A Type Theory for Strictly Associative Infinity Categories

Eric Finster   Alex Rice   Jamie Vicary

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Outline

1. Strict Infinity Categories
2. CaTT
3. Strict Associators

Globular Sets

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Ordinary 1-categories

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Globular sets

$$\begin{tikzcd} \vdots \\ G_2 \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "t_1"] \\ G_1 \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "t_0"] \\ G_0 \end{tikzcd}$$

Globular Sets

Definition

A *globular set* $\mathcal{G}$ consists of sets G_n for each n and maps $s_n, t_n : G_{n+1} \to G_n$ for each n such that the following *globularity conditions* hold:

$$s_n \circ s_{n+1} = s_n \circ t_{n+1}$$
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Terminal globular set

The terminal globular set has one cell at each dimension and all source and target maps are uniquely defined.

Compostition in Globular Sets

Composition of 1 cells

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

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Composition along a 1-boundary:
$$\begin{tikzcd}
\bullet \arrow[r, bend left=70, ""{name=T}] \arrow[r, ""{name=M}] \arrow[r, bend right=70, ""{name=B}] & \bullet
\arrow[from=M, to=T, phantom, "\beta\Uparrow"]
\arrow[from=B, to=M, phantom, "\alpha\Uparrow"]
\end{tikzcd}$$

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Composition along a 0-boundary:
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Uparrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\beta\Uparrow"] & \bullet \end{tikzcd}$$

Strict Infinity Categories - Composition

In a strict infinity category we have binary composition of n-cells for along a k boundary for all $k < n$.

Composition

If f and g are n-cells with the k-target of f equalling the k-source of g then there is an n-cell $f \circ_k g$.

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Identities

For each n-cell f there is an $(n+1)$-cell $\mathrm{id}_f : f \to f$.

Strict Infinity Categories - Associativity

Associativity: if $0 \leq k < n$ and f, g, and h are n-cells then:

$$f \circ_k (g \circ_k h) = (f \circ_k g) \circ_k h$$

Strict Infinity Categories - Identities

Identities: if $0 \leq k < n$ and f is an n-cell with k-source x and k-target y then:

$$\mathrm{id}(x) \circ_k f = f = f \circ_k \mathrm{id}(y)$$

Strict Infinity Categories - Interchange

Interchange: if $0 \leq q < p < n$ and a, b, c, d are n-cells then:

$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$

$$
\begin{array}{c}
\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow c"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow d"] & \bullet
\end{tikzcd} \\
\circ \\
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\end{tikzcd}
\end{array}
=
\begin{tikzcd}
\bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Uparrow c"] \arrow[r, phantom, shift right=3, "\Uparrow a"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Uparrow d"] \arrow[r, phantom, shift right=3, "\Uparrow b"] & \bullet
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Further if $f \circ_k g$ is well defined then:

$$\mathrm{id}_f \circ_k \mathrm{id}(g) = \mathrm{id}(f \circ_k g)$$

Monoidal Categories

Monoidal categories are instances of infinity categories.

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A monoidal category is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object I satisfying some conditions.

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A monoidal category is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object I satisfying some conditions.

A strict infinity category with one object and no non-identity n-cells for n higher than 2 is a strict monoidal category.

Pasting Diagrams

A pasting diagram represents a composition that can be done in an infinity category. More precisely it is an object of the free strict infinity category on the terminal globular set.

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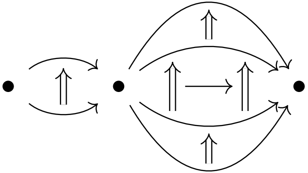
The compositions we have already seen form pasting diagrams.

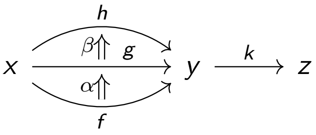
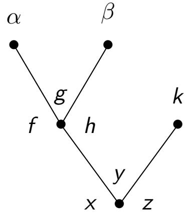
$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Uparrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\beta\Uparrow"] & \bullet \end{tikzcd}$$

We can also form more complicated compositions as pasting diagrams.



Trees

<table>
<tr><th>Pasting diagram</th><th>Batanin tree</th></tr>
<tr><td>
</td><td>
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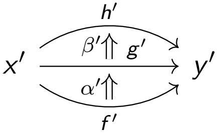
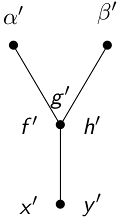
<tr><td>
</td><td>
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The monoidal product in **Set** is *not* strict.

Strictification

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All weak monoidal categories and all weak 2-categories are equivalent to a strict version of themselves.
However this is no longer possible at dimensions 3 and higher.

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- Types: A type contains all the information of the sources and targets for a term.
- Substitutions: A substitution is a morphism between contexts.

