

# A Type Theory for Strictly Unital $\infty$ -Categories

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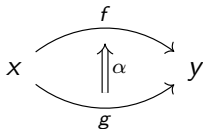
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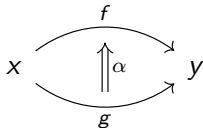
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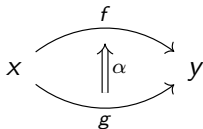
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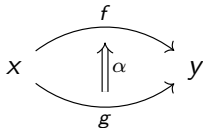
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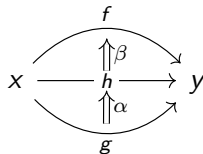
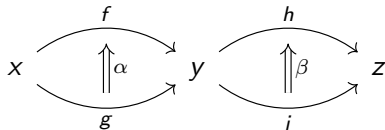
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Compositions:





# An overview of (globular) infinity categories

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- Objects  $x, y, z$
- 1-arrows:

$$x \xrightarrow{f} y$$

- 2-arrows:

A diagram showing two objects,  $x$  and  $y$ , connected by two curved arrows:  $f$  (top) and  $g$  (bottom). A vertical double arrow labeled  $\alpha$  points from  $g$  to  $f$ , representing a 2-arrow between the 1-arrows.

- ...

Our arrows are *Globular*.

Compositions:

A diagram showing the composition of two 1-arrows. On the left,  $x$  and  $y$  are connected by  $f$  (top) and  $g$  (bottom). On the right,  $y$  and  $z$  are connected by  $h$  (top) and  $i$  (bottom). A vertical double arrow labeled  $\alpha$  connects  $f$  and  $g$ , and another labeled  $\beta$  connects  $h$  and  $i$ .

A diagram showing the composition of two 2-arrows. On the left,  $x$  and  $y$  are connected by  $f$  (top) and  $g$  (bottom). A horizontal arrow labeled  $h$  points from  $x$  to  $y$ . A vertical double arrow labeled  $\beta$  connects  $f$  and  $h$ , and another labeled  $\alpha$  connects  $h$  and  $g$ .

Identities:

$$x \xrightarrow{\text{id}_x} x$$

# Strict Infinity Categories

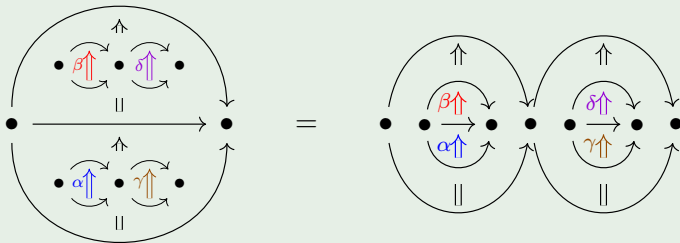
## Associativity

$$w \xrightarrow{w \xrightarrow{f} x \xrightarrow{g} y} y \xrightarrow{h} z = w \xrightarrow{f} x \xrightarrow{x \xrightarrow{g} y \xrightarrow{h} z} z$$

