

A Syntax for Strictly Associative and Unital ∞ -Categories

Eric Finster Alex Rice Jamie Vicary

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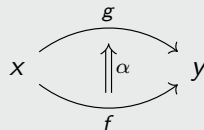
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Globular ∞ -categories contain higher dimensional arrows.

2-cell $\alpha : f \rightarrow g$



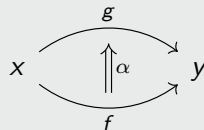
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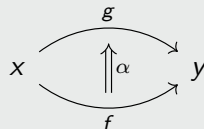
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A type gives the boundary of a term.

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Terms in CATT

Terms represent the possible operations in a globular ∞ -category.

Terms built over *pasting diagrams*.

Compound terms using substitutions.

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$$\text{coh}(\Gamma : s \rightarrow t)$$

$$\Gamma := \textcolor{red}{x} \xrightarrow{f} y \xrightarrow{g} \textcolor{red}{z}$$

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$$\Delta := x \begin{array}{c} \curvearrowright^h \\ \Uparrow \alpha \\ \Downarrow f \\ \curvearrowright \end{array} y \begin{array}{c} \curvearrowright^i \\ \Uparrow \beta \\ \Downarrow g \\ \curvearrowright \end{array} z$$

$$\alpha \otimes \beta := \text{coh}(\Delta : f * g \rightarrow h * i)$$

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Compound terms using substitutions.

$$s[\![\sigma]\!]$$

$$\sigma := \langle f \mapsto a, g \mapsto (b * c) \rangle$$

$$a * (b * c) := (f * g)[\![\sigma]\!]$$

$$\Delta := x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w$$

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$$\alpha_{a,b,c} := \text{coh} (\Delta : (a * b) * c \rightarrow a * (b * c))$$

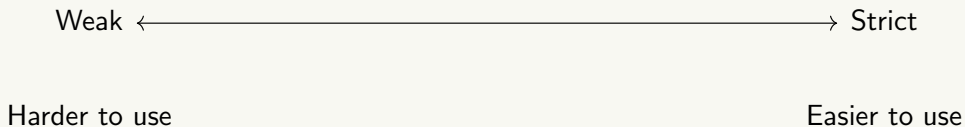
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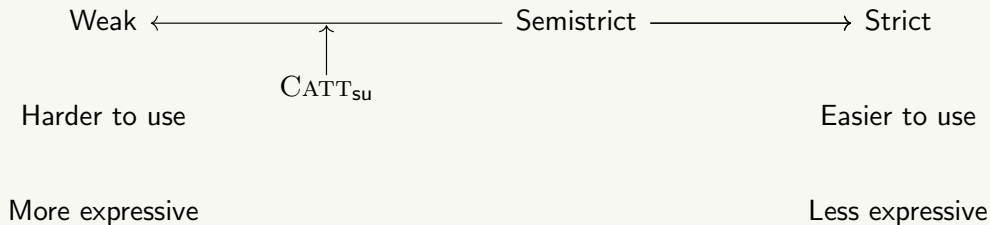
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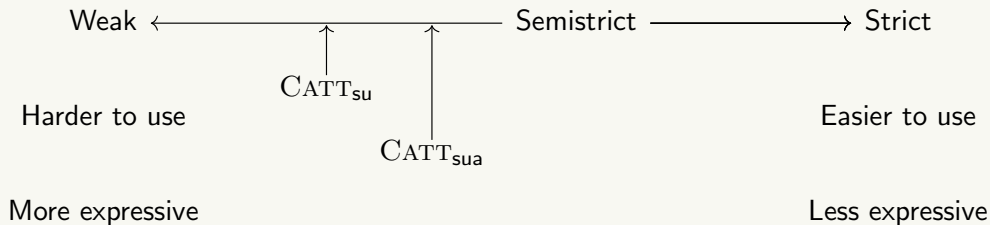
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Semistrictness

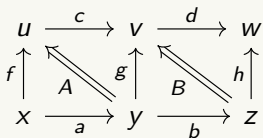
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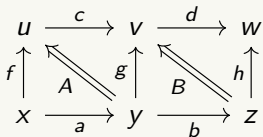


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$$(\text{id}_a \otimes B) * (A \otimes \text{id}_d) : (a * b) * h \rightarrow f * (c * d)$$

CATT has trivial equality.

CATT_{su} has disc removal, endo-coherence removal, and pruning.

