Biased whisker-based composition in higher categories

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Outline

What do we mean by a higher category?

A regular (1-)category consists of objects and arrows.

In higher category theory we expand this to allow arrows of higher dimensions between lower dimensional arrows.

What should these higher dimensional arrows look like?

Globular Sets

Globular sets are one natural shape of higher categories:

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Ordinary 1-categories

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Globular sets

$$\begin{tikzcd} \vdots \\ G_2 \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "t_1"] \\ G_1 \arrow[d, shift right, "s_0"'] \arrow[d, shift left, "t_0"] \\ G_0 \end{tikzcd}$$

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Definition

A *globular set* $\mathcal{G}$ consists of sets G_n for each n and maps $s_n, t_n : G_{n+1} \to G_n$ for each n such that the following *globularity conditions* hold:

$$s_n \circ s_{n+1} = s_n \circ t_{n+1}$$
$$t_n \circ s_{n+1} = t_n \circ t_{n+1}$$

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We will write $f : x \to y$ for an object f in G_{n+1} with $s(f) = x$ and $t(f) = y$.
Let the objects of the globular set be it's 0-cells, morphisms between these be 1-cells, ...

Examples

- Ordinary 1-categories
- 2-categories, 3-categories, ...
- Monoidal categories, braided monoidal categories, ...
- Martin Löf Type Theory/non-Cubical Homotopy Type Theory

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Unbiased vs Biased

In an *unbiased* definition, we allow every possible composition operation.
In a *biased* definition, we only allow a subset of these operations.

Composition in infinity categories

Composition of 2 cells

Composition along a 1-boundary:
$$\begin{tikzcd} \bullet \arrow[r, bend left=80, ""{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right=80, ""{name=D}] & \bullet \arrow[from=M, to=U, phantom, "\beta\Uparrow"] \arrow[from=D, to=M, phantom, "\alpha\Uparrow"] \end{tikzcd}$$

Composition in infinity categories

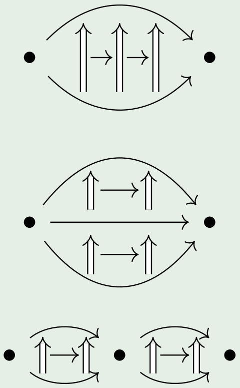
Composition of 2 cells

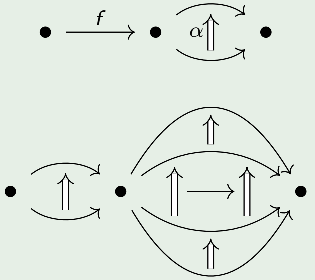
Composition along a 1-boundary:
$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=T, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=B, above}] & \bullet \arrow[from=M, to=T, phantom, "\beta\Uparrow"] \arrow[from=B, to=M, phantom, "\alpha\Uparrow"] \end{tikzcd}$$

Codimension along a 0-boundary:
$$\begin{tikzcd} \bullet \arrow[r, bend left=50, ""{name=A1, below}] \arrow[r, bend right=50, ""{name=A2, above}] & \bullet \arrow[r, bend left=50, ""{name=B1, below}] \arrow[r, bend right=50, ""{name=B2, above}] & \bullet \arrow[from=A2, to=A1, Rightarrow, "\alpha"] \arrow[from=B2, to=B1, Rightarrow, "\beta"] \end{tikzcd}$$

Higher compositions and pasting diagrams

3-cell composition





Stable Compositions

Definition

A *Stable* binary composition is a composition of an n-cell a and an m-cell b along their $(\min(n, m) - 1)$-boundary. We write this composition $a \cdot_k b$ where k is the boundary dimension.

Whisker-based composition scheme

Claim

A definition of ∞-categories which only allows stable compositions is valid and equivalent to a fully unbiased definition.

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Theorem

Every pasting diagram can be realised as a tree of stable binary composites. Furthermore this realisation respects source and target maps.

Inductive Characterisation of Pasting Diagrams

Pasting diagrams "with a focus" are uniquely generated by the following rules.

- The singleton pasting diagram x, is a pasting diagram with focus x.
- If Γ is a pasting diagram with focus x, then $\Gamma, y, f : x \to y$ is a pasting diagram with focus f.
- If Γ is a pasting diagram with focus $f : x \to y$, then it is also a pasting diagram with focus y.

A pasting diagram is a pasting diagram with a 0-dimensional focus.

Proof Sketch

Definition

If a is a cell and x is a variable, the *principle replacement* $a\langle x \rangle$ is given recursively by:

- If $a = b \cdot c$ then $a\langle x \rangle = b \cdot (c\langle x \rangle)$.
- If a is a variable then $a\langle x \rangle = x$.

Proof Sketch

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If a is a cell and x is a variable, the *principle replacement* $a\langle x\rangle$ is given recursively by:

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Definition

For pasting diagram Γ, define its *stabilised form* $S(\Gamma)$ by induction:

- If Γ is a singleton x, then $S(\Gamma) = x$.
- If $\Gamma = \Delta, y, f$ and $\dim(f) > \dim(\Delta)$ then $S(\Gamma) = S(\Delta)\langle f\rangle$.
- If $\Gamma = \Delta, y, f$ and $\dim(f) \leq \dim(\Delta)$ then $S(\Gamma) = S(\Delta) \cdot (\delta^+_{\dim(y)}(S(\Delta)))\langle f\rangle$.

Example

Conclusions

- We introduced the notion of a stable composite.
- We define a translation S from an arbitrary pasting diagram to a tree of stable binary composites.
- This translation S respects boundaries.
- Future aims include:
  - Proving the existence of an equivalence.
  - Generalise the work to prove the viability of many composition schemes.