

A Type Theoretic Approach to Semistrict Higher Categories

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12th May 2022

- 1 Globular Infinity Categories
- 2 Weak Infinity Categories
- 3 Semistrict infinity categories

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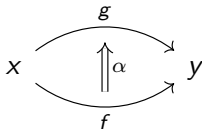
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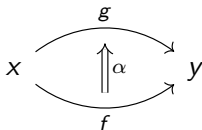
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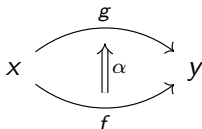
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Definition

A *Globular Set* is a set G with a globular set $G_{x,y}$ for each pair of objects $x, y \in G$.

Composition of 1 cells

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

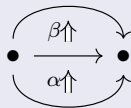
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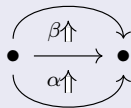
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In a *strict infinity category* we have binary composition of n -cells for along a k boundary for all $k < n$.

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If f and g are n -cells with the k -target of f equalling the k -source of g then there is an n -cell $f \circ_k g$.

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Identities

For each n -cell f there is an $(n + 1)$ -cell $\text{id}_f : f \rightarrow f$.

If $0 \leq k < n$ and f , g , and h are n -cells then:

$$f \circ_k (g \circ_k h) = (f \circ_k g) \circ_k h$$

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Associativity of 1-cells

Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ we have:

$$f \circ_0 (g \circ_0 h) = (f \circ_0 g) \circ_0 h$$

If $0 \leq k < n$ and f is an n -cell with k -source x and k -target y then:

$$\mathrm{id}^{n-k}(x) \circ_k f = f = f \circ_k \mathrm{id}^{n-k}(y)$$

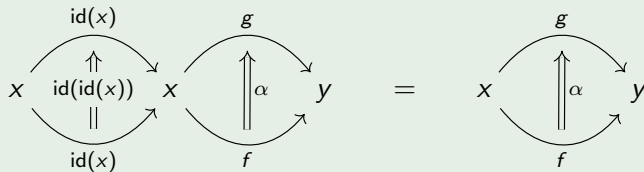
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Identity on 2-cell

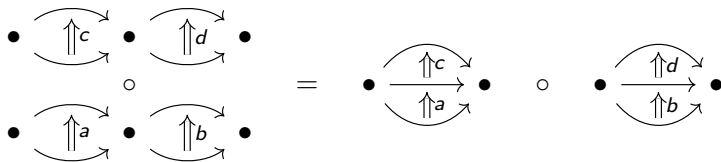
Given $f, g : x \rightarrow y$ and $\alpha : f \rightarrow g$ we have:



Strict Infinity Categories - Interchange

If $0 \leq q < p < n$ and a, b, c, d are n -cells then:

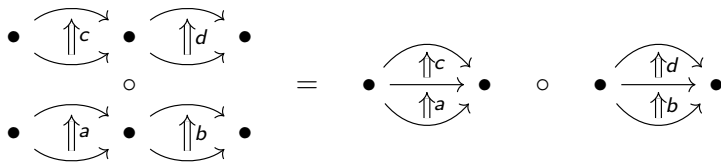
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$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$



Further if $f \circ_k g$ is well defined then:

$$\text{id}_f \circ_k \text{id}(g) = \text{id}(f \circ_k g)$$

Monoidal categories are instances of infinity categories.

Definition (Monoidal category)

A *monoidal category* is a category \mathcal{C} equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and a unit object I satisfying some conditions.

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A strict infinity category with one object and no non-identity n -cells for n higher than 2 is a strict monoidal category.

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The category **Set** is a monoidal category with \otimes given by cartesian product and unit object given by the singleton set.

The monoidal product in **Set** is *not* strict.

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However many isomorphisms can exist between two cells. We require that these isomorphisms be *coherent*.

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However many isomorphisms can exist between two cells. We require that these isomorphisms be *coherent*.

- For a 1-cell $f : x \rightarrow y$, there are unitors $\lambda_f : \text{id}_x \circ f \rightarrow f$ and $\rho_f : f \circ \text{id}_y$.
- λ_{id_x} and ρ_{id_x} are both arrows $\text{id}_x \circ \text{id}_x \rightarrow \text{id}_x$. We can ask that they be isomorphic.
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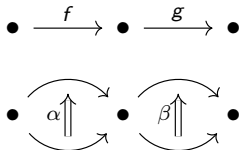
It quickly becomes apparent that we need a more uniform way to package this coherence data.

A *pasting diagram* represents a composition that can be done in an infinity category.

