A Type Theory for Strictly Associative Infinity Categories

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Outline

Globular Sets

Globular sets are one natural shape of higher categories:

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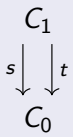
Ordinary 1-categories



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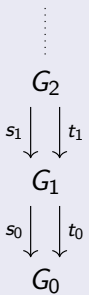

Globular Sets

Definition

A *globular set* $\mathcal{G}$ consists of sets G_n for each n and maps $s_n, t_n : G_{n+1} \to G_n$ for each n such that the following *globularity conditions* hold:

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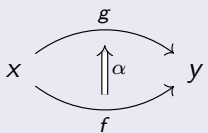

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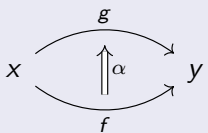

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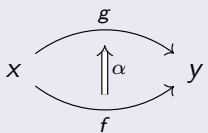

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Terminal globular set

The *terminal globular set* has one cell at each dimension and all source and target maps are uniquely defined.

Compostition in Globular Sets

Composition of 1 cells

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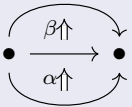
Composition of 2 cells

Composition along a 1-boundary:



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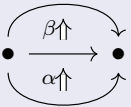

Composition along a 0-boundary:

Strict Infinity Categories - Composition

In a *strict infinity category* we have binary composition of n-cells for along a k boundary for all $k < n$.

Composition

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Identities

For each n-cell f there is an $(n+1)$-cell $\mathrm{id}_f : f \to f$.

Strict Infinity Categories - Associativity

If $0 \leq k < n$ and f, g, and h are n-cells then:

$$f \circ_k (g \circ_k h) = (f \circ_k g) \circ_k h$$

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Associativity of 1-cells

Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ we have:

$$f \circ_0 (g \circ_0 h) = (f \circ_0 g) \circ_0 h$$

Strict Infinity Categories - Identities

If $0 \leq k < n$ and f is an n-cell with k-source x and k-target y then:

$$\mathrm{id}^{n-k}(x) \circ_k f = f = f \circ_k \mathrm{id}(y)$$

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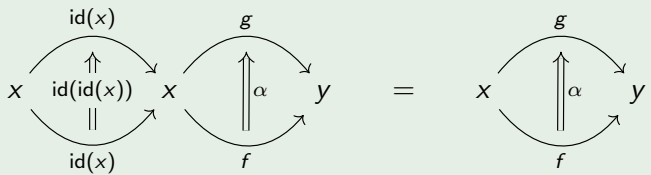
Identity on 2-cell

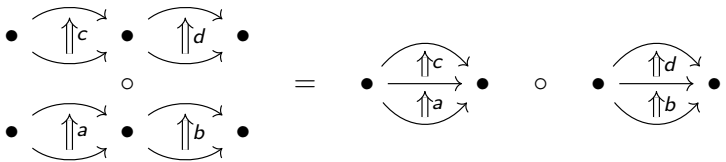
Given $f, g : x \to y$ and $\alpha : f \to g$ we have:



Strict Infinity Categories - Interchange

If $0 \leq q < p < n$ and a, b, c, d are n-cells then:

$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$



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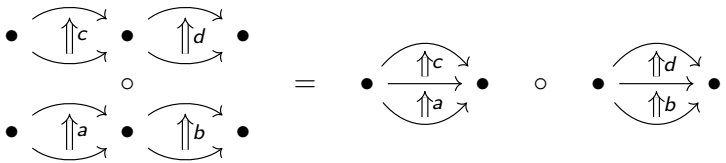

Further if $f \circ_k g$ is well defined then:

$$\mathrm{id}_f \circ_k \mathrm{id}(g) = \mathrm{id}(f \circ_k g)$$

Monoidal Categories

Monoidal categories are instances of infinity categories.

Definition (Monoidal category)

A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object I satisfying some conditions.

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A strict infinity category with one object and no non-identity n-cells for n higher than 2 is a strict monoidal category.

Weakness

If a category has all products and a terminal object, then it can be given the structure of a monoidal category.

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However this is no longer possible at dimensions 3 and higher.

Semistrictness

Since full strictification is not possible, we want to do the best possible.

