A Type Theory for Strictly Associative ∞-Categories

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SYCO 10

UNIVERSITY OF CAMBRIDGE

Outline

1. Weak Globular Infinity Categories
2. Type Theories for Infinity Categories
3. Strict Associators

Globular Sets

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Composition of 1 cells

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$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

Composition of 2 cells

Composition along a 1-boundary: $\begin{tikzcd} \bullet \arrow[r, bend left=80, ""{name=T}] \arrow[r, ""{name=M}] \arrow[r, bend right=80, ""{name=B}] & \bullet \arrow[from=M, to=T, phantom, "\beta\Uparrow"] \arrow[from=B, to=M, phantom, "\alpha\Uparrow"] \end{tikzcd}$

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Composition along a 1-boundary:
$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet
\arrow[from=M, to=U, phantom, "\beta\Uparrow"]
\arrow[from=D, to=M, phantom, "\alpha\Uparrow"]
\end{tikzcd}$$

Composition along a 0-boundary:
$$\begin{tikzcd}
\bullet \arrow[r, bend left=50, ""{name=U1, below}] \arrow[r, bend right=50, ""{name=D1, above}] & \bullet \arrow[r, bend left=50, ""{name=U2, below}] \arrow[r, bend right=50, ""{name=D2, above}] & \bullet
\arrow[from=D1, to=U1, phantom, "\alpha\Uparrow"]
\arrow[from=D2, to=U2, phantom, "\beta\Uparrow"]
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Coherence

- For a 1-cell $f : x \to y$, there are unitors $\lambda_f : \mathrm{id}_x \circ f \to f$ and $\rho_f : f \circ \mathrm{id}_y$.
- λ_{id_x} and ρ_{id_x} are both arrows $\mathrm{id}_x \circ \mathrm{id}_x \to \mathrm{id}_x$.
- These should be equivalent.

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- However this is no longer possible at dimensions 3 and higher.

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	Strict ∞-**Cat**
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	Strict ∞-**Cat**	Simpson
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	Strict ∞-**Cat**	Simpson	Grey	$\mathsf{CaTT}_{\mathsf{su}}$[1]
Associators	✓	✓	✓	
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[1] Finster, Reutter, et al., *A Type Theory for Strictly Unital ∞-Categories*
[2] Finster, R., and Vicary, *A Type Theory for Strictly Associative Infinity Categories*

CaTT is a type theory for *weak infinity categories*[3].

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There are 4 pieces of syntax, all defined by mutual induction:

- Contexts: Generating data of an infinity category.

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- Contexts: Generating data of an infinity category.
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- Types: Source and Target for a term.
- Substitutions: A mapping from variables of one context to terms of another.

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$$\alpha : f \to_{x \to_\star y} g$$

Contexts

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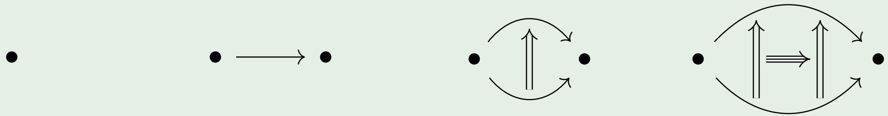
Disc contexts

For each natural number we can define the *disc context* D_n.

D_0 D_1 D_2 D_3



$$D_2 := x : \star, y : \star, f : x \to_\star y, g : x \to_\star y, \alpha : f \to_{x \to_\star y} g$$

Composition in CaTT

Composition can be done with the coh constructor.

coh constructor

Given:

- A context Γ - the shape of the composition,
- A type A in Γ - the boundary of the composition,
- A substitution $\sigma : \Gamma \to \Delta$ - the terms to be composed,

we get a term in Δ:

$$\mathsf{coh}\ (\Gamma : A)[\sigma]$$

The contexts for which the coh constructor is well typed are called *pasting contexts*

Example composition

Suppose we have:

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Let $\Gamma = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$. Γ is a pasting context. Then:

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$$\begin{aligned} (f \cdot g) \cdot h := \mathrm{coh}\ (\Gamma : x \to z)[a \mapsto f \cdot g, \\ b \mapsto h] \end{aligned}$$

Type theories for semistrict languages

- CaTT as we have presented it has no non-trivial equality and no computation.
- The idea is to implement a reduction relation that unifies the operations we want to strictify.
- By doing this we obtain a type theory for which the models are semistrict categories.

