A Type Theoretic Approach to Semistrict Higher Categories

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12$^{\text{th}}$ May 2022

Outline

1. Globular Infinity Categories
2. Weak Infinity Categories
3. Semistrict infinity categories

Globular Sets

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- For each pair of parallel arrows f, g, a set of 2-arrows from f to g.

$$\begin{tikzcd} x \arrow[r, bend left=50, "g"{name=G}] \arrow[r, bend right=50, "f"'{name=F}] & y \arrow[from=F, to=G, Rightarrow, "\alpha"'] \end{tikzcd}$$

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$$\begin{tikzcd} x \arrow[r, bend left=50, "g"{name=U}] \arrow[r, bend right=50, "f"'{name=D}] & y \arrow[from=D, to=U, Rightarrow, "\alpha"'] \end{tikzcd}$$

- ...

Definition

A *Globular Set* is a set G with a globular set $G_{x,y}$ for each pair of objects $x, y \in G$.

Compostition in Globular Sets

Composition of 1 cells

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

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Composition of 2 cells

Composition along a 1-boundary:
$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D}] & \bullet
\arrow[from=M, to=U, phantom, "\beta\Uparrow"] \arrow[from=D, to=M, phantom, "\alpha\Uparrow"]
\end{tikzcd}$$

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Composition along a 1-boundary:
$$\begin{tikzcd} \bullet \arrow[r, bend left=80, ""{name=T}] \arrow[r, ""{name=M}] \arrow[r, bend right=80, ""{name=B}] & \bullet \arrow[from=M, to=T, phantom, "\beta\Uparrow"] \arrow[from=B, to=M, phantom, "\alpha\Uparrow"] \end{tikzcd}$$

Composition along a 0-boundary:
$$\begin{tikzcd} \bullet \arrow[r, bend left=50, ""{name=A1}] \arrow[r, bend right=50, ""{name=A2}] \arrow[from=A2, to=A1, phantom, "\alpha\Uparrow"] & \bullet \arrow[r, bend left=50, ""{name=B1}] \arrow[r, bend right=50, ""{name=B2}] \arrow[from=B2, to=B1, phantom, "\beta\Uparrow"] & \bullet \end{tikzcd}$$

In a *strict infinity category* we have binary composition of n-cells for along a k boundary for all $k < n$.

Composition

If f and g are n-cells with the k-target of f equalling the k-source of g then there is an n-cell $f \circ_k g$.

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Identities

For each n-cell f there is an $(n+1)$-cell $\mathrm{id}_f : f \to f$.

Strict Infinity Categories - Associativity

If $0 \leq k < n$ and f, g, and h are n-cells then:

$$f \circ_k (g \circ_k h) = (f \circ_k g) \circ_k h$$

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Associativity of 1-cells

Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ we have:

$$f \circ_0 (g \circ_0 h) = (f \circ_0 g) \circ_0 h$$

Strict Infinity Categories - Identities

If $0 \leq k < n$ and f is an n-cell with k-source x and k-target y then:

$$\mathrm{id}^{n-k}(x) \circ_k f = f = f \circ_k \mathrm{id}^{n-k}(y)$$

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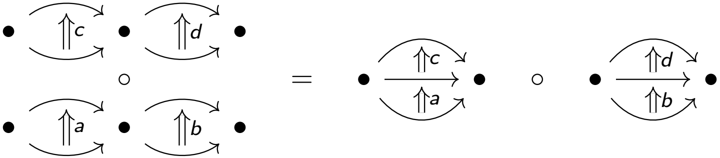
Identity on 2-cell

Given $f, g : x \to y$ and $\alpha : f \to g$ we have:

$$\begin{tikzcd} x \arrow[r, bend left=50, "\mathrm{id}(x)"] \arrow[r, bend right=50, "\mathrm{id}(x)"'] \arrow[r, phantom, "\Uparrow \mathrm{id}(\mathrm{id}(x))"] & x \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Uparrow \alpha"] & y \end{tikzcd} = \begin{tikzcd} x \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Uparrow \alpha"] & y \end{tikzcd}$$

If $0 \leq q < p < n$ and a, b, c, d are n-cells then:

$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$



Strict Infinity Categories - Interchange

If $0 \leq q < p < n$ and a, b, c, d are n-cells then:

$$(a \circ_p b) \circ_q (c \circ_p d) = (a \circ_q c) \circ_p (b \circ_q d)$$

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow c"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow d"] & \bullet \\
 & \circ & \\
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow a"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow b"] & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2ex, "\Uparrow c"] \arrow[r, phantom, shift right=2ex, "\Uparrow a"] & \bullet & \circ & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2ex, "\Uparrow d"] \arrow[r, phantom, shift right=2ex, "\Uparrow b"] & \bullet
\end{tikzcd}$$

Further if $f \circ_k g$ is well defined then:

$$\mathrm{id}_f \circ_k \mathrm{id}(g) = \mathrm{id}(f \circ_k g)$$

Monoidal Categories

Monoidal categories are instances of infinity categories.

Definition (Monoidal category)

A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object I satisfying some conditions.

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A *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a unit object I satisfying some conditions.

A strict infinity category with one object and no non-identity n-cells for n higher than 2 is a strict monoidal category.

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The monoidal product in **Set** is *not* strict.

Weak Infinity Categories

In a weak infinity category, we only require that the various laws hold up to isomorphism.
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However many isomorphisms can exist between two cells. We require that these isomorphisms be *coherent*.

- For a 1-cell $f : x \to y$, there are unitors $\lambda_f : \mathrm{id}_x \circ f \to f$ and $\rho_f : f \circ \mathrm{id}_y$.
- λ_{id_x} and ρ_{id_x} are both arrows $\mathrm{id}_x \circ \mathrm{id}_x \to \mathrm{id}_x$. We can ask that they be isomorphic.
- This isomorphism will also be subject to coherence conditions.

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It quickly becomes apparent that we need a more uniform way to package this coherence data.

Pasting Diagrams

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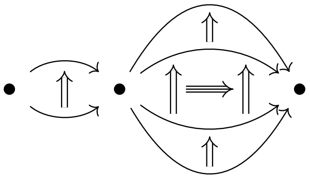
The compositions we have already seen form pasting diagrams.

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Uparrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\beta\Uparrow"] & \bullet
\end{tikzcd}$$

We can also form more complicated compositions as pasting diagrams.



Motto for Pasting Diagrams

- Pasting diagrams for 1-categories are simply chains of 1-cells:

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- Further, there is exactly 1 composite.
- In a strict infinity category, every (higher dimensional) pasting diagram has exactly one composite.
- For weak infinity categories, we weaken the exactness condition to uniqueness up to isomorphism.

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Taking the composite of the diagram:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$$

gives the composite $f \circ g$.

Over the singleton pasting diagram

$$x$$

and taking $s = x$ and $t = x$ we get a term from x to x representing the identity on x.

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- Substitutions: A substitution is a *morphism* between contexts.

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- If a term is a 0-cell in our infinity category, then it has type $\star$.
- Otherwise a term is an $(n+1)$-cell between parallel n-cells f and g, in which case it has type:

$$f \to_A g$$

  where A is the (common) type of f and g.

The Coh Constructor

The crucial part of CaTT is the Coh constructor, which captures the motto for weak composition.

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- s and t are two parallel terms, which can be represented as a type.
- σ labels the pasting diagram with (compatible) terms, and can be represented as a substitution.

Examples

Identity

Let t be a 1 dimensional term. The identity on t is:

$$\mathsf{coh}\ (x \xrightarrow{f} y : f \to_{x \to_{\star} y} f)[\sigma]$$

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1-composition

Let $s : x \to_\star y$ and $t : y \to_\star z$ be terms. Their composite is given by:

$$\mathsf{coh}\ (x \xrightarrow{f} y \xrightarrow{g} z : x \to_\star z)[\sigma]$$

where $\sigma(x) = x$, $\sigma(y) = y$, $\sigma(z) = z$, $\sigma(f) = s$, $\sigma(g), = t$.

Examples

Take the context $\Gamma = w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z$.

The *associator* is given by:

$$\mathrm{coh}\ (\Gamma : (f \circ g) \circ h \to_{w \to_\star z} f \circ (g \circ h))[\mathrm{id}]$$

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However this is no longer possible at dimensions 3 and higher.

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We can strictify:

	Strict ∞-**Cat**	Simpson	Grey
Associators	✓	✓	✓
Unitors	✓		✓
Interchangers	✓	✓	

Type theories for semistrict languages

CaTT as we have presented it has no non-trivial equality and no computation.

The idea is to implement a reduction relation that unifies the operations we want to strictify.

By doing this we obtain a type theory for which the models are semistrict categories. Further by showing our reduction is terminating and confluent, we obtain a language for the operations which has decidable type checking and equality.

- $\mathsf{CaTT}_{\mathsf{su}}$: Has strict units. Generated by the pruning operation.
- $\mathsf{CaTT}_{\mathsf{sa}}$: Has strict associators. Generated by the insertion operation.
- $\mathsf{CaTT}_{\mathsf{sua}}$ (Work in Progress): Combines the previous two theories.

Example - Syllepsis

- Given two scalars $a, b : id_x \to id_x$, by the Eckmann Hilton argument we have an isomorphism $\mathrm{EH}_{f,g} : a \circ_1 b \simeq b \circ_1 a$.
- In fact, there are two such isomorphisms, $\mathrm{EH}_{a,b}$ and $\mathrm{EH}^{-1}_{b,a}$, that need not be themselves isomorphic.
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	CaTT	$\mathrm{CaTT}_{\mathrm{su}}$	$\mathrm{CaTT}_{\mathrm{sua}}$
Eckmann-Hilton	297	15	15
Syllepsis	NA	675	397

Figure: Coh constructors in Eckmann-Hilton and Syllepsis

Further work

- Finish proving metatheorems for $\mathsf{CaTT}_{\mathsf{sua}}$.
- Equivalence of Theories.
- More semistrict type theories, including one for Simpson-like semistrictness.
- Bridging the gap between CaTT and graphical methods.