Associators
Unitors
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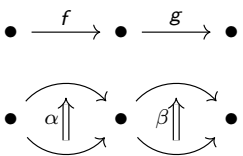
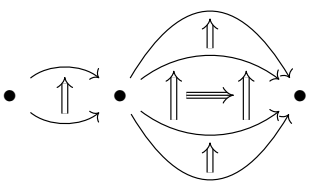
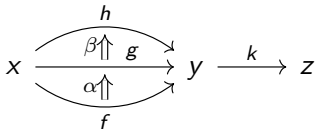
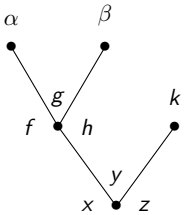
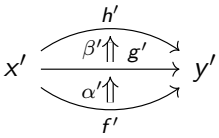
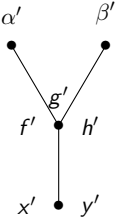
The compositions we have already seen form pasting diagrams.



We can also form more complicated compositions as pasting diagrams.



Trees

Pasting diagram	Batanin tree
	
	

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- Types: A type contains all the information of the *sources* and *targets* for a term.
- Substitutions: A substitution is a *morphism* between contexts.

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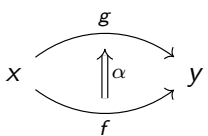

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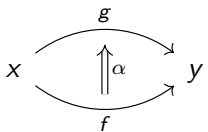

$$\alpha : f \to_{x \to_\star y} g$$

Contexts and Substitutions

Contexts consist of a list of pairs of variable names and types.

All *pasting diagrams* describe a context.

$$x \xrightarrow{f} y \xrightarrow{g} z$$

gives the context

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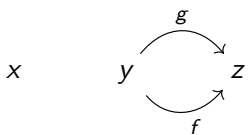
Contexts can represent non-pasting diagrams.



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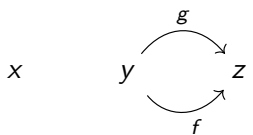

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Substitutions $\sigma : \Gamma \to \Delta$ map *variables* of Γ to *terms* of Δ, this operation can be extended to all terms, types, and substitutions.

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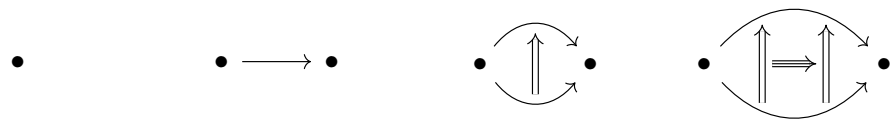

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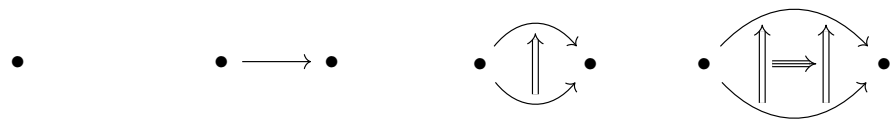

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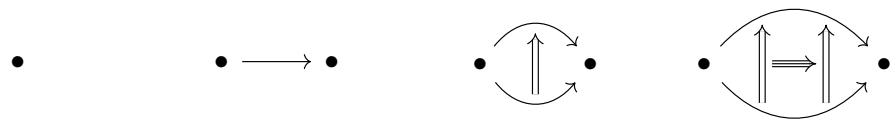

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- Their trees are all *linear*.
- A well-typed substitution from a disc context has the same data as a term.

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Taking the composite of the diagram:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

gives the composite $f \circ g$.

Over the singleton pasting diagram

$$x$$

and taking $s = x$ and $t = x$ we get a term from x to x representing the identity on x.

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- A pasting diagram, which we can represent as a context satisfying a certain typing judgment.

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This gives us the coherence constructor

$$\text{coh}\ (\Gamma : A)[\sigma]$$

which takes a pasting context Γ, a type A over Γ and $\sigma : \Gamma \to \Delta$ to form a term in Δ.

Examples

Identity

Let t be a 1 dimensional term. The identity on t is:

$$\text{coh}\ (x \xrightarrow{f} y : f \to_{x \to_\star y} f)[\sigma]$$

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1-composition

Let $s : x \to_\star y$ and $t : y \to_\star z$ be terms. Their composite is given by:

$$\text{coh}\ (x \xrightarrow{f} y \xrightarrow{g} z : x \to_\star z)[\sigma]$$

where $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(z) = z$, $\sigma(f) = s$, $\sigma(g), = t$.

Examples

Take the context $\Gamma = w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$.

