

A Type Theory for Strictly Unital ∞ -Categories

Eric Finster David Reutter Alex Rice Jamie Vicary

LICS 2022



An overview of (globular) infinity categories

Infinity categories contain:

An overview of (globular) infinity categories

Infinity categories contain:

- Objects x, y, z

An overview of (globular) infinity categories

Infinity categories contain:

- Objects x, y, z
- 1-arrows:

$$x \xrightarrow{f} y$$

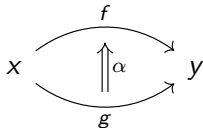
An overview of (globular) infinity categories

Infinity categories contain:

- Objects x, y, z
- 1-arrows:

$$x \xrightarrow{f} y$$

- 2-arrows:



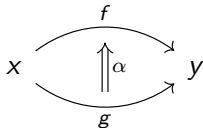
An overview of (globular) infinity categories

Infinity categories contain:

- Objects x, y, z
- 1-arrows:

$$x \xrightarrow{f} y$$

- 2-arrows:



- ...

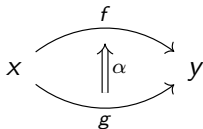
An overview of (globular) infinity categories

Infinity categories contain:

- Objects x, y, z
- 1-arrows:

$$x \xrightarrow{f} y$$

- 2-arrows:



- ...

Our arrows are *Globular*.

An overview of (globular) infinity categories

Infinity categories contain:

- Objects x, y, z
- 1-arrows:

$$x \xrightarrow{f} y$$

- 2-arrows:

A diagram showing two objects, x and y , connected by two curved arrows: f (top) and g (bottom). A vertical double arrow labeled α points from g to f , representing a 2-arrow between the 1-arrows.

- ...

Our arrows are *Globular*.

Compositions:

A diagram showing the composition of two 1-arrows. On the left, x and y are connected by f (top) and g (bottom). On the right, y and z are connected by h (top) and i (bottom). A vertical double arrow labeled α connects f and g , and another labeled β connects h and i .

A diagram showing the composition of two 2-arrows. On the left, x and y are connected by f (top) and g (bottom). A horizontal arrow labeled h points from x to y . A vertical double arrow labeled β points from h to f , and another labeled α points from h to g .

An overview of (globular) infinity categories

Infinity categories contain:

- Objects x, y, z
- 1-arrows:

$$x \xrightarrow{f} y$$

- 2-arrows:

A diagram showing two objects, x and y , connected by two curved arrows: f (top) and g (bottom). A vertical double arrow labeled α points from g to f , representing a 2-arrow between the 1-arrows.

- ...

Our arrows are *Globular*.

Compositions:

A diagram showing the composition of two 1-arrows. On the left, x and y are connected by f (top) and g (bottom). On the right, y and z are connected by h (top) and i (bottom). A vertical double arrow labeled α connects f and g , and another labeled β connects h and i .

A diagram showing the composition of two 2-arrows. On the left, x and y are connected by f (top) and g (bottom). A horizontal arrow labeled h points from x to y . A vertical double arrow labeled β points from h to f , and another labeled α points from h to g .

Identities:

$$x \xrightarrow{\text{id}_x} x$$

Strict Infinity Categories

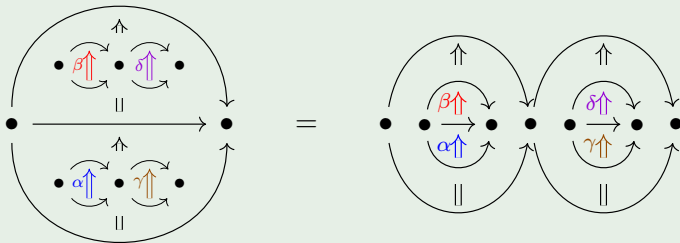
Associativity

$$w \xrightarrow{w \xrightarrow{f} x \xrightarrow{g} y} y \xrightarrow{h} z = w \xrightarrow{f} x \xrightarrow{x \xrightarrow{g} y \xrightarrow{h} z} z$$

