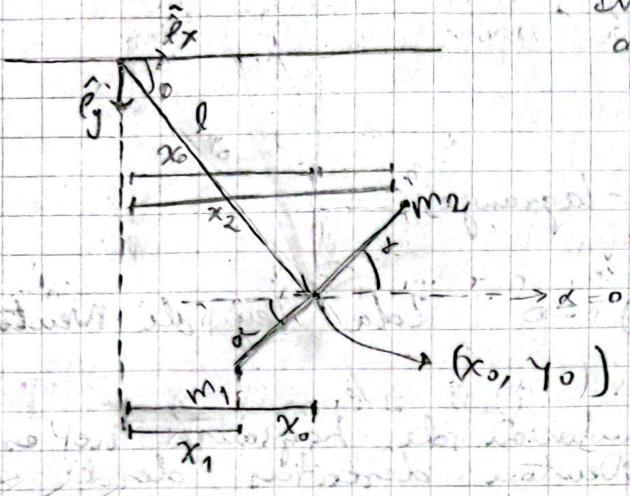
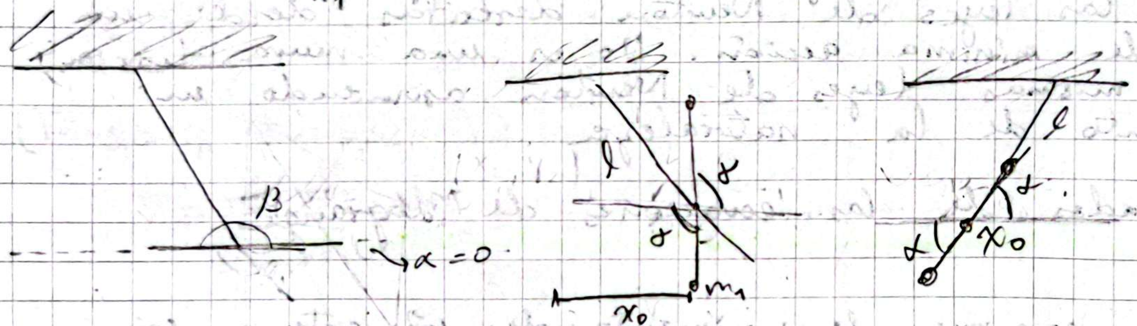
Problemas de los viernes

1) Encontrar las ecuaciones de movimiento





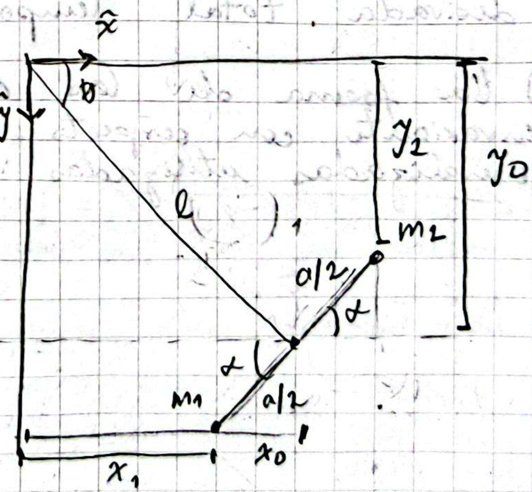
Coordenadas de x_0, y_0 en términos de θ,

$$x_0 = l\cos\theta$$
$$y_0 = l\,\text{sen}\,\theta$$

$$x_0 - x_1 = a/2 \cos\alpha$$
$$y_1 - y_0 = a/2\,\text{sen}\,\alpha$$
$$x_2 - x_0 = a/2 \cos\alpha$$
$$y_0 - y_2 = a/2\,\text{sen}\,\alpha$$

$$\Rightarrow \quad x_1 = x_0 - a/2 \cos\alpha$$
$$y_1 = a/2\,\text{sen}\,\alpha + y_0$$

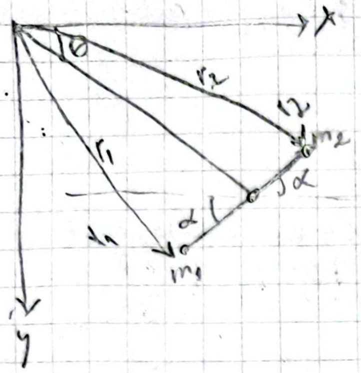
$$\Rightarrow \quad x_2 = a/2 \cos\alpha + x_0$$
$$y_2 = -a/2\,\text{sen}\,\alpha + y_0$$



$$x_1 = l\cos\theta - a/2\cos\alpha$$
$$y_1 = a/2\,\text{sen}\,\alpha + l\,\text{sen}\,\theta$$
$$x_2 = a/2\cos\alpha + l\cos\theta$$
$$y_2 = l\,\text{sen}\,\theta - a/2\,\text{sen}\,\alpha$$

2 ligaduras: $|r_1| - |r_2| = a$ (f1)

movimiento circular, cuyo centro se desplaza como un péndulo (f2).