Types and Variables

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The arrow constructor takes 2 terms and a type as arguments. A variable of type $s \rightarrow_A t$ has source s, target t and lower dimensional sources and targets given by A.

$$
\begin{tikzcd}
x \arrow[r, bend left=50, "g"{name=G}] \arrow[r, bend right=50, "f"'{name=F}] & y
\arrow[from=F, to=G, Rightarrow, "\alpha"']
\end{tikzcd}
$$

Contexts and Substitutions

Contexts consist of a list of pairs of variable names and types.

All pasting diagrams describe a context.

$$x \xrightarrow{f} y \xrightarrow{g} z$$

gives the context

$$\begin{aligned} &x : \star, \\ &y : \star, \\ &f : x \to_\star y, \\ &z : \star, \\ &g : y \to_\star z \end{aligned}$$

Contexts can represent non-pasting diagrams.

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Substitutions $\sigma : \Gamma \to \Delta$ map variables of Γ to terms of Δ, this

Disc contexts

We can define disc contexts by mutual induction as follows:

$$\begin{aligned} D_0 &= (d_0^- : \star) & A_0 &= \star \\ D_{n+1} &= D_n, d_n^+ : A_n, d_{n+1} & A_{n+1} &= d_n^- \to_{A_n} d_n^+ \end{aligned}$$

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These are all pasting diagrams.
A well-typed substitution from a disc context has the same data as a term.

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Taking the composite of the diagram:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

gives the composite $f \circ g$.

Over the singleton pasting diagram

$$x$$

and taking $s = x$ and $t = x$ we get a term from x to x representing the identity on x.

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This gives us the coherence constructor $\mathsf{coh}\ (\Gamma : A)[\sigma]$ which takes a pasting context Γ, a type A over Γ and $\sigma : \Gamma \to \Delta$ to form a term in Δ.

Examples

Identity

Let t be a term. The identity on t is:

$$\mathrm{coh}\ (D_1 : d_1^- \to_{A_1} d_1^-)[\sigma]$$

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1-composition

Let $s : x \to_\star y$ and $t : y \to_\star z$ be terms. Their composite is given by:

$$\mathsf{coh}\ (x : \star, y : \star, f : x \to_\star y, z : \star, g : y \to_\star z : x \to_\star z)[\sigma]$$

where $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(z) = z$, $\sigma(f) = s$, $\sigma(g), = t$.

Examples

Take the context
$\Gamma = w : \star, x : \star, f : w \to_\star x, y : \star, g : x \to_\star y, z : \star, h : y \to_\star z.$
The associator is given by:

$$\mathrm{coh}\ (\Gamma : (f \circ g) \circ h \to_{w \to_\star z} f \circ (g \circ h))[\mathrm{id}]$$

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By showing this is confluent and terminating, we can create a type theory where both the source target of the associator are the same but retain decidable type checking and equality.

Insertion

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We propose an operation on terms which we call *Insertion*. This collapses certain compound composites into a single composite by "inserting" the inner composite into the outer composite.

1-Associator

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[r, "g"] & z & & \color{red}{x'} \arrow[r, red, "f'"] & \color{red}{y'} \arrow[r, red, "g'"] \arrow[llll, Rightarrow, bend left] & \color{red}{z'} \end{tikzcd}$$

is sent to:

$$\begin{tikzcd} x \arrow[r, "f"] & \color{red}{x'} \arrow[r, red, "f'"] & \color{red}{y'} \arrow[r, red, "g'"] & \color{red}{z'} \end{tikzcd}$$

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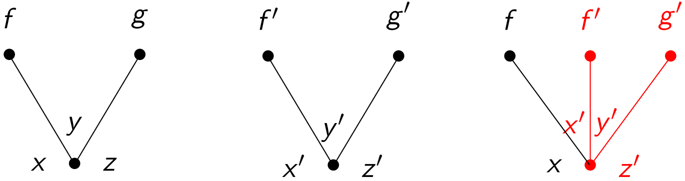

