Quantum Software Lab

THE UNIVERSITY *of* EDINBURGH

Quantum Circuits are Just a Phase

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Overview

- Quantum circuit model of quantum computation
- A quantum “if let”.
- The quantum phase language.

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```
if let |1⟩ = a {
  if let |−⟩ = b {
    Ph(π)
  }
}
```

Qubits and Unitaries

Classical Bits	Quantum Qubits
$\{\text{false}, \text{true}\}$	unit vectors in $\mathbb{C}^2$
false	$\lvert 0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
true	$\lvert 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
N/A	$\lvert +\rangle = \frac{1}{\sqrt{2}}(\lvert 0\rangle + \lvert 1\rangle)$
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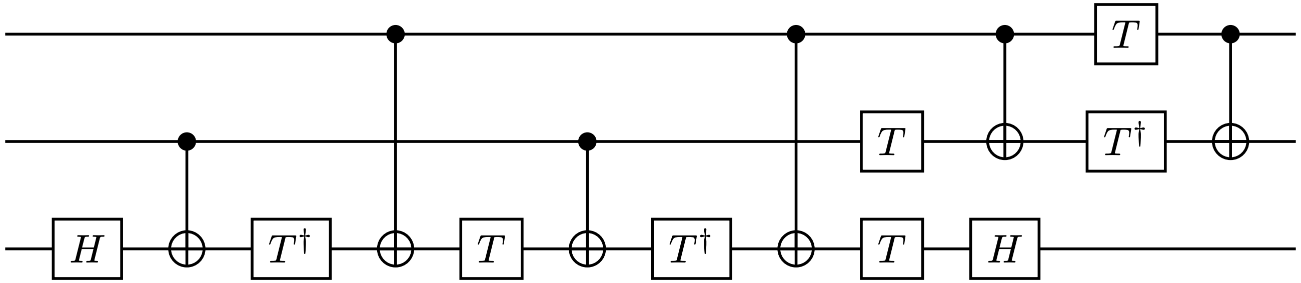
(Measurement-free) quantum computations correspond to *unitary maps*.

$$U \text{ is unitary} \Leftrightarrow UU^\dagger = U^\dagger U = I$$

What programming constructions can we use for unitary maps?

Quantum circuits

Quantum computations are often graphically represented as circuits.



- Wires represent qubits
- Each symbol is a primitive *gate*
- Gates are composed in sequence and parallel

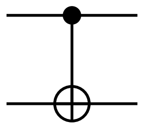
Quantum gates

Each gate is a “black boxed” unitary.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}$$

$\mathrm{CX} =$



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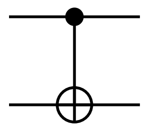


$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Claim: Quantum gates sit at an awkward level of abstraction

Just a phase

Perhaps the simplest unitary map takes the following form:

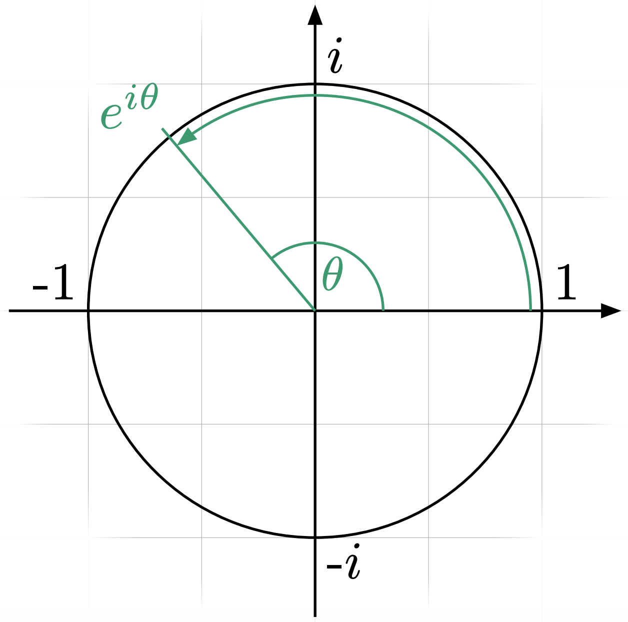
$$v \mapsto e^{i\theta} v$$

We call such a map a *phase* and represent it by the term:

```
Ph(θ)
```

Our language consists of just:

- this phase operator
- a quantum pattern matching construction



Pattern matching for unitary maps.

Maps from classical data can be specified by pattern matching:

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match x {
  false => { /* Do something */ }
  true => { /* Do something else */ }
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Linear maps only need to be specified on a basis.

```
match x {
  |0⟩ => { /* Do something */ }
  |1⟩ => { /* Do something else */ }
}
```

Z gate

We can already define the Z gate. Recall:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Z|0\rangle = |0\rangle \quad Z|1\rangle = -|1\rangle$$

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match q {
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  |1⟩ => { Ph(π) }
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“if let” construction

To simplify the syntax, we borrow Rust’s “if let” expression.

```
match q {
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if let |1⟩ = q {
  /* Do something */
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if let |1⟩ = q {
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```

```
Z(q) = if let |1⟩ = q { Ph(π) }
S(q) = if let |1⟩ = q { Ph(0.5π) }
T(q) = if let |1⟩ = q { Ph(0.25π) }
```

X as an “if let”

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$$X|-\rangle = X\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) = \frac{1}{\sqrt{2}}(X|0\rangle - X|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = -|-\rangle$$

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X(q) = if let |−⟩ = q { Ph(π) }
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Applying a unitary to a pattern

Consider $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

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$$Y|L\rangle = |L\rangle \qquad Y|R\rangle = -|R\rangle$$

We could add $|L\rangle$ and $|R\rangle$ as patterns.

