

# A Syntax for Strictly Associative and Unital $\infty$ -Categories

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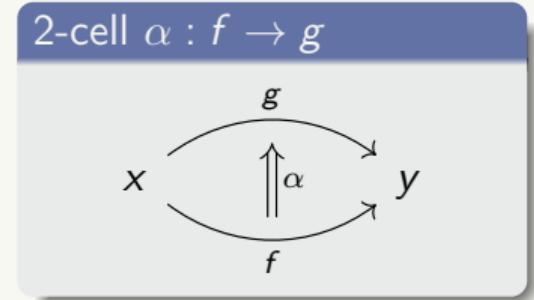
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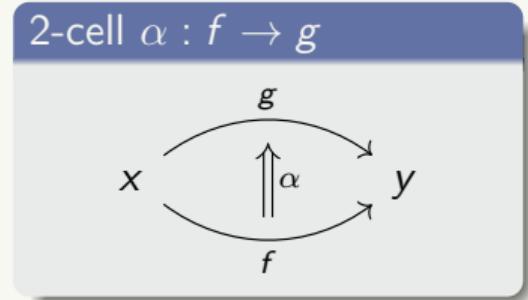


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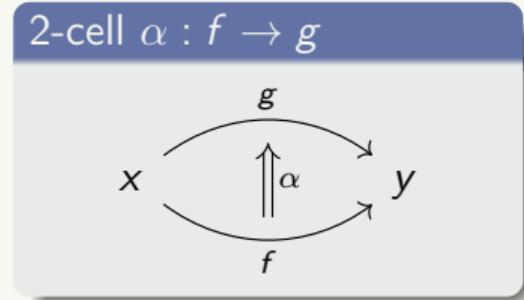
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A type gives the boundary of a term.



## Terms in CATT

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Terms built over *pasting diagrams*.

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$$\Gamma := \underset{\textcolor{red}{x}}{x} \xrightarrow{f} \underset{\textcolor{red}{y}}{y} \xrightarrow{g} \underset{\textcolor{red}{z}}{z}$$

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$$\Delta := x \begin{array}{c} \nearrow h \\ \uparrow \alpha \\ \searrow f \end{array} y \begin{array}{c} \nearrow i \\ \uparrow \beta \\ \searrow g \end{array} z$$

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$$s[\![\sigma]\!]$$

$$\sigma := \langle f \mapsto a, g \mapsto (b * c) \rangle$$

$$a * (b * c) := (f * g)[\![\sigma]\!]$$

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$$\alpha_{a,b,c} := \text{coh } (\Delta : (a * b) * c \rightarrow a * (b * c))$$

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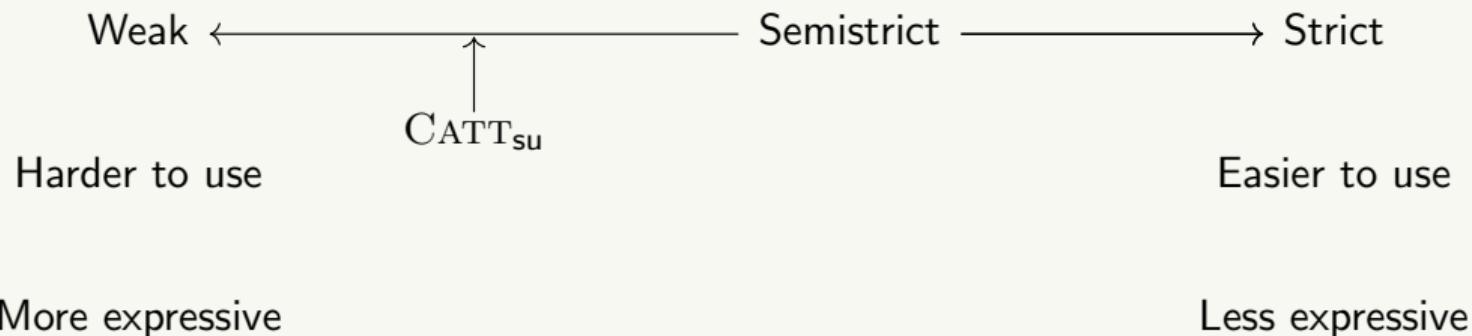
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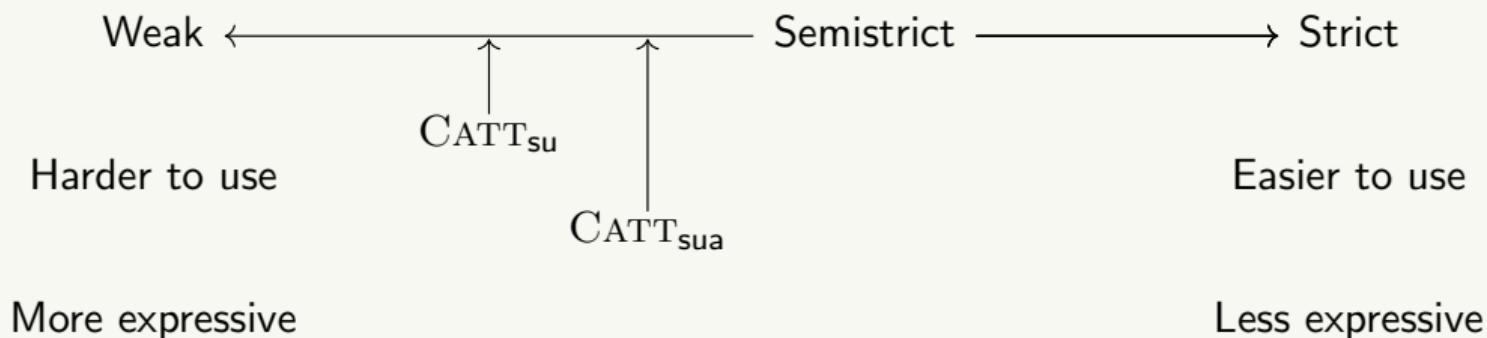
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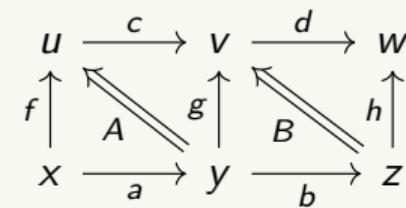


