Coinductive Invertibility in Higher Categories

Alex Rice

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Abstract

Invertibility is an important concept in category theory. In higher category theory, it becomes less obvious what the correct notion of invertibility is, as extra coherence conditions can become necessary for invertible structures to have desirable properties.

We define some properties we expect to hold in any reasonable definition of a weak ω-category. With these properties we define three notions of invertibility inspired by homotopy type theory. These are *quasi-invertibility*, where a two sided inverse is required, *bi-invertibility*, where a separate left and right inverse is given, and *half-adjoint inverse*, which is a quasi-inverse with an extra coherence condition. These definitions take the form of coinductive data structures. Using coinductive proofs we are able to show that these three notions are all equivalent in that given any one of these invertibility structures, the others can be obtained. The methods used to do this are generic and it is expected that the results should be applicable to any reasonable model of higher category theory.

Many of the results of the paper have been formalised in Agda using coinductive records and the machinery of sized types.

1 Introduction

In the study of higher category theory, the notion of invertibility is central. It is common within higher category theory to not specify that two objects are equal, or that two sides of an equation are equal, but rather specify that there is a higher level equivalence between these two objects. It is usual to say that two objects are equivalent exactly when there is an invertible morphism between them.

This idea can be seen even in the simplest examples of higher categories. Consider the definition of a monoidal category. It is our intention that given objects A, B and C that $A \otimes (B \otimes C)$ and $(A \otimes B) \otimes C$ represent the same object. However it is often the case that requiring equality between these objects (as in a strict monoidal category) is too restrictive and so the more general definition states that there is a natural isomorphism between them.

This is our first example of invertibility. An isomorphism between A and B is a morphism $f : A \to B$ such that there exists a morphisms $g : B \to A$ with $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. This notion of isomorphism works well throughout standard category theory and even in monoidal categories.

However if higher categories are considered, it can be seen that the problem has just been pushed into the higher structure of the category. We have specified that $g \circ f$ and id_A should be equal, yet have already stated that equality is not the best notion of two objects in a higher category being the same. Ideally we want a collection of data that specifies that f has an inverse g, and that there is a higher morphism $\alpha : g \circ f \to \mathrm{id}_A$ and an inverse $\beta : \mathrm{id}_A \to g \circ f$, and that there exists a morphism $\alpha \circ \beta \to \mathrm{id}_{\mathrm{id}_A}$ and continuing in this fashion. However, this generates an unwieldy set of data that is difficult to describe.

A natural way we can talk about infinite data structures like this is through coinduction. The definition of equivalence above can be written nicely by saying that $f : A \to B$ is an equivalence if there exists a $g : B \to A$ and equivalences $f \circ g \to \mathrm{id}_B$ and $g \circ f \to \mathrm{id}_A$. This is a neat method of defining an equivalence, but also has the advantage of opening up more proof techniques to us. In particular, it allows us to structure proofs coinductively.

Given this coinductive machinery, the definition of equivalence above seems very natural. If we start with our definition of isomorphism, then realise that the equalities introduced should themselves be weakened to equivalences, we end up with the proposed definition. However this is not the only way that invertibility could have been defined. In type theory and in particular in HoTT [19], there are three notions of invertibility defined:

1. a *quasi-inverse*, which is similar to the notion of isomorphism presented already;
2. a *bi-inverse*, where instead of specifying an inverse to the function f, we instead present separate left and right inverses;
3. a *half-adjoint inverse*, defined to be the same as the quasi-inverse but with an extra coherence condition.

There are certain properties of these invertibility conditions that are important in type theory. In the language of HoTT, all three of these definitions are equivalent [19, Corollary 4.3.3], in that a function f has one of these three types of inverses, then it also has the other two. This can be viewed as the $(\infty, 1)$ case of what is proved in this paper, as all equalities are naturally invertible in HoTT. Secondly, definitions 2 and 3 are propositions [19, Theorem 4.2.13, Theorem 4.3.2]. This means that there is at most one way in which they hold up to propositional equality.

Although it seems that these three are not applicable to higher categories due to the use of equality in their definitions, they can nonetheless be modified into coinductive definitions suitable for such a category.

1.1 Contributions and further work

This paper explores the equivalence of various types of invertible cells, using coinductive proof techniques. The aim is that these techniques make managing the data of a cell being invertible easier and the proofs simpler.

Given a setting of higher category theory which we believe is sufficiently general, we have managed to prove many properties of bi-invertible structures, which we believe have not been studied in great depth before. We have shown that, using coinductive techniques, one can work with these structures using only 2-dimensional reasoning. Further we have shown that these structures work nicely with inverses and composition as expected.

These results have been used to show that the three definitions of invertibility given in the paper all imply each other, in that given one of these structures we can construct the others. This mirrors similar proofs in lower dimensional category theory that were already known well.

Going further, we would like to be able to prove some of the contractibility results that have been conjectured. It is possible that this either requires a lot more structure than we have postulated that any weak higher category should have. Nevertheless, we believe that contractibility results for bi-invertible structures could have large implications on ways in which higher categories can be defined and worked with. It would also have implications on the study of ∞-groupoids, where every morphism is invertible.