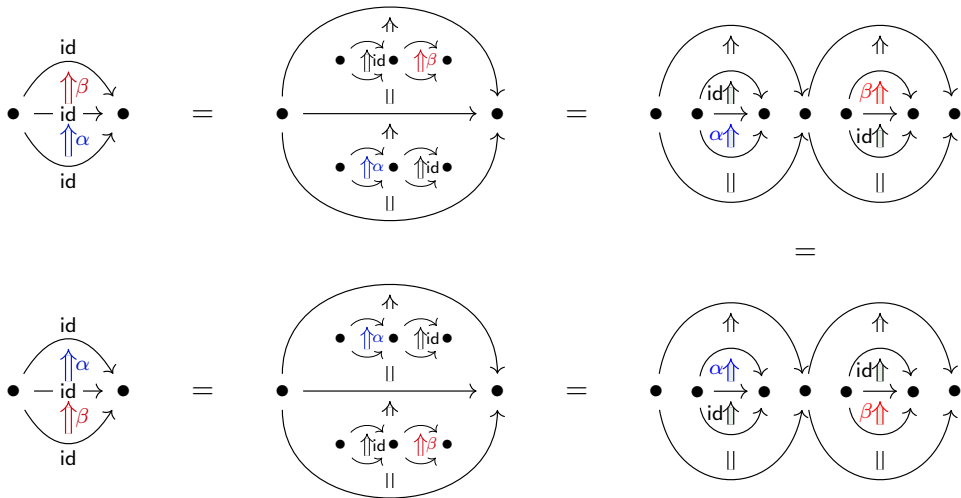
Unitality

The diagram illustrates the unitality property. On the left, a composition of two 2-cells is shown. The first 2-cell has boundary arrows $\text{id}(x)$ and $\text{id}(x)$ and an internal arrow $\text{id}(\text{id}(x))$. The second 2-cell has boundary arrows g and f and an internal arrow α . The right side shows a single 2-cell with boundary arrows g and f and an internal arrow α .

Interchange



Example: Eckmann-Hilton



We have only so far talked about strict ∞ -categories.

We have only so far talked about strict ∞ -categories.

In higher categories, non-equal objects can be isomorphic.

In a weak higher category, the laws are only required to hold up to isomorphism.

We have only so far talked about strict ∞ -categories.

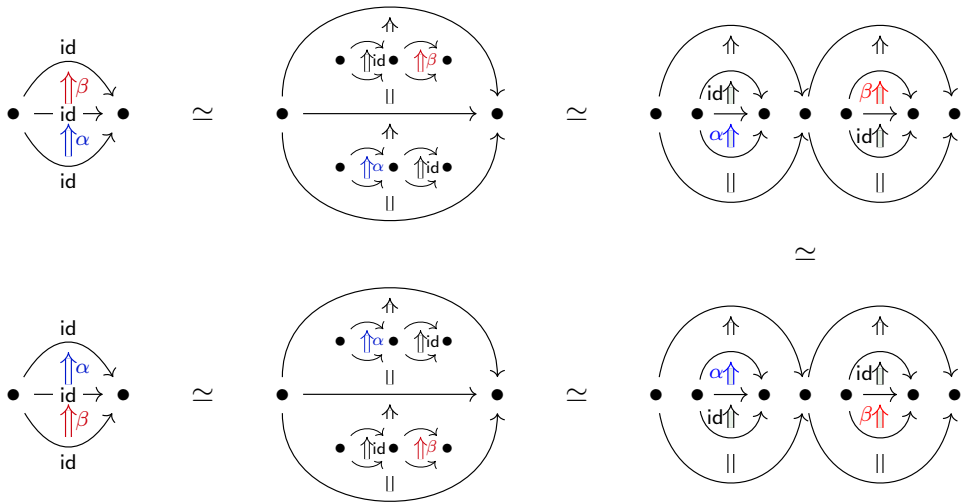
In higher categories, non-equal objects can be isomorphic.

In a weak higher category, the laws are only required to hold up to isomorphism.

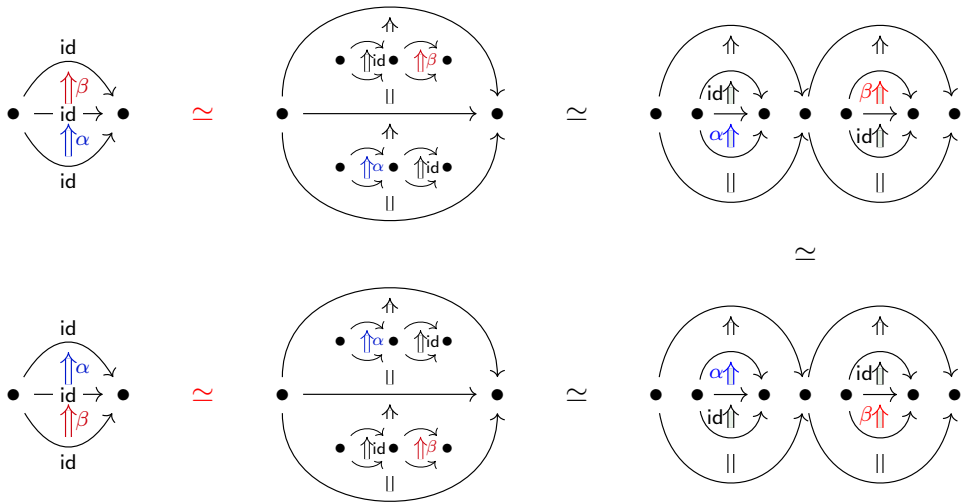
Many examples of higher categories are weak:

- Homotopy groups of topological spaces.
- Equality types in HoTT.
- Bicategory of categories and profunctors.

