Coinductive Invertibility in Higher Categories

Alex Rice

19/03/2020

Outline

What is higher category?

In a regular category there are:

- A collection of objects;
- Between each pair objects there is a collection of morphisms between them.

What is higher category?

In a regular category there are:

- A collection of objects;
- Between each pair objects there is a collection of morphisms between them.

In higher category theory, we study cases where there is more structure on the collections of morphisms.

ω-categories

In a 2-category, the collections of morphisms form categories.

ω-categories

In a 2-category, the collections of morphisms form categories.
In a 3-category, the collections of morphisms form 2-categories.

ω-categories

In a 2-category, the collections of morphisms form categories.
In a 3-category, the collections of morphisms form 2-categories.
In an ω-category, this continues.

Globular Sets

ω-categories can take the shape of globular sets.

Globular Sets

ω-categories can take the shape of globular sets.
These are usually defined as presheaves.

Globular Sets

ω-categories can take the shape of globular sets.
These are usually defined as presheaves.

Definition

A *globular set* $\mathcal{G}$ is a collection of objects $|\mathcal{G}|$ and for each $x, y \in \mathcal{G}$ a globular set $\mathcal{G}_{x,y}$.

Globular Sets

ω-categories can take the shape of globular sets.
These are usually defined as presheaves.

Definition

A *globular set* $\mathcal{G}$ is a collection of objects $|\mathcal{G}|$ and for each $x, y \in \mathcal{G}$ a globular set $\mathcal{G}_{x,y}$.

An object $f \in |\mathcal{G}_{x,y}|$ will be called a morphism between x and y, and will be written $f : x \to y$.

Globular Sets

ω-categories can take the shape of globular sets.
These are usually defined as presheaves.

Definition

A *globular set* $\mathcal{G}$ is a collection of objects $|\mathcal{G}|$ and for each $x, y \in \mathcal{G}$ a globular set $\mathcal{G}_{x,y}$.

An object $f \in |\mathcal{G}_{x,y}|$ will be called a morphism between x and y, and will be written $f : x \to y$.
Let the objects of the globular set be it's 0-cells, morphisms between these be 1-cells, ...

Composition in infinity categories

Composition of 1 cells

$x \xrightarrow{f} y \xrightarrow{g} z$ written $f \star_1 g$.

Composition in infinity categories

Composition of 1 cells

$x \xrightarrow{f} y \xrightarrow{g} z$ written $f \star_1 g$.

Composition of 2 cells

Codimension 1: $$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, bend left=30, "\beta\Uparrow"] \arrow[r, phantom, bend right=30, "\alpha\Uparrow"] & \bullet \end{tikzcd}$$ written $\alpha \star_1 \beta$.

Composition in infinity categories

Composition of 1 cells

$x \xrightarrow{f} y \xrightarrow{g} z$ written $f \star_1 g$.

Composition of 2 cells

Codimension 1:
$$\begin{tikzcd} \bullet \arrow[r, bend left=80] \arrow[r] \arrow[r, bend right=80] \arrow[r, phantom, shift left=3ex, "\beta\Uparrow"] \arrow[r, phantom, shift right=3ex, "\alpha\Uparrow"] & \bullet \end{tikzcd}$$
written $\alpha \star_1 \beta$.

Codimension 2:
$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Uparrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\beta\Uparrow"] & \bullet \end{tikzcd}$$
written $\alpha \star_2 \beta$.

Coherence for infinity categories

Infinity categories also have identity cells.
For each n-cell f there is an $(n+1)$-cell $\mathrm{id}_f : f \to f$.

Coherence for infinity categories

Infinity categories also have identity cells.
For each n-cell f there is an $(n+1)$-cell $\mathrm{id}_f : f \to f$.
Regular categories have associativity and unit laws. These are also present in ω-categories.

Topological spaces

Topological spaces are nice examples of ω-categories. Take a topological space X.

Topological spaces

Topological spaces are nice examples of ω-categories. Take a topological space X.

- 0 cells: X
- 1 cells: Paths between points
- 2 cells: Homotopies between paths
- ...