Lastly, this paper also contributes a large amount of Agda formalisation for all the structures mentioned. With a little work, much of the code could be reused for proving similar facts about ω-categories. The power of this formalisation technique would be largely increased if the condition of "respecting the graphical calculus" could be removed, which is hard to express in Agda. We believe that just as the correctness of string diagrams for bi-categories follows from the existence of associators, unitors and some coherence conditions, there should be a similar set of conditions on ω-category which at least allows up to 2-dimensional graphical reasoning. The ability to automate a translation from the graphical calculus to a proof system such as Agda would allow very complex proofs, including all the proofs in this paper, to be formalised with ease.

1.2 Related Work

A lot of work has been done relating to invertible structures in higher categories. The idea of defining a notion of pre-category is largely inspired by Cheng [3]. Kansangian, Metere, and Vitale [11] discuss the equivalence of some forms of weak inverse in strict n-categories. The result that an equivalence implies an adjoint equivalence in bicategories is well known and this has been extended to tricategories by Gurski [7]. Lafont et al. [13] construct the idea of a quasi-invertible structure in precisely the same way as in this paper, and proves some of the same results, though these are done in the setting of a strict ω-category, where some of the coinductive arguments become easier.

While the idea of viewing higher categories as coinductive data structures is not new [4, 9], we believe the set up used in this paper is novel. The work of Hirschowitz, Hirschowitz, and Tabareau is similar enough that it was possible to adapt it to form the basis for the formalisation [17] in Section 1.3.

1.3 Formalisation

It may at first appear that a lot of the coinductive proofs in this paper work by "magic" in that they seem like non-well founded induction. To remedy this all the definitions and results from Sections 1.4.2 and 2.1 have been formalised in the proof assistant Agda[1] with use of the standard library[2]. The formalisation uses sized types [1] to ensure productivity of the functions defined and makes heavy use of coinductive record types and copatterns [2].

The notion of higher pre-category introduced in this paper is very set-theoretic and becomes messy in the world of type theory. Because of this, alternative coinductive definitions of globular sets and composition are taken from "Wild omega-Categories for the Homotopy Hypothesis in Type Theory" [9]. This allows a lot of the proofs to become neater as coinduction tends to work nicely with coinductively defined structures. We believe the use of a different definition is not consequential as the main purpose of the formalisation was to show that coinductive elements of the arguments hold.

The code is publicly available [17] and has been tested with Agda version 2.6.1 and standard library version 1.3.

[1] `https://github.com/agda/agda`

[2] `https://github.com/agda/agda-stdlib`

1.4 Background

This background section will give a quick introduction to coinduction and some of the ideas behind ω-categories required for the paper as well as introducing the first notion on invertibility.

1.4.1 Coinduction

Throughout the paper, we want to use coinductive data structures and many functions will be generated by corecursion, and many proofs made using coinduction. Like inductive data structures, coinductive data structures are able to reference themselves in their definition. Many of the coinductive results in this paper are proved in a style similar to that given in "Practical coinduction" [12], and are justified by *guardedness* which is a well-known concept [5, 6]. Below, a summary of coinduction is given, all of which is standard.

Categorically, coinduction forms the categorical dual of induction. Whereas inductive data structures can be represented as initial algebras, coinductive data structures may be represented as terminal coalgebras [10]. To see the differences clearly between these two constructions and how they relate, it will be helpful to go through an example. Since a lot of the language used throughout this paper is type-theoretic, induction and coinduction will be explained in the same way.

When a type is introduced in type theory, there are four parts of its definition: type formation, term formation, a recursion principle, and an induction principle. These will be demonstrated with the example of lists.

- Type formation tells us how to build the type. For lists, given a type A, the type $\mathsf{List}(A)$ can be formed.
- Term formation tells us how terms of the type can be constructed. $\mathsf{List}(A)$ has multiple constructors. We have $\mathsf{nil} : \mathsf{List}(A)$, which is a constructor with no arguments and $\mathsf{cons} : A \to \mathsf{List}(A) \to \mathsf{List}(A)$, which takes two.
- Recursion lets us form functions from an inductive type. For lists this says that to form a function $f : \mathsf{List}(A) \to B$, it suffices to define $f(\mathsf{nil}) : B$ and define $f(\mathsf{cons}(a)(xs))$ given the value of $f(xs)$.
- Induction is a generalisation of recursion and allows us to define functions $f : \Pi_{xs:\mathsf{List}(A)} P(xs)$ where P is a type family of type $\mathsf{List}(A) \to \mathcal{U}$, with $\mathcal{U}$ being a universe of types. In other words P assigns a type to each list.

A further thing to note is that we will require the use of a generalisation of this known as *inductive type families* [19, Section 5.7]. These allow our inductive types to be parameterised. A good example of this is the vector type, which is a list of specified length. Here our type formation rule says that given a type A we can get a type $\mathsf{Vec}(A) : \mathbb{N} \to \mathcal{U}$. Further the term formation rules become $\mathsf{nil} : \mathsf{Vec}(A)(0)$ and $\mathsf{cons} : \Pi_{n:\mathbb{N}} A \to \mathsf{Vec}(A)(n) \to \mathsf{Vec}(A)(n+1)$. Notice that unlike with lists, which also took a parameter, vectors can not be defined by considering each element of the parameter separately. The recursion and induction rules also need to modified.

Now it will be seen how this differs for coinductive types. For this we will use one of the most common coinductive types, the stream, which are infinite lists.

- Type formation remains the same as before. Given any type A we can form $\mathsf{Stream}(A)$.