$2(2) - 2 = 2 \longrightarrow$ grados de libertad

$$\left.\begin{aligned} x_1 &= l\cos\theta - a/2\cos\alpha \\ y_1 &= a/2\,\text{sen}\,\alpha + l\,\text{sen}\,\theta \\ x_2 &= a/2\cos\alpha + l\cos\theta \\ y_1 &= l\,\text{sen}\,\theta - a/2\,\text{sen}\,\alpha \end{aligned}\right\}$$ las coordenadas generalizadas son θ y α.

velocidad: (potencial sencilla)

$$\vec{v_1} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[(l\cos\theta - a/2\cos\alpha)\hat{e_x} + (a/2\,\text{sen}\,\alpha + l\,\text{sen}\,\theta)\hat{e_y}\right]$$

$$\vec{v_1} = (-l\,\text{sen}\,\theta\,\dot\theta + a/2\,\text{sen}\,\alpha\,\dot\alpha)\hat{e_x} + (a/2\cos\alpha\,\dot\alpha + l\cos\theta\,\dot\theta)\hat{e_y}$$

$$v_1^2 = (-l\,\text{sen}\,\theta\,\dot\theta + a/2\,\text{sen}\,\alpha\,\dot\alpha)^2 + (a/2\cos\alpha\,\dot\alpha + l\cos\theta\,\dot\theta)^2$$

$$= \left[(l\,\text{sen}\,\theta\,\dot\theta)^2 - \cancel{2l\,a/2\,\text{sen}\,\theta\,\dot\theta\,\text{sen}\,\alpha\,\dot\alpha} + (a/2\,\text{sen}\,\alpha\,\dot\alpha)^2\right]$$
$$+ \left[(a/2\cos\alpha\,\dot\alpha)^2 + \cancel{2l\,a/2\cos\alpha\,\dot\alpha\cos\theta\,\dot\theta} + (l\cos\theta\,\dot\theta)^2\right]$$

$$= l^2\dot\theta^2(\underbrace{\text{sen}^2\theta + \cos^2\theta}_{1}) + a^2/4\,\dot\alpha^2(\underbrace{\text{sen}^2\alpha + \cos^2\alpha}_{1})$$
$$= l^2\dot\theta^2 + a^2/4\,\dot\alpha^2$$

$$\vec{v_2} = \frac{d}{dt}\left(a/2\cos\alpha + l\cos\theta\right)\hat{e_x} + (l\,\text{sen}\,\theta - a/2\,\text{sen}\,\alpha)\hat{e_y}$$

$$= (-a/2\,\text{sen}\,\alpha\,\dot\alpha - l\cos\theta\,\dot\theta)\hat{e_x} + (l\cos\theta\,\dot\theta - a/2\,\text{sen}\,\alpha\,\dot\alpha)\hat{e_y}$$

$$v_2^2 = (a/2\,\text{sen}\,\alpha\,\dot\alpha + l\cos\theta\,\dot\theta)^2 + (l\cos\theta\,\dot\theta - a/2\,\text{sen}\,\alpha\,\dot\alpha)^2$$
$$= (a/2\,\text{sen}\,\alpha\,\dot\alpha)^2 + 2\,a/2\,\text{sen}\,\alpha\,\dot\alpha\,l\cos\theta\,\dot\theta + (l\cos\theta\,\dot\theta)^2$$
$$+ (l\cos\theta\,\dot\theta)^2 - 2l\cos\theta\,\dot\theta\,a/2\,\text{sen}\,\alpha\,\dot\alpha + (a/2\,\text{sen}\,\alpha\,\dot\alpha)^2$$
$$= l^2\dot\theta^2 + a^2/4\,\dot\alpha^2$$

