A Type Theory for Strictly Associative ∞-Categories

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SYCO 10

UNIVERSITY OF CAMBRIDGE

Outline

1. Weak Globular Infinity Categories

2. Type Theories for Infinity Categories

3. Strict Associators

Globular Sets

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- For each pair of parallel arrows f, g, a set of 2-arrows (or 2-cells) from f to g.

$$\begin{tikzcd} x \arrow[r, bend left=50, "g"{name=G}] \arrow[r, bend right=50, "f"'{name=F}] & y \arrow[from=F, to=G, Rightarrow, "\alpha"'] \end{tikzcd}$$

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$$\begin{tikzcd}
x \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "f"'{name=D}] & y
\arrow[Rightarrow, from=D, to=U, "\alpha"']
\end{tikzcd}$$

- . . .

Composition in Globular Sets

Composition of 1 cells

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Composition of 2 cells

Composition along a 1-boundary:

$$\begin{tikzcd}\bullet \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=3ex, "\beta\Uparrow"] \arrow[r, phantom, shift right=3ex, "\alpha\Uparrow"] & \bullet\end{tikzcd}$$

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Composition of 1 cells

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

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Composition along a 1-boundary:
$$\begin{tikzcd}
\bullet \arrow[r, bend left=70, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=N, below}] \arrow[r, bend right=70, ""{name=D, above}] & \bullet
\arrow[Rightarrow, from=N, to=U, "\beta"]
\arrow[Rightarrow, from=D, to=M, "\alpha"]
\end{tikzcd}$$

Composition along a 0-boundary:
$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=U1, below}] \arrow[r, bend right=60, ""{name=D1, above}] & \bullet \arrow[r, bend left=60, ""{name=U2, below}] \arrow[r, bend right=60, ""{name=D2, above}] & \bullet
\arrow[Rightarrow, from=D1, to=U1, "\alpha"]
\arrow[Rightarrow, from=D2, to=U2, "\beta"]
\end{tikzcd}$$

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Coherence

- For a 1-cell $f : x \to y$, there are unitors $\lambda_f : \mathrm{id}_x \circ f \to f$ and $\rho_f : f \circ \mathrm{id}_y$.
- λ_{id_x} and ρ_{id_x} are both arrows $\mathrm{id}_x \circ \mathrm{id}_x \to \mathrm{id}_x$.
- These should be equivalent.

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- However this is no longer possible at dimensions 3 and higher.

Semistrictness

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	Strict ∞-**Cat**
Associators	✓
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Associators	✓	✓	✓		✓
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[1] Finster, Reutter, et al., *A Type Theory for Strictly Unital ∞-Categories*

[2] Finster, R., and Vicary, *A Type Theory for Strictly Associative Infinity Categories*

CaTT

CaTT is a type theory for *weak infinity categories*[3].

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There are 4 pieces of syntax, all defined by mutual induction:

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- Contexts: Generating data of an infinity category.
- Terms: Operations in an infinity category.
- Types: Source and Target for a term.
- Substitutions: A mapping from variables of one context to terms of another.

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Types in CaTT

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$$\begin{tikzcd} x \arrow[r, bend left=40, "g"{name=G}] \arrow[r, bend right=40, "f"'{name=F}] & y \arrow[from=F, to=G, Rightarrow, "\alpha"'] \end{tikzcd}$$

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$$\begin{tikzcd}
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$$\alpha : f \to_{x \to_\star y} g$$

Contexts

Contexts consist of a list of pairs of variable names and types.