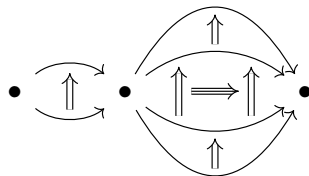
Pasting Diagrams

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The compositions we have already seen form pasting diagrams.



We can also form more complicated compositions as pasting diagrams.



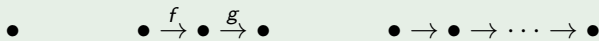
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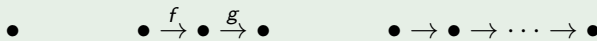
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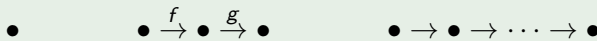
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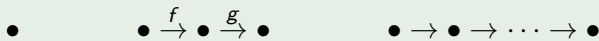
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- In a 1-category, any pasting diagram has a composite.
- Further, there is exactly 1 composite.
- In a strict infinity category, every (higher dimensional) pasting diagram has exactly one composite.
- For weak infinity categories, we weaken the exactness condition to uniqueness up to isomorphism.

Motto for weak infinity categories

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Taking the composite of the diagram:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

gives the composite $f \circ g$.

Over the singleton pasting diagram

$$x$$

and taking $s = x$ and $t = x$ we get a term from x to x representing the identity on x .

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- Substitutions: A substitution is a *morphism* between contexts.

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- If a term is a 0-cell in our infinity category, then it has type \star .
- Otherwise a term is an $(n + 1)$ -cell between parallel n -cells f and g , in which case it has type:

$$f \rightarrow_A g$$

where A is the (common) type of f and g .

The crucial part of CaTT is the Coh constructor, which captures the motto for weak composition.

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- σ labels the pasting diagram with (compatible) terms, and can be represented as a substitution.

Identity

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1-composition

Let $s : x \rightarrow_* y$ and $t : y \rightarrow_* z$ be terms. Their composite is given by:

$$\text{coh } (x \xrightarrow{f} y \xrightarrow{g} z : x \rightarrow_* z)[\sigma]$$

where $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(z) = z$, $\sigma(f) = s$, $\sigma(g) = t$.

Take the context $\Gamma = w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$.

The *associator* is given by:

$$\text{coh}(\Gamma : (f \circ g) \circ h \rightarrow_{w \rightarrow_* z} f \circ (g \circ h))[\text{id}]$$

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However this is no longer possible at dimensions 3 and higher.

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	Strict ∞ - Cat
Associators	✓
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We can strictify:

	Strict ∞ - Cat	Simpson	Grey
Associators	✓	✓	✓
Unitors	✓		✓
Interchangers	✓	✓	

CaTT as we have presented it has no non-trivial equality and no computation.

The idea is to implement a reduction relation that unifies the operations we want to strictify.

By doing this we obtain a type theory for which the models are semistrict categories. Further by showing our reduction is terminating and confluent, we obtain a language for the operations which has decidable type checking and equality.

- CaTT_{su} : Has strict units. Generated by the pruning operation.
- CaTT_{sa} : Has strict associators. Generated by the insertion operation.
- CaTT_{sua} (Work in Progress): Combines the previous two theories.

Example - Syllepsis

- Given two scalars $a, b : id_X \rightarrow id_X$, by the Eckmann Hilton argument we have an isomorphism $EH_{f,g} : a \circ_1 b \simeq b \circ_1 a$.
- In fact, there are two such isomorphisms, $EH_{a,b}$ and $EH_{b,a}^{-1}$, that need not be themselves isomorphic.
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	CaTT	CaTT _{su}	CaTT _{sua}
Eckmann-Hilton	297	15	15
Syllepsis	N/A	675	397

Figure: Coh constructors in Eckmann-Hilton and Syllepsis

- Finish proving metatheorems for CaTT_{SUA} .
- Equivalence of Theories.
- More semistrict type theories, including one for Simpson-like semistrictness.
- Bridging the gap between CaTT and graphical methods.

- [1] Eric Finster and Samuel Mimram. “A Type-Theoretical Definition of Weak ω -Categories”. In: *Proceedings of LICS 2017*. 2017. DOI: 10.1109/lics.2017.8005124. eprint: 1706.02866.
- [2] Eric Finster, Alex Rice, and Jamie Vicary. *A Type Theory for Strictly Associative Infinity Categories*. 2021. arXiv: 2109.01513.
- [3] Eric Finster et al. “A Type Theory for Strictly Unital ∞ -Categories”. In: (2020). DOI: 10.1145/3531130.3533363. arXiv: 2007.08307.