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$$T = \frac{1}{2} m_1 (\ell^2 \dot{\theta}^2 + a^2/4\, \dot{\alpha}^2) + \frac{1}{2} m_2 (\ell^2 \dot{\theta}^2 + a^2/4\, \dot{\alpha}^2)$$

$$U = m_1 g y_1 + m_2 g y_2 = m_1 g (a/2 \,\text{sen}\,\alpha + \ell \,\text{sen}\,\theta) + m_2 g (\ell \,\text{sen}\,\theta - a/2 \,\text{sen}\,\alpha)$$

El lagrangiano es:

$$\mathcal{L} = \frac{1}{2} (\ell^2 \dot{\theta}^2 + a^2/4\, \dot{\alpha}^2) \underbrace{(m_1 + m_2)}_{m_T} - m_1 g (a/2 \,\text{sen}\,\alpha + \ell \,\text{sen}\,\theta) - m_2 g (\ell \,\text{sen}\,\theta - a/2 \,\text{sen}\,\alpha)$$

* Ecuación de euler-lagrange para θ:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0.$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -m_1 g \ell \cos\theta - m_2 g \ell \cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m_T \ell^2 \dot{\theta} \quad , \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m_T \ell^2 \ddot{\theta}$$

$$\Rightarrow \quad m_T \ell^2 \ddot{\theta} - m_T g \ell \cos\theta = 0.$$

* Ecuación de euler-lagrange para α:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} = 0.$$

$$\frac{\partial \mathcal{L}}{\partial \alpha} = -m_1 g \frac{a}{2} \cos\alpha + m_2 \frac{a}{2} \cos\alpha$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = m_T \frac{a^2}{4} \dot{\alpha} \quad , \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) = m_T \frac{a^2}{4} \ddot{\alpha}$$

$$\Rightarrow \quad m_T \frac{a^2}{4} \ddot{\alpha} - g \frac{a}{2} \cos\alpha (m_2 - m_1) = 0.$$

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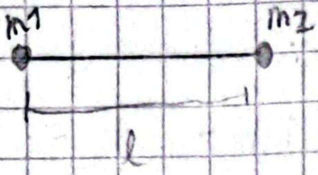
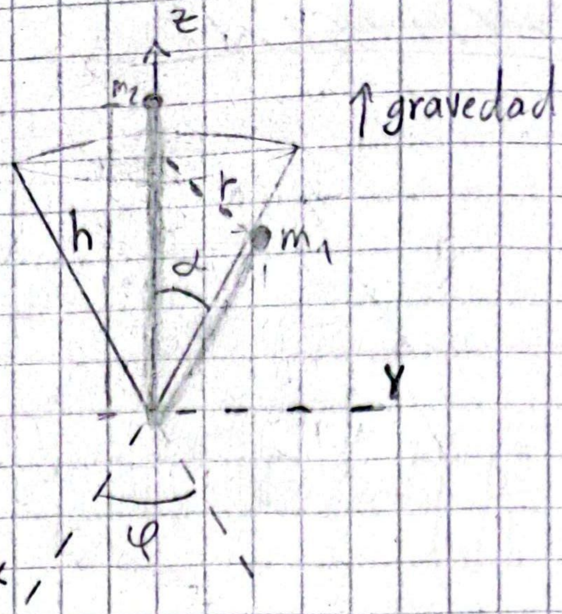
— Coordenadas cíclicas y cantidades conservadas

Si la coordenada cíclica no aparece en el lagrangiano, este es invariante sobre transformaciones de dicha coordenada (def. simetría)

J da las cantidades conservadas

2 Dos masas m1 y m2





ligaduras

superficie del cono para m_1

m_2 se mueve por z

$$|r_1| + |r_2| = l$$

$$\Rightarrow |x_1\hat{i} + y_1\hat{j} + z_1\hat{z}| + |z\hat{z}| = l$$

$$\sqrt{x_1^2 + y_1^2 + z_1^2} + \sqrt{z_2^2} = l$$

$$x_1 = r\cos\varphi,\quad y_1 = r\,\text{sen}\,\varphi,\quad z_1 = r\cot\alpha \;\text{ y }\; z_2 = h$$

$$\Rightarrow \left(r^2\cos^2\varphi + r^2\text{sen}^2\varphi + r^2\cot^2\alpha\right)^{1/2} + h = l$$

$$r(1+\cot^2\alpha)^{1/2} + h = l \xrightarrow[\text{al tiempo}]{\text{derivando respecto}} \dot{r}(1+\cot^2\alpha)^{1/2} + \dot{h} = 0$$

