

A Syntax for Strictly Associative and Unital ∞ -Categories

Eric Finster Alex Rice Jamie Vicary

LICS 2024

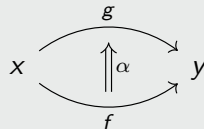


\mathbf{CATT} is a type theory for ∞ -categories.

\mathbf{CATT} is a type theory for ∞ -categories.

Globular ∞ -categories contain higher dimensional arrows.

2-cell $\alpha : f \rightarrow g$



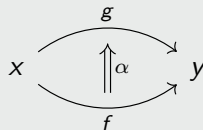
The Type Theory CATT

CATT is a type theory for ∞ -categories.

Globular ∞ -categories contain higher dimensional arrows.

Terms of CATT correspond to operations.

2-cell $\alpha : f \rightarrow g$



The Type Theory \mathbf{CATT}

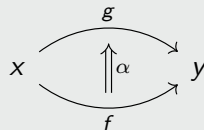
\mathbf{CATT} is a type theory for ∞ -categories.

Globular ∞ -categories contain higher dimensional arrows.

Terms of \mathbf{CATT} correspond to operations.

A type gives the boundary of a term.

2-cell $\alpha : f \rightarrow g$



Terms in CATT

Terms represent the possible operations in a globular ∞ -category.

Terms built over *pasting diagrams*.

Compound terms using substitutions.

Terms in CATT

Terms represent the possible operations in a globular ∞ -category.

Terms built over *pasting diagrams*.

Compound terms using substitutions.

$\text{coh}(\Gamma : s \rightarrow t)$

Terms in \mathbf{CATT}

Terms represent the possible operations in a globular ∞ -category.

Terms built over *pasting diagrams*.

$$\text{coh}(\Gamma : s \rightarrow t)$$

$$\Gamma := \textcolor{red}{x} \xrightarrow{f} y \xrightarrow{g} \textcolor{red}{z}$$

$$f * g := \text{coh}(\Gamma : x \rightarrow z)$$

Compound terms using substitutions.

Terms in CATT

Terms represent the possible operations in a globular ∞ -category.

Terms built over *pasting diagrams*.

Compound terms using substitutions.

$$\text{coh}(\Gamma : s \rightarrow t)$$

$$\Gamma := x \xrightarrow{f} y \xrightarrow{g} z$$

$$f * g := \text{coh}(\Gamma : x \rightarrow z)$$

$$\Delta := x \begin{array}{c} \curvearrowright^h \\ \Uparrow \alpha \\ \Downarrow f \\ \curvearrowright \end{array} y \begin{array}{c} \curvearrowright^i \\ \Uparrow \beta \\ \Downarrow g \\ \curvearrowright \end{array} z$$

$$\alpha \otimes \beta := \text{coh}(\Delta : f * g \rightarrow h * i)$$

Terms in CATT

Terms represent the possible operations in a globular ∞ -category.

Terms built over *pasting diagrams*.

$$\text{coh}(\Gamma : s \rightarrow t)$$

$$\Gamma := x \xrightarrow{f} y \xrightarrow{g} z$$

$$f * g := \text{coh}(\Gamma : x \rightarrow z)$$

$$\Delta := x \begin{array}{c} \curvearrowright^h \\ \Uparrow \alpha \\ \Downarrow f \\ \curvearrowright \end{array} y \begin{array}{c} \curvearrowright^i \\ \Uparrow \beta \\ \Downarrow g \\ \curvearrowright \end{array} z$$

$$\alpha \otimes \beta := \text{coh}(\Delta : f * g \rightarrow h * i)$$

Compound terms using substitutions.

$$s[\![\sigma]\!]$$

Terms in CATT

Terms represent the possible operations in a globular ∞ -category.