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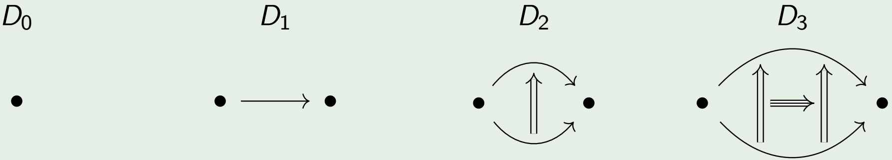
Disc contexts

For each natural number we can define the *disc context* D_n.



$$D_2 := x : \star, y : \star, f : x \to_\star y, g : x \to_\star y, \alpha : f \to_{x \to_\star y} g$$

Composition in CaTT

Composition can be done with the coh constructor.

coh constructor

Given:

- A context Γ - the shape of the composition,
- A type A in Γ - the boundary of the composition,
- A substitution $\sigma : \Gamma \to \Delta$ - the terms to be composed,

we get a term in Δ:

$$\mathsf{coh}\ (\Gamma : A)[\sigma]$$

The contexts for which the coh constructor is well typed are called *pasting contexts*

Example composition

Suppose we have:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$$

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$$\begin{aligned} f \cdot g := \mathsf{coh}\ (\Gamma : x \to z)[a \mapsto f, \\ b \mapsto g] \end{aligned}$$

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$$\begin{aligned} f \cdot g := \mathsf{coh}\ (\Gamma : x \to z)[a \mapsto f, \\ b \mapsto g] \end{aligned}$$

$$\begin{aligned} (f \cdot g) \cdot h := \mathsf{coh}\ (\Gamma : x \to z)[a \mapsto f \cdot g, \\ b \mapsto h] \end{aligned}$$

Type theories for semistrict languages

- CaTT as we have presented it has no non-trivial equality and no computation.
- The idea is to implement a reduction relation that unifies the operations we want to strictify.
- By doing this we obtain a type theory for which the models are semistrict categories.

Insertion

$\mathrm{CaTT}_{\mathrm{sa}}$ has a definitional equality based on an operation we call insertion.

1-associator

$$\begin{tikzcd}
x \arrow[r, "f"] & y \arrow[r, "g"] & z & x' \arrow[r, "f'"] & y' \arrow[r, "g'"] \arrow[ll, Rightarrow, bend left] & z'
\end{tikzcd}$$

is sent to:

$$\begin{tikzcd}
x \arrow[r, "f"] & x' \arrow[r, "f'"] & y' \arrow[r, "g'"] & z'
\end{tikzcd}$$

Components of insertion

$$\Delta = \begin{tikzcd} x \arrow[r, bend left=50, "h"] \arrow[r, "g"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, shift left=2.5ex, "\beta\Uparrow"] \arrow[r, phantom, shift right=2.5ex, "\alpha\Uparrow"] & y \arrow[r, "k"] & z \end{tikzcd}$$

$$\Theta = \begin{tikzcd} x' \arrow[r, bend left=50, "h'"] \arrow[r, "g'"] \arrow[r, bend right=50, "f'"'] \arrow[r, phantom, shift left=2.5ex, "\beta'\Uparrow"] \arrow[r, phantom, shift right=2.5ex, "\alpha'\Uparrow"] & y' \end{tikzcd}$$

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$$\Delta \ll\alpha\ \Theta = \begin{tikzcd} x' \arrow[r, bend left=70, "h"] \arrow[r, bend left=20, "h'"] \arrow[r, bend right=20, "g'"] \arrow[r, bend right=70, "f'"'] \arrow[r, phantom, bend left=45, "\beta\Uparrow"] \arrow[r, phantom, "\beta'\Uparrow"] \arrow[r, phantom, bend right=45, "\alpha'\Uparrow"] & y' \arrow[r, "k"] & z \end{tikzcd}$$

Components of insertion

$$\begin{tikzcd}[column sep=large] \Delta = x \arrow[r, bend left=60, "h"] \arrow[r, "g"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, shift left=3, "\beta\Uparrow"] \arrow[r, phantom, shift right=3, "\alpha\Uparrow"] & y \arrow[r, "k"] & z \end{tikzcd}$$