Topological spaces

Topological spaces are nice examples of ω-categories. Take a topological space X.

- 0 cells: X
- 1 cells: Paths between points
- 2 cells: Homotopies between paths
- ...

Identities are given by constant paths/homotopies.

Topological spaces

Topological spaces are nice examples of ω-categories. Take a topological space X.

- 0 cells: X
- 1 cells: Paths between points
- 2 cells: Homotopies between paths
- ...

Identities are given by constant paths/homotopies.
Composition is given by path composition.

Fundamental ω-groupoid

Similar to the topological space example, any type X forms an ω-category.

Fundamental ω-groupoid

Similar to the topological space example, any type X forms an ω-category.

- 0 cells: terms of type X;
- Higher cells: terms of equality types.

Fundamental ω-groupoid

Similar to the topological space example, any type X forms an ω-category.

- 0 cells: terms of type X;
- Higher cells: terms of equality types.

Identities given by reflexivity proofs.

Fundamental ω-groupoid

Similar to the topological space example, any type X forms an ω-category.

- 0 cells: terms of type X;
- Higher cells: terms of equality types.

Identities given by reflexivity proofs.
Composition is transitivity of equality.

ω-**Cat**

Cat, the category of (small) categories, forms a 2-category with:

- 0-cells: Categories;
- 1-cells: Functors;
- 2-cells: Natural transformations.

ω-**Cat**

Cat, the category of (small) categories, forms a 2-category with:

- 0-cells: Categories;
- 1-cells: Functors;
- 2-cells: Natural transformations.

Similarly **2-Cat**, the category of 2-categories, forms a 3-category.

ω-**Cat**

Cat, the category of (small) categories, forms a 2-category with:

- 0-cells: Categories;
- 1-cells: Functors;
- 2-cells: Natural transformations.

Similarly **2-Cat**, the category of 2-categories, forms a 3-category.
The category of ω-categories, ω-**Cat**, is itself an ω-category.

Isomorphism

In categories talking about whether two objects are the same or whether an object is unique is often the incorrect perspective.

Isomorphism

In categories talking about whether two objects are the same or whether an object is unique is often the incorrect perspective. Instead, it is usual to talk about two objects being isomorphic, or an object being unique up to isomorphism.

Definition

Objects X and Y are *isomorphic* if there are morphisms $f : X \to Y$ and $g : Y \to X$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Equivalence

When comparing categories, the notion of isomorphism is too restrictive. Instead the notion of equivalence is used.

Equivalence

When comparing categories, the notion of isomorphism is too restrictive. Instead the notion of equivalence is used.

Definition

An *equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$

Equivalence

When comparing categories, the notion of isomorphism is too restrictive. Instead the notion of equivalence is used.

Definition

An *equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$

The need for equivalence here arises as **Cat** is a 2-category.

Equivalence in higher categories

When using n-categories for n larger than 2 or ω-categories, an equivalence also becomes too restrictive because of its use of natural isomorphisms.

Equivalence in higher categories

When using n-categories for n larger than 2 or ω-categories, an equivalence also becomes too restrictive because of its use of natural isomorphisms.
Ideally, a version of equivalence where the natural isomorphisms are themselves equivalences is required. This leads naturally to a coinductive definition.

Quasi-invertibility

Definition

Given an n-cell $f : x \to y$, a *quasi-invertible* structure on f is a tuple $(f^{-1}, f_R, f_L, f_R I, f_L I)$ where:

- f^{-1} is an n-cell $y \to x$;
- f_R is an $(n+1)$-cell $f \star_1 f^{-1} \to \mathrm{id}_x$;
- f_L is an $(n+1)$-cell $f^{-1} \star_1 f \to \mathrm{id}_y$.
- $f_R I$ is a quasi-invertible structure on f_R.
- $f_L I$ is a quasi-invertible structure on f_L.