$$\Rightarrow h = l - r\csc\alpha \Big\} \text{Ec } 2 \qquad\qquad \Rightarrow \dot{r}^2\csc^2\alpha + \dot{h}^2 = 0$$

$$\Rightarrow \underbrace{\dot{h}^2 = -\dot{r}^2\csc^2\alpha}_{\text{Ec } 1.}$$

$$\dot{x}_1 = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \qquad \dot{z}_2 = \dot{h}$$

$$\dot{y}_1 = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$$

$$\dot{z}_1 = \dot{r}\cot\alpha$$

$T = T_1 + T_2$

$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2} m_2 (\dot{h}^2)$

$T = \frac{1}{2} m_1 (\dot{r}^2 (1 + \cot^2\alpha) + r^2 \dot{\varphi}^2) + \frac{1}{2} m_2 \dot{h}^2$

$\downarrow$ Eq 1

$T = \frac{1}{2} m_1 (\dot{r}^2 \csc^2\alpha + r^2 \dot{\varphi}^2) - \frac{1}{2} m_2 \dot{r}^2 \csc^2\alpha$

$V = V_1 + V_2$

$V = m_1 g r \cot\alpha + m_2 g h = m_1 g r \cot\alpha + m_2 g l - m_2 g r \csc\alpha$

Eq 2

$\mathcal{L} = T - V$

$\mathcal{L} = \frac{1}{2} m_1 (\dot{r}^2 \csc^2\alpha + r^2 \dot{\varphi}^2) - \frac{1}{2} m_2 \dot{r}^2 \csc^2\alpha$
$\quad - m_1 g r \cot\alpha - m_2 g l + m_2 g r \csc\alpha$

Eq Euler - Lagrange para φ

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$\Rightarrow m_1 r^2 \dot{\varphi} = C_1$

$\Rightarrow \dot{\varphi} = \frac{C_1}{m_1 r^2} \Rightarrow \dot{\varphi}^2 = \frac{C_1^2}{m_1 r^4} = Eq. 3$

Ec de Euler-Lagrange para r

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right)-\frac{\partial \mathcal{L}}{\partial r}=0$$

$$\frac{\partial \mathcal{L}}{\partial r}=m_1 r \dot{\varphi}^2-m_1 g \cot\alpha+m_2 g \csc\alpha$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}}=m_1 \dot{r}\csc^2\alpha-m_2 \dot{r}\csc^2\alpha$$

$$\frac{\partial \mathcal{L}}{\partial t}=m_1 \ddot{r}\csc^2\alpha-m_2 \ddot{r}\csc^2\alpha$$

$$m_1\ddot{r}\csc^2\alpha-m_2\ddot{r}\csc^2\alpha-m_1 r\dot{\varphi}^2+m_1 g\cot\alpha-m_2 g\csc\alpha=0$$

Donde hay equilibrio?

de la Ec de Euler-Lagrange para r, como hay equilibrio $\dot{r}=0$, $\ddot{r}=0$; por lo cual la ecuación de Euler-Lagrange para r es:

$$-m_1 r\underbrace{\dot{\varphi}^2}_{\text{Ec. 3}}+m_1 g\cot\alpha-m_2 g\csc\alpha=0$$

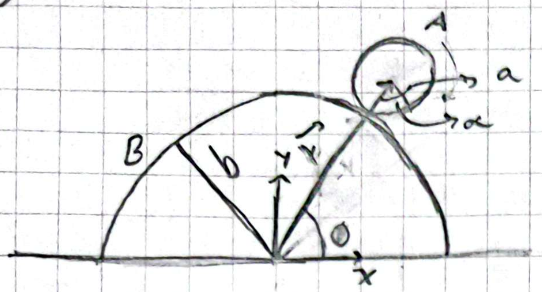
$$-\frac{c_1^2}{m_1 r^3}+m_1 g\cot\alpha-m_2 g\csc\alpha=0$$

$$\frac{-C_1^2}{m_1 r^3} = m_2 g \csc\alpha - m_1 g \cot\alpha$$

$$-C_1^2 = r^3(m_1 m_2 g \csc\alpha - m_1^2 g \cot\alpha)$$

$$\sqrt[3]{\frac{-C_1^2}{m_1 m_2 g \csc\alpha - m_1^2 g \cot\alpha}} = r$$

3)



considerando $b > a$, A rueda sin deslizar sobre B.