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$$\begin{tikzcd}[column sep=large] \Delta \lll\alpha\ \Theta = x' \arrow[r, bend left=70, "h"] \arrow[r, bend left=20, "h'"] \arrow[r, bend right=20, "g'"] \arrow[r, bend right=70, "f'"'] \arrow[r, phantom, shift left=5, "\beta\Uparrow"] \arrow[r, phantom, "\beta'\Uparrow"] \arrow[r, phantom, shift right=5, "\alpha'\Uparrow"] & y' \arrow[r, "k"] & z \end{tikzcd}$$

$$\iota : \Theta \to \Delta \lll\alpha\ \Theta$$

Components of insertion

$$\Delta = \begin{tikzcd} x \arrow[r, bend left=50, "h"{name=H}] \arrow[r, "g"{near end, name=G}] \arrow[r, bend right=50, "f"'{name=F}] \arrow[from=G, to=H, phantom, "\beta\Uparrow"] \arrow[from=F, to=G, phantom, "\alpha\Uparrow"] & y \arrow[r, "k"] & z \end{tikzcd}$$

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$$\iota : \Theta \to \Delta \ll\alpha\ \Theta$$

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$$\Delta \ll_\alpha \Theta = \begin{tikzcd} \textcolor{red}{x'} \arrow[r, bend left=70, "h"] \arrow[r, red, bend left=25, "h'"] \arrow[r, red, bend right=10, "g'"] \arrow[r, red, bend right=60, "f'"'] \arrow[r, phantom, shift left=5, "\beta\Uparrow"] \arrow[r, phantom, red, shift left=1, "\beta'\Uparrow"] \arrow[r, phantom, red, shift right=4, "\alpha'\Uparrow"] & \textcolor{red}{y'} \arrow[r, "k"] & z \end{tikzcd}$$

$$\iota : \Theta \to \Delta \ll_\alpha \Theta$$

$$\kappa : \Delta \to \Delta \ll_\alpha \Theta$$

Given $\sigma : \Delta \to \Gamma$ and $\tau : \Theta \to \Gamma$ we get:

$$\sigma \ll_\alpha \tau : \Delta \ll_\alpha \Theta \to \Gamma$$

Universal Property of Insertion

Insertion also satisfies a *universal property*. Suppose we have $\mathsf{coh}\ (\Delta : A)[\sigma]$ where $\sigma(\alpha) = \mathsf{coh}\ (\Theta : B)[\tau]$.

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$$\begin{tikzcd}
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\Theta \arrow[drr, bend right, "\tau"'] & & \\
 & & \Gamma
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$$\begin{tikzcd}
D_n \arrow[r, "\alpha"] \arrow[d, "\text{coh}\ (\Theta:B)[\text{id}]"'] & \Delta \arrow[d, "\kappa"] \arrow[rdd, bend left, "\sigma"] & \\
\Theta \arrow[r, "\iota"'] \arrow[rrd, bend right, "\tau"'] & \Delta \ll\alpha \Theta \arrow[rd, "\sigma \ll\alpha \tau" description] & \\
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\Theta \arrow[r, "\iota"'] \arrow[drr, bend right, "\tau"'] & \Delta \ll\alpha \Theta \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "\sigma \ll\alpha \tau"] & \\
 & & \Gamma
\end{tikzcd}$$

Properties of Insertion

Insertion generates a reduction relation for $\mathsf{Catt}_{\mathsf{sa}}$:

$$\mathsf{coh}\ (\Delta : A)[\sigma] \rightsquigarrow \mathsf{coh}\ (\Delta \ll_\alpha \Theta : A[\![\kappa]\!])[\sigma \ll_\alpha \tau]$$

where $\sigma(\alpha) = \mathsf{coh}\ (\Delta : B)[\tau]$.

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This reduction has been proven to have the following properties:

- Subject reduction
- Termination
- Confluence

References

- Finster, Eric and Samuel Mimram. *A Type-Theoretical Definition of Weak ω-Categories*. 2017. DOI: 10.1109/lics.2017.8005124. eprint: 1706.02866.
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- Finster, Eric, David Reutter, et al. *A Type Theory for Strictly Unital ∞-Categories*. Proceedings of the Thirty-Seventh Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2022). 2020. DOI: 10.1145/3531130.3533363. arXiv: 2007.08307.