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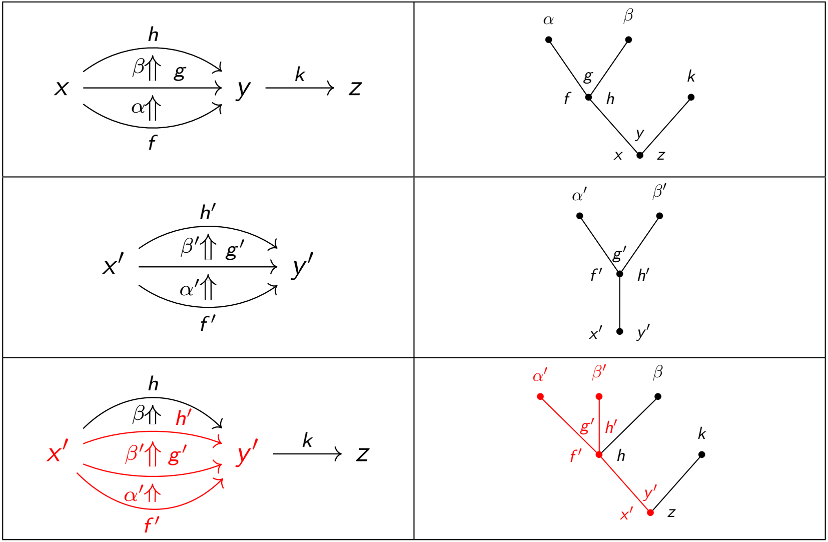

Insertion Rule

Suppose we have a coherence $\mathsf{coh}\ (\Gamma : A)[\sigma]$ where $\sigma(x) \equiv \mathsf{coh}\ (\Delta : B)[\tau]$ for some locally maximal x in Γ. So far we have described the action on pasting diagrams but not on the entire term.

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- A new substitution assigning arguments to the new composite, which we call $\sigma \ll_x \tau$

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To make a new coherence term we need:

- A new pasting diagram, which we call $\Gamma \ll_x \Delta$.
- A new type over this pasting diagram. We form a substitution $\kappa : \Gamma \to \Gamma \ll_x \Delta$ and then use $A[\![\kappa]\!]$.
- A new substitution assigning arguments to the new composite, which we call $\sigma \ll_x \tau$

We therefore get the following generator for our equality relation:

$$\text{coh } (\Gamma : A)[\sigma] = \text{coh } ((\Gamma \ll_x \Delta) : (A[\![\kappa]\!]))[(\sigma \ll_x \tau)]$$

Example

Take the term $a \circ (b \circ c)$. Written in full this is:

$$\mathsf{coh}\ (x \xrightarrow{f} y \xrightarrow{g} z : x \to_\star z)[\sigma]$$

with $\sigma(f) = a$ and:

$$\sigma(g) = \mathsf{coh}\ (x' \xrightarrow{f'} y' \xrightarrow{g'} z' : x' \to_\star z')[\tau]$$

with $\tau(f') = b$ and $\tau(g') = c$.

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In this case the inserted context is $x \xrightarrow{a} x' \xrightarrow{a'} y' \xrightarrow{b'} z'$ with $\kappa(a) = a$ and $\kappa(b) = a' \circ b'$ which gives us final term:

$$\mathsf{coh}\ (x \xrightarrow{a} x' \xrightarrow{a'} y' \xrightarrow{b'} z' : x \to_\star z')[\sigma \ll_x \tau]$$

Universal Property of Insertion

Insertion also satisfies a universal property. Suppose we have coh $(\Delta : A)[\sigma]$ where $\sigma(x) = \text{coh}\ (\Theta : B)[\tau]$.

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\Theta \arrow[drr, bend right, "\tau"'] & & \\
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$$\begin{tikzcd}
 & \Delta \arrow[d, "\kappa"] \arrow[dr, bend left, "\sigma"] & \\
\Theta \arrow[rr, bend right, "\tau"'] & \Delta \ll_x \Theta & \\
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 & \Delta \arrow[d, "\kappa"] \arrow[ddr, bend left, "\sigma"] & \\
\Theta \arrow[r, "\iota"'] \arrow[drr, bend right, "\tau"] & \Delta \ll_x \Theta & \\
 & & \Gamma
\end{tikzcd}$$

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 & \Delta \arrow[d, "\kappa"] \arrow[ddr, bend left, "\sigma"] & \\
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\end{tikzcd}
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$$\begin{tikzcd}
D_n \arrow[r, "x"] \arrow[d, "\text{coh } (\Theta : B)[\text{id}]"'] \arrow[dr, phantom, "\ulcorner", very near end] & \Delta \arrow[d, "\kappa"] \arrow[ddr, bend left, "\sigma"] & \\
\Theta \arrow[r, "\iota"'] \arrow[drr, bend right, "\tau"'] & \Delta \ll_x \Theta \arrow[dr, dashed, "\sigma \ll_x \tau"] & \\
 & & \Gamma
\end{tikzcd}$$

Equality generated from Insertion

Our reduction scheme generates an equality that:

- trivialises all associativity equations,
- is terminating,
- is confluent,
- and has a decidable algorithm for type-checking.

Future Work

- Formalise all results in the paper.
- Combine this with the reduction for strict units to get a type theory for strictly unital and associative categories.
- Create a general framework for CaTT based type theories with definitional equality.
- Show that models of the strict versions of CaTT are equivalent to the models of the original version.