- Instead of specifying how to form terms, we specify how to deconstruct terms. Streams have two deconstructors: $\mathsf{head} : \mathsf{Stream}(A) \to A$ and $\mathsf{tail} : \mathsf{Stream}(A) \to \mathsf{Stream}(A)$.

- Instead of recursion, we have corecursion, which specifies how to build functions $f : B \to \mathsf{Stream}(A)$. Whereas in recursion, we specified the behaviour of the function on each constructor, in corecursion we specify how to deconstruct the value returned. Therefore it suffices to provide $\mathsf{head}(f(b)) : A$ and to provide $\mathsf{tail}(f(b)) : \mathsf{Stream}(A)$, where $\mathsf{tail}(f(b))$ can be defined by either by providing a stream or by providing a $c : B$ (which may or may not equal b) such that $\mathsf{tail}(f(b)) = f(c)$.

- Coinduction does not work quite the same as induction, as flipping the arrow no longer works due to the dependency in the type. Instead, if we are trying to prove a predicate $P : \mathsf{Stream}(A) \to \mathcal{U}$, we can define the predicate coinductively, and then provide a witness to it by corecursion. Coinductive type families will likely be needed to define the predicate.

As an example of coinduction, suppose we wanted to prove every element of a stream was even. First we construct the predicate $P : \mathsf{Stream}(\mathbb{N}) \to \mathcal{U}$ as having deconstructors $\mathsf{headProof}(xs) = \mathsf{is\text{-}even}(\mathsf{head}(xs))$ and $\mathsf{tailProof}(xs) = P(\mathsf{tail}(xs))$. An element of $P(xs)$ (a proof that every element of the stream is even) could then be formed by corecursion.

1.4.2 ω-Categories

In this paper we want to talk about different structures on ω-categories, however there are many different definitions of ω-categories [14]. Therefore, similar in style to a paper by Cheng [3], we give a set of conditions that we expect to hold in any reasonable definition of a weak infinity category. Results are then proved using only these conditions, and we reason that given a precise definition of an ω-category, if one could show that these conditions hold, the results should follow for free.

Definition 1. The *globe category* $\mathbb{G}$ is the category where the objects are the natural numbers and morphisms are generated by

$$\sigma_n : n \to n+1$$
$$\tau_n : n \to n+1$$

subject to the conditions

$$\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n$$
$$\sigma_{n+1} \circ \tau_n = \tau_{n+1} \circ \tau_n$$

Definition 2. A *globular set* G is a presheaf on $\mathbb{G}$. We refer to $G(n)$ as the set of n-cells and for two n-cells x and y we write $f : x \to y$ to mean that f is an $(n+1)$-cell and

$$G(\sigma_n)(f) = x$$
$$G(\tau_n)(f) = y$$

where we call x the source of f and y the target of f.

Definition 3. A *globular set with identities and composition* is a globular set G with the following:

- For each n-cell x, there is an $(n+1)$-cell, $\mathrm{id}_x : x \to x$.
- Inductively define composition as follows:
  - Given n-cells $f : x \to y$ and $g : y \to z$ there is an n-cell $g \star_0 f : x \to z$.
  - Given $\alpha : f \to g$ and $\beta : h \to j$, where the composites $f \star_n h$ and $g \star_n j$ are well-defined, there is a morphism $\alpha \star_{n+1} \beta : (f \star_n h) \to (g \star_n j)$

Once we have a globular set with identities and composition, we have enough to define a notion of equivalence. The most basic notion of invertibility will be given here, as it will be needed to state the remainder of properties that we expect an higher category to obey.

Definition 4. Given a globular set G with identities and composition, with an n-cell $f : x \to y$, a *quasi-invertible* structure on f is a tuple $(f^{-1}, f_R, f_L, f_R^{QI}, f_L^{QI})$ where:

- f^{-1} is an n-cell $y \to x$;
- f_R is an $(n+1)$-cell $f \star_0 f^{-1} \to \mathrm{id}_y$;
- f_L is an $(n+1)$-cell $f^{-1} \star_0 f \to \mathrm{id}_x$.
- f_R^{QI} is a quasi-invertible structure on f_R.
- f_L^{QI} is a quasi-invertible structure on f_L.

Remark 5. The previous definition is a coinductive one. Formally, we have defined a coinductive data type, which we could call $\mathsf{QuasiInvertible}(f)$, which references itself (by saying that f_R and f_L themselves have a quasi-invertible structure). Note that as $\mathsf{QuasiInvertible}$ is dependent on the parameter f, it is in fact a coinductive type family as described in Section 1.4.1.

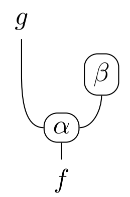
Further, it should be noted that, as a coinductive structure, $\mathsf{QuasiInvertible}(f)$ contains an infinite stack of data, and in particular contains the data witnessing the invertibility of f at all dimensions.

Remark 6. Notice that in a globular set G with identities and composition, given two n-cells x and y with the same source and target, a new globular set $G_{x,y}$ can be defined:

- the 0-cells are the $(n+1)$-cells in G with source x and target y;
- the m-cells are the $(n+m+1)$-cells in G whose source and target lie in the $(m-1)$-cells of $G_{x,y}$.

It is clear that we can carry over identities and composition to this new globular set.