The *associator* is given by:

$$\mathrm{coh}\ (\Gamma : (f \circ g) \circ h \to_{w \to_\star z} f \circ (g \circ h))[\mathrm{id}]$$

Strategy

CaTT as we have presented it has no non-trivial equality and no computation.

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By showing this is confluent and terminating, we can create a type theory where both the source target of the associator are the same but retain decidable type checking and equality.

Insertion

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We propose an operation on terms which we call *Insertion*. This collapses certain compound composites into a single composite by "inserting" the inner composite into the outer composite.

1-Associator

is sent to:

$$x \xrightarrow{f} x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

Insertion on Trees

The *insertion* operation on pasting diagrams is more easily described as an operation on trees.

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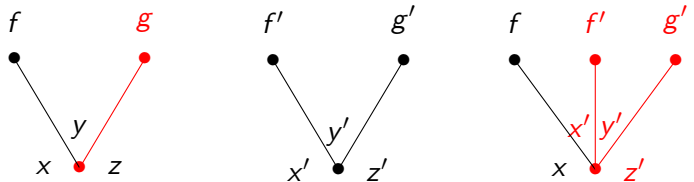

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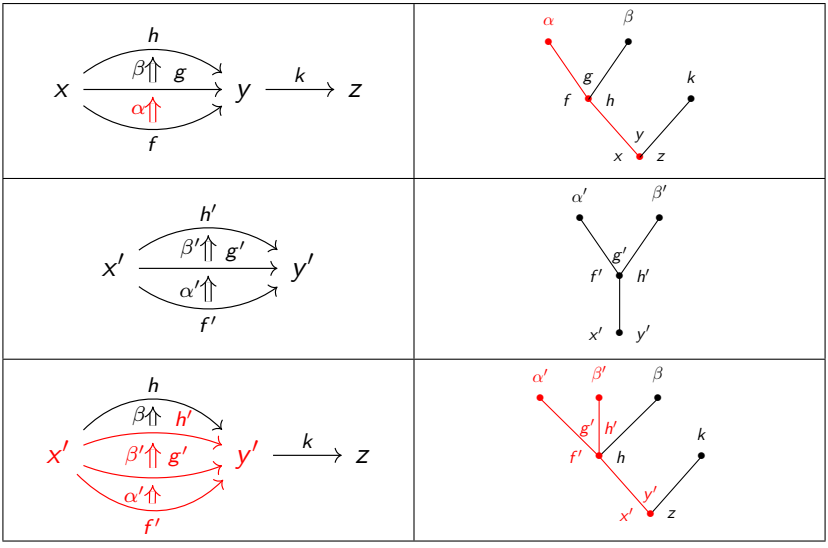

Insertion Rule

Suppose we have a coherence $\mathsf{coh}\ (\Gamma : A)[\sigma]$ where $\sigma(x) \equiv \mathsf{coh}\ (\Delta : B)[\tau]$ for some locally maximal x in Γ. So far we have described the action on pasting diagrams but not on the entire term.

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- A new pasting diagram, which we call $\Gamma \ll_x \Delta$.

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We therefore get the following generator for our reduction:

$$\mathsf{coh}\ (\Gamma : A)[\sigma] \rightsquigarrow \mathsf{coh}\ ((\Gamma \ll x\ \Delta) : (A[\![\kappa]\!]))[(\sigma \ll x\ \tau)]$$

Example

Over the context $\Gamma = w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$ we have that:

$$f \circ_0 (g \circ_0 h) \rightsquigarrow f \circ_0 g \circ_0 h = \mathrm{coh}\ (\Gamma : w \to_\star z)[\mathrm{id}]$$

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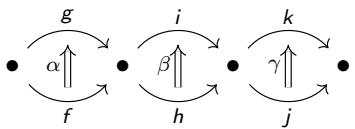

We have:

$$\begin{aligned} \alpha \circ_0 (\beta \circ_0 \gamma) &\rightsquigarrow \mathrm{coh}\ (\Delta : f \circ_0 (h \circ_0 j) \to g \circ_0 (i \circ_0 k))[\mathrm{id}] \\ &\rightsquigarrow \mathrm{coh}\ (\Delta : f \circ_0 h \circ_0 j \to g \circ_0 i \circ_0 k)[\mathrm{id}] \end{aligned}$$

Universal Property of Insertion

Insertion also satisfies a *universal property*. Suppose we have coh $(\Delta : A)[\sigma]$ where $\sigma(x) = \text{coh } (\Theta : B)[\tau]$.

