Coinductive Invertibility in Higher Categories

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Outline

What is higher category?

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In higher category theory, we study cases where there is more structure on the collections of morphisms.

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In an ω-category, this continues.

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Let the objects of the globular set be it's 0-cells, morphisms between these be 1-cells, ...

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$x \xrightarrow{f} y \xrightarrow{g} z$ written $f \star_1 g$.

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Codimension 2:
$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\alpha\Uparrow"] & \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\beta\Uparrow"] & \bullet \end{tikzcd}$$
written $\alpha \star_2 \beta$.

Coherence for infinity categories

Infinity categories also have identity cells.
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Regular categories have associativity and unit laws. These are also present in ω-categories.

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Composition is given by path composition.

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Composition is transitivity of equality.

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Similarly **2-Cat**, the category of 2-categories, forms a 3-category.
The category of ω-categories, ω-**Cat**, is itself an ω-category.

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In categories talking about whether two objects are the same or whether an object is unique is often the incorrect perspective. Instead, it is usual to talk about two objects being isomorphic, or an object being unique up to isomorphism.

Definition

Objects X and Y are *isomorphic* if there are morphisms $f : X \to Y$ and $g : Y \to X$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

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Definition

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The need for equivalence here arises as **Cat** is a 2-category.

Equivalence in higher categories

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Ideally, a version of equivalence where the natural isomorphisms are themselves equivalences is required. This leads naturally to a coinductive definition.

Quasi-invertibility

Definition

Given an n-cell $f : x \to y$, a *quasi-invertible* structure on f is a tuple $(f^{-1}, f_R, f_L, f_R I, f_L I)$ where:

- f^{-1} is an n-cell $y \to x$;
- f_R is an $(n+1)$-cell $f \star_1 f^{-1} \to \mathrm{id}_x$;
- f_L is an $(n+1)$-cell $f^{-1} \star_1 f \to \mathrm{id}_y$.
- $f_R I$ is a quasi-invertible structure on f_R.
- $f_L I$ is a quasi-invertible structure on f_L.

Properties of higher categories

ω-categories have coherence properties. Here we specify a minimal set of properties required:

- For $n > 0$ and each n-cell $f : x \to y$, there are $(n+1)$-cells, known as unitors, $\lambda_f : \mathrm{id}_x \star_1 f \to f$ and $\rho_f : f \star_1 \mathrm{id}_y \to f$.
- Given f, g, h, $n > 1$-cells with suitable composition defined, we have an associator $a_{f,g,h} : (f \star_1 g) \star_1 h \to f \star_1 (g \star_1 h)$.
- For compatible morphisms f, g, h, j, we have an interchanger $i_{f,g,h,j} : (f \star_n g) \star_1 (h \star_n j) \to (f \star_1 h) \star_n (g \star_1 j)$.
- For suitable f, g and $n > 1$, there is a cell $\mathrm{id}_f \star_{n+1} \mathrm{id}_g \to \mathrm{id}_{f \star_n g}$.

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Further it is required that the ω-category "respects the graphical calculus".

String diagrams

	Pasting diagrams	String diagrams
0 cells	points	areas
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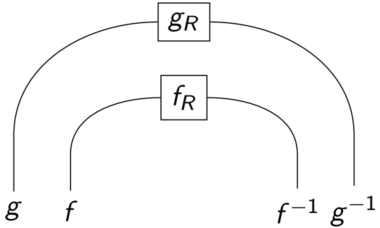

Respecting the graphical calculus

Theorem

In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.

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Definition

An ω-category *respects the graphical calculus* if, for any pair of string diagrams with a planar isotopy between them, there is a quasi-invertible 3-cell from the cell represented by the first to the cell represented by the second.

Properties of quasi-invertible structures

- Given a quasi-invertible structure of f, there exists a quasi-invertible structure on f^{-1}.
- There is a quasi-invertible structure on any identity morphism.

Limitations of quasi-invertibility

Take the ω-category generated by 0-cells x and y and a quasi-invertible morphism $f : x \to y$.

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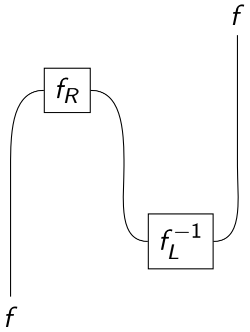

Invertibility in type theory

Inverses of $f : A \to B$:

- Quasi-invertible:

$$\mathrm{qinv}(f) : \Sigma_{g:B \to A}\, f \circ g \sim \mathrm{id}_B \times g \circ f \sim \mathrm{id}_A$$

- Bi-invertible:

$$\begin{aligned}\mathrm{binv}(f) &: \mathrm{linv}(f) \times \mathrm{rinv}(f)\\ \mathrm{linv}(f) &: \Sigma_{g:B \to A}\, g \circ f \sim \mathrm{id}_A\\ \mathrm{rinv}(f) &: \Sigma_{g:B \to A}\, f \circ f \sim \mathrm{id}_B\end{aligned}$$

- Half-adjoint invertible:

$$\mathrm{ishai}(f) : \Sigma_{g:B \to A} \Sigma_{\eta : g \circ f \sim \mathrm{id}_A} \Sigma_{\epsilon : f \circ g \sim \mathrm{id}_B} \Pi_{x:A}\, f(\eta x) = \epsilon(fx)$$

Bi-invertibility

Given an n-cell $f : x \to y$, a *bi-invertible* structure on f is a tuple $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ where:

- $f^\star$ is an n-cell $y \to x$;
- ${}^\star f$ is an n-cell $y \to x$;
- f_R is an $(n+1)$-cell $f \star_1 f^\star \to \mathrm{id}_x$;
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- $f_R BI$ is a bi-invertible structure on f_R.
- $f_L BI$ is a bi-invertible structure on f_L.

