

# A Type Theoretic Approach to Semistrict Higher Categories

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- 1 Globular Infinity Categories
- 2 Weak Infinity Categories
- 3 Semistrict infinity categories

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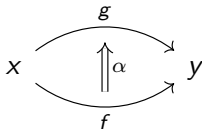
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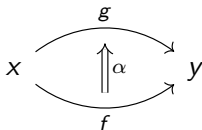
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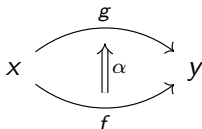
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## Definition

A *Globular Set* is a set  $G$  with a globular set  $G_{x,y}$  for each pair of objects  $x, y \in G$ .



## Composition of 1 cells

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

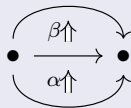
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Composition along a 1-boundary:



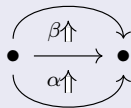
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Composition along a 1-boundary:



Composition along a 0-boundary:



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## Composition

If  $f$  and  $g$  are  $n$ -cells with the  $k$ -target of  $f$  equalling the  $k$ -source of  $g$  then there is an  $n$ -cell  $f \circ_k g$ .

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## Identities

For each  $n$ -cell  $f$  there is an  $(n + 1)$ -cell  $\text{id}_f : f \rightarrow f$ .

If  $0 \leq k < n$  and  $f$ ,  $g$ , and  $h$  are  $n$ -cells then:

$$f \circ_k (g \circ_k h) = (f \circ_k g) \circ_k h$$

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## Associativity of 1-cells

Given  $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$  we have:

$$f \circ_0 (g \circ_0 h) = (f \circ_0 g) \circ_0 h$$

If  $0 \leq k < n$  and  $f$  is an  $n$ -cell with  $k$ -source  $x$  and  $k$ -target  $y$  then:

$$\mathrm{id}^{n-k}(x) \circ_k f = f = f \circ_k \mathrm{id}^{n-k}(y)$$



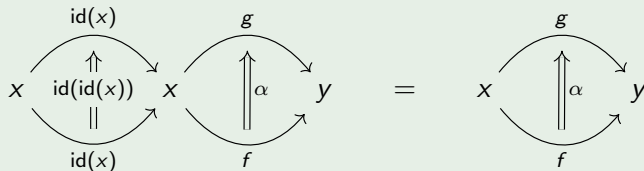
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## Identity on 2-cell

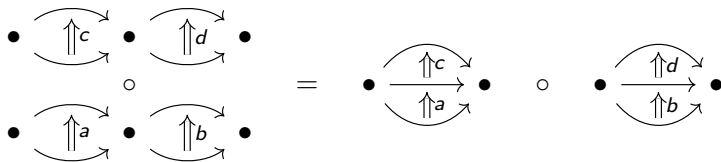
Given  $f, g : x \rightarrow y$  and  $\alpha : f \rightarrow g$  we have:



# Strict Infinity Categories - Interchange

If  $0 \leq q < p < n$  and  $a, b, c, d$  are  $n$ -cells then:

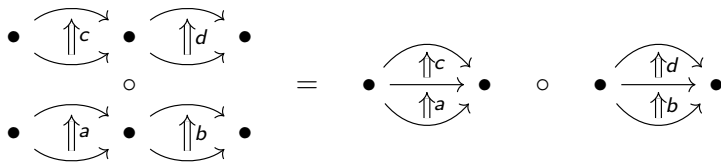
$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$



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$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$



Further if  $f \circ_k g$  is well defined then:

$$\text{id}_f \circ_k \text{id}(g) = \text{id}(f \circ_k g)$$

*Monoidal categories* are instances of infinity categories.

## Definition (Monoidal category)

A *monoidal category* is a category  $\mathcal{C}$  equipped with a functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  and a unit object  $I$  satisfying some conditions.

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A strict infinity category with one object and no non-identity  $n$ -cells for  $n$  higher than 2 is a strict monoidal category.

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The monoidal product in **Set** is *not* strict.



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However many isomorphisms can exist between two cells. We require that these isomorphisms be *coherent*.

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However many isomorphisms can exist between two cells. We require that these isomorphisms be *coherent*.

- For a 1-cell  $f : x \rightarrow y$ , there are unitors  $\lambda_f : \text{id}_x \circ f \rightarrow f$  and  $\rho_f : f \circ \text{id}_y$ .
- $\lambda_{\text{id}_x}$  and  $\rho_{\text{id}_x}$  are both arrows  $\text{id}_x \circ \text{id}_x \rightarrow \text{id}_x$ . We can ask that they be isomorphic.
- This isomorphism will also be subject to coherence conditions.

# Weak Infinity Categories

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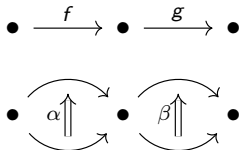
It quickly becomes apparent that we need a more uniform way to package this coherence data.

A *pasting diagram* represents a composition that can be done in an infinity category.

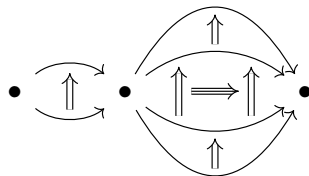
# Pasting Diagrams

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The compositions we have already seen form pasting diagrams.