Properties of higher categories

ω-categories have coherence properties. Here we specify a minimal set of properties required:

- For $n > 0$ and each n-cell $f : x \to y$, there are $(n+1)$-cells, known as unitors, $\lambda_f : \mathrm{id}_x \star_1 f \to f$ and $\rho_f : f \star_1 \mathrm{id}_y \to f$.
- Given f, g, h, $n > 1$-cells with suitable composition defined, we have an associator $a_{f,g,h} : (f \star_1 g) \star_1 h \to f \star_1 (g \star_1 h)$.
- For compatible morphisms f, g, h, j, we have an interchanger $i_{f,g,h,j} : (f \star_n g) \star_1 (h \star_n j) \to (f \star_1 h) \star_n (g \star_1 j)$.
- For suitable f, g and $n > 1$, there is a cell $\mathrm{id}_f \star_{n+1} \mathrm{id}_g \to \mathrm{id}_{f \star_n g}$.

Properties of higher categories

ω-categories have coherence properties. Here we specify a minimal set of properties required:

- For $n > 0$ and each n-cell $f : x \to y$, there are $(n+1)$-cells, known as unitors, $\lambda_f : \mathrm{id}_x \star_1 f \to f$ and $\rho_f : f \star_1 \mathrm{id}_y \to f$.
- Given f, g, h, $n > 1$-cells with suitable composition defined, we have an associator $a_{f,g,h} : (f \star_1 g) \star_1 h \to f \star_1 (g \star_1 h)$.
- For compatible morphisms f, g, h, j, we have an interchanger $i_{f,g,h,j} : (f \star_n g) \star_1 (h \star_n j) \to (f \star_1 h) \star_n (g \star_1 j)$.
- For suitable f, g and $n > 1$, there is a cell $\mathrm{id}_f \star_{n+1} \mathrm{id}_g \to \mathrm{id}_{f \star_n g}$.

It is required that all these morphisms have quasi-invertible structures.

Properties of higher categories

ω-categories have coherence properties. Here we specify a minimal set of properties required:

- For $n > 0$ and each n-cell $f : x \to y$, there are $(n+1)$-cells, known as unitors, $\lambda_f : \mathrm{id}_x \star_1 f \to f$ and $\rho_f : f \star_1 \mathrm{id}_y \to f$.
- Given f, g, h, $n > 1$-cells with suitable composition defined, we have an associator $a_{f,g,h} : (f \star_1 g) \star_1 h \to f \star_1 (g \star_1 h)$.
- For compatible morphisms f, g, h, j, we have an interchanger $i_{f,g,h,j} : (f \star_n g) \star_1 (h \star_n j) \to (f \star_1 h) \star_n (g \star_1 j)$.
- For suitable f, g and $n > 1$, there is a cell $\mathrm{id}_f \star_{n+1} \mathrm{id}_g \to \mathrm{id}_{f \star_n g}$.

It is required that all these morphisms have quasi-invertible structures.

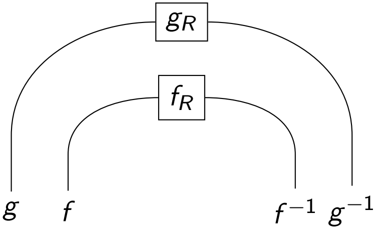
Further it is required that the ω-category "respects the graphical calculus".

String diagrams

	Pasting diagrams	String diagrams
0 cells	points	areas
1 cells	arrows	lines
2 cells	arrows between arrows	points

String diagrams

	Pasting diagrams	String diagrams
0 cells	points	areas
1 cells	arrows	lines
2 cells	arrows between arrows	points



Respecting the graphical calculus

Theorem

In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.

Respecting the graphical calculus

Theorem

In a 2-category, if two string diagrams are planar isotopic then the two morphisms they represent are equal.

Definition

An ω-category *respects the graphical calculus* if, for any pair of string diagrams with a planar isotopy between them, there is a quasi-invertible 3-cell from the cell represented by the first to the cell represented by the second.

Properties of quasi-invertible structures

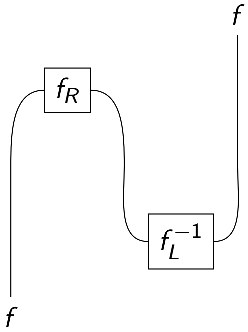
- Given a quasi-invertible structure of f, there exists a quasi-invertible structure on f^{-1}.
- There is a quasi-invertible structure on any identity morphism.

