

A Type Theory for Strictly Associative Infinity Categories

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Outline

- 1 Strict Infinity Categories
- 2 CaTT
- 3 Strict Associators

Globular Sets

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$$\begin{array}{c} \vdots \\ G_2 \\ s_1 \downarrow \quad \downarrow t_1 \\ G_1 \\ s_0 \downarrow \quad \downarrow t_0 \\ G_0 \end{array}$$

Globular Sets

Definition

A *globular set* \mathcal{G} consists of sets G_n for each n and maps $s_n, t_n : G_{n+1} \rightarrow G_n$ for each n such that the following *globularity conditions* hold:

$$s_n \circ s_{n+1} = s_n \circ t_{n+1}$$

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Terminal globular set

The terminal globular set has one cell at each dimension and all source and target maps are uniquely defined.

Composition in Globular Sets

Composition of 1 cells

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

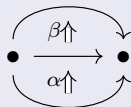
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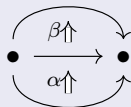
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Strict Infinity Categories - Composition

In a strict infinity category we have binary composition of n -cells for along a k boundary for all $k < n$.

Composition

If f and g are n -cells with the k -target of f equalling the k -source of g then there is an n -cell $f \circ_k g$.

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Identities

For each n -cell f there is an $(n + 1)$ -cell $\text{id}_f : f \rightarrow f$.

Strict Infinity Categories - Associativity

Associativity: if $0 \leq k < n$ and f , g , and h are n -cells then:

$$f \circ_k (g \circ_k h) = (f \circ_k g) \circ_k h$$

Strict Infinity Categories - Identities

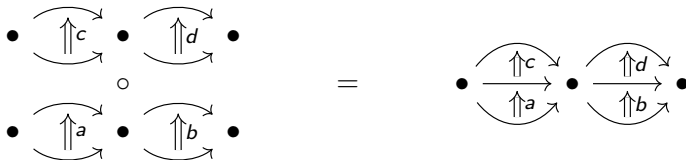
Identities: if $0 \leq k < n$ and f is an n -cell with k -source x and k -target y then:

$$\text{id}(x) \circ_k f = f = f \circ_k \text{id}(y)$$

Strict Infinity Categories - Interchange

Interchange: if $0 \leq q < p < n$ and a, b, c, d are n -cells then:

$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$



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Further if $f \circ_k g$ is well defined then:

$$\text{id}_f \circ_k \text{id}(g) = \text{id}(f \circ_k g)$$

Monoidal Categories

Monoidal categories are instances of infinity categories.

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A monoidal category is a category \mathcal{C} equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and a unit object I satisfying some conditions.

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A strict infinity category with one object and no non-identity n -cells for n higher than 2 is a strict monoidal category.

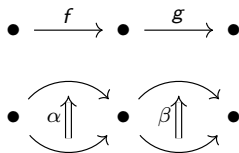
Pasting Diagrams

A pasting diagram represents a composition that can be done in an infinity category. More precisely it is an object of the free strict infinity category on the terminal globular set.

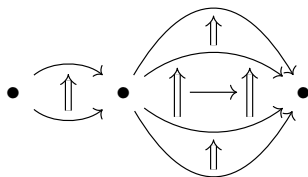
Pasting Diagrams

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The compositions we have already seen form pasting diagrams.



We can also form more complicated compositions as pasting diagrams.



Trees

Pasting diagram	Batanin tree
<p>A pasting diagram showing a composition of two 2-cells. A horizontal arrow g goes from x to y. Above it is a 2-cell h with boundary arrows β (up) and α (down). Below it is another 2-cell f with boundary arrows α (up) and β (down). An arrow k goes from y to z.</p>	<p>A Batanin tree corresponding to the first pasting diagram. It is a rooted tree with root x. x has two children: f and y. f has two children: α and g. g has two children: β and h. h has one child: k. k has one child: z.</p>
<p>A pasting diagram showing a composition of two 2-cells. A horizontal arrow g' goes from x' to y'. Above it is a 2-cell h' with boundary arrows β' (up) and α' (down). Below it is another 2-cell f' with boundary arrows α' (up) and β' (down).</p>	<p>A Batanin tree corresponding to the second pasting diagram. It is a rooted tree with root x'. x' has two children: f' and y'. f' has two children: α' and g'. g' has two children: β' and h'. h' has one child: y'.</p>

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The monoidal product in **Set** is *not* strict.

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However this is no longer possible at dimensions 3 and higher.

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- Substitutions: A substitution is a morphism between contexts.

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The arrow constructor takes 2 terms and a type as arguments. A variable of type $s \rightarrow_A t$ has source s , target t and lower dimensional sources and targets given by A .