We can also form more complicated compositions as pasting diagrams.



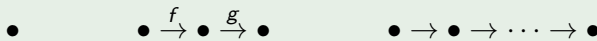
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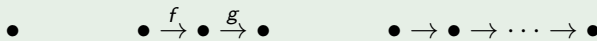
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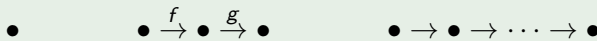


- In a 1-category, any pasting diagram has a composite.
- Further, there is exactly 1 composite.



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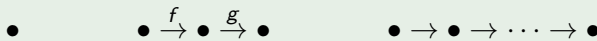
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- Pasting diagrams for 1-categories are simply chains of 1-cells:



- In a 1-category, any pasting diagram has a composite.
- Further, there is exactly 1 composite.
- In a strict infinity category, every (higher dimensional) pasting diagram has exactly one composite.
- For weak infinity categories, we weaken the exactness condition to uniqueness up to isomorphism.

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Taking the composite of the diagram:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

gives the composite  $f \circ g$ .

Over the singleton pasting diagram

$$x$$

and taking  $s = x$  and  $t = x$  we get a term from  $x$  to  $x$  representing the identity on  $x$ .

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- Types: A type contains all the information of the *sources* and *targets* for a term.
- Substitutions: A substitution is a *morphism* between contexts.

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- If a term is a 0-cell in our infinity category, then it has type  $\star$ .
- Otherwise a term is an  $(n + 1)$ -cell between parallel  $n$ -cells  $f$  and  $g$ , in which case it has type:

$$f \rightarrow_A g$$

where  $A$  is the (common) type of  $f$  and  $g$ .

The crucial part of CaTT is the Coh constructor, which captures the motto for weak composition.

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- $\Gamma$  is a pasting diagram, and is represented by a context.
- $s$  and  $t$  are two parallel terms, which can be represented as a type.
- $\sigma$  labels the pasting diagram with (compatible) terms, and can be represented as a substitution.

## Identity

Let  $t$  be a 1 dimensional term. The identity on  $t$  is:

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## 1-composition

Let  $s : x \rightarrow_* y$  and  $t : y \rightarrow_* z$  be terms. Their composite is given by:

$$\text{coh } (x \xrightarrow{f} y \xrightarrow{g} z : x \rightarrow_* z)[\sigma]$$

where  $\sigma(x) = x$ ,  $\sigma(y) = y$ ,  $\sigma(z) = z$ ,  $\sigma(f) = s$ ,  $\sigma(g) = t$ .

Take the context  $\Gamma = w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$ .

The *associator* is given by:

$$\text{coh}(\Gamma : (f \circ g) \circ h \rightarrow_{w \rightarrow_* z} f \circ (g \circ h))[\text{id}]$$

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However this is no longer possible at dimensions 3 and higher.



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	Strict $\infty$ - <b>Cat</b>	Simpson	Grey
Associators	✓	✓	✓
Unitors	✓		✓
Interchangers	✓	✓	

CaTT as we have presented it has no non-trivial equality and no computation.

The idea is to implement a reduction relation that unifies the operations we want to strictify.

By doing this we obtain a type theory for which the models are semistrict categories. Further by showing our reduction is terminating and confluent, we obtain a language for the operations which has decidable type checking and equality.



- $\text{CaTT}_{\text{su}}$ : Has strict units. Generated by the pruning operation.
- $\text{CaTT}_{\text{sa}}$ : Has strict associators. Generated by the insertion operation.
- $\text{CaTT}_{\text{sua}}$  (Work in Progress): Combines the previous two theories.

## Example - Syllepsis

- Given two scalars  $a, b : id_X \rightarrow id_X$ , by the Eckmann Hilton argument we have an isomorphism  $EH_{f,g} : a \circ_1 b \simeq b \circ_1 a$ .
- In fact, there are two such isomorphisms,  $EH_{a,b}$  and  $EH_{b,a}^{-1}$ , that need not be themselves isomorphic.
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	CaTT	CaTT <sub>su</sub>	CaTT <sub>sua</sub>
Eckmann-Hilton	297	15	15
Syllepsis	N/A	675	397

Figure: Coh constructors in Eckmann-Hilton and Syllepsis

- Finish proving metatheorems for  $\text{CaTT}_{\text{SUA}}$ .
- Equivalence of Theories.
- More semistrict type theories, including one for Simpson-like semistrictness.
- Bridging the gap between CaTT and graphical methods.

- [1] Eric Finster and Samuel Mimram. “A Type-Theoretical Definition of Weak  $\omega$ -Categories”. In: *Proceedings of LICS 2017*. 2017. DOI: 10.1109/lics.2017.8005124. eprint: 1706.02866.
- [2] Eric Finster, Alex Rice, and Jamie Vicary. *A Type Theory for Strictly Associative Infinity Categories*. 2021. arXiv: 2109.01513.
- [3] Eric Finster et al. “A Type Theory for Strictly Unital  $\infty$ -Categories”. In: (2020). DOI: 10.1145/3531130.3533363. arXiv: 2007.08307.