A Type Theory for Strictly Unital ∞-Categories

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An overview of (globular) infinity categories

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Compositions:

$$\begin{tikzcd} x \arrow[r, bend left=50, "f", ""{name=U1, below}] \arrow[r, bend right=50, "g"', ""{name=D1}] & y \arrow[r, bend left=50, "h", ""{name=U2, below}] \arrow[r, bend right=50, "i"', ""{name=D2}] & z \arrow[Rightarrow, from=D1, to=U1, "\alpha"'] \arrow[Rightarrow, from=D2, to=U2, "\beta"'] \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, bend left=60, "f", ""{name=U, below}] \arrow[r, "h" description, ""{name=M}] \arrow[r, bend right=60, "g"', ""{name=D}] & y \arrow[Rightarrow, from=M, to=U, "\beta"'] \arrow[Rightarrow, from=D, to=M, "\alpha"'] \end{tikzcd}$$

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Identities:

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] & x \end{tikzcd}$$

Strict Infinity Categories

Associativity

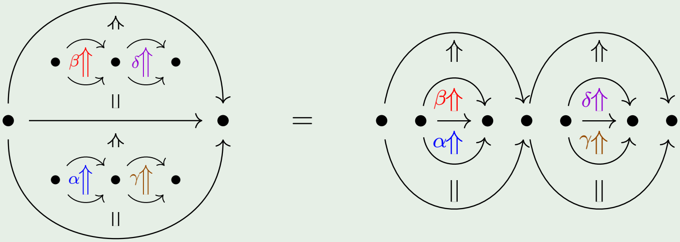
$$w \xrightarrow{w \xrightarrow{f} x \xrightarrow{g} y} y \xrightarrow{h} z \quad = \quad w \xrightarrow{f} x \xrightarrow{x \xrightarrow{g} y \xrightarrow{h} z} z$$

Unitality

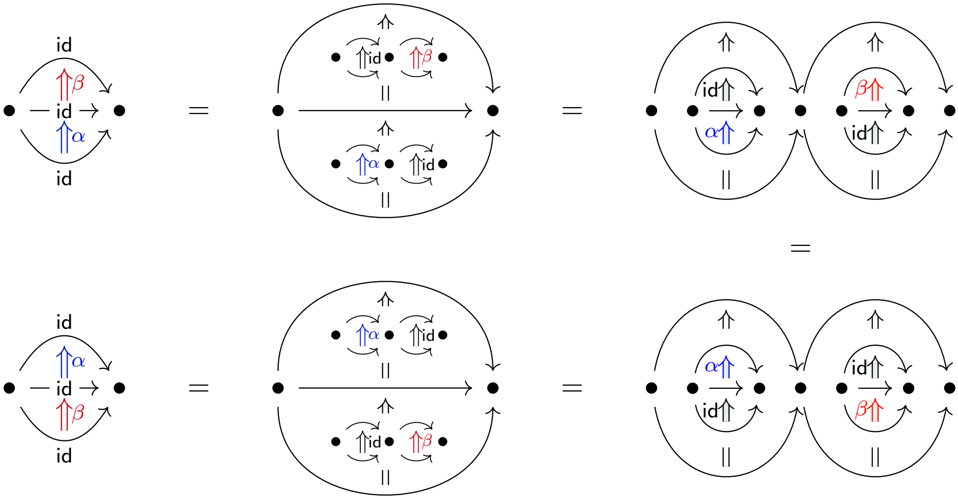
$$\begin{tikzcd}
x \arrow[r, bend left=50, "\mathrm{id}(x)", ""{name=U1, below}] \arrow[r, bend right=50, "\mathrm{id}(x)"', ""{name=D1}] & x \arrow[r, bend left=50, "g", ""{name=U2, below}] \arrow[r, bend right=50, "f"', ""{name=D2}] & y & = & x \arrow[r, bend left=50, "g", ""{name=U3, below}] \arrow[r, bend right=50, "f"', ""{name=D3}] & y
\arrow[Rightarrow, from=D1, to=U1, "\mathrm{id}(\mathrm{id}(x))" description]
\arrow[Rightarrow, from=D2, to=U2, "\alpha"']
\arrow[Rightarrow, from=D3, to=U3, "\alpha"']
\end{tikzcd}$$

Strict Infinity Categories

Interchange



Example: Eckmann-Hilton



Weakness

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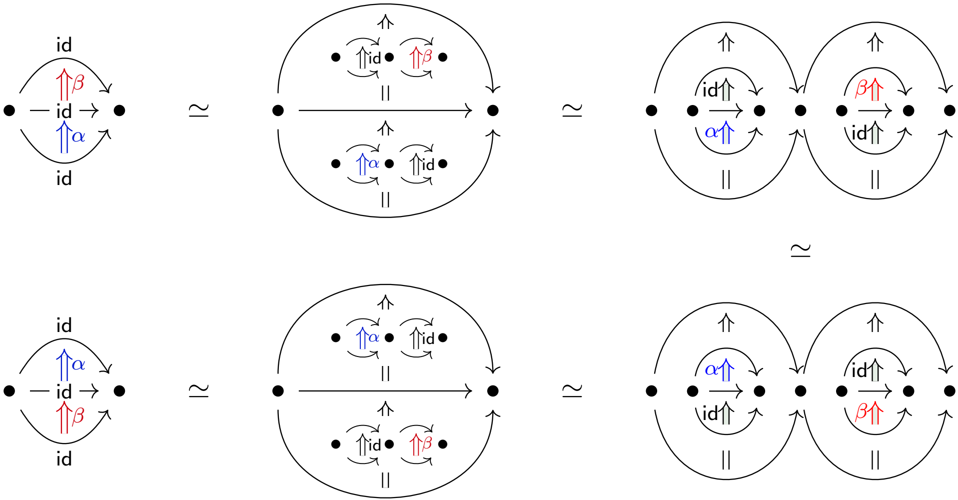
In a weak higher category, the laws are only required to hold up to isomorphism.