Properties of bi-invertible structures

- Any quasi-invertible structure can be converted to a bi-invertible structure.
- Given a bi-invertible structures on a pair of compatible morphisms, there is a bi-invertible structure on their composite.
- Given a bi-invertible structure on f, $f, f^{\star}, {}^{\star}f, \dots$ there are bi-invertible structures on both $f^{\star}$ and ${}^{\star}f$.

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These are proved using coinduction and the results have been formalised in Agda using sized types.

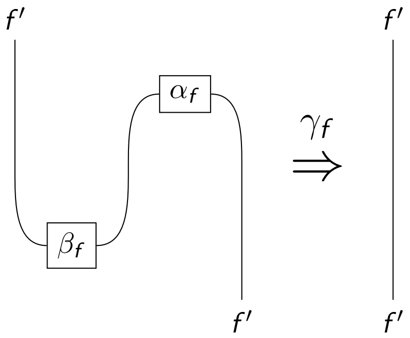
Half-adjoint invertibility

Definition

Given an n-cell $f : x \to y$, a *half-adjoint invertible* structure on f is a tuple $(f', \alpha_f, \beta_f, \gamma_f, \alpha_f HAI, \beta_f HAI, \gamma_f HAI)$ where:

- f' is an n-cell $y \to x$;
- α_f is an $(n+1)$-cell $f \star_1 f' \to \mathrm{id}_y$;
- β_f is an $(n+1)$-cell $\mathrm{id}_x \to f' \star_1 f$;
- γ_f is an $(n+2)$-cell $(\lambda_{f'}^{-1} \star_1 (\beta_f \star_2 \mathrm{id}_{f'}) \star_1 a_{f',f,f'} \star_1 (\mathrm{id}_{f'} \star_2 \alpha_f) \star_1 \rho_{f'}) \to \mathrm{id}_{f'}$;
- $\alpha_f HAI$ is a half-adjoint invertible structure on α_f;
- $\beta_f HAI$ is a half-adjoint invertible structure on β_f;
- $\gamma_f HAI$ is a half-adjoint invertible structure on γ_f.

Half-adjoint invertibility



Adjoint equivalence

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An adjoint equivalence is precisely a half-adjoint invertible structure in **Cat**.

Main theorem

Theorem

Let G be a globular set with the given higher category properties. Let $n > 0$ and f be an n-cell of G. Then the following are equivalent:

- *f has a bi-invertible structure.*
- *f has a quasi-invertible structure.*
- *f has a half-adjoint invertible structure.*

Further work

- A limitation of quasi-invertible structures was presented earlier. Do bi-invertible structures and half-adjoint invertible structures have the same limitation?
- The "respects the graphical calculus" condition is slightly mysterious. It would be good to find a set of more concrete conditions from which it follows.
- Can coinduction be used to nicely describe other parts of higher category theory?

Bi-invertibility implies half-adjoint invertibility

Theorem

A bi-invertible structure $(f^\star, {}^\star f, f_R, f_L, \dots)$ *of a cell* f *induces a half-adjoint invertible structure* $(f^\star, f_R, \dots)$ *on* f*.*

Proof

Let $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ be a bi-invertible structure on f. Then we give the right-adjoint invertible structure $(f^\star, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)$ where:

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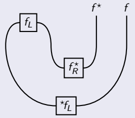

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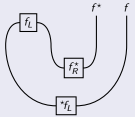

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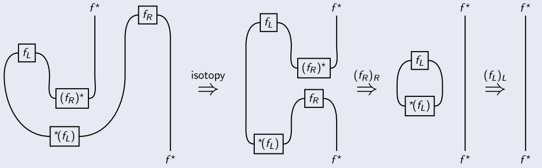

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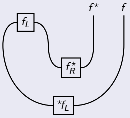

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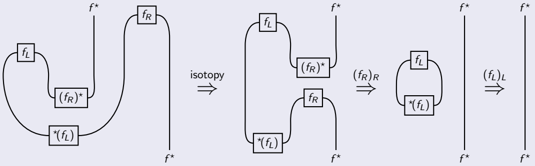

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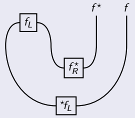

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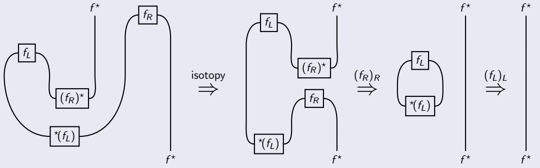

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