Terms built over *pasting diagrams*.

$$\text{coh}(\Gamma : s \rightarrow t)$$

$$\Gamma := x \xrightarrow{f} y \xrightarrow{g} z$$

$$f * g := \text{coh}(\Gamma : x \rightarrow z)$$

$$\Delta := \begin{array}{ccccc} & & h & & \\ & \nearrow & & \searrow & \\ x & & \Uparrow \alpha & & y & \nearrow & i & & \searrow & & z \\ & \searrow & & \nearrow & \\ & & f & & g & & \end{array}$$

$$\alpha \otimes \beta := \text{coh}(\Delta : f * g \rightarrow h * i)$$

Compound terms using substitutions.

$$s[\![\sigma]\!]$$

$$\sigma := \langle f \mapsto a, g \mapsto (b * c) \rangle$$

$$a * (b * c) := (f * g)[\![\sigma]\!]$$

$$\Delta := x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w$$

$$\Delta := x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w$$

$$\alpha_{a,b,c} := \text{coh} (\Delta : (a * b) * c \rightarrow a * (b * c))$$

The theory described by \mathbf{CATT} is *Weak*.

Laws of categories are given by equivalence.

Weak \longleftrightarrow Strict

Semistrictness

The theory described by \mathbf{CATT} is *Weak*.

Laws of categories are given by equivalence.



Semistrictness

The theory described by \mathbf{CATT} is *Weak*.

Laws of categories are given by equivalence.

Weak \longleftrightarrow Strict

Harder to use

Easier to use

More expressive

Less expressive

Semistrictness

The theory described by \mathbf{CATT} is *Weak*.

Laws of categories are given by equivalence.

Weak \longleftrightarrow Semistrict \longrightarrow Strict

Harder to use

Easier to use

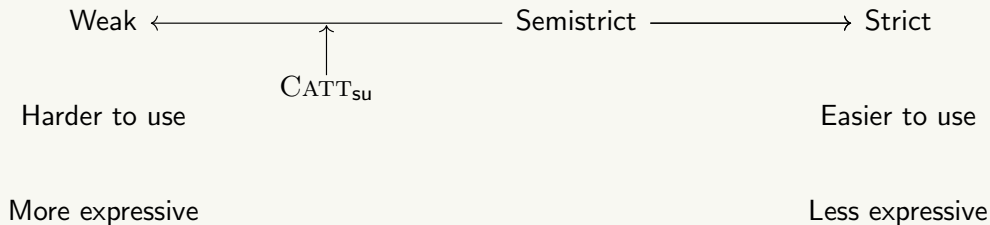
More expressive

Less expressive

Semistrictness

The theory described by CATT is *Weak*.

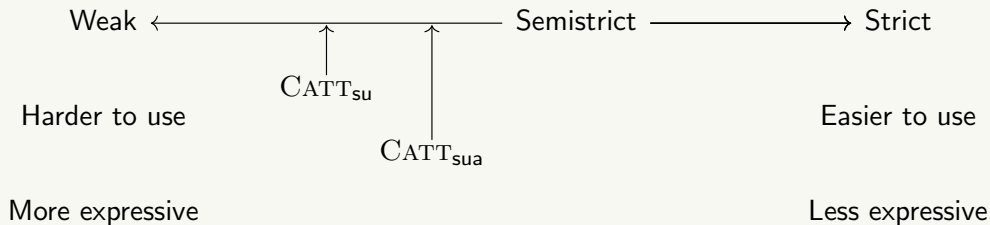
Laws of categories are given by equivalence.



Semistrictness

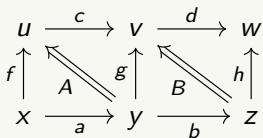
The theory described by $CATT$ is *Weak*.

Laws of categories are given by equivalence.

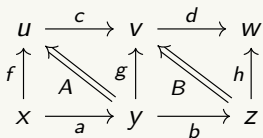


(Semi)strictness allows more operations to be defined.

(Semi)strictness allows more operations to be defined.