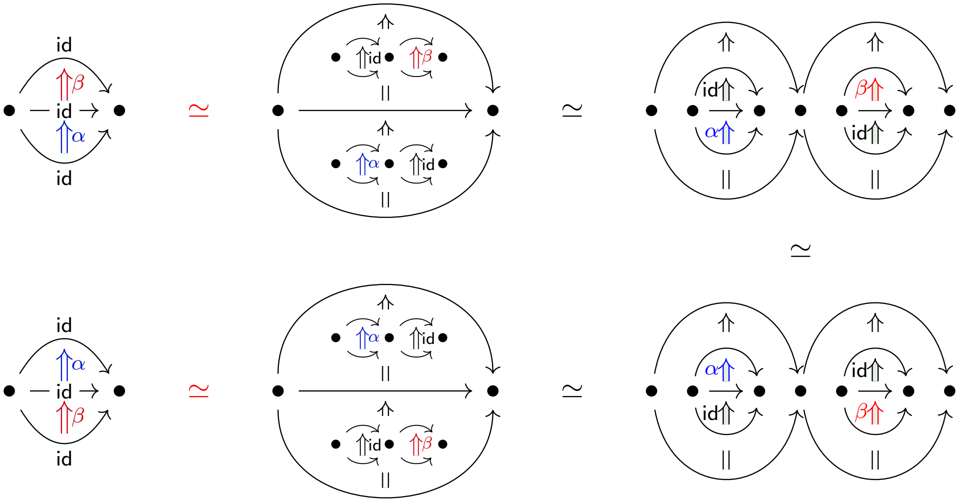
Many examples of higher categories are weak:

- Homotopy groupoids of topological spaces.
- Equality types in HoTT.
- Bicategory of categories and profunctors.

Example: Eckmann-Hilton



Example: Eckmann-Hilton



Weakness vs Strictness

Weak ⟷ Strict

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Harder to use

Easier to use

Weakness vs Strictness

Weak ⟷ Strict

Harder to use	Easier to use
More expressive	Less expressive

Weakness vs Strictness

Weak	⟷	Strict
Harder to use	?	Easier to use
More expressive		Less expressive

Weakness vs Strictness

Weak ← Semistrict → Strict

Harder to use		Easier to use
More expressive		Less expressive

Weakness vs Strictness

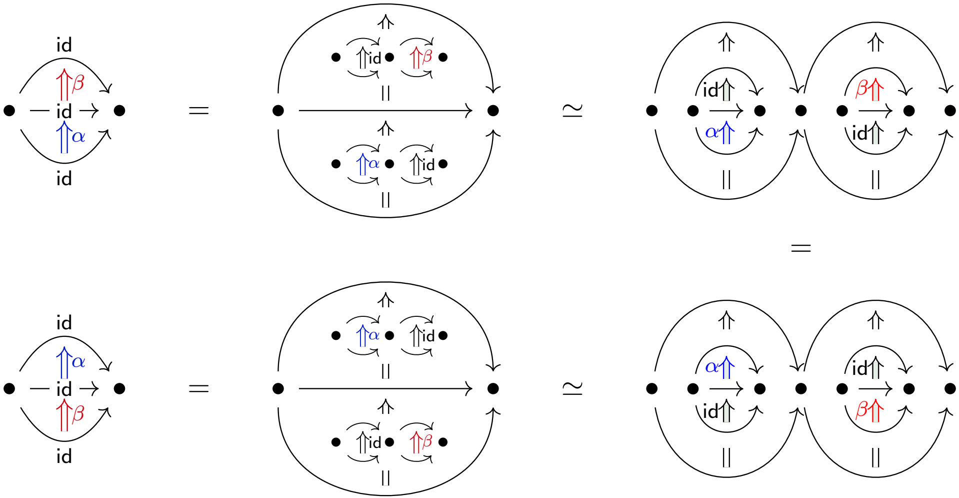
Weak ⟵ Semistrict ⟶ Strict

Harder to use	Lack of definitions	Easier to use
More expressive	Retains expressiveness?	Less expressive

$\mathsf{Catt}_{\mathsf{su}}$

- Catt is a type theory for weak ∞-categories.
- Its terms are the possible operations in an ∞-category.
- By adding a definitional equality to Catt, we can unify certain operations.
- $\mathsf{Catt}_{\mathsf{su}}$ is a new type theory based on Catt with strict units.

Example: Eckmann-Hilton



Eckmann-Hilton in Catt$_{su}$

```
coh id C (x) : x => x
coh id2 C (x(f)y) : f => f
coh comp C (x(f)y(g)z) : x => z
coh vert C (x(f(a)g(b)h)y) : f => h
coh horiz C (x(f(a)g)y(h(b)k)z) : comp f h => comp g k

coh swap3 C (x(f(a)g)y(h(b)k)z)
  : vert (horiz a (id2 h)) (horiz (id2 g) b) =>
    vert (horiz (id2 f) b) (horiz a (id2 k))

let eh {C : Cat} {x :: C} (a :: id x => id x) (b :: id x => id x)
  : [ vert a b => vert b a ]
  = swap3 a b
```

Properties of $\mathsf{Catt}_{\mathsf{su}}$

- Equality in $\mathsf{Catt}_{\mathsf{su}}$ preserves typing.
- Equality is generated by a strongly-terminating, confluent reduction relation.
- Equality and type checking are decidable.
- All terms (of the same dimension) in a disc context are identified.
- Eckmann-Hilton and the Syllepsis have been formalised in $\mathsf{Catt}_{\mathsf{su}}$.