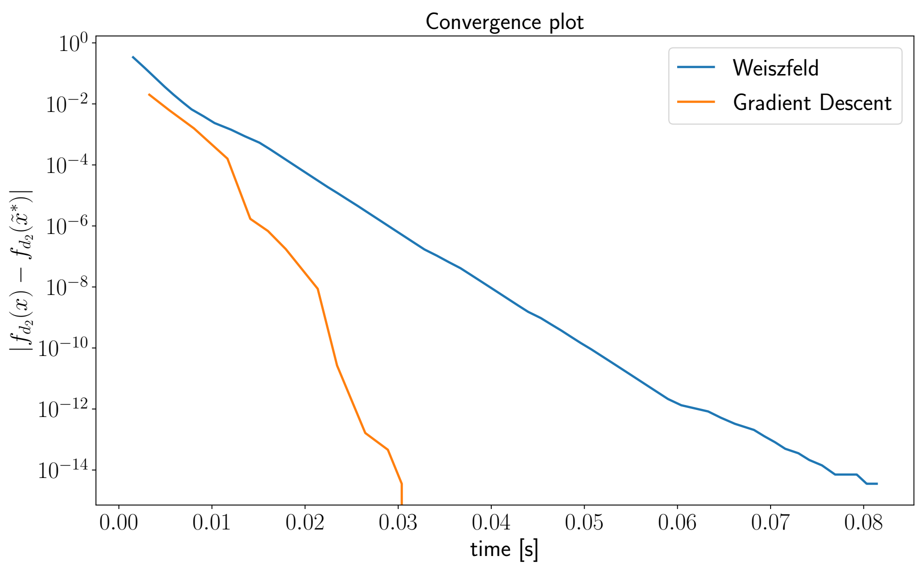
As observed in [1], the Weiszfeld algorithm is in fact a gradient descent method. Letting $L(x):=\sum_{i=1}^m v^i/d_2(a^i,x)$, the iteration scheme can be written as

$$x^{new}=x-\frac{1}{L(x)}\nabla f_{d_2}(x), \tag{20}$$

so the difference between the two methods is how the step length is determined. In the Weiszfeld algorithm, this is done with $L(x)$, while in the gradient descent method used above, this is done with backtracking. Based on the test examples, it seems like the Weiszfeld algorithm becomes "overly careful" when it closes in on a minimizer. This is linked to the

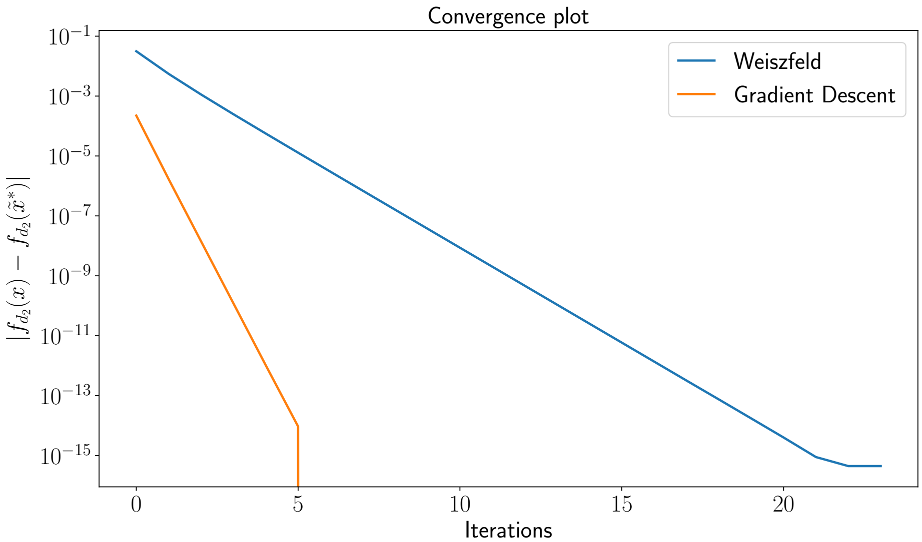


(a)

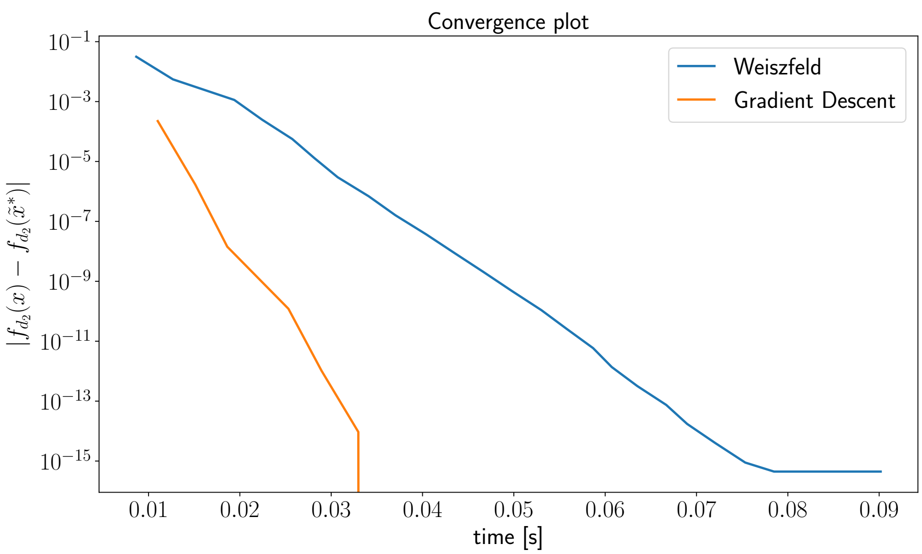


(b)

Figure 4: Test example 1. A reference solution $\tilde{x}^*$ is computed for Example 2.2 from Love and Morris [4]: $A = \{(1,1)^\top, (1,4)^\top, (2,2)^\top, (4,5)^\top\}$, with weights $1, 2, 2$, and 4, respectively. Based on the reference solution, gradient descent with backtracking and the Weiszfeld algorithm are compared with respect to change in $\mathcal{X}$ with respect to (a) number of iterations, and (b) time usage.



(a)



(b)

Figure 5: Test example 2. A reference solution $\tilde{x}^*$ is computed with the Weiszfeld algorithm with $\epsilon = 10^{-14}$ for the location problem $A = \{(2,0)^\top, (0,1)^\top, (-1,0)^\top, (0,-1)^\top\}$ with respective weights 0.5, 1, 1 and 1. Based on the reference solution, gradient descent with backtracking and the Weiszfeld algorithm are compared with respect to change in $\mathcal{X}$ with respect to (a) number of iterations, and (b) time usage.

fact that $L(x)$ should be relatively large close to a minimum and hence the step size, $\frac{1}{L(x)}$, becomes small. Gradient descent with backtracking does not seem to suffer from the same problem.

Recall that [1] states that the iteration scheme (20) converges to an optimal solution, as long as $x^{new} \notin A$. Even though we have no proof of convergence for the gradient descent method with backtracking in this case, it can probably be constructed with a similar approach as in [2]. To conclude, we observe that, even though we have no proof of convergence for gradient descent with backtracking on problem (15), it has proved superior to the Weiszfeld algorithm with respect to all parameters in the tests conducted. We therefore recommend gradient descent with backtracking, based on these examples.

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