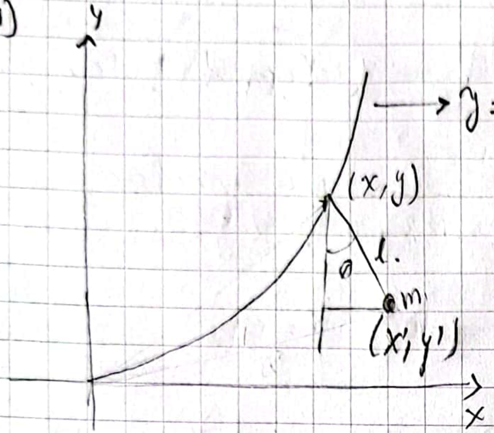
Mecánica Lagrangiana

Problemas viernes.

1)



$\longrightarrow y = ax^2$. Encontrar el hamiltoniano.

$x' = l\,\text{sen}\,\theta + x$,

$y' = y - l\cos\theta$
$= ax^2 - l\cos\theta$

} Coordenadas generalizadas.

Lagrangeano:

$$L = T - V$$

$$= \frac{1}{2} m \dot{x}'^2 - mgh.$$

$$\dot{x}'^2 = \left(l\cos\theta\,\dot{\theta} + \dot{x},\ 2ax\dot{x} + l\,\text{sen}\,\theta\,\dot{\theta}\right)^2$$

$$= l^2\cos^2\theta\,\dot{\theta}^2 + \dot{x}^2 + 2l\cos\theta\,\dot{\theta}\,\dot{x}$$
$$+ 4a^2x^2\dot{x}^2 + l^2\,\text{sen}^2\theta\,\dot{\theta}^2 + 4ax\dot{x}\,l\,\text{sen}\,\theta\,\dot{\theta}$$

$$= l^2\dot{\theta}^2(\underbrace{\cos^2\theta + \text{sen}^2\theta}_{\to 1}) + \dot{x}^2(1 + 4a^2x^2)$$
$$+ 2l\dot{x}\dot{\theta}(\cos\theta + 4ax\,\text{sen}\,\theta)$$

$$\Rightarrow L = \frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{x}^2(1 + 4a^2x^2) + 2l\dot{x}\dot{\theta}(\cos\theta + 4ax\,\text{sen}\,\theta)\right)$$
$$- mg(ax^2 - l\cos\theta)$$

momentos conjugados:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{1}{2} m \left(2\dot{x}(1+4a^2x^2) + 2l\dot{\theta}(\cos\theta + 2ax\,\text{sen}\,\theta) \right)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m \left(2l^2\dot{\theta} + 2l\dot{x}(\cos\theta + 2ax\,\text{sen}\,\theta) \right)$$

Resolviendo el sistema,

$$\left[\frac{2p_x}{m} - 2l\dot{\theta}(\cos\theta + 2ax\,\text{sen}\,\theta) \right] \frac{1}{2(1+4a^2x^2)} = \dot{x}$$

$$\Rightarrow \dot{x} = \frac{p_x}{m(1+4a^2x^2)} - \frac{l\dot{\theta}(\cos\theta + 2ax\,\text{sen}\,\theta)}{(1+4a^2x^2)} \longrightarrow 1$$

$$\left[\frac{2p_\theta}{m} - 2l\dot{x}(\cos\theta + 2ax\,\text{sen}\,\theta) \right] \frac{1}{2l^2} = \dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{p_\theta}{ml^2} - \frac{\dot{x}}{l}(\cos\theta + 2ax\,\text{sen}\,\theta) \longrightarrow 2$$

Reemplazando 2 en 1,

$$\dot{x} = \frac{p_x}{m(1+4a^2x^2)} - \frac{l(\cos\theta + 2ax\,\text{sen}\,\theta)}{(1+4a^2x^2)} \left[\frac{p_\theta}{ml^2} - \frac{\dot{x}}{l}(\cos\theta + 2ax\,\text{sen}\,\theta) \right]$$

$$\Rightarrow \dot{x} = \frac{p_x}{m(1+4a^2x^2)} - \frac{p_\theta(\cos\theta + 2ax\,\text{sen}\,\theta)}{ml(1+4a^2x^2)} + \frac{\dot{x}(\cos\theta + 2ax\,\text{sen}\,\theta)^2}{(1+4a^2x^2)}$$

$$\dot{x}\left(1-\frac{(\cos\theta+2ax\,\text{sen}\,\theta)^2}{(1+4a^2x^2)}\right)=\frac{p_x}{m(1+4a^2x^2)}-\frac{p_\theta(\cos\theta+2ax\,\text{sen}\,\theta)}{ml(1+4a^2x^2)}$$

$$\Rightarrow \dot{x}\left(\frac{(1+4a^2x^2)-(\cos\theta+2ax\,\text{sen}\,\theta)^2}{(1+4a^2x^2)}\right)=\frac{p_x}{m(1+4a^2x^2)}-\frac{p_\theta(\cos\theta+2ax\,\text{sen}\,\theta)}{ml(1+4a^2x^2)}$$

$$\Rightarrow \dot{x}=\left[\frac{p_x}{m}-\frac{p_\theta}{ml}(\cos\theta+2ax\,\text{sen}\,\theta)\right]\frac{1}{(1+4a^2x^2)-(\cos\theta+2ax\,\text{sen}\,\theta)^2}$$