Consider the standard string diagram diagrammatic calculus for bicategories [8, 18]. We can draw diagrams containing nodes, lines, and areas where areas represent 0-cells, a line between areas representing cells x and y represents a 1-cell between x and y, and node between lines represent 2-cells between the cells which those lines represent. These diagrams still represent well formed morphisms in an globular set with identities and composition (at least in the presence of unitors and associators, which we introduce below). In a bicategory there exists the theorem that given a planar isotopy of string diagrams, the source and target morphisms are equal. In this paper, string diagrams are written bottom to top and right to left, like in Fig. 1.



$$f : x \to y$$
$$g : x \to y$$
$$h : y \to y$$
$$\alpha : f \to g \star_0 h$$
$$\beta : h \to \mathrm{id}_y$$

Figure 1: The morphism $\alpha \star_0 (\mathrm{id} \star_1 \beta)$

Definition 7. Say that a globular set *respects the graphical calculus* if for any well typed string diagram and any planar isotopy of this diagram, there exists a 3-cell from the source of the isotopy to the target. Further, it is required that there is a quasi-invertible structure on this morphism.

We define a *higher pre-category* to be a globular set with identities and composition such that any globular set generated as in Theorem 6 respects the graphical calculus. Further we assume the existence of the following morphisms:

- For $n > 0$ and each n-cell $f : x \to y$, there are $n+1$-cells, known as unitors, $\lambda_f : \mathrm{id}_y \star_0 f \to f$ and $\rho_f : f \star_0 \mathrm{id}_x \to f$.
- Given f, g, h, $n > 1$-cells with suitable composition defined, we have an associator $a_{f,g,h}$: $(f \star_0 g) \star_0 h \to f \star_0 (g \star_0 h)$.
- For compatible morphisms f, g, h, j, we have an interchanger $i_{f,g,h,j} : (f \star_n g) \star_0 (h \star_n j) \to (f \star_0 h) \star_n (g \star_0 j)$.
- For suitable f, g and n, there is a cell $\mathrm{id}_f \star_{n+1} \mathrm{id}_g \to \mathrm{id}_{f \star_n g}$.

All these further morphisms must be equipped with a quasi-invertible structure.

Remark 8. It is worth stressing that it is not intended that a higher pre-category is in any way a definition of an ω-category. Instead, this is just a list of properties that are expected to hold in any realistic definition of a higher category. The definition is made to be as general as possible so that the results can be as applicable as possible. Further, while this definition seems to imply that we require our definition of ω-category to be enriched over ω-categories, this is not the case. If G is a globular set then $G_{x,y}$ will also be a globular set with no extra conditions. From here it makes sense to separately require that each of these globular sets respects the graphical calculus.

To end this section we give two small lemmas, which demonstrates the use of these higher pre-categories and will also be invaluable later.

Lemma 9. *Any identity morphism in a higher pre-category G has a quasi-invertible structure.*

Proof. Let x be any cell. Take $\mathrm{id}_x^{-1} = \mathrm{id}_x$ and $(\mathrm{id}_x)_R = (\mathrm{id}_x)_L = \lambda_{id_x}$. By assumption, λ_{id_x} has a quasi-invertible structure which can be used to form a structure on id_x. □

Lemma 10. *Suppose f has a quasi-invertible structure. This induces a quasi-invertible structure on f^{-1}.*

Proof. Let $(f^{-1}, f_R, f_L, f_R^{QI}, f_L^{QI})$ be a quasi-invertible structure on f. Then $(f, f_L, f_R, f_L^{QI}, f_R^{QI})$ is a quasi-invertible structure on f^{-1}. □

2 Types of Invertibility

In this section we introduce two more types of invertibility. Some lemmas are proved that help us to work with them and finally all three types of invertibility are shown to imply each other. Throughout this section it will be assumed that we are working in a higher pre-category G.

2.1 Bi-Invertibility

Normally throughout mathematics, when saying a map is invertible a single inverse is specified, and it is shown that this cancels the function when composed on the left and the right. Instead we can consider a function that has a left inverse and right inverse, an inverse only cancels the function on when composed on the left (or right respectively). It usually does not make sense to consider functions which have both a separate left and right inverse as in most scenarios it can be proved that these are equal and any computations will be simplified by treating them as the same.

However, the concept of bi-invertibility, having both a left and right inverse, plays a role in type theory. It can be shown (perhaps not surprisingly) that a morphism being bi-invertible implies that it is invertible in the usual sense. More surprisingly, the data for being bi-invertible is in some ways simpler than the data for being invertible, in that it can be shown that any two ways of showing a function is bi-invertible turn out to be equal.

Before these can be studied, we must define what we mean for a morphism in a higher category to be bi-invertible, as a generalisation of the idea in type theory.

Definition 11. Given a globular set G with identities and composition, with an $n > 0$ cell $f : x \to y$, a *bi-invertible* structure on f is a tuple $(f^\star, {}^\star f, f_R, f_L, f_R^{BI}, f_L^{BI})$ where:

- $f^\star$ is an n-cell $y \to x$;
- ${}^\star f$ is an n-cell $y \to x$;
- f_R is an $(n+1)$-cell $f \star_0 f^\star \to \mathrm{id}_y$;
- f_L is an $(n+1)$-cell ${}^\star f \star_0 f \to \mathrm{id}_x$.
- f_R^{BI} is a bi-invertible structure on f_R.
- f_L^{BI} is a bi-invertible structure on f_L.

Next it is shown that invertibility implies bi-invertibility. This is fairly trivial though it will be written out in full to demonstrate proof by coinduction.