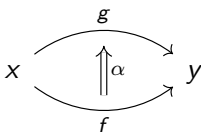
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Contexts and Substitutions

Contexts consist of a list of pairs of variable names and types.

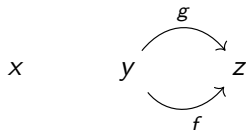
All pasting diagrams describe a context.

$$x \xrightarrow{f} y \xrightarrow{g} z$$

gives the context

$$\begin{aligned} x &: \star, \\ y &: \star, \\ f &: x \rightarrow_{\star} y, \\ z &: \star, \\ g &: y \rightarrow_{\star} z \end{aligned}$$

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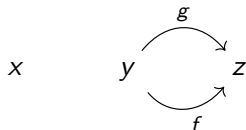
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Substitutions $\sigma : \Gamma \rightarrow \Delta$ map variables of Γ to terms of Δ , this

Disc contexts

We can define disc contexts by mutual induction as follows:

$$\begin{aligned} D_0 &= (d_0^- : \star) & A_0 &= \star \\ D_{n+1} &= D_n, d_n^+ : A_n, d_{n+1} & A_{n+1} &= d_n^- \rightarrow_{A_n} d_n^+ \end{aligned}$$

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A well-typed substitution from a disc context has the same data as a term.

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Taking the composite of the diagram:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

gives the composite $f \circ g$.

Over the singleton pasting diagram

x

and taking $s = x$ and $t = x$ we get a term from x to x representing the identity on x .

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This gives us the coherence constructor $\text{coh}(\Gamma : A)[\sigma]$ which takes a pasting context Γ , a type A over Γ and $\sigma : \Gamma \rightarrow \Delta$ to form a term in Δ .

Examples

Identity

Let t be a term. The identity on t is:

$$\text{coh } (D_1 : d_1^- \rightarrow_{A_1} d_1^-)[\sigma]$$

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1-composition

Let $s : x \rightarrow_\star y$ and $t : y \rightarrow_\star z$ be terms. Their composite is given by:

$$\text{coh } (x : \star, y : \star, f : x \rightarrow_\star y, z : \star, g : y \rightarrow_\star z : x \rightarrow_\star z)[\sigma]$$

where $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(z) = z$, $\sigma(f) = s$, $\sigma(g) = t$.

Examples

Take the context

$\Gamma = w : \star, x : \star, f : w \rightarrow_{\star} x, y : \star, g : x \rightarrow_{\star} y, z : \star, h : y \rightarrow_{\star} z.$

The associator is given by:

$$\text{coh } (\Gamma : (f \circ g) \circ h \rightarrow_{w \rightarrow_{\star} z} f \circ (g \circ h))[\text{id}]$$

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By showing this is confluent and terminating, we can create a type theory where both the source target of the associator are the same but retain decidable type checking and equality.

Insertion

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We propose an operation on terms which we call *Insertion*. This collapses certain compound composites into a single composite by “inserting” the inner composite into the outer composite.

1-Associator

$$\begin{array}{ccccc}
 x & \xrightarrow{f} & y & \xrightarrow{g} & z \\
 & & & \swarrow & \nearrow \\
 & & & & x' \xrightarrow{f'} y' \xrightarrow{g'} z'
 \end{array}$$

is sent to:

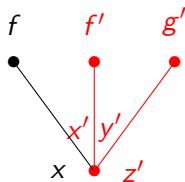
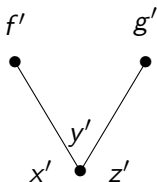
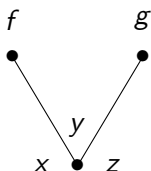
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Insertion on Trees

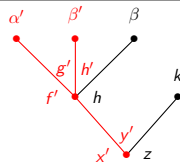
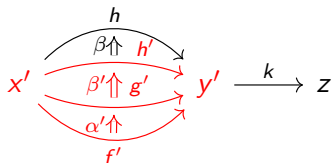
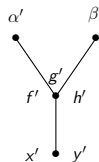
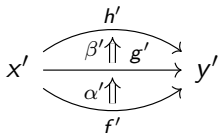
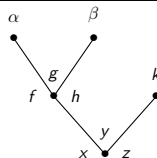
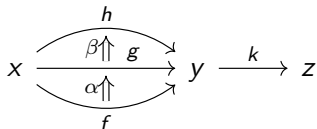
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Insertion Rule

Suppose we have a coherence $\text{coh}(\Gamma : A)[\sigma]$ where $\sigma(x) \equiv \text{coh}(\Delta : B)[\tau]$ for some locally maximal x in Γ . So far we have described the action on pasting diagrams but not on the entire term.

