A Syntax for Strictly Associative and Unital ∞-Categories

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The Type Theory CATT

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2-cell $\alpha : f \to g$

$$\begin{tikzcd} x \arrow[r, bend left=50, "g"{name=G}] \arrow[r, bend right=50, "f"'{name=F}] \arrow[from=F, to=G, Rightarrow, "\alpha"'] & y \end{tikzcd}$$

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A type gives the boundary of a term.

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Terms built over *pasting diagrams*.

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$$\Delta := \quad \begin{tikzcd} x \arrow[r, bend left, "h"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & y \arrow[r, bend left, "i"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow\beta"] & z \end{tikzcd}$$

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$$s[\![\sigma]\!]$$

$$\sigma := \langle f \mapsto a, g \mapsto (b * c)\rangle$$

$$a * (b * c) := (f * g)[\![\sigma]\!]$$

The Associator

$$\Delta := \quad x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w$$

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$$\alpha_{a,b,c} := \mathsf{coh}\,(\Delta : (a * b) * c \to a * (b * c))$$

Semistrictness

The theory described by CATT is *Weak*.

Laws of categories are given by equivalence.

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Weak $\longleftarrow\!\!\!\longrightarrow$ Semistrict $\longrightarrow$ Strict

$\uparrow$

$\textsc{Catt}_{\mathsf{su}}$

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CATT_{su}

CATT_{sua}

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\begin{tikzcd}
u \arrow[r, "c"] & v \arrow[r, "d"] & w \\
x \arrow[u, "f"] \arrow[r, "a"'] & y \arrow[u, "g"'] \arrow[ul, Rightarrow, "A"] \arrow[r, "b"'] & z \arrow[u, "h"'] \arrow[ul, Rightarrow, "B"]
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$$

$$(\mathrm{id}_a \otimes B) * (A \otimes \mathrm{id}_d) : (a * b) * h \to f * (c * d)$$

$\textsc{Catt}$ has trivial equality.

$\textsc{Catt}_{\mathsf{su}}$ has disc removal, endo-coherence removal, and pruning.

In $\textsc{Catt}_{\mathsf{sua}}$, pruning is replaced by insertion.

$$\textsc{Catt}_{\mathsf{sua}} := \textsc{Catt} + \text{insertion} + \text{disc removal} + \text{endo-coherence removal}$$

Insertion Rule

$$(a * b) * c \ =_{\mathsf{sua}} \ a * b * c \equiv \mathsf{coh}\,(x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w : x \to w)$$

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Recalling $(a * b) * c \equiv (f * g)[\![\langle f \mapsto a, g \mapsto b * c \rangle]\!]$:

$$x \xrightarrow{\;f\;} y \xrightarrow{\;b*c\;} z \qquad\qquad x' \xrightarrow{\;f'\;} y' \xrightarrow{\;g'\;} z'$$

is sent to:

$$x \xrightarrow{\;f\;} x' \xrightarrow{\;f'\;} y' \xrightarrow{\;g'\;} z'$$

Universal Property of Insertion

$$\mathsf{coh}\ (\Delta : s \to t)[\![\sigma]\!] \quad x \in \Delta \quad x[\![\sigma]\!] \equiv \mathsf{coh}\ (\Theta : u \to v)[\![\tau]\!]$$

$$\begin{tikzcd} & \Delta \arrow[dr, bend left, "\sigma"] & \\ \Theta \arrow[rr, bend right, "\tau"'] & & \Gamma \end{tikzcd}$$

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$$\begin{tikzcd}
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$$\begin{tikzcd}
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\Theta \arrow[rr, bend right, "\tau"'] & {\color{red}\Delta \ll_x \Theta} & \Gamma
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 & \Delta \arrow[d, "\color{red}\kappa"] \arrow[dr, bend left, "\sigma"] & \\
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$$\begin{tikzcd}
D \arrow[r, "x"] \arrow[d, "\mathsf{coh}\ (\Theta : u \to v)"'] & \Delta \arrow[d, "\kappa"] \arrow[ddr, bend left, "\sigma"] & \\
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\Theta \arrow[r, "\iota"'] \arrow[drr, bend right, "\tau"'] \arrow[ur, phantom, "\ulcorner" very near end] & \Delta \ll_x \Theta \arrow[dr, "\sigma \ll_x \tau" description] & \\
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\Theta \arrow[r, swap, "\iota"] \arrow[drr, bend right, swap, "\tau"] & \Delta \ll_x \Theta \arrow[dr, "\sigma \ll_x \tau" description] & \\
& & \Gamma
\end{tikzcd}$$

$$\mathsf{coh}\ (\Delta : s \to t)[\![\sigma]\!] = \mathsf{coh}\ (\Delta \ll_x \Theta : s[\![\kappa]\!] \to t[\![\kappa]\!])[\![\sigma \ll_x \tau]\!]$$

Normalisation

Equality in $\textsc{Catt}_{\mathsf{sua}}$ is decidable.

$\textsc{Catt}_{\mathsf{sua}}$ has unique normal forms.

Obtained by reduction system.

Termination: Assign *syntactic complexity* to terms.

Confluence: Encode various constructions in Agda.

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Type checking is decidable.

We provide a interpreter which:

- Provides tools for construction terms.
- Type checks terms.
- Reduces terms to $\textsc{Catt}_{\mathsf{sua}}$ normal form.

Summary

We introduce the type theory $\textsc{Catt}_{\mathsf{sua}}$.

$\textsc{Catt}_{\mathsf{sua}}$ models strictly unital and associative ∞-categories.

$\textsc{Catt}_{\mathsf{sua}}$ terms are simpler than their $\textsc{Catt}$ or $\textsc{Catt}_{\mathsf{su}}$ equivalents.

Normal forms for $\textsc{Catt}_{\mathsf{sua}}$ are obtained via a reduction system:

This reduction system is strongly terminating and confluent.

Try our interpreter for $\textsc{Catt}_{\mathsf{sua}}$:

`https://github.com/alexarice/catt-strict`

Thank you for listening.