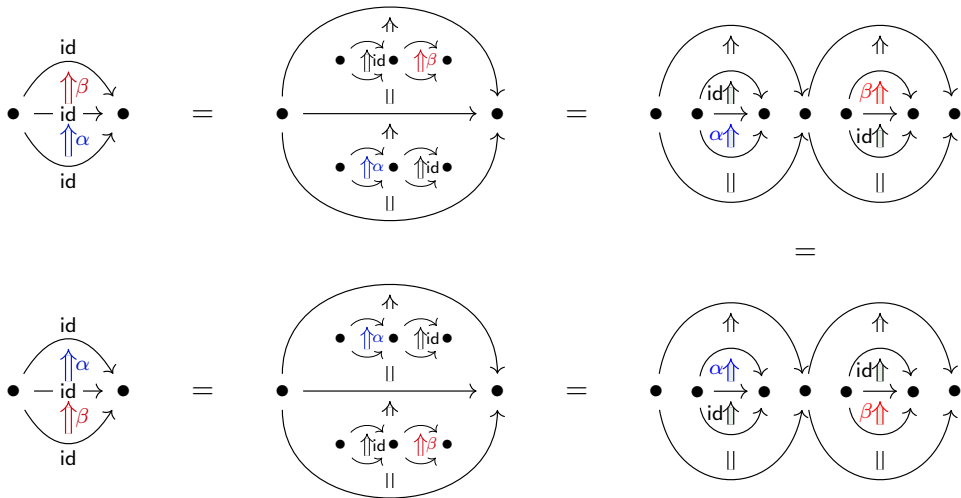
## Unitality

The diagram illustrates the unitality property. On the left, a composition of two 2-cells is shown. The first 2-cell has boundary maps  $\text{id}(x)$  and  $\text{id}(x)$  and a 2-cell  $\text{id}(\text{id}(x))$ . The second 2-cell has boundary maps  $g$  and  $f$  and a 2-cell  $\alpha$ . The right side shows a single 2-cell with boundary maps  $g$  and  $f$  and a 2-cell  $\alpha$ .

## Interchange



# Example: Eckmann-Hilton



We have only so far talked about strict  $\infty$ -categories.

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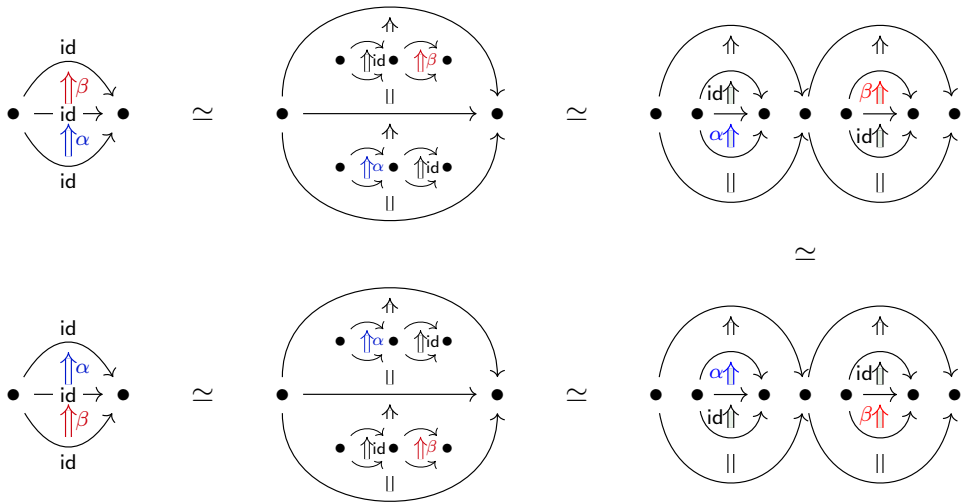
In higher categories, non-equal arrows can be isomorphic.

In a weak higher category, the laws are only required to hold up to isomorphism.

Many examples of higher categories are weak:

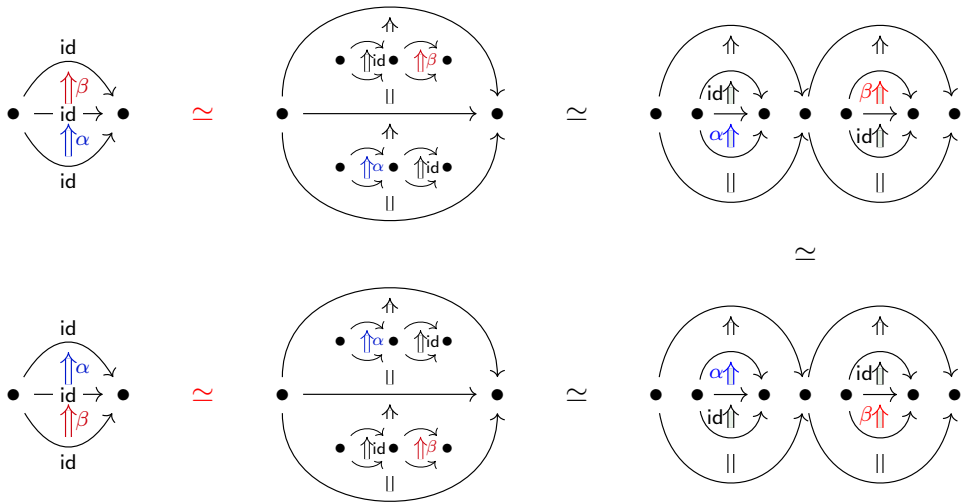
- Homotopy groupoids of topological spaces.
- Equality types in HoTT.
- Bicategory of categories and profunctors.