Lemma 12. *Let $f : x \to y$ be invertible. Then f is bi-invertible.*

Proof. This is proven by constructing a corecursive function $\mathsf{invToBiInv} : \Pi_{f:x\to y}(\mathsf{Invertible}(f) \to \mathsf{BiInvertible}(f))$. Given $I = (f^{-1}, f_R^i, f_L^i, f_R^{QI}, f_L^{QI}) : \mathsf{Invertible}(f)$ we can construct $\mathsf{invToBiInv}(f)(I)$:

- $f^\star = f^{-1}$;
- ${}^\star f = f^{-1}$;
- $f_R = f_R^i$;
- $f_L = f_L^i$;

- corecursively, we let $f_R^{BI} = \mathsf{invToBiInv}(f_R)(f_R^{QI})$;
- similarly, let $f_L^{BI} = \mathsf{invToBiInv}(f_L)(f_L^{QI})$.

Then $\mathsf{invToBiInv}(f)(I) : \mathsf{BiInvertible}(f)$ as required. $\square$

Next, one of the most important and complex theorems of the paper is proved.

Theorem 13. *For any $n \in \mathbb{N}$ and cells f, g where $g \star_n f$ is well-defined, given a bi-invertible structure on f and g, we can generate a bi-invertible structure on $g \star_n f$.*

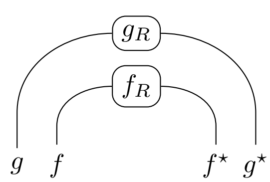
Proof. This will be proved by corecursion on the bi-invertible structure that is being generated. We will split into cases on n.

Suppose $n = 0$. Let $(f^\star, {}^\star f, f_R, f_L, f_R^{BI}, f_L^{BI})$ and $(g^\star, {}^\star g, g_R, g_L, g_R^{BI}, g_L^{BI})$ be bi-invertible structures for f and g respectively. We define a bi-invertible structure on $g \star_0 f$ where:

- $(g \star_0 f)^\star = f^\star \star_0 g^\star$
- ${}^\star(g \star_0 f) = {}^\star f \star_0 {}^\star g$
- $(g \star_0 f)_R : g \star_0 f \star_0 f^\star \star_0 g^\star \to \mathrm{id}_z$ is the morphism

$$a_{g,f,f^\star \star_0 g^\star} \star_0 (\mathrm{id}_g \star_1 a^{-1}_{f,f^\star,g^\star}) \star_0 (\mathrm{id}_g \star_1 (f_R \star_1 \mathrm{id}_{g^\star})) \star_0 (\mathrm{id}_g \star_1 \lambda_{g^\star}) \star_0 g_R$$

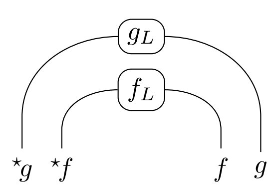
This may be easier to understand from its string diagram, which is given below:



- $(g \star_0 f)_L : {}^\star g \star_0 {}^\star f \star_0 f \star_0 g \to \mathrm{id}_x$ is the morphism

$$a_{{}^\star g,{}^\star f,f \star_0 g} \star_0 (\mathrm{id}_{{}^\star g} \star_1 a) \star_0 (\mathrm{id}_{{}^\star g} \star_1 (f_L \star_1 \mathrm{id}_g)) \star_0 (\mathrm{id}_{{}^\star g} \star_1 \lambda_g) \star_0 g_L$$

which is given by the diagram:



From the bi-invertible structures on f and g, bi-invertible structures on f_L, g_L, f_R, and g_R can be obtained. The associators and unitors used have bi-invertible structures by Theorem 12 and 10. All identity morphisms can be equipped with a bi-invertible structure using Theorems 9 and 12. Then by coinductive hypothesis we can generate bi-invertible structures on $(g \star_0 f)_R$ and $(g \star_0 f)_L$.

Instead suppose $n > 0$. Suppose we have $\alpha : f \to g$ and $\beta : h \to j$ where $f \star_{n-1} h$ and $g \star_{n-1} j$ are both well-defined. Further suppose:

- $(\alpha^\star, {}^\star\alpha, \alpha_R, \alpha_L, \alpha_R^{BI}, \alpha_L^{BI})$ is a bi-invertible structure on α;
- $(\beta^\star, {}^\star\beta, \beta_R, \beta_L, \beta_R^{BI}, \beta_R^{BI})$ is a bi-invertible structure for β.

Then we can define the following bi-invertible structure on $\alpha \star_n \beta$:

- $(\alpha \star_n \beta)^\star = \alpha^\star \star_n \beta^\star$
- ${}^\star(\alpha \star_n \beta) = {}^\star\alpha \star_n {}^\star\beta$
- Let $(\alpha \star_n \beta)_R$ be the following composition:

$$(\alpha \star_n \beta) \star_0 (\alpha^\star \star_n \beta^\star) \overset{i_{\alpha,\beta,\alpha^\star,\beta^\star}}{\to} (\alpha \star_0 \alpha^\star) \star_n (\beta \star_0 \beta^\star) \overset{\alpha_R \star_{n+1} \beta_R}{\to} \mathrm{id}_g \star_n \mathrm{id}_j \to \mathrm{id}_{g \star_{n-1} j}$$

