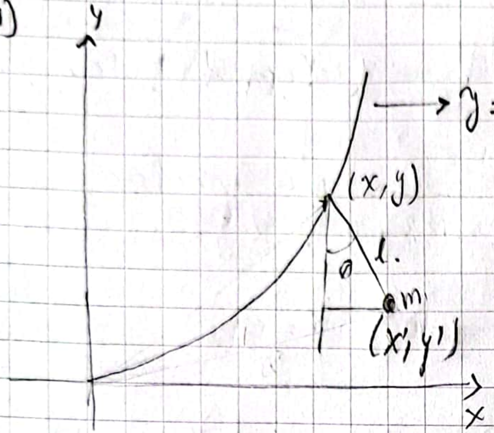
Mecánica Lagrangiana

Problemas viernes

1)



$\rightarrow y = ax^2$. Encontrar el hamiltoniano.

$$x' = l\,\text{sen}\,\theta + x,$$
$$y' = y - l\cos\theta$$
$$= ax^2 - l\cos\theta$$

Coordenadas generalizadas.

Lagrangeano:

$$\mathcal{L} = T - V$$
$$= \frac{1}{2} m \dot{x}'^2 - mgh.$$

$$\dot{x}'^2 = \left(l\cos\theta\,\dot\theta + \dot x,\ 2ax\dot x + l\,\text{sen}\,\theta\,\dot\theta\right)^2$$

$$= l^2\cos^2\theta\,\dot\theta^2 + \dot x^2 + 2l\cos\theta\,\dot\theta\,\dot x$$
$$+ 4a^2x^2\dot x^2 + l^2\text{sen}^2\theta\,\dot\theta^2 + 4ax\dot x\,l\,\text{sen}\,\theta\,\dot\theta$$

$$= l^2\dot\theta^2(\underbrace{\cos^2\theta + \text{sen}^2\theta}_{1}) + \dot x^2(1+4a^2x^2)$$
$$+ 2l\dot x\dot\theta(\cos\theta + 4ax\,\text{sen}\,\theta)$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}m\left(l^2\dot\theta^2 + \dot x^2(1+4a^2x^2) + 2l\dot x\dot\theta(\cos\theta + 2ax\,\text{sen}\,\theta)\right)$$
$$- mg(ax^2 - l\cos\theta)$$

momentos conjugados:

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{1}{2} m \left(2\dot{x}(1+4a^2x^2) + 2l\dot{\theta}(\cos\theta + 2ax\,\text{sen}\,\theta) \right)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m \left(2l^2\dot{\theta} + 2l\dot{x}(\cos\theta + 2ax\,\text{sen}\,\theta) \right)$$

Resolviendo el sistema,

$$\left[\frac{2p_x}{m} - 2l\dot{\theta}(\cos\theta + 2ax\,\text{sen}\,\theta) \right] \frac{1}{2(1+4a^2x^2)} = \dot{x}$$

$$\Rightarrow \dot{x} = \frac{p_x}{m(1+4a^2x^2)} - \frac{l\dot{\theta}(\cos\theta + 2ax\,\text{sen}\,\theta)}{(1+4a^2x^2)} \longrightarrow 1$$

$$\left[\frac{2p_\theta}{m} - 2l\dot{x}(\cos\theta + 2ax\,\text{sen}\,\theta) \right] \frac{1}{2l^2} = \dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{p_\theta}{ml^2} - \frac{\dot{x}}{l}(\cos\theta + 2ax\,\text{sen}\,\theta) \longrightarrow 2$$

Reemplazando 2 en 1,

$$\dot{x} = \frac{p_x}{m(1+4a^2x^2)} - \frac{l(\cos\theta + 2ax\,\text{sen}\,\theta)}{(1+4a^2x^2)} \left[\frac{p_\theta}{ml^2} - \frac{\dot{x}}{l}(\cos\theta + 2ax\,\text{sen}\,\theta) \right]$$

$$\Rightarrow \dot{x} = \frac{p_x}{m(1+4a^2x^2)} - \frac{p_\theta(\cos\theta + 2ax\,\text{sen}\,\theta)}{ml(1+4a^2x^2)} + \frac{\dot{x}(\cos\theta + 2ax\,\text{sen}\,\theta)^2}{(1+4a^2x^2)}$$

$$\dot{x}\left(1-\frac{(\cos\theta+2ax\,\text{sen}\,\theta)^2}{(1+4a^2x^2)}\right)=\frac{p_x}{m(1+4a^2x^2)}-\frac{p_\theta(\cos\theta+2ax\,\text{sen}\,\theta)}{ml(1+4a^2x^2)}$$

$$\Rightarrow \dot{x}\left(\frac{(1+4a^2x^2)-(\cos\theta+2ax\,\text{sen}\,\theta)^2}{(1+4a^2x^2)}\right)=\frac{p_x}{m(1+4a^2x^2)}-\frac{p_\theta(\cos\theta+2ax\,\text{sen}\,\theta)}{ml(1+4a^2x^2)}$$

$$\Rightarrow \dot{x}=\left[\frac{p_x}{m}-\frac{p_\theta}{ml}(\cos\theta+2ax\,\text{sen}\,\theta)\right]\frac{1}{(1+4a^2x^2)-(\cos\theta+2ax\,\text{sen}\,\theta)^2}$$