# Strictness

(Semi)strictness allows more operations to be defined.

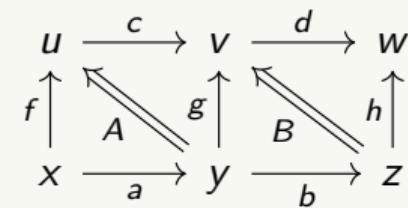
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$$(\text{id}_a \otimes B) * (A \otimes \text{id}_d) : (a * b) * h \rightarrow f * (c * d)$$

## CATT<sub>sua</sub>

CATT has trivial equality.

CATT<sub>su</sub> has disc removal, endo-coherence removal, and pruning.

In CATT<sub>sua</sub>, pruning is replaced by insertion.

$$\text{CATT}_{\text{sua}} := \text{CATT} + \text{insertion} + \text{disc removal} + \text{endo-coherence removal}$$

## Insertion Rule

$$(a * b) * c =_{\text{sua}} a * b * c \equiv \text{coh} (x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w : x \rightarrow w)$$

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Recalling  $(a * b) * c \equiv (f * g)[\langle f \mapsto a, g \mapsto b * c \rangle]$ :

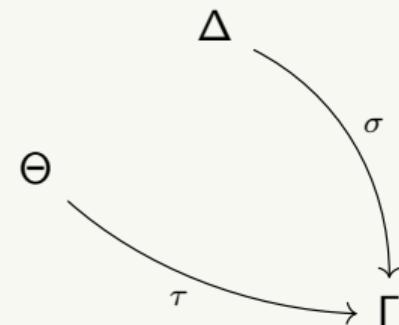


is sent to:

$$x \xrightarrow{f} x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

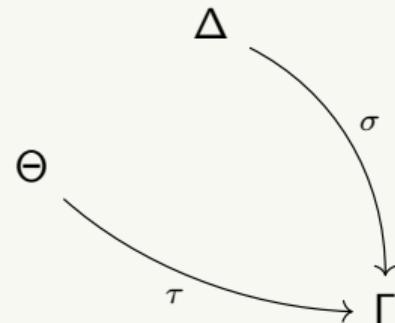
# Universal Property of Insertion

$$\text{coh } (\Delta : s \rightarrow t) [\![\sigma]\!] \quad x \in \Delta \quad x [\![\sigma]\!] \equiv \text{coh } (\Theta : u \rightarrow v) [\![\tau]\!]$$