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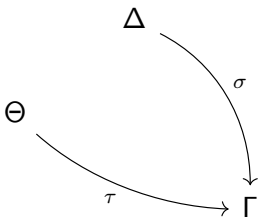

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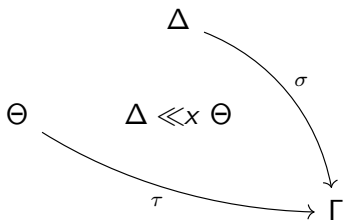

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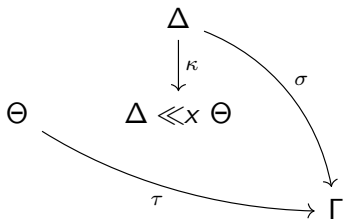

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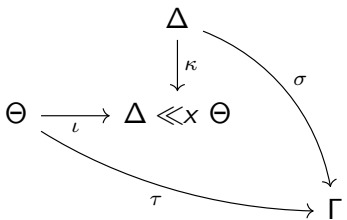

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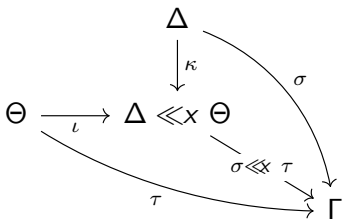

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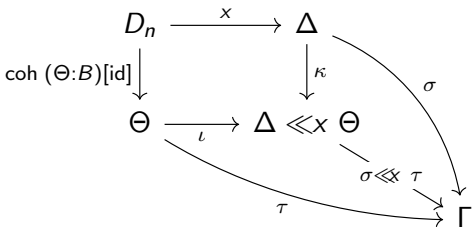

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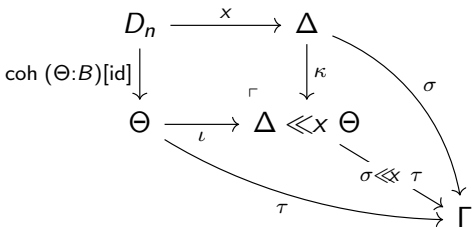

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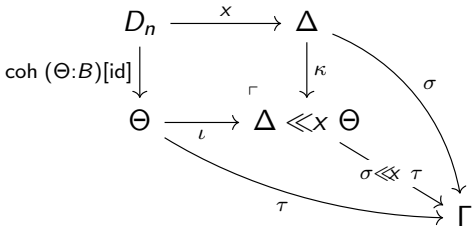

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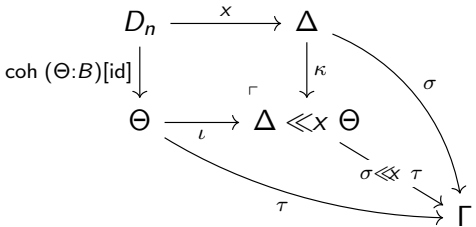

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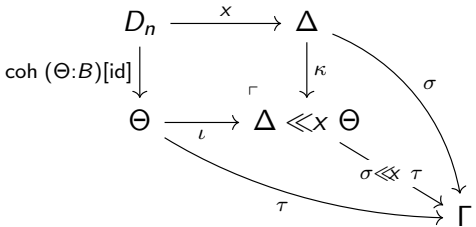

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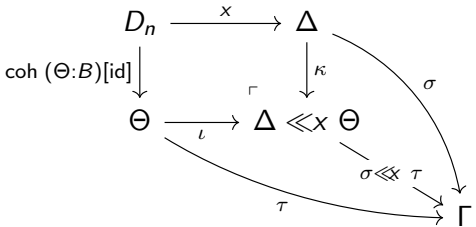

The universal property provides intuition for insertion, and also provides us with another proof technique for working with insertion.

Equality generated from Insertion

Our reduction scheme generates an equality that:

- trivialises all *associativity* equations,
- is *terminating*,
- is *confluent*,
- and has a *decidable* algorithm for type-checking.

Future Work

- Formalise all results in the paper.
- Combine this with the reduction for strict units to get a type theory for strictly unital and associative categories.
- Create a general framework for CaTT based type theories with definitional equality.
- Show that models of the strict versions of CaTT are equivalent to the models of the original version.