# New Minimal Linear Inferences in Boolean Logic Independent of Switch and Medial

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# Outline

1. Background
    - Linear Inferences
    - Relation Webs
2. Algorithm and Implementation
3. Results

# Linear Inferences in Classical Logic

We consider formulae to built with connectives $\wedge$ and $\vee$, constants $\bot$ and $\top$, and negation of variables.

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**Definition**

A linear inference is a valid implication $\varphi \to \psi$, where $\varphi$ and $\psi$ are linear formulae.

The set of linear inferences is **coNP**-complete.
The linear inferences are just the multiplicative fragments of Blass' game semantics for linear logic.

# Switch and Medial

### Switch

$$x \wedge (y \vee z) \;\rightarrow\; (x \wedge y) \vee z$$

Switch underlies multiplicative linear logic.

### Medial

$$(w \wedge x) \vee (y \wedge z) \;\rightarrow\; (w \vee y) \wedge (x \vee z)$$

Switch and medial are the logical fragment of deep inference proof theory.

Medial allows locality of contraction.

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## Definition

Let $\sim_{\mathsf{acu}}$ be the smallest congruence containing associativity, commutativity, and unit laws.

Therefore we have

$$\begin{array}{ll} \varphi \vee \psi \sim_{\mathsf{ac}} \psi \vee \varphi & \quad \varphi \wedge (\psi \wedge \chi) \sim_{\mathsf{ac}} (\varphi \wedge \psi) \wedge \chi \\ \varphi \wedge \psi \sim_{\mathsf{ac}} \psi \wedge \varphi & \quad \varphi \vee (\psi \vee \chi) \sim_{\mathsf{ac}} (\varphi \vee \psi) \vee \chi \end{array}$$

for associativity and commutativity and

$$\begin{array}{llll} \varphi \wedge \top \sim_{\mathsf{u}} \varphi & \varphi \vee \bot \sim_{\mathsf{u}} \varphi & \top \wedge \varphi \sim_{\mathsf{u}} \varphi & \bot \vee \varphi \sim_{\mathsf{u}} \varphi \\ \varphi \wedge \bot \sim_{\mathsf{u}} \bot & \varphi \vee \top \sim_{\mathsf{u}} \top & \bot \wedge \varphi \sim_{\mathsf{u}} \bot & \top \vee \varphi \sim_{\mathsf{u}} \top \end{array}$$

for unitality.

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## Definition

Write $\varphi \rightsquigarrow_{\mathsf{msu}} \psi$ if there are linear formulae $\varphi'$ and $\psi'$ with $\varphi \sim_{\mathsf{acu}} \varphi' \rightarrow_{\mathsf{ms}} \psi' \sim_{\mathsf{acu}} \psi$. Further write $\overset{*}{\rightsquigarrow}_{\mathsf{msu}}$ for the reflexive transitive closure of $\rightsquigarrow_{\mathsf{msu}}$.

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- This was improved to a 10 variable inference which cannot be derived from switch and medial (Das, 2013).

$$\begin{array}{rl} & (z \vee (w \wedge w')) \wedge (y \vee y') \wedge (u \vee u') \wedge ((x \wedge x') \vee z') \\ \to & (z \wedge (x \vee y)) \vee (u \wedge x') \vee (w' \wedge u') \vee ((w \vee y') \wedge z') \end{array}$$

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**Question**

What is the smallest inference that cannot be derived with switch and medial?

# Relation Webs

It turns out that it is sufficient to consider constant-free (unit-free), negation-free formulae.
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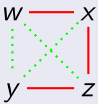
## Definition (Relation Web)

A relation web is an undirected graph which is $P_4$-free, meaning none of its induced subgraphs are isomorphic to:



## Operations on Relation Webs

The web for a formula $\varphi$, $\mathcal{W}(\varphi)$ has:

- Nodes given by the variables of $\varphi$.
- There is an edge between $x$ and $y$ if the smallest subformula containing $x$ and $y$ has an $\wedge$ as the top most connective.

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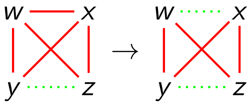

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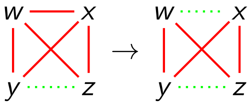


To determine whether $\varphi \to \psi$ is valid, it suffices to know the maximum cliques of $\mathcal{W}(\varphi)$ and $\mathcal{W}(\psi)$

# Overview

Our main contribution is an algorithm that is able to search for linear inferences and determine whether they are derivable. We have also written an implementation capable of running this algorithm on linear inferences with up to 8 variables.

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- Check remaining inferences.

# Results

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Our implementation was also able to run on 8 variables, and found two minimal underivable inferences.

**Answer**

The smallest inference that is not derivable from switch and medial has 8 variables.

# 8-Variable Inferences

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\begin{aligned}
& (z \vee (w \wedge w')) \wedge ((x \wedge x') \vee ((y \vee y') \wedge z')) \\
\rightarrow \quad & (z \wedge (x \vee y)) \vee ((w \vee y') \wedge ((w' \wedge x') \vee z'))
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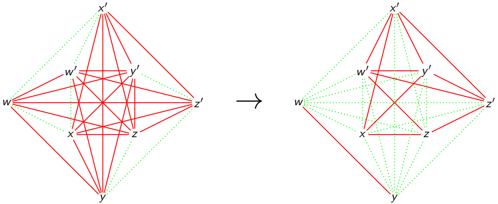

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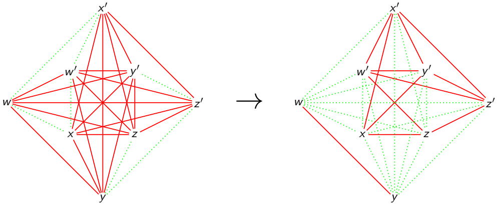


## Corollary

*The second inference contradicts a previous conjecture (Das and Straßburger, 2016).*

# Conclusions

- Our implementation is able to search linear inferences for derivability, crucially leveraging graph theoretic tools.
- The implementation is written in a generic way which could allow it to be applied to other problems (including those where graphs are treated as first class objects such as (Nguyên and Seiller, 2018),(Acclavio, Horne, and Straßburger, 2020),(Calk, Das, and Waring, 2020)).
- This was used to solve an open problem of the size of the smallest inference which could not be derived from switch and medial.
- We further classified the 8-variable inferences, including finding one that contradicted a previous conjecture.

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