# Example: Eckmann-Hilton





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# Weakness vs Strictness

Weak ←————→ Strict

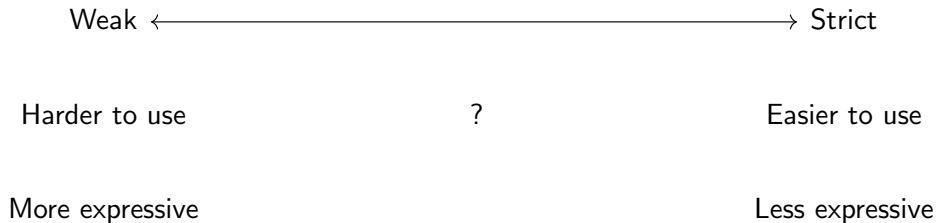
# Weakness vs Strictness



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# Weakness vs Strictness

Weak ←———— Semistrict —————→ Strict

Harder to use

Easier to use

More expressive

Less expressive

# Weakness vs Strictness

Weak ←———— Semistrict —————→ Strict

Harder to use

Lack of definitions

Easier to use

More expressive

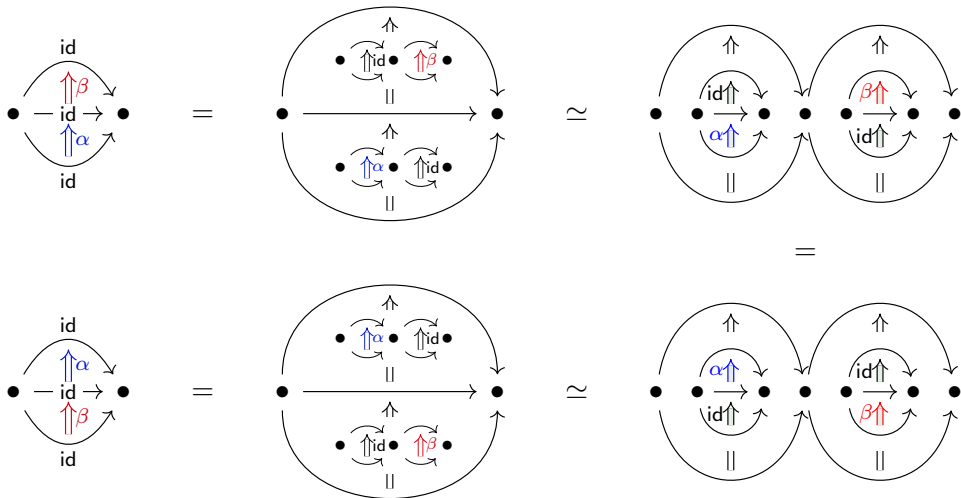
Retains expressiveness?

Less expressive

- Catt [1] is a type theory for weak  $\infty$ -categories.
- Its terms are the possible operations in an  $\infty$ -category.
- By adding a definitional equality to Catt, we can unify certain operations.
- Catt<sub>SU</sub> is a new type theory based on Catt with strict units.



# Example: Eckmann-Hilton



```
coh id C (x) : x => x
coh id2 C (x(f)y) : f => f
coh comp C (x(f)y(g)z) : x => z
coh vert C (x(f(a)g(b)h)y) : f => h
coh horiz C (x(f(a)g)y(h(b)k)z) : comp f h => comp g k

coh swap3 C (x(f(a)g)y(h(b)k)z)
  : vert (horiz a (id2 h)) (horiz (id2 g) b) =>
    vert (horiz (id2 f) b) (horiz a (id2 k))

let eh {C : Cat} {x :: C} (a :: id x => id x) (b :: id x => id x)
  : [ vert a b => vert b a ]
  = swap3 a b
```

- Equality in  $\text{Catt}_{\text{su}}$  preserves typing.
- Equality is generated by a strongly-terminating, confluent reduction relation.
- Equality and type checking are decidable.
- All terms (of the same dimension) in a disc context are identified.
- Eckmann-Hilton and the Syllepsis have been formalised in  $\text{Catt}_{\text{su}}$ .

- [1] Eric Finster and Samuel Mimram. “A Type-Theoretical Definition of Weak  $\omega$ Categories”. In: *Proceedings of LICS 2017*. arXiv:1706.02866. 2017.