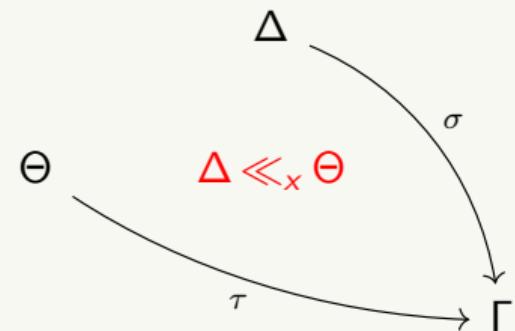
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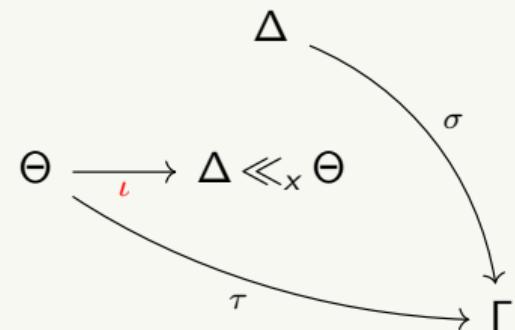
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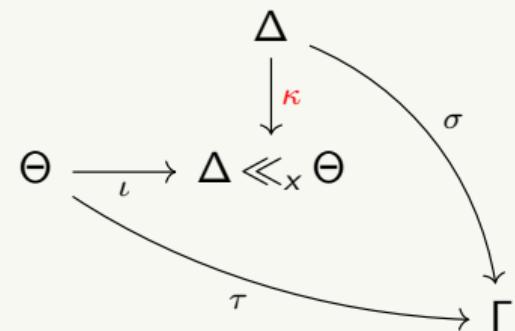
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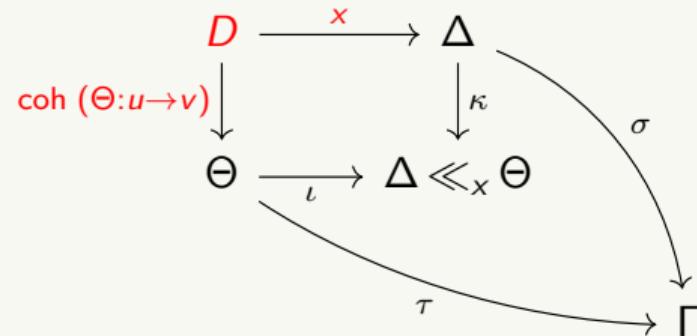
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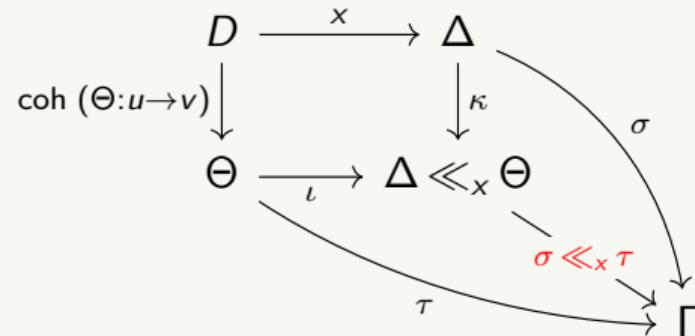
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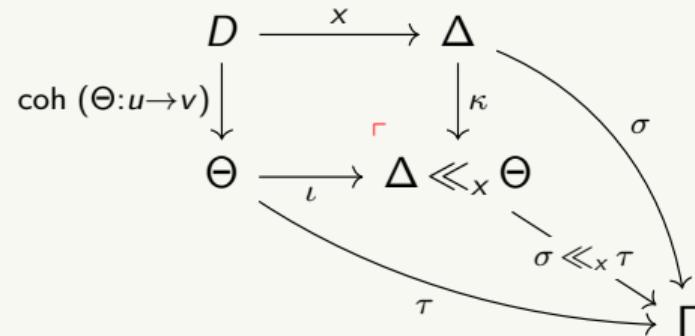
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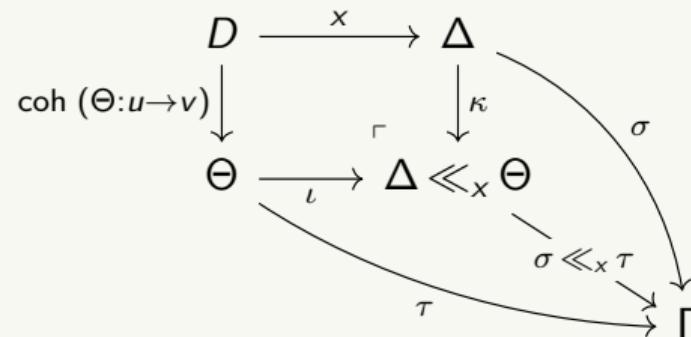
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$$\text{coh } (\Delta : s \rightarrow t)[\sigma] = \text{coh } (\Delta \ll_x \Theta : s[\kappa] \rightarrow t[\kappa])[\sigma \ll_x \tau]$$

Equality in  $\text{CATT}_{\text{sua}}$  is decidable.

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Confluence: Encode various constructions in Agda.

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Type checking is decidable.

We provide a interpreter which:

- Provides tools for construction terms.
- Type checks terms.
- Reduces terms to  $\text{CATT}_{\text{sua}}$  normal form.

We introduce the type theory  $\text{CATT}_{\text{sua}}$ .

$\text{CATT}_{\text{sua}}$  models strictly unital and associative  $\infty$ -categories.

$\text{CATT}_{\text{sua}}$  terms are simpler than their  $\text{CATT}$  or  $\text{CATT}_{\text{su}}$  equivalents.

Normal forms for  $\text{CATT}_{\text{sua}}$  are obtained via a reduction system:

This reduction system is strongly terminating and confluent.

Try our interpreter for  $\text{CATT}_{\text{sua}}$ :

<https://github.com/alexarice/catt-strict>

Thank you for listening.