- Similarly, let $(\alpha \star_n \beta)_L$ be the following composition:

$$({}^\star\alpha \star_n {}^\star\beta) \star_0 (\alpha \star_n \beta) \overset{i_{{}^\star\alpha,{}^\star\beta,\alpha,\beta}}{\to} ({}^\star\alpha \star_0 \alpha) \star_n ({}^\star\beta \star_0 \beta) \overset{\alpha_L \star_{n+1} \beta_L}{\to} \mathrm{id}_f \star_n \mathrm{id}_h \to \mathrm{id}_{f \star_n h}$$

Now, both $(\alpha \star_n \beta)_R$ and $(\alpha \star_n \beta)_L$ are the composition of cells with quasi-invertible structures given by the higher pre-category structure (and so have bi-invertible structures given by Theorem 12) and α_L, α_R, β_L, and β_R which have structures α_L^{BI}, α_R^{BI}, β_L^{BI}, and β_R^{BI}. Therefore by coinductive hypothesis there are bi-invertible structures on both $(\alpha \star_n \beta)^\star$ and ${}^\star(\alpha \star_n \beta)$. □

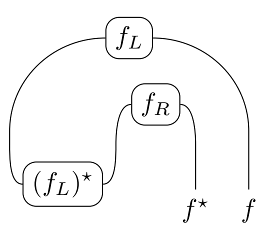
Lemma 14. *Let f be a cell. Given a bi-invertible structure on f, one can be generated on both ${}^\star f$ and $f^\star$.*

Proof. A structure is given for $f^\star$ is bi-invertible and ${}^\star f$ will follow by symmetry. As before let $(f^\star, {}^\star f, f_R, f_L, f_R^{BI}, f_L^{BI})$ be a bi-invertible structure on f. Then a bi-invertible structure on $f^\star$ can be generated by:

- $(f^\star)^\star = f$
- ${}^\star(f^\star) = f$
- $(f^\star)_R : f^\star \star_0 f \to \mathrm{id}$ is the morphism:

  $\lambda^{-1}_{f^\star \star_0 f} \star_0 (f_L^\star \star_1 \mathrm{id}_{f^\star \star_0 f}) \star_0 a_{{}^\star f, f, f^\star \star_0 f} \star_0 (\mathrm{id}_{{}^\star f} \star_1 a^{-1}_{f, f^\star, f}) \star_0 (\mathrm{id}_{{}^\star f} \star_1 (f_R \star_1 \mathrm{id}_f)) \star_0 (\mathrm{id}_{{}^\star f} \star_1 \lambda_f) \star_0 f_L$

  given by the string diagram:



- $(f^\star)_L : f \star_0 f^\star \to \mathrm{id}$ is given by f_R

Let $(f^\star)_L$ be given f_R^{BI} as its bi-invertible structure. A bi-invertible structure on $(f^\star)_R$ can be formed using Theorem 13, using that $f_L^\star$ has a bi-invertible structure given by coinductive hypothesis. □

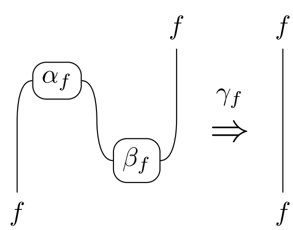
2.2 Half-Adjoint Inverses

Another type of equivalence is a half-adjoint equivalence. Whereas a bi-invertible structure was a weakening of a quasi-invertible structure, a half-adjoint invertible structure is a strict strengthening of a quasi-invertible structure. This is done by adding a coherence condition which effectively enforces f_L and f_R to work "nicely" together. It turns out that there are two such coherence conditions, known as snake equations or zigzag identities, that can be added, yet each of these implies the other and so it sufficient to provide one of these [15, Lemma 3.2]. This is what gives rise to the name *half*-adjoint invertible.

Definition 15. Given a globular set G with identities and composition, with an n-cell $f : x \to y$, a *half-adjoint invertible* structure on f is a tuple $(f', \alpha_f, \beta_f, \gamma_f, \alpha_f^{HAI}, \beta_f^{HAI}, \gamma_f^{HAI})$ where:

- f' is an n-cell $y \to x$;
- α_f is an $(n+1)$-cell $f \star_0 f' \to \mathrm{id}_y$;
- β_f is an $(n+1)$-cell $\mathrm{id}_x \to f' \star_0 f$;
- γ_f is an $(n+2)$-cell $(\rho_f^{-1} \star_0 (\mathrm{id}_f \star_1 \beta_f) \star_0 a_{f,f',f} \star_0 (\alpha_f \star_1 \mathrm{id}_f) \star_0 \lambda_f) \to \mathrm{id}_f$;
- α_f^{HAI} is a half-adjoint invertible structure on α_f;
- β_f^{HAI} is a half-adjoint invertible structure on β_f;
- γ_f^{HAI} is a half-adjoint invertible structure on γ_f.

Where γ_f can be graphically represented by the following diagram:



Theorem 16. *A half-adjoint invertible structure can be restricted to a quasi-invertible structure on the same morphism.*

Proof. Suppose f has half-adjoint invertible structure $(f', \alpha_f, \beta_f, \gamma_f, \alpha_f^{HAI}, \beta_f^{HAI}, \gamma_f^{HAI})$. Then let:

- $f^{-1} = f'$;
- $f_R = \alpha_f$;
- $f_L = \beta'_f$.