CaTT$_{\mathsf{sa}}$ has a definitional equality based on an operation we call insertion.

1-associator

$$x \xrightarrow{f} y \xrightarrow{g} z \qquad\qquad x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

is sent to:

$$x \xrightarrow{f} x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

Components of insertion

$$\Delta = \begin{tikzcd} x \arrow[r, bend left=50, "h"] \arrow[r, "g" description] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, shift left=3, "\beta\Uparrow"] \arrow[r, phantom, shift right=3, "\color{red}\alpha\Uparrow"] & y \arrow[r, "k"] & z \end{tikzcd}$$

$$\Theta = \color{red}\begin{tikzcd} x' \arrow[r, bend left=50, "h'"] \arrow[r, "g'" description] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, shift left=3, "\beta'\Uparrow"] \arrow[r, phantom, shift right=3, "\alpha'\Uparrow"] & y' \end{tikzcd}$$

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$$\Delta \ll_\alpha \Theta = \begin{tikzcd} x' \arrow[r, bend left=70, "h"] \arrow[r, bend left=20, "h'" description] \arrow[r, bend right=20, "g'" description] \arrow[r, bend right=70, "f'"'] \arrow[r, phantom, shift left=5, "\beta\Uparrow"] \arrow[r, phantom, "\beta'\Uparrow"] \arrow[r, phantom, shift right=5, "\alpha'\Uparrow"] & y' \arrow[r, "k"] & z \end{tikzcd}$$

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$$\Delta = \begin{tikzcd} x \arrow[r, bend left=50, "h"{name=H}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=50, "f"'{name=F}] \arrow[from=G, to=H, phantom, "\beta\Uparrow"] \arrow[from=F, to=G, phantom, "\alpha\Uparrow"] & y \arrow[r, "k"] & z \end{tikzcd} \qquad \Theta = \begin{tikzcd} x' \arrow[r, bend left=50, "h'"{name=H}] \arrow[r, "g'"{name=G, description}] \arrow[r, bend right=50, "f'"'{name=F}] \arrow[from=G, to=H, phantom, "\beta'\Uparrow"] \arrow[from=F, to=G, phantom, "\alpha'\Uparrow"] & y' \end{tikzcd}$$

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$$\kappa : \Delta \to \Delta \ll\alpha\ \Theta$$

Given $\sigma : \Delta \to \Gamma$ and $\tau : \Theta \to \Gamma$ we get:

$$\sigma \ll\alpha\ \tau : \Delta \ll\alpha\ \Theta \to \Gamma$$

Universal Property of Insertion

Insertion also satisfies a *universal property*. Suppose we have $\mathsf{coh}\ (\Delta : A)[\sigma]$ where $\sigma(\alpha) = \mathsf{coh}\ (\Theta : B)[\tau]$.

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\Theta \arrow[dr, bend right, "\tau"'] & \\
 & \Gamma
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 & & \Gamma
\end{tikzcd}
$$

Properties of Insertion

Insertion generates a reduction relation for $\mathsf{Catt}_{\mathsf{sa}}$:

$$\mathsf{coh}\ (\Delta : A)[\sigma] \rightsquigarrow \mathsf{coh}\ (\Delta \ll_\alpha \Theta : A[\![\kappa]\!])[\sigma \ll_\alpha \tau]$$

where $\sigma(\alpha) = \mathsf{coh}\ (\Delta : B)[\tau]$.

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This reduction has been proven to have the following properties:

- Subject reduction
- Termination
- Confluence

References

- Finster, Eric and Samuel Mimram. *A Type-Theoretical Definition of Weak ω-Categories*. 2017. DOI: 10.1109/lics.2017.8005124. eprint: 1706.02866.
- Finster, Eric, Alex R., and Jamie Vicary. *A Type Theory for Strictly Associative Infinity Categories*. 2021. arXiv: 2109.01513.
- Finster, Eric, David Reutter, et al. *A Type Theory for Strictly Unital ∞-Categories*. Proceedings of the Thirty-Seventh Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2022). 2020. DOI: 10.1145/3531130.3533363. arXiv: 2007.08307.