Determinar las fuerzas de ligadura y encontrar la posición en la que ambos cilindros se separan.

la energía cinética: La traslación y la rotación del cilindro.

$$T = \frac{1}{2} m v^2 + \frac{1}{2} I_c \omega^2$$

donde $I_c = \frac{1}{2} M R^2 = \frac{1}{2} m_A a^2$, $\omega = \dot{\alpha}$

$$\vec{r} = r \cos\theta \, \hat{e_x} + r \,\text{sen}\,\theta \, \hat{e_y}$$

$$\vec{v} = (\dot{r} \cos\theta - r\,\text{sen}\,\theta \,\dot{\theta}) \hat{e_x} + (\dot{r}\,\text{sen}\,\theta + r \cos\theta \,\dot{\theta}) \hat{e_y}$$

$$v^2 = (\dot{r}\cos\theta - r\,\text{sen}\,\theta\,\dot{\theta})^2 + (\dot{r}\,\text{sen}\,\theta + r\cos\theta\,\dot{\theta})^2$$

$$= \dot{r}^2 \cos^2\theta - 2\dot{r}\cos\theta \, r\,\text{sen}\,\theta\,\dot{\theta} + r^2 \text{sen}^2\theta\,\dot{\theta}^2$$

$$+ \dot{r}^2 \text{sen}^2\theta + 2\dot{r}\cos\theta \, r\,\text{sen}\,\theta\,\dot{\theta} + r^2\cos^2\theta\,\dot{\theta}^2$$

$$= \dot{r}^2 + r^2\dot{\theta}^2$$

$$\Rightarrow T = \frac{1}{2} m_A (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m_A a^2 \dot{\alpha}$$

La energía potencial es:

$$U = mgr\cos\theta$$

El lagrangiano es

$$\mathcal{L} = \frac{1}{2}m_A(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_A a^2(\dot{\alpha}^2) - mgr\cos\theta$$

Las ligaduras son:

$$R\dot{\theta} + a\dot{\alpha} = 0$$
$$R\ddot{\theta} + a\ddot{\alpha} = 0$$
$$\ddot{\alpha} = \frac{R\ddot{\theta}}{a}$$

$$(a+b) - r = 0 \qquad (f_1)$$
$$\Rightarrow \dot{r} = 0$$

$$b\theta - a\alpha = 0$$
$$(b+a)\ddot{\theta} - a\ddot{\alpha} = 0 \rightarrow \ddot{\alpha} = \frac{(b+a)\ddot{\theta}}{a} \quad (f_2)$$

Las ecuaciones de euler-lagrange para r:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} = \lambda_1\frac{\partial f_1}{\partial r} + \lambda_2\frac{\partial f_2}{\partial r}$$

$$\Rightarrow \frac{\partial\mathcal{L}}{\partial r} = m_A r\dot{\theta}^2 - mg\cos\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot{r}} = m_A\dot{r} \Rightarrow \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = m_A\ddot{r} \quad ; \quad a_{r_1} = 1.$$

$$\Rightarrow m_A\ddot{r} - m_A r\dot{\theta}^2 + m_A g\cos\theta = \lambda_1$$
$$m_A(\ddot{r} - r\dot{\theta}^2 + g\cos\theta) = \lambda_1 \qquad (1)$$

La ecuación de euler-lagrange para θ:

$$\frac{\partial\mathcal{L}}{\partial\theta} = mgr\,\text{sen}\,\theta \quad ; \quad a_{\theta 2} = b.$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m_A r^2\dot{\theta} \;; \Rightarrow \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = m_A r^2\ddot{\theta} + 2m_A r\dot{r}\dot{\theta}$$

$$\Rightarrow m_A r^2\ddot{\theta} + 2m_A r\dot{r}\dot{\theta} + mgr\,\text{sen}\,\theta = (b+a)\lambda_2 \qquad (2)$$

La ecuación de euler-lagrange para α:

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = m_A a^2 \dot{\alpha} \quad , \quad \frac{d}{dt}\left(m_A a^2 \dot{\alpha}\right) = m_A a^2 \ddot{\alpha}$$

$$\Rightarrow \quad m_A a^2 \ddot{\alpha} = -\lambda_2 a. \qquad (3)$$

Reemplazando las ligaduras en las ecuaciones,

$$-m_A (a+b)\dot{\theta}^2 + m_A g \cos\theta = \lambda_1 \qquad 1^*$$

$$m(a+b)^2 \ddot{\theta} + mg(a+b)\,\text{sen}\,\theta = (b+a)\lambda_2 \qquad 2^*$$

$$\boxed{-m_A (b+a)\ddot{\theta} = \lambda_2} \qquad (4)$$ fuerza de rozamiento entre los cilindros

Reemplazando (4) en 2^*

$$m(a+b)^2 \ddot{\theta} + mg(a+b)\,\text{sen}\,\theta = (b+a)\left[-m(b+a)\ddot{\theta}\right]$$

$$\ddot{\theta}\left(2m(a+b)^2\right) + mg(a+b)\,\text{sen}\,\theta = 0$$

$$\Rightarrow \quad \ddot{\theta} = -\frac{g\,\text{sen}\,\theta}{2(a+b)}$$

multiplicando por $\dot{\theta}$ la expresión, queda:

$$\ddot{\theta}\dot{\theta} = -\frac{g\,\text{sen}\,\theta\,\dot{\theta}}{2(a+b)}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right) = \frac{d}{dt}\left(-\frac{g\cos\theta}{2(a+b)}\right)$$

$$\Rightarrow \quad \frac{\dot{\theta}^2}{2} = \frac{-g}{2(b+a)}\cos\theta + c$$

Para hallar c, consideramos condiciones iniciales del problema: $\theta = \pi/2$, $\dot{\theta} = 0$.

$$c = \frac{g}{2(b+a)} \quad \Rightarrow \quad \dot{\theta}^2 = \frac{g}{b+a}\left(1 - \cos\theta\right)$$

Esta expresión la reemplazamos en 1^*, nos queda:

$$\lambda_1 = -m(a+b)\frac{g}{(a+b)}(1-\cos\theta) + mg\cos\theta$$

$$\Rightarrow \boxed{\lambda_1 = mg(2\cos\theta - 1)} \longrightarrow$$ Fuerza normal entre los cilindros

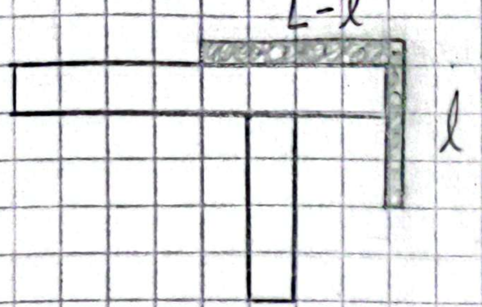
De aquí, podemos obtener el ángulo en que ambos cilindros se separan si: $\lambda_1 = 0$.

$$0 = mg(2\cos\theta - 1)$$

$$2\cos\theta - 1 = 0 \Rightarrow \cos\theta = 1/2$$

$$\Rightarrow \theta = 60°$$

4)



Cuerda uniforme de Masa M, longitud L y densidad lineal ρ

En un determinado instante t, la longitud del segmento vertical de la cuerda que cuelga del borde de la mesa es "l", la fuerza que actúa sobre toda la cuerda es el peso en la zona vertical. El peso en la zona horizontal L - l se equilibra con la relación de la mesa. La segunda ley de Newton es:

$$M\frac{d^2l}{dt^2} = \rho l g \Rightarrow \rho L \frac{d^2 l}{dt^2} = \rho l g$$

$$\Rightarrow L\ddot{l} = lg$$

la solución de esta ecuación diferencial es:

$$l = A\cosh\left(\sqrt{\frac{g}{L}}\,t\right) + B\,\mathrm{senh}\left(\sqrt{\frac{g}{L}}\,t\right)$$

$$v = \frac{dl}{dt} = \sqrt{\frac{g}{L}}\left[A\,\mathrm{senh}\left(\sqrt{\frac{g}{L}}\,t\right) + B\cosh\left(\sqrt{\frac{g}{L}}\,t\right)\right]$$

Ya que la cuerda se suelta desde el reposo cuando la sección de longitud l está colgando.

$t = 0$, $l = l_0$, $v = 0$

$$\boxed{l = l_0 \cosh\left(\sqrt{\frac{g}{L}}\, t\right) \qquad v = \sqrt{\frac{g}{L}}\, l_0 \operatorname{Senh}\left(\sqrt{\frac{g}{L}}\, t\right)}$$