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$$Y|L\rangle = |L\rangle \qquad Y|R\rangle = -|R\rangle$$

We could add $|L\rangle$ and $|R\rangle$ as patterns. But:

$$|L\rangle = S|+\rangle \qquad |R\rangle = S|-\rangle$$

So we instead add the pattern `s . P` for unitary `s` and pattern `P`.

```
Y(q) = if let S . |−⟩ = q { Ph(π) }
```

Quantum phase language

These two constructions form our universal quantum phase language.

Terms represent unitary maps:

```
s, t ::= Ph(θ) | s ; t | s ⊗ t | if let P = q { s }
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Patterns represent isometries, allowing subspace selection:

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P, Q ::= |0⟩ | |1⟩ | |+⟩ | |−⟩ | s . P | P ⊗ Q | q
```

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Patterns represent isometries, allowing subspace selection:

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P, Q ::= |0⟩ | |1⟩ | |+⟩ | |−⟩ | s . P | P ⊗ Q | q
```

From just these we can derive a universal gate set.

```
if let |1⟩ = a {
  if let |−⟩ = b {
    Ph(π)
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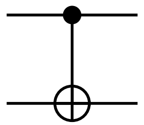
$$\mathrm{CX} =$$  $$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Metaoperations

Our language admits two interesting metaoperations:

Inverses

Can be defined inductively:

- `(s ; t)`† `=` `t`† `;` `s`†
- `(s ⊗ t)`† `=` `s`† `⊗` `t`†
- `Ph(θ)`† `= Ph(-θ)`
- `(if let P = q { e })`†
  `= if let P = q {` `e`† `}`

Exponentiation

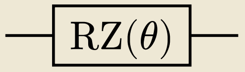
"Composition-free" terms can be exponentiated:

- `(s ⊗ t)`^{x} `=` `s`^{x} `⊗` `t`^{x}
- `Ph(θ)`^{x} `= Ph(x*θ)`
- `(if let P = q { e })`^{x}
  `= if let P = q {` `e`^{x} `}`

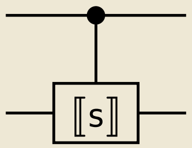
Circuit compilation

A term s can be reduced down to a circuit ⟦s⟧.

⟦if let |1⟩ = q { Ph(θ) }⟧



⟦if let |1⟩ = q { s }⟧



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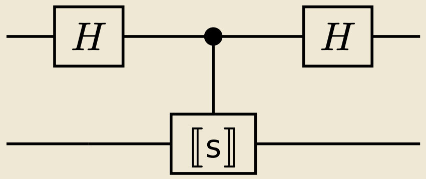
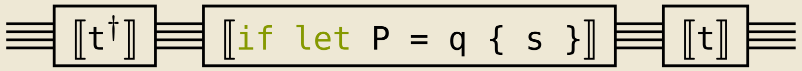


⟦if let t . P = q { s }⟧



Compiling the Y gate

$$[\![\texttt{Y(q)}]\!]$$

$$= [\![\texttt{if let S . }|-\rangle\texttt{ = q \{ Ph(}\pi\texttt{) \}}]\!]$$

$$= -\boxed{[\![\texttt{S}^\dagger]\!]}-\boxed{[\![\texttt{if let }|-\rangle\texttt{ = q \{ Ph(}\pi\texttt{) \}}]\!]}-\boxed{[\![\texttt{S}]\!]}-$$

$$= -\boxed{[\![\texttt{S}^\dagger]\!]}-\boxed{H}-\boxed{[\![\texttt{if let }|1\rangle\texttt{ = q \{ Ph(}\pi\texttt{) \}}]\!]}-\boxed{H}-\boxed{[\![\texttt{S}]\!]}-$$

$$= -\boxed{S^\dagger}-\boxed{H}-\boxed{Z}-\boxed{H}-\boxed{S}-$$

Compiling the Y gate

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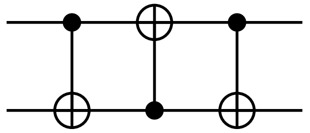
$$= -\boxed{S^\dagger}-\boxed{H}-\boxed{Z}-\boxed{H}-\boxed{S}-$$

$$\approx -\boxed{S^\dagger}-\boxed{X}-\boxed{S}-$$

The swap gate

$$\mathrm{Swap} =$$


Can we decompile this to a term?

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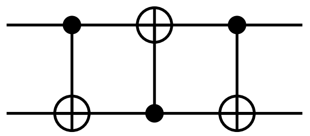

Can we decompile this to a term?

```
  Swap(q1, q2)
= CX(q1 ⊗ q2) ; CX(q2 ⊗ q1) ; CX(q1 ⊗ q2)
= if let CX(a ⊗ b) = q1 ⊗ q2 { CX(b ⊗ a) }
= if let CX(a ⊗ b) = q1 ⊗ q2 { if let |1⟩ ⊗ |−⟩ = b ⊗ a { Ph(π) }}
= if let CX(|−⟩ ⊗ |1⟩) = q1 ⊗ q2 { Ph(π) }
= if let "|01⟩ - |10⟩" = q1 ⊗ q2 { Ph(π) }
```

Summary

In the paper

- Defined a type system.
- Gave a categorical semantics.
- Defined an evaluation to circuits.
- Proved semantic equalities on terms.
- Defined well-known algorithms.

In the future

- Measurement.
- Equational theory.
- Operational semantics.
- Optimisations.

Implementation (https://github.com/alexarice/phase-rs)

- Parsing
- Typechecking
- Compiling to circuits
- Evaluating to unitary matrix