In CATT_{sua}, pruning is replaced by insertion.

$$\text{CATT}_{\text{sua}} := \text{CATT} + \text{insertion} + \text{disc removal} + \text{endo-coherence removal}$$

$$(a * b) * c =_{\text{sua}} a * b * c \equiv \text{coh} (x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w : x \rightarrow w)$$

Insertion Rule

$$(a * b) * c =_{\text{SUA}} a * b * c \equiv \text{coh} (x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w : x \rightarrow w)$$

Recalling $(a * b) * c \equiv (f * g)[\langle f \mapsto a, g \mapsto b * c \rangle]$:

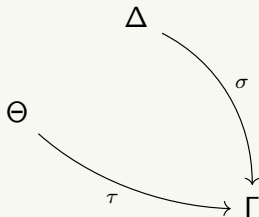


is sent to:

$$x \xrightarrow{f} x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

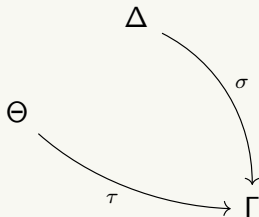
Universal Property of Insertion

$$\text{coh } (\Delta : s \rightarrow t) \llbracket \sigma \rrbracket \quad x \in \Delta \quad x \llbracket \sigma \rrbracket \equiv \text{coh } (\Theta : u \rightarrow v) \llbracket \tau \rrbracket$$



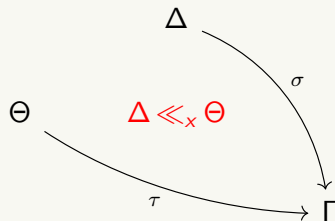
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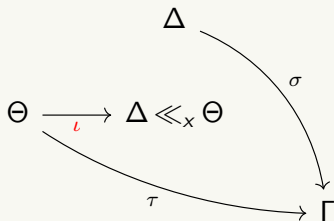
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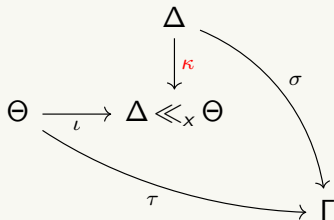
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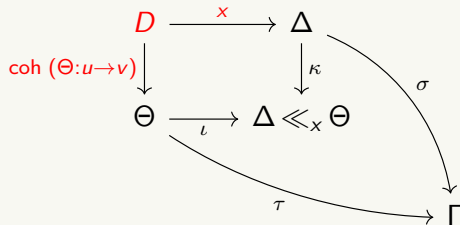
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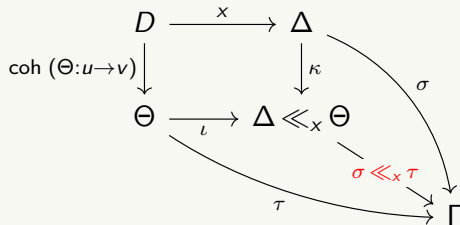
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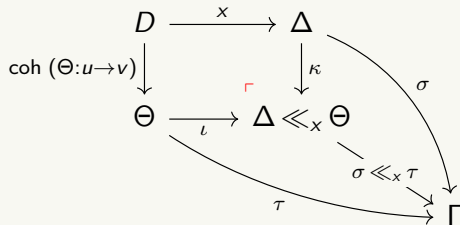
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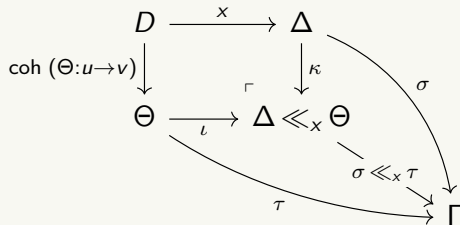
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Universal Property of Insertion

$\text{coh}(\Delta : s \rightarrow t)[\sigma] \quad x \in \Delta \quad x[\sigma] \equiv \text{coh}(\Theta : u \rightarrow v)[\tau] \quad (+ \text{ syntactic side condition})$



$$\text{coh}(\Delta : s \rightarrow t)[\sigma] = \text{coh}(\Delta \ll_x \Theta : s[\kappa] \rightarrow t[\kappa])(\sigma \ll_x \tau)$$

Equality in CATT_{sua} is decidable.

CATT_{sua} has unique normal forms.

Obtained by reduction system.

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Confluence: Encode various constructions in Agda.

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Type checking is decidable.

We provide a interpreter which:

- Provides tools for construction terms.
- Type checks terms.
- Reduces terms to CATT_{sua} normal form.

We introduce the type theory CATT_{sua} .

CATT_{sua} models strictly unital and associative ∞ -categories.

CATT_{sua} terms are simpler than their CATT or CATT_{su} equivalents.

Normal forms for CATT_{sua} are obtained via a reduction system:

This reduction system is strongly terminating and confluent.

Try our interpreter for CATT_{sua} :

<https://github.com/alexarice/catt-strict>

Thank you for listening.