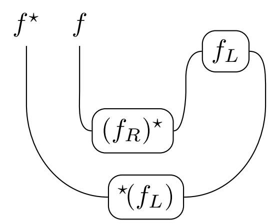
By coinduction, α_f^{HAI} can be restricted to a quasi-invertible structure on f_R and β_f^{HAI} can be restricted to quasi-invertible structure on β_f which induces a structure on β_f^{-1} by Theorem 10. □

Corollary 17. *A half-adjoint invertible structure can be restricted to a bi-invertible structure on the same morphism.*

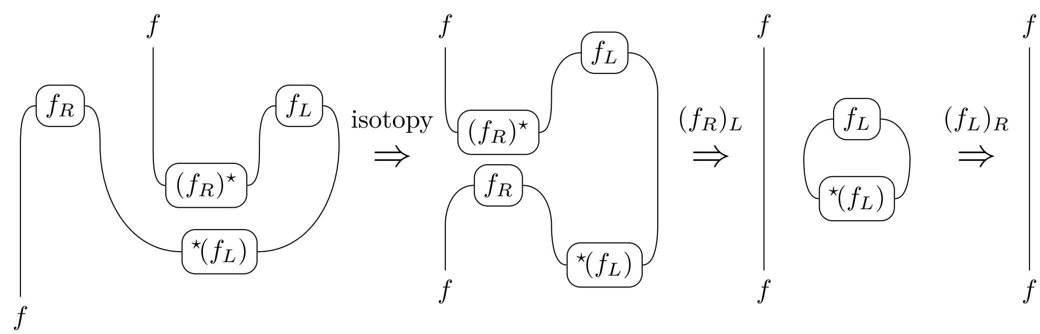
Theorem 18. *A bi-invertible structure* $(f^\star, {}^\star f, f_R, f_L, \dots)$ *on a cell* f *induces a half-adjoint invertible structure* $(f^\star, f_R, \dots)$ *on* f*.*

Proof. Let $(f^\star, {}^\star f, f_R, f_L, f_R^{BI}, f_L^{BI})$ be a bi-invertible structure on f. Then we give the right-adjoint invertible structure $(f^\star, f_R, \beta_f, \gamma_f, f_R^{HAI}, \beta_f^{HAI}, \gamma_f^{HAI})$ where:

- β_f is given by the following diagram:



- γ_f is given by the following diagram:



- f_R^{HAI} can be generated by coinduction hypothesis on f_R^{BI}.
- A bi-invertible structure can be formed for β_f as it is a composition of identities, morphisms with given bi-invertible structures and inverses of those morphisms. Then β_f^{HAI} can be formed coinductively.
- A bi-invertible structure can be put on γ_f as it is the composite of morphisms that have a given bi-invertible structure and an isotopy, which has a quasi-invertible structure given by G being a higher pre-category. Therefore, as before, γ_f^{HAI} can be formed by coinduction.

Hence, $(f^\star, f_R, \beta_f, \gamma_f, f_R^{HAI}, \beta_f^{HAI}, \gamma_f^{HAI})$ is in the form required. □

Corollary 19. *Let* G *be a higher pre-category. Let* $n > 0$ *and* f *be an* n*-cell of* G*. Then the following are equivalent:*

- f *has a bi-invertible structure.*

- *f has a quasi-invertible structure.*
- *f has a half-adjoint invertible structure.*

It should be stressed that this is an equivalence in that each structure can be obtained from the others, and not that the various transformations are in any way inverses to each other.

3 Towards Contractibility

Section 2.2 (in particular Theorem 19) achieves one of the goals set out in the introduction. The other property stated was that bi-invertibility and half-adjoint invertibility should be contractible types. In type theory, a type is contractible if there is an element in that type to which all other terms of that type are propositionally equal. Any contractible type is then equivalent to the unit type. The natural analogue to the higher categorical setting would be to say that a category is contractible if it is equivalent to the terminal category.

Definition 20. The *terminal globular set* T is the globular set with exactly one n-cell for each n. There is then only one choice for the source and target of each cell and all required equations hold by this uniqueness. We can further add identities and composition to this globular set. There is only one way these could be defined due to uniqueness.

What this does not answer is what a suitable notion of equivalence should be. As this could be dependent of the specific definition of a higher category, we do not try to answer this here. Instead, we give a notion of contractibility based on the work that has already been done. Due to the simplicity of the terminal category it is highly likely that this is a sufficient condition for contractibility.

Definition 21. Let G be a globular set. G is *contractible* if given any parallel cells f and g there is a cell $f \to g$.

In this section we aim to define a higher category of cells between invertibility data. While the task of showing contractibility falls beyond the scope of this paper, we do manage to prove partial contractibility results for the bi-invertible case, and suggest how this could be continued.

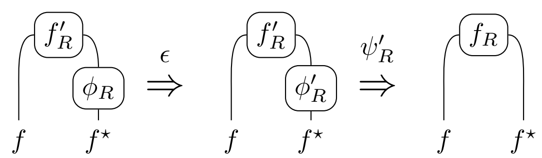
The definition of the cells between these types is largely inspired by [19, Lemma 4.2.5]. This gives us a compositional way to think about equivalence between these two types. It can be formulated as follows: Suppose we have a cell $f : x \to y$ and two bi-invertible structures on it $(f^\star, {}^\star f, f_R, f_L, f_R^{BI}, f_L^{BI})$ and $(f^{\star\prime}, {}^\star f', f'_R, f'_L, f_R^{BI\prime}, f_L^{BI\prime})$. A *bi-invertible 1-morphism* is a tuple consisting of the following data:

- $\phi_R : f^\star \to f^{\star\prime}$;
- ϕ_R^{BI} is a bi-invertible structure on ϕ_R;
- $\psi_R : (\mathrm{id}_f \star_1 \phi_R) \star_0 f'_R \to f_R$;
- ψ_R^{BI} is a bi-invertible structure on ψ_R;
- BIC_R is a bi-invertible 1-morphism from the induced bi-invertible structure on $(\mathrm{id}_f \star_1 \phi_R) \star_0 f'_R$ to f_R^{BI};

- ϕ_L, ϕ_L^{BI}, ψ_L, ψ_L^{BI}, and BIC_L are similar but symmetric to above.