Reemplazando $\dot{x}$ en ②,

$$\dot{\theta}=\frac{p_\theta}{ml^2}-\frac{(\cos\theta+2ax\,\text{sen}\,\theta)}{l}\left[\frac{p_x}{m}-\frac{p_\theta}{ml}(\cos\theta+2ax\,\text{sen}\,\theta)\right]\left[\frac{1}{(1+4a^2x^2)-(\cos\theta+2ax\,\text{sen}\,\theta)^2}\right]$$

$$\Rightarrow \dot{\theta}=\frac{p_\theta}{ml^2}-\frac{(\cos\theta+2ax\,\text{sen}\,\theta)}{l(1+4a^2x^2-(\cos\theta+2ax\,\text{sen}\,\theta)^2)}\left[\frac{p_x}{m}-\frac{p_\theta}{ml}(\cos\theta+2ax\,\text{sen}\,\theta)\right]$$

El Hamiltoniano es:

$$H(p_\theta,p_x,\theta,x)=p_\theta\dot{\theta}+p_x\dot{x}-\mathcal{L}(\theta,\dot{\theta},x,\dot{x})$$

llamando:

$A=(1+4a^2x^2)$ $\Rightarrow$ $\dot{x}=\left[\frac{p_x}{m}-\frac{p_\theta B}{ml}\right]\frac{1}{A-B^2}$

$B=\cos\theta+2ax\,\text{sen}\,\theta$

$$\dot{\theta}=\frac{p_\theta}{ml^2}-\frac{B}{l(A-B^2)}\left[\frac{p_x}{m}-\frac{p_\theta B}{ml}\right]$$

$$H = \frac{p_\theta^2}{ml^2} - \frac{p_\theta B}{l(A-B^2)}\left[\frac{p_x}{m} - \frac{p_\theta}{ml}B\right]$$

$$+ \frac{p_x}{A-B^2}\left[\frac{p_x}{m} - \frac{p_\theta}{ml}B\right]$$

$$- \frac{1}{2}m\left\{\left[\frac{p_\theta}{m} - \frac{B}{A-B^2}\left(\frac{p_x}{m} - \frac{p_\theta}{ml}B\right)\right]^2\right.$$

$$+ A\left[\left(\frac{p_x}{m} - \frac{p_\theta}{ml}B\right)\left(\frac{1}{A-B^2}\right)\right]^2$$

$$\left.+ 2lB\left[\left(\frac{p_x}{m} - \frac{p_\theta}{ml}B\right)\left(\frac{1}{A-B^2}\right)\right]\left[\frac{p_\theta}{ml^2} - \frac{B}{l(A-B^2)}\left(\frac{p_x}{m} - \frac{p_\theta B}{ml}\right)\right]\right\}$$

$$+ mg(ax^2 - l\cos\theta)$$

3) $H = q + te^{p}$

$Q = q + e^{p}, \quad P = p$

desarrollo:

Primero a mostrar que es una transformación canónica

$$\{Q,Q\} = \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q} = (1)(e^{p}) - (e^{p})(1) = 0$$

$$\{P,P\} = \frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q} = (0)(1) - (1)(0) = 0$$

$$\{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = (1)(1) - (e^{p})(0) = 1$$

$\therefore$ Es una transformación canónica

$$\{Q,Q\} = 0, \ \{P,P\} = 0, \ \{Q,P\} = 1.$$

Encontrar la función generatriz:

Se propone $F_2(q, P)$ donde:

$$\boxed{p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P}} \quad a)$$

nuestra transformación canónica es

$$\underbrace{Q = q + e^{p}}_{b)}, \quad \underbrace{P = p}_{c)}$$

reemplazando c) en b)

$$\underbrace{Q = q + e^{P}}_{d)}$$

y si recordamos a) la función generatriz que satisface estas condiciones:

$$\underbrace{F_2 = qP + e^{P}}_{\uparrow}$$

ya que precisamente F_2 cumple con a)

$$p = \frac{\partial F_2}{\partial q} = \underbrace{P}_{c} \checkmark, \quad Q = \frac{\partial F_2}{\partial P} = \underbrace{q + e^{P}}_{d)}$$

$\therefore$ $F_2 = qP + e^{P}$ es la función generatriz de esta transformación canónica.

Encontrar el nuevo Hamiltoniano:

El original:

$$\mathcal{H} = q + e^{p}$$

la transformación canónica

$$Q = q + e^{P}, \quad P = p$$

$$\Rightarrow q = Q - e^{P}, \quad P = p$$

$$\therefore \mathcal{H} = q + e^{p} = (Q - e^{P}) + e^{P}$$

$$\tilde{\mathcal{H}} = Q \; \} \text{ El nuevo Hamiltoniano}$$

Resolver el nuevo hamiltoniano

$\dot{Q} = \frac{\partial \tilde{\mathcal{H}}}{\partial P}$	$\dot{P} = -\frac{\partial \tilde{\mathcal{H}}}{\partial Q}$
$\dot{Q} = 0$	$\dot{P} = -1$
$Q = C_1$	$P = -t + C_2$
$Q(t) = C_1$	$P(t) = C_2 - t$

Esto representa cómo evoluciona el sistema con el tiempo en las nuevas coordenadas. En estas coordenadas Q permanece constante en el tiempo y P decrece linealmente con el tiempo.