Limitations of quasi-invertibility

Take the ω-category generated by 0-cells x and y and a quasi-invertible morphism $f : x \to y$.

Limitations of quasi-invertibility

Take the ω-category generated by 0-cells x and y and a quasi-invertible morphism $f : x \to y$.



Invertibility in type theory

Inverses of $f : A \to B$:

- Quasi-invertible:

$$\mathrm{qinv}(f) : \Sigma_{g:B\to A}\, f \circ g \sim \mathrm{id}_B \times g \circ f \sim \mathrm{id}_A$$

- Bi-invertible:

$$\begin{aligned}\mathrm{binv}(f) &: \mathrm{linv}(f) \times \mathrm{rinv}(f)\\ \mathrm{linv}(f) &: \Sigma_{g:B\to A}\, g \circ f \sim \mathrm{id}_A\\ \mathrm{rinv}(f) &: \Sigma_{g:B\to A}\, f \circ f \sim \mathrm{id}_B\end{aligned}$$

- Half-adjoint invertible:

$$\mathrm{ishai}(f) : \Sigma_{g:B\to A}\,\Sigma_{\eta:g\circ f\sim\mathrm{id}_A}\,\Sigma_{\epsilon:f\circ g\sim\mathrm{id}_B}\,\Pi_{x:A}\, f(\eta x) = \epsilon(fx)$$

Bi-invertibility

Given an n-cell $f : x \to y$, a *bi-invertible* structure on f is a tuple $(f^\star, {}^\star\!f, f_R, f_L, f_R BI, f_L BI)$ where:

- $f^\star$ is an n-cell $y \to x$;
- ${}^\star\!f$ is an n-cell $y \to x$;
- f_R is an $(n+1)$-cell $f \star_1 f^\star \to \mathrm{id}_x$;
- f_L is an $(n+1)$-cell ${}^\star\!f \star_1 f \to \mathrm{id}_y$.
- $f_R BI$ is a bi-invertible structure on f_R.
- $f_L BI$ is a bi-invertible structure on f_L.

Properties of bi-invertible structures

- Any quasi-invertible structure can be converted to a bi-invertible structure.
- Given a bi-invertible structures on a pair of compatible morphisms, there is a bi-invertible structure on their composite.
- Given a bi-invertible structure on f, $f, f^\star, {}^\star f, \dots$ there are bi-invertible structures on both $f^\star$ and ${}^\star f$.

Properties of bi-invertible structures

- Any quasi-invertible structure can be converted to a bi-invertible structure.
- Given a bi-invertible structures on a pair of compatible morphisms, there is a bi-invertible structure on their composite.
- Given a bi-invertible structure on f, $f, f^\star, {}^\star f, \ldots$ there are bi-invertible structures on both $f^\star$ and ${}^\star f$.

These are proved using coinduction and the results have been formalised in Agda using sized types.

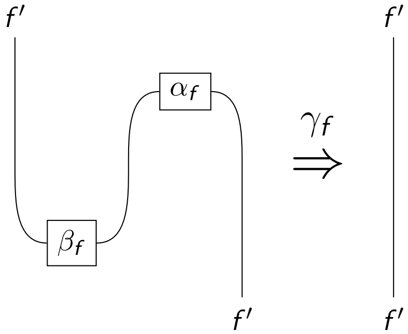
Half-adjoint invertibility

Definition

Given an n-cell $f : x \to y$, a *half-adjoint invertible* structure on f is a tuple $(f', \alpha_f, \beta_f, \gamma_f, \alpha_f HAI, \beta_f HAI, \gamma_f HAI)$ where:

- f' is an n-cell $y \to x$;
- α_f is an $(n+1)$-cell $f \star_1 f' \to \mathrm{id}_y$;
- β_f is an $(n+1)$-cell $\mathrm{id}_x \to f' \star_1 f$;
- γ_f is an $(n+2)$-cell $(\lambda_{f'}^{-1} \star_1 (\beta_f \star_2 \mathrm{id}_{f'}) \star_1 a_{f',f,f'} \star_1 (\mathrm{id}_{f'} \star_2 \alpha_f) \star_1 \rho_{f'}) \to \mathrm{id}_{f'}$;
- $\alpha_f HAI$ is a half-adjoint invertible structure on α_f;
- $\beta_f HAI$ is a half-adjoint invertible structure on β_f;
- $\gamma_f HAI$ is a half-adjoint invertible structure on γ_f.