Reemplazando $\dot{x}$ en ②,

$$\dot{\theta}=\frac{p_\theta}{ml^2}-\frac{(\cos\theta+2ax\,\text{sen}\,\theta)}{l}\left[\frac{p_x}{m}-\frac{p_\theta}{ml}(\cos\theta+2ax\,\text{sen}\,\theta)\right]$$

$$\left[\frac{1}{(1+4a^2x^2)-(\cos\theta+2ax\,\text{sen}\,\theta)^2}\right]$$

$$\Rightarrow \dot{\theta}=\frac{p_\theta}{ml^2}-\frac{(\cos\theta+2ax\,\text{sen}\,\theta)}{l(1+4a^2x^2-(\cos\theta+2ax\,\text{sen}\,\theta))^2}\left[\frac{p_x}{m}-\frac{p_\theta}{ml}(\cos\theta+2ax\,\text{sen}\,\theta)\right]$$

El Hamiltoniano es:

$$H(p_\theta,p_x,\theta,x)=p_\theta\dot{\theta}+p_x\dot{x}-\mathcal{L}(\theta,\dot{\theta},x,\dot{x})$$

llamando:

$$A=(1+4a^2x^2)\qquad\Rightarrow\qquad \dot{x}=\left[\frac{p_x}{m}-\frac{p_\theta B}{ml}\right]\frac{1}{A-B^2}$$

$$B=\cos\theta+2ax\,\text{sen}\,\theta$$

$$\dot{\theta}=\frac{p_\theta}{ml^2}-\frac{B}{l(A-B^2)}\left[\frac{p_x}{m}-\frac{p_\theta B}{ml}\right]$$

$$H = \frac{p_\theta^2}{ml^2} - \frac{p_\theta B}{l(A-B^2)}\left[\frac{p_x}{m} - \frac{p_\theta}{ml}B\right]$$

$$+ \frac{p_x}{A-B^2}\left[\frac{p_x}{m} - \frac{p_\theta}{ml}B\right]$$

$$-\frac{1}{2}m\left\{\left[\frac{p_\theta}{m} - \frac{B}{A-B^2}\left(\frac{p_x}{m} - \frac{p_\theta}{ml}B\right)\right]^2\right.$$

$$+A\left[\left(\frac{p_x}{m} - \frac{p_\theta}{ml}B\right)\left(\frac{1}{A-B^2}\right)\right]^2$$

$$\left.+2lB\left[\left(\frac{p_x}{m} - \frac{p_\theta}{ml}B\right)\left(\frac{1}{A-B^2}\right)\right]\left[\frac{p_\theta}{ml^2} - \frac{B}{l(A-B^2)}\left(\frac{p_x}{m} - \frac{p_\theta B}{ml}\right)\right]\right\}$$

$$+ mg(ax^2 - l\cos\theta)$$

(2) $H = \frac{p^2}{2m} - A\left(\frac{p}{m}\cos\gamma t + \gamma q \sin\gamma t\right) + \frac{1}{2}Kq^2$

A, γ y $K \rightarrow$ ctes.

$H = \sum_{i=1}^{n} \dot{q}_i p_i - \mathcal{L}(q, \dot{q}, t)$, $p_i = m\dot{q}_i$

$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$, $\dot{q} = \frac{\partial H}{\partial p}$

$\dot{q} = \frac{p}{m} - \frac{A}{m}\cos\gamma t \longrightarrow p = \dot{q}m + A\cos\gamma t$

$\mathcal{L} = p\dot{q} - H(q, p, t)$

$p\dot{q} = (\dot{q}m + A\cos\gamma t)\dot{q} = \dot{q}^2 m + \dot{q}A\cos\gamma t$

Reemplazo p en los terminos del Hamiltoniano que lo contengan

- $\frac{p^2}{2m} = \frac{1}{2m}(\dot{q}m + A\cos\gamma t)^2 = \frac{\dot{q}^2 m^{\cancel{2}}}{2\cancel{m}} + \frac{A^2\cos^2\gamma t}{2m} + \frac{\cancel{2}\dot{q}\cancel{m}A\cos\gamma t}{\cancel{2m}}$

$\frac{p^2}{2m} = \frac{\dot{q}^2 m}{2} + \frac{A^2\cos^2\gamma t}{2m} + A\dot{q}\cos\gamma t$

- $-\frac{Ap}{m}\cos\gamma t = -\frac{A}{m}(\dot{q}m + A\cos\gamma t)\cos\gamma t$

$= -A\dot{q}\cos\gamma t - \frac{A^2}{m}\cos^2\gamma t$

$\mathcal{L} = \dot{q}p - H(q, p, t)$

$\mathcal{L} = \underbrace{\dot{q}^2 m} + \cancel{\dot{q}A\cos\gamma t} - \underbrace{\frac{\dot{q}^2 m}{2}} - \underbrace{\frac{A^2\cos^2\gamma t}{2m}} - \cancel{A\dot{q}\cos\gamma t}$