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We therefore get the following generator for our equality relation:

$$\text{coh} (\Gamma : A)[\sigma] = \text{coh} ((\Gamma \ll_x \Delta) : (A[\llbracket \kappa \rrbracket]))[(\sigma \ll_x \tau)]$$

Example

Take the term $a \circ (b \circ c)$. Written in full this is:

$$\text{coh } (x \xrightarrow{f} y \xrightarrow{g} z : x \rightarrow_{\star} z)[\sigma]$$

with $\sigma(f) = a$ and:

$$\sigma(g) = \text{coh } (x' \xrightarrow{f'} y' \xrightarrow{g'} z' : x' \rightarrow_{\star} z')[\tau]$$

with $\tau(f') = b$ and $\tau(g') = c$.

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with $\sigma(f) = a$ and:

$$\sigma(g) = \text{coh } (x' \xrightarrow{f'} y' \xrightarrow{g'} z' : x' \rightarrow_* z')[\tau]$$

with $\tau(f') = b$ and $\tau(g') = c$.

In this case the inserted context is $x \xrightarrow{a} x' \xrightarrow{a'} y' \xrightarrow{b'} z'$ with $\kappa(a) = a$ and $\kappa(b) = a' \circ b'$ which gives us final term:

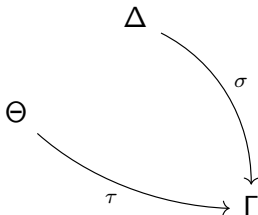
$$\text{coh } (x \xrightarrow{a} x' \xrightarrow{a'} y' \xrightarrow{b'} z' : x \rightarrow_* z')[\sigma \ll x \tau]$$

Universal Property of Insertion

Insertion also satisfies a universal property. Suppose we have $\text{coh}(\Delta : A)[\sigma]$ where $\sigma(x) = \text{coh}(\Theta : B)[\tau]$.

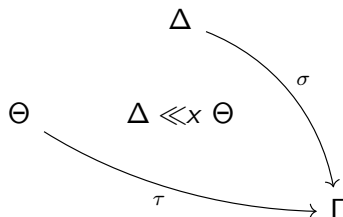
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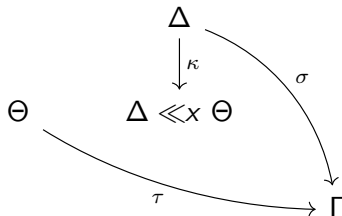
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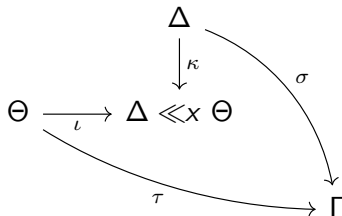
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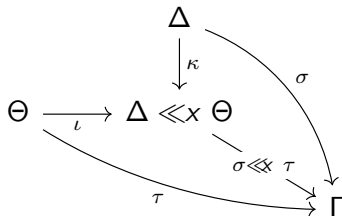
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$$\begin{array}{ccccc}
 D_n & \xrightarrow{x} & \Delta & & \\
 \text{coh}(\Theta:B)[\text{id}] \downarrow & & \downarrow \kappa & & \\
 \Theta & \xrightarrow{\iota} & \Delta \llcorner x \Theta & \xrightarrow{\sigma \llcorner x \tau} & \Gamma \\
 & \searrow \tau & & \nearrow & \\
 & & & &
 \end{array}$$

Diagram illustrating the universal property of Insertion. The diagram shows a commutative structure involving objects D_n , Δ , Θ , $\Delta \llcorner x \Theta$, and Γ , connected by arrows labeled with x , κ , ι , σ , τ , and $\sigma \llcorner x \tau$. The vertical arrow from D_n to Θ is labeled $\text{coh}(\Theta:B)[\text{id}]$. The curved arrow from Δ to Γ is labeled σ . The curved arrow from Θ to Γ is labeled τ . The arrow from $\Delta \llcorner x \Theta$ to Γ is labeled $\sigma \llcorner x \tau$.

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 & \searrow \tau & & & \uparrow \sigma
 \end{array}$$

Equality generated from Insertion

Our reduction scheme generates an equality that:

- trivialises all associativity equations,
- is terminating,
- is confluent,
- and has a decidable algorithm for type-checking.

Future Work

- Formalise all results in the paper.
- Combine this with the reduction for strict units to get a type theory for strictly unital and associative categories.
- Create a general framework for CaTT based type theories with definitional equality.
- Show that models of the strict versions of CaTT are equivalent to the models of the original version.