Example: Eckmann-Hilton



Example: Eckmann-Hilton



Weakness vs Strictness

Weak ←————→ Strict

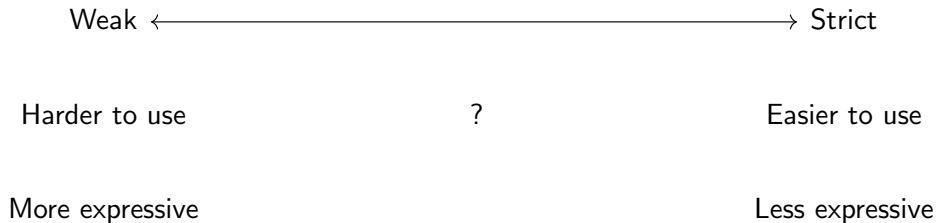
Weakness vs Strictness



Weakness vs Strictness



Weakness vs Strictness



Weakness vs Strictness

Weak ←———— Semistrict —————→ Strict

Harder to use

Easier to use

More expressive

Less expressive

Weakness vs Strictness

Weak ←———— Semistrict —————→ Strict

Harder to use

Lack of definitions

Easier to use

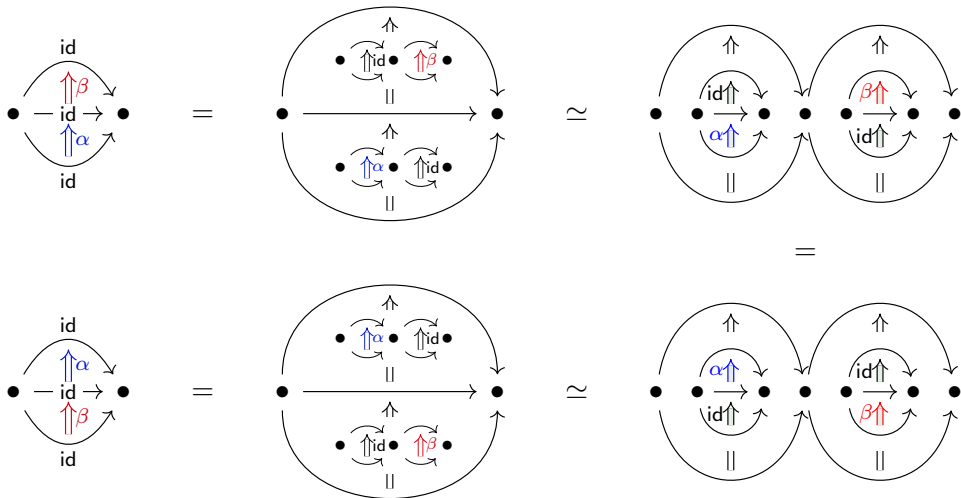
More expressive

Retains expressiveness?

Less expressive

- Catt is a type theory for weak ∞ -categories.
- Its terms are the possible operations in an ∞ -category.
- By adding a definitional equality to Catt, we can unify certain operations.
- Catt_{su} is a new type theory based on Catt with strict units.

Example: Eckmann-Hilton



```

coh id C (x) : x => x
coh id2 C (x(f)y) : f => f
coh comp C (x(f)y(g)z) : x => z
coh vert C (x(f(a)g(b)h)y) : f => h
coh horiz C (x(f(a)g)y(h(b)k)z) : comp f h => comp g k

coh swap3 C (x(f(a)g)y(h(b)k)z)
  : vert (horiz a (id2 h)) (horiz (id2 g) b) =>
    vert (horiz (id2 f) b) (horiz a (id2 k))

let eh {C : Cat} {x :: C} (a :: id x => id x) (b :: id x => id x)
  : [ vert a b => vert b a ]
  = swap3 a b

```

- Equality in Catt_{su} preserves typing.
- Equality is generated by a strongly-terminating, confluent reduction relation.
- Equality and type checking are decidable.
- All terms (of the same dimension) in a disc context are identified.
- Eckmann-Hilton and the Syllepsis have been formalised in Catt_{su} .