(Semi)strictness allows more operations to be defined.



$$(\text{id}_a \otimes B) * (A \otimes \text{id}_d) : (a * b) * h \rightarrow f * (c * d)$$

CATT has trivial equality.

CATT_{su} has disc removal, endo-coherence removal, and pruning.

In CATT_{sua}, pruning is replaced by insertion.

$$\text{CATT}_{\text{sua}} := \text{CATT} + \text{insertion} + \text{disc removal} + \text{endo-coherence removal}$$

$$(a * b) * c =_{\text{sua}} a * b * c \equiv \text{coh} (x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w : x \rightarrow w)$$

Insertion Rule

$$(a * b) * c =_{\text{Sua}} a * b * c \equiv \text{coh} (x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w : x \rightarrow w)$$

Recalling $(a * b) * c \equiv (f * g)[\langle f \mapsto a, g \mapsto b * c \rangle]$:

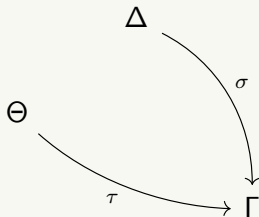


is sent to:

$$x \xrightarrow{f} x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

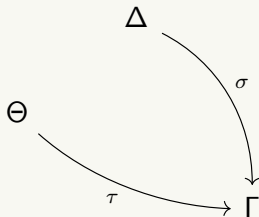
Universal Property of Insertion

$$\text{coh } (\Delta : s \rightarrow t) \llbracket \sigma \rrbracket \quad x \in \Delta \quad x \llbracket \sigma \rrbracket \equiv \text{coh } (\Theta : u \rightarrow v) \llbracket \tau \rrbracket$$



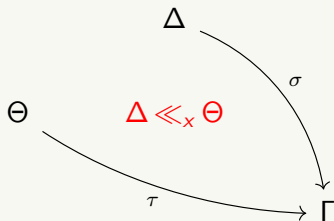
Universal Property of Insertion

$\text{coh}(\Delta : s \rightarrow t) \llbracket \sigma \rrbracket \quad x \in \Delta \quad x \llbracket \sigma \rrbracket \equiv \text{coh}(\Theta : u \rightarrow v) \llbracket \tau \rrbracket \quad (+ \text{ syntactic side condition})$



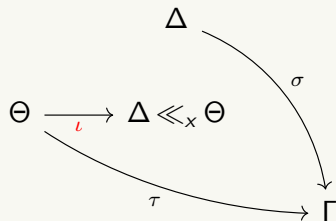
Universal Property of Insertion

$\text{coh}(\Delta : s \rightarrow t) \llbracket \sigma \rrbracket \quad x \in \Delta \quad x \llbracket \sigma \rrbracket \equiv \text{coh}(\Theta : u \rightarrow v) \llbracket \tau \rrbracket \quad (+ \text{ syntactic side condition})$



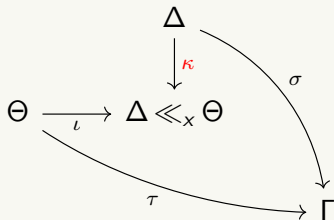
Universal Property of Insertion

$\text{coh}(\Delta : s \rightarrow t) \llbracket \sigma \rrbracket \quad x \in \Delta \quad x \llbracket \sigma \rrbracket \equiv \text{coh}(\Theta : u \rightarrow v) \llbracket \tau \rrbracket \quad (+ \text{ syntactic side condition})$



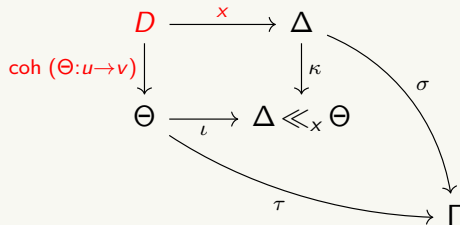
Universal Property of Insertion

$\text{coh}(\Delta : s \rightarrow t) \llbracket \sigma \rrbracket \quad x \in \Delta \quad x \llbracket \sigma \rrbracket \equiv \text{coh}(\Theta : u \rightarrow v) \llbracket \tau \rrbracket \quad (+ \text{ syntactic side condition})$