$+ A\dot{q}\cos\gamma t + \underbrace{\frac{A^2}{m}\cos^2\gamma t} + A\gamma q\sin\gamma t - \frac{1}{2}Kq^2$

$\left[\mathcal{L} = \frac{\dot{q}^2 m}{2} + \frac{A^2}{2m}\cos^2\gamma t + A\gamma q\sin\gamma t - \frac{1}{2}Kq^2\right]$ (a)

3) $H = q + t e^{p}$

$Q = q + e^{p}, \quad P = p$

desarrollo:

Primero a mostrar que es una transformación canónica

$$\{Q,Q\} = \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial q} = (1)(e^{p}) - (e^{p})(1) = 0$$

$$\{P,P\} = \frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q} = (0)(1) - (1)(0) = 0$$

$$\{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = (1)(1) - (e^{p})(0) = 1$$

$\therefore$ Es una transformación canónica

$$\{Q,Q\} = 0, \ \{P,P\} = 0, \ \{Q,P\} = 1.$$

Powered by CamScanner

Encontrar la función generatriz:

Se propone $F_2(q, P)$ donde:

$$\boxed{p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P}} \quad a)$$

nuestra transformación canonica es

$$\underbrace{Q = q + e^p}_{b)}, \quad \underbrace{P = p}_{c)}$$

reemplazando c) en b)

$$\underbrace{Q = q + e^P}_{d)}$$

y si recordamos a) la función generatriz que satisface estas condiciones:

$$\underbrace{F_2 = qP + e^P}_{\uparrow}$$

ya que precisamente F_2 cumple con a)

$$p = \frac{\partial F_2}{\partial q} = \underbrace{p}_{c} \checkmark, \quad Q = \frac{\partial F_2}{\partial P} = \underbrace{q + e^P}_{d)}$$

$\therefore$ $F_2 = qP + e^P$ es la función generatriz de esta transformación canónica.

Encontrar el nuevo Hamiltoniano:

El original:

$$\mathcal{H} = q + e^{p}$$

la transformación canónica

$$Q = q + e^{P} , \quad P = p$$

$$\Rightarrow q = Q - e^{P} , \quad P = p$$

$$\therefore \mathcal{H} = q + e^{p} = (Q - e^{P}) + e^{P}$$

$$\tilde{\mathcal{H}} = Q \ \} \text{ El nuevo Hamiltoniano}$$

Resolver el nuevo hamiltoniano

$$\dot{Q} = \frac{\partial \tilde{\mathcal{H}}}{\partial P} \qquad \dot{P} = -\frac{\partial \tilde{\mathcal{H}}}{\partial Q}$$

$$\dot{Q} = 0 \qquad \dot{P} = -1$$

$$Q = C_1 \qquad P = -t + C_2$$

$$Q(t) = C_1 \qquad P(t) = C_2 - t$$

Esto representa cómo evoluciona el sistema con el tiempo en las nuevas coordenadas. En estas coordenadas Q permanece constante en el tiempo y P decrece linealmente con el tiempo.

(4) $\mathcal{H} = \frac{1}{2}\left(q p^3 + \frac{q}{p}\right)$ Ec. de movimiento

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \quad , \quad \frac{\partial \mathcal{H}}{\partial q} = -\dot{p}$$

$$\dot{q} = \frac{1}{2} q (3p^2) - \frac{q}{2p^2} \quad ; \quad -\dot{p} = \frac{p^3}{2} + \frac{1}{2p}$$

$$\underbrace{\dot{q} = \frac{3}{2} q p^2 - \frac{q}{2p^2}} \quad , \quad \underbrace{-\dot{p} = \frac{p^3}{2} + \frac{1}{2p}}$$

$$\dot{p} = -\frac{p^4 + 1}{2p} \rightarrow \frac{p\dot{p}}{(p^4+1)} = -\frac{1}{2}$$

$$\frac{p}{(p^4+1)} dp = -\frac{1}{2} dt$$

$$\frac{1}{2} \arctan(p^2) = -\frac{t}{2} + c$$

$$p^2 = \tan(-t + c_1)$$

$$\boxed{p = \sqrt{\tan(-t + c_1)}}$$

$$\dot q = \frac{1}{2}\left(3q(\cancel{A}\tan(-t+c_1))^{\cancel{2}} - \frac{q}{(\cancel{A}\tan(-t+c_1))^{\cancel{2}}}\right)$$

$$\dot q = \frac{q}{2}\left(3\tan(-t+c_1) - \frac{1}{\tan(-t+c_1)}\right)$$

$$\int \frac{1}{q}\,dq = \int \frac{1}{2}\left(3\tan(-t+c_1) - \frac{1}{\tan(-t+c_1)}\right)dt$$

$$\ln|q| = \frac{3}{2}\left(-\ln|\cos(-t+c_1)|\right) - \ln|\sin(-t+c_1)|$$

$$\boxed{q = -\cos^{3/2}(-t+c_1)\sin(-t+c_1)}$$