

# Biased whisker-based composition in higher categories

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# Outline

- 1 Globular sets
- 2 Composition in Globular sets
- 3 Whisker-based composition

# What do we mean by a higher category?

A regular (1-)category consists of objects and arrows.

In higher category theory we expand this to allow arrows of higher dimensions between lower dimensional arrows.

What should these higher dimensional arrows look like?

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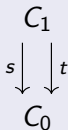
## Ordinary 1-categories

$$\begin{array}{c} C_1 \\ s \downarrow \quad \downarrow t \\ C_0 \end{array}$$

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## Ordinary 1-categories



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$$s_n \circ s_{n+1} = s_n \circ t_{n+1}$$

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Let the objects of the globular set be it's 0-cells, morphisms between these be 1-cells, ...

# Examples

- Ordinary 1-categories
- 2-categories, 3-categories, ...
- Monoidal categories, braided monoidal categories, ...
- Martin L f Type Theory/non-Cubical Homotopy Type Theory

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## Unbiased vs Biased

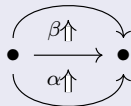
In an *unbiased* definition, we allow every possible composition operation.

In a *biased* definition, we only allow a subset of these operations.

# Composition in infinity categories

## Composition of 2 cells

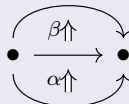
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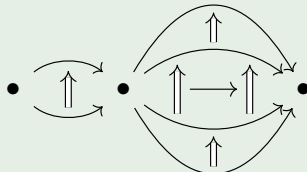
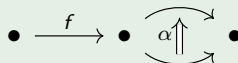
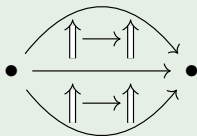
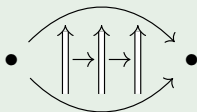
Codimension along a 0-boundary:





# Higher compositions and pasting diagrams

## 3-cell composition



# Stable Compositions

## Definition

A *Stable* binary composition is a composition of an  $n$ -cell  $a$  and an  $m$ -cell  $b$  along their  $(\min(n, m) - 1)$ -boundary. We write this composition  $a \cdot_k b$  where  $k$  is the boundary dimension.

# Whisker-based composition scheme

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A definition of  $\infty$ -categories which only allows stable compositions is valid and equivalent to a fully unbiased definition.

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## Theorem

*Every pasting diagram can be realised as a tree of stable binary composites. Furthermore this realisation respects source and target maps.*

# Inductive Characterisation of Pasting Diagrams

Pasting diagrams “with a focus” are uniquely generated by the following rules.

- The singleton pasting diagram  $x$ , is a pasting diagram with focus  $x$ .
- If  $\Gamma$  is a pasting diagram with focus  $x$ , then  $\Gamma, y, f : x \rightarrow y$  is a pasting diagram with focus  $f$ .
- If  $\Gamma$  is a pasting diagram with focus  $f : x \rightarrow y$ , then it is also a pasting diagram with focus  $y$ .

A pasting diagram is a pasting diagram with a 0-dimensional focus.

# Proof Sketch

## Definition

If  $a$  is a cell and  $x$  is a variable, the *principle replacement*  $a\langle x \rangle$  is given recursively by:

- If  $a = b \cdot c$  then  
 $a\langle x \rangle = b \cdot (c\langle x \rangle)$ .
- If  $a$  is a variable then  
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## Definition

For pasting diagram  $\Gamma$ , define its *stabilised form*  $S(\Gamma)$  by induction:

- If  $\Gamma$  is a singleton  $x$ , then  $S(\Gamma) = x$ .
- If  $\Gamma = \Delta, y, f$  and  $\dim(f) > \dim(\Delta)$  then  $S(\Gamma) = S(\Delta)\langle f \rangle$ .
- If  $\Gamma = \Delta, y, f$  and  $\dim(f) \leq \dim(\Delta)$  then  $S(\Gamma) = S(\Delta) \cdot (\delta_{\dim(y)}^+(S(\Delta)))\langle f \rangle$ .

# Example



# Conclusions

- We introduced the notion of a stable composite.
- We define a translation  $S$  from an arbitrary pasting diagram to a tree of stable binary composites.
- This translation  $S$  respects boundaries.
- Future aims include:
  - Proving the existence of an equivalence.
  - Generalise the work to prove the viability of many composition schemes.