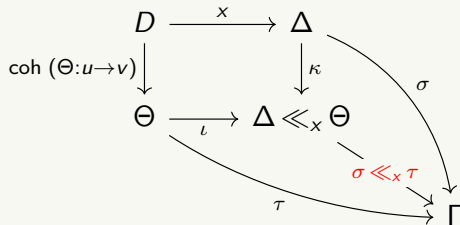
Universal Property of Insertion

$\text{coh}(\Delta : s \rightarrow t) \llbracket \sigma \rrbracket \quad x \in \Delta \quad x \llbracket \sigma \rrbracket \equiv \text{coh}(\Theta : u \rightarrow v) \llbracket \tau \rrbracket \quad (+ \text{ syntactic side condition})$



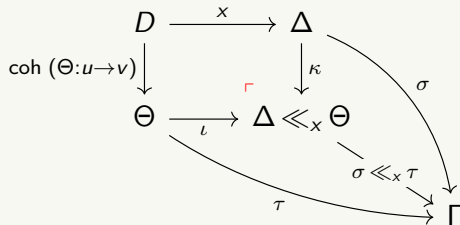
Universal Property of Insertion

$\text{coh}(\Delta : s \rightarrow t) \llbracket \sigma \rrbracket \quad x \in \Delta \quad x \llbracket \sigma \rrbracket \equiv \text{coh}(\Theta : u \rightarrow v) \llbracket \tau \rrbracket \quad (+ \text{ syntactic side condition})$



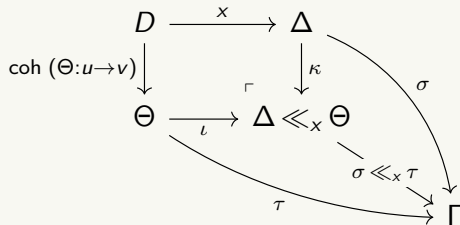
Universal Property of Insertion

$\text{coh}(\Delta : s \rightarrow t) \llbracket \sigma \rrbracket \quad x \in \Delta \quad x \llbracket \sigma \rrbracket \equiv \text{coh}(\Theta : u \rightarrow v) \llbracket \tau \rrbracket \quad (+ \text{ syntactic side condition})$



Universal Property of Insertion

$\text{coh}(\Delta : s \rightarrow t)[\sigma] \quad x \in \Delta \quad x[\sigma] \equiv \text{coh}(\Theta : u \rightarrow v)[\tau] \quad (+ \text{ syntactic side condition})$



$$\text{coh}(\Delta : s \rightarrow t)[\sigma] = \text{coh}(\Delta \ll_x \Theta : s[\kappa] \rightarrow t[\kappa])(\sigma \ll_x \tau)$$

Equality in CATT_{sua} is decidable.

CATT_{sua} has unique normal forms.

Obtained by reduction system.

Termination: Assign *syntactic complexity* to terms.

Confluence: Encode various constructions in Agda.

Equality in CATT_{sua} is decidable.

CATT_{sua} has unique normal forms.

Obtained by reduction system.

Termination: Assign *syntactic complexity* to terms.

Confluence: Encode various constructions in Agda.

Type checking is decidable.

We provide a interpreter which:

- Provides tools for construction terms.
- Type checks terms.
- Reduces terms to CATT_{sua} normal form.

We introduce the type theory CATT_{sua} .

CATT_{sua} models strictly unital and associative ∞ -categories.

CATT_{sua} are simpler than their CATT or CATT_{su} equivalents.

Normal forms for CATT_{sua} are obtained via a reduction system:

This reduction system is strongly terminating and confluent.

Try our interpreter for CATT_{sua} :

<https://github.com/alexarice/catt-strict>

Thank you for listening.