It should be noted that this is quite a natural way to define a cell between these structures. We simply defined a cell for each part of the structure separately, with pretty much only one way of defining each such cell. The only oddity may be that we require these cells to be bi-invertible. This is simply so that $(\mathrm{id}_f \star_1 \phi_R) \star_0 f'_R$ has a canonical bi-invertible structure. Given this, the construction can be continued. Suppose we have $(\phi_R, \psi_R, \dots)$ and $(\phi'_R, \psi'_R, \dots)$. Then a *bi-invertible 2-morphism* can be defined as:

- a bi-invertible 1-morphism $\epsilon : \phi_R \to \phi'_R$;
- a bi-invertible 1-morphism from:



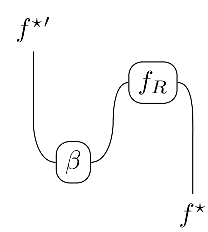
  to ψ_R;
- similar constructions for the remainder of the parts.

We conjecture that continuing in this way will generate a higher category structure. Next is the main result of this section.

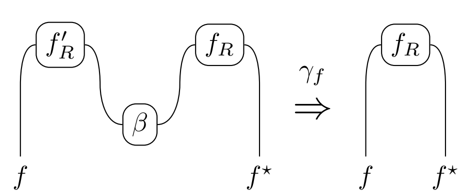
Theorem 22. *Given a cell $f : x \to y$, there is a bi-invertible 1-morphism between any two bi-invertible structures on f.*

Proof. Take structures $(f^\star, {}^\star f, f_R, f_L, f_R^{BI}, f_L^{BI})$ and $(f^{\star\prime}, {}^\star f', f'_R, f'_L, f_R^{BI\prime}, f_L^{BI\prime})$. It will be sufficient to show that ϕ_R and ψ_R can be constructed. Then the cell between bi-invertible structures can be obtained by coinductive hypothesis and the rest of the data follows from symmetry.

Using Theorem 18, a half-adjoint invertible structure $(f^{\star\prime}, f'_R, \beta_{f'}, \gamma_{f'}, {f'_R}^{HAI}, \beta_{f'}^{HAI}, \gamma_{f'}^{HAI})$ on f can be obtained. Then let ϕ_R be the cell given by the diagram:



There is a clear bi-invertible structure on this morphism as it a composition of various bi-invertible cells. We now let ψ_R be:



Which is bi-invertible by composition of bi-invertible structures γ_f and the bi-invertible structure on identities. □

This theorem effectively proves the first "layer" of contractibility, that there is a cell between each pair of 0-cells. Whereas this layer used the adjoint coherences in its proof, the next layer, of which the proof is omitted as it requires 3 dimensional reasoning that is not rigorous with the tools we have here, can be constructed with swallowtail equivalences [16]. These can be thought of as the next coherence up. It is expected that the layer after this could be nicely proved with the coherence condition after this one. We conjecture that given a suitable notion of higher category, it should be possible to show this structure is fully contractible.

One might ask why a similar proof will not work for invertible cells. The reason for this is that the cells considered above are not the canonical type of cell one would use to compare invertible cells. The difference is that in the case of invertible cells, ϕ_L and ϕ_R can be forced to be equal. In the above proof this is not the case. In fact, given that invertibility is not a contractible type in the type theory setting, we should expect that there is no such proof.

It is expected that adding the structure of the half adjoint inverse adds the exactly the kind of coherence we need to get the proof to work. In the half-adjoint case we also have that ϕ_L and ϕ_R must be equal. However a coherence for γ would also be necessary. This higher dimensional cell makes it hard to work with the tools we have, and so a proof is not attempted.

Another contractibility question that can be asked is the following: Let $\mathcal{C}$ be the infinity category freely generated by two 0-cells, a single 1-cell between them and an invertibility structure on this morphism. Is $\mathcal{C}$ contractible?

We know that this is certainly not true for quasi-invertible structures. Suppose we start with 0-cells x and y and a morphism $f : x \to y$ and quasi-invertibility structure $(f^{-1}, f_R, f_L, f_R^{QI}, f_L^{QI})$. Then the contractibility condition effectively tells us that the cell required for a half-adjoint invertibility structure should exist. However, this cell need not exist. If we consider **Cat** as a ω-category (by considering the 2-category and letting each higher globular set be either empty or terminal), then we get that a quasi-invertible structure is an equivalence and a half-adjoint invertible structure is an adjoint equivalence. It is well known that not all equivalences are adjoint equivalences.

However this problem does not arise in the bi-invertible case, and the existence of the cell is enforced in the half-adjoint invertible case. This gives good reason to believe that the infinity category freely generated from these structures may be contractible though we have no progress towards a proof of this. It is possible that similar coinductive procedures can be employed to prove this.

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