Half-adjoint invertibility



Adjoint equivalence

Definition

A *adjoint equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) such that $F \dashv G$ with unit η and counit ϵ

Adjoint equivalence

Definition

A *adjoint equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) such that $F \dashv G$ with unit η and counit ϵ

An adjoint equivalence is precisely a half-adjoint invertible structure in **Cat**.

Main theorem

Theorem

Let G be a globular set with the given higher category properties. Let $n > 0$ and f be an n-cell of G. Then the following are equivalent:

- *f has a bi-invertible structure.*
- *f has a quasi-invertible structure.*
- *f has a half-adjoint invertible structure.*

Further work

- A limitation of quasi-invertible structures was presented earlier. Do bi-invertible structures and half-adjoint invertible structures have the same limitation?
- The "respects the graphical calculus" condition is slightly mysterious. It would be good to find a set of more concrete conditions from which it follows.
- Can coinduction be used to nicely describe other parts of higher category theory?

Bi-invertibility implies half-adjoint invertibility

Theorem

A bi-invertible structure $(f^\star, {}^\star f, f_R, f_L, \dots)$ *of a cell* f *induces a half-adjoint invertible structure* $(f^\star, f_R, \dots)$ *on* f*.*

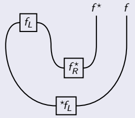
Proof

Let $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ be a bi-invertible structure on f. Then we give the right-adjoint invertible structure $(f^\star, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)$ where:

Proof

Let $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ be a bi-invertible structure on f. Then we give the right-adjoint invertible structure $(f^\star, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)$ where:

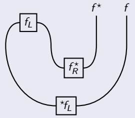
β_f



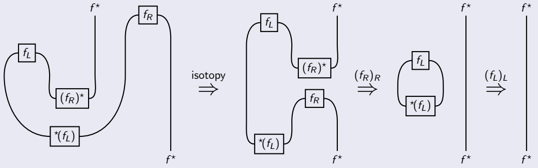
Proof

Let $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ be a bi-invertible structure on f. Then we give the right-adjoint invertible structure $(f^\star, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)$ where:

β_f



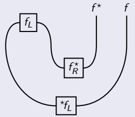
γ_f



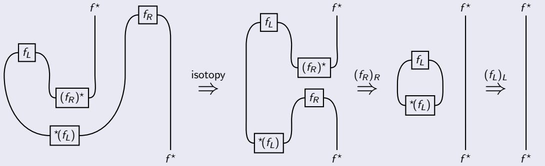
Proof

Let $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ be a bi-invertible structure on f. Then we give the right-adjoint invertible structure $(f^\star, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)$ where:

β_f



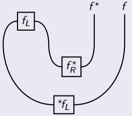
γ_f



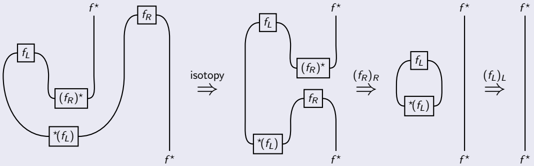
Proof

Let $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ be a bi-invertible structure on f. Then we give the right-adjoint invertible structure $(f^\star, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)$ where:

β_f



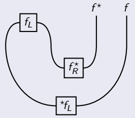
γ_f



Proof

Let $(f^\star, {}^\star f, f_R, f_L, f_R BI, f_L BI)$ be a bi-invertible structure on f.
Then we give the right-adjoint invertible structure
$(f^\star, f_R, \beta_f, \gamma_f, f_R HAI, \beta_f HAI, \gamma_f HAI)$ where:

β_f



γ_f