

A Type Theory for Strictly Associative ∞ -Categories

Alex Rice Eric Finster Jamie Vicary

SYCO 10



- 1 Weak Globular Infinity Categories
- 2 Type Theories for Infinity Categories
- 3 Strict Associators

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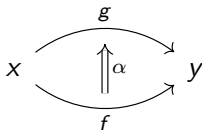
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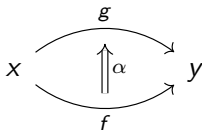
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Composition of 1 cells

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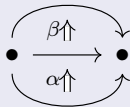
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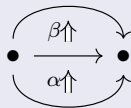
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Composition along a 0-boundary:



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Coherence

- For a 1-cell $f : x \rightarrow y$, there are unitors $\lambda_f : \text{id}_x \circ f \rightarrow f$ and $\rho_f : f \circ \text{id}_y$.
- λ_{id_x} and ρ_{id_x} are both arrows $\text{id}_x \circ \text{id}_x \rightarrow \text{id}_x$.
- These should be equivalent.

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²Finster, R., and Vicary, *A Type Theory for Strictly Associative Infinity Categories*

CaTT is a type theory for *weak infinity categories*³.

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- Types: Source and Target for a term.
- Substitutions: A mapping from variables of one context to terms of another.

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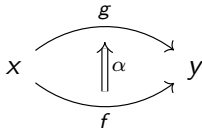
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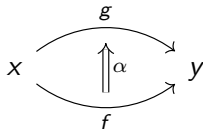
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$$\alpha : f \rightarrow_{x \rightarrow \star y} g$$

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Disc contexts

For each natural number we can define the *disc context* D_n .

D_0



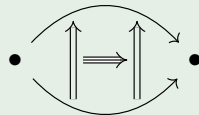
D_1



D_2



D_3



$$D_2 := x : \star, y : \star, f : x \rightarrow_{\star} y, g : x \rightarrow_{\star} y, \alpha : f \rightarrow_{x \rightarrow_{\star} y} g$$

Composition can be done with the `coh` constructor.

`coh` constructor

Given:

- A context Γ - the shape of the composition,
- A type A in Γ - the boundary of the composition,
- A substitution $\sigma : \Gamma \rightarrow \Delta$ - the terms to be composed,

we get a term in Δ :

$$\text{coh } (\Gamma : A)[\sigma]$$

The contexts for which the `coh` constructor is well typed are called *pasting contexts*

Example composition

Suppose we have:

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$$(f \cdot g) \cdot h := \text{coh } (\Gamma : x \rightarrow z) [a \mapsto f \cdot g, \\ b \mapsto h]$$

- CaTT as we have presented it has no non-trivial equality and no computation.
- The idea is to implement a reduction relation that unifies the operations we want to strictify.
- By doing this we obtain a type theory for which the models are semistrict categories.

CaTT_{sa} has a definitional equality based on an operation we call insertion.

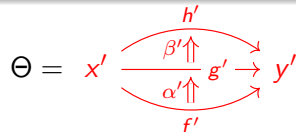
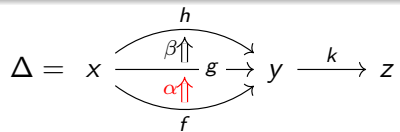
1-associator

$$x \xrightarrow{f} y \xrightarrow{g} z \qquad x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$


is sent to:

$$x \xrightarrow{f} x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

Components of insertion



Components of insertion

$$\Delta = x \begin{array}{c} \xrightarrow{h} \\ \beta \uparrow \uparrow \\ \xrightarrow{g} \\ \alpha \uparrow \uparrow \\ \xrightarrow{f} \end{array} y \xrightarrow{k} z$$

$$\Theta = x' \begin{array}{c} \xrightarrow{h'} \\ \beta' \uparrow \uparrow \\ \xrightarrow{g'} \\ \alpha' \uparrow \uparrow \\ \xrightarrow{f'} \end{array} y' \xrightarrow{k} z$$

$$\Delta \ll_{\alpha} \Theta = x' \begin{array}{c} \xrightarrow{h} \\ \beta \uparrow \uparrow \\ \xrightarrow{h'} \\ \beta' \uparrow \uparrow \\ \xrightarrow{g'} \\ \alpha' \uparrow \uparrow \\ \xrightarrow{f'} \end{array} y' \xrightarrow{k} z$$

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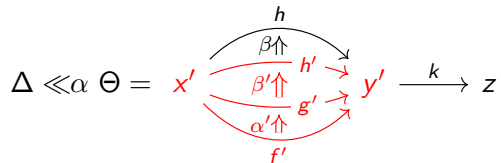
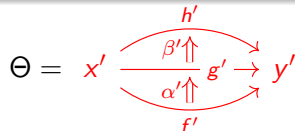
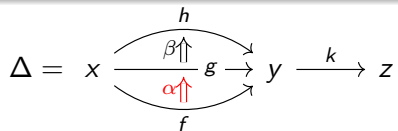
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Given $\sigma : \Delta \rightarrow \Gamma$ and $\tau : \Theta \rightarrow \Gamma$ we get:

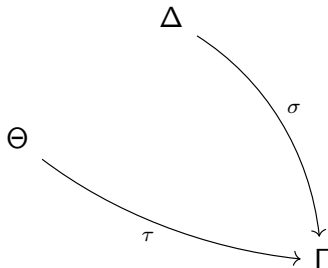
$$\sigma \ll_{\alpha} \tau : \Delta \ll_{\alpha} \Theta \rightarrow \Gamma$$

Universal Property of Insertion

Insertion also satisfies a *universal property*. Suppose we have $\text{coh } (\Delta : A)[\sigma]$ where $\sigma(\alpha) = \text{coh } (\Theta : B)[\tau]$.

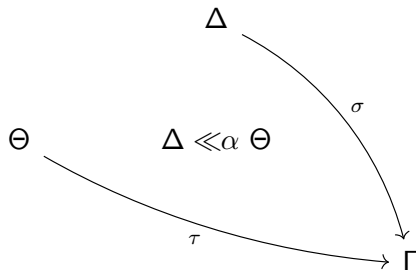
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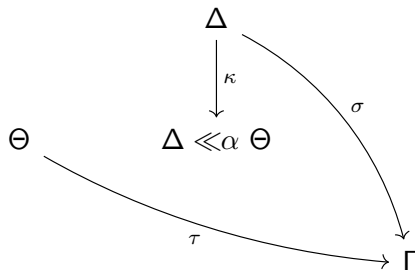
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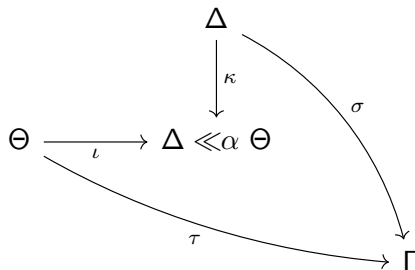
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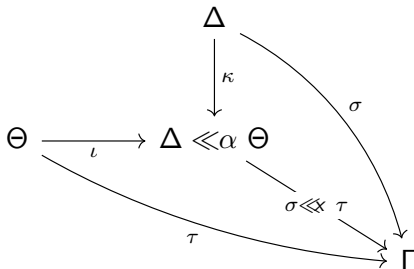
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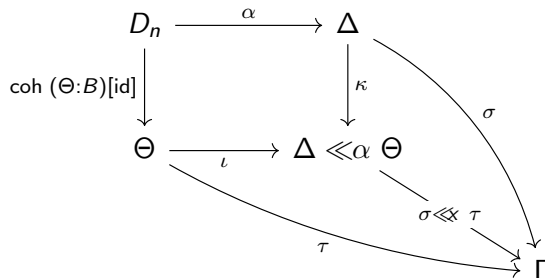
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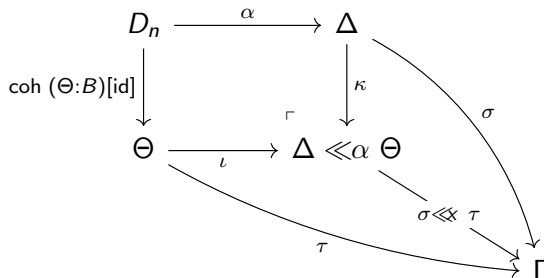
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Insertion generates a reduction relation for Catt_{sa} :

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This reduction has been proven to have the following properties:

- Subject reduction
- Termination
- Confluence

-  Finster, Eric and Samuel Mimram. *A Type-Theoretical Definition of Weak ω -Categories*. 2017. DOI: [10.1109/lics.2017.8005124](https://doi.org/10.1109/lics.2017.8005124). eprint: [1706.02866](https://arxiv.org/abs/1706.02866).
-  Finster, Eric, Alex R., and Jamie Vicary. *A Type Theory for Strictly Associative Infinity Categories*. 2021. arXiv: [2109.01513](https://arxiv.org/abs/2109.01513).
-  Finster, Eric, David Reutter, et al. *A Type Theory for Strictly Unital ∞ -Categories*. Proceedings of the Thirty-Seventh Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2022). 2020. DOI: [10.1145/3531130.3533363](https://doi.org/10.1145/3531130.3533363). arXiv: [2007.08307](https://arxiv.org/abs/2007.08307).