A Syntax for Strictly Associative and Unital ∞-Categories

Eric Finster Alex Rice Jamie Vicary

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The Type Theory CATT

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Globular ∞-categories contain higher dimensional arrows.

2-cell $\alpha : f \to g$

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A type gives the boundary of a term.

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$$\begin{tikzcd} x \arrow[r, bend left=50, "g"{name=G}] \arrow[r, bend right=50, "f"'{name=F}] & y \arrow[from=F, to=G, phantom, "\Uparrow\alpha"] \end{tikzcd}$$

Terms in CATT

Terms represent the possible operations in a globular ∞-category.

Terms built over *pasting diagrams*.

Compound terms using substitutions.

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$$\Delta := \quad \begin{tikzcd} x \arrow[r, bend left, "h"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\alpha"] & y \arrow[r, bend left, "i"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow\beta"] & z \end{tikzcd}$$

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$$s[\![\sigma]\!]$$

$$\sigma := \langle f \mapsto a, g \mapsto (b * c) \rangle$$

$$a * (b * c) := (f * g)[\![\sigma]\!]$$

The Associator

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$$\alpha_{a,b,c} := \mathrm{coh}\,(\Delta : (a * b) * c \to a * (b * c))$$

Semistrictness

The theory described by CATT is *Weak*.

Laws of categories are given by equivalence.

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$$\begin{array}{ccccc} \text{Weak} & \longleftarrow\! & \text{Semistrict} & \longrightarrow & \text{Strict} \end{array}$$

$$\textsc{Catt}_{\mathsf{su}} \uparrow$$

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$$\begin{tikzcd} u \arrow[r, "c"] & v \arrow[r, "d"] & w \\ x \arrow[u, "f"] \arrow[r, "a"'] & y \arrow[u, "g"] \arrow[ul, Rightarrow, "A"] \arrow[r, "b"'] & z \arrow[u, "h"'] \arrow[ul, Rightarrow, "B"] \end{tikzcd}$$

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\end{tikzcd}$$

$$(\mathrm{id}_a \otimes B) * (A \otimes \mathrm{id}_d) : (a * b) * h \to f * (c * d)$$

$\textsc{Catt}_{\mathsf{sua}}$

$\textsc{Catt}$ has trivial equality.

$\textsc{Catt}_{\mathsf{su}}$ has disc removal, endo-coherence removal, and pruning.

In $\textsc{Catt}_{\mathsf{sua}}$, pruning is replaced by insertion.

$$\textsc{Catt}_{\mathsf{sua}} := \textsc{Catt} + \text{insertion} + \text{disc removal} + \text{endo-coherence removal}$$

Insertion Rule

$$(a * b) * c \ =_{\mathsf{sua}}\ a * b * c \equiv \mathsf{coh}\,(x \xrightarrow{a} y \xrightarrow{b} z \xrightarrow{c} w : x \to w)$$

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Recalling $(a * b) * c \equiv (f * g)[\![\langle f \mapsto a, g \mapsto b * c\rangle]\!]$:

$$x \xrightarrow{f} y \xrightarrow{b*c} z \qquad\qquad x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

is sent to:

$$x \xrightarrow{f} x' \xrightarrow{f'} y' \xrightarrow{g'} z'$$

Universal Property of Insertion

$$\mathsf{coh}\ (\Delta : s \to t)[\![\sigma]\!] \quad x \in \Delta \quad x[\![\sigma]\!] \equiv \mathsf{coh}\ (\Theta : u \to v)[\![\tau]\!]$$

$$\begin{tikzcd}
 & \Delta \arrow[dd, bend left, "\sigma"] \\
\Theta \arrow[dr, bend right, "\tau"'] & \\
 & \Gamma
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$$\begin{tikzcd}
 & \Delta \arrow[d, "\color{red}\kappa"] \arrow[dr, bend left, "\sigma"] & \\
\Theta \arrow[r, "\iota"'] \arrow[rr, bend right, "\tau"'] & \Delta \ll_x \Theta & \Gamma
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$$\begin{tikzcd}
{\color{red}D} \arrow[r, "{\color{red}x}"] \arrow[d, "{\color{red}\mathsf{coh}\ (\Theta : u \to v)}"'] & \Delta \arrow[d, "\kappa"] \arrow[ddr, bend left, "\sigma"] & \\
\Theta \arrow[r, "\iota"'] \arrow[drr, bend right, "\tau"'] & \Delta \ll_x \Theta & \\
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\Theta \arrow[r, "\iota"'] \arrow[drr, bend right, "\tau"'] & \Delta \ll_x \Theta \arrow[dr, "\sigma \ll_x \tau" description] & \\
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& & \Gamma
\end{tikzcd}$$

$$\mathsf{coh}\ (\Delta : s \to t)[\![\sigma]\!] = \mathsf{coh}\ (\Delta \ll_x \Theta : s[\![\kappa]\!] \to t[\![\kappa]\!])[\![\sigma \ll_x \tau]\!]$$

Normalisation

Equality in $\textsc{Catt}_{\mathsf{sua}}$ is decidable.

$\textsc{Catt}_{\mathsf{sua}}$ has unique normal forms.

Obtained by reduction system.

Termination: Assign *syntactic complexity* to terms.

Confluence: Encode various constructions in Agda.

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Type checking is decidable.

We provide a interpreter which:

- Provides tools for construction terms.
- Type checks terms.
- Reduces terms to $\textsc{Catt}_{\mathsf{sua}}$ normal form.

Summary

We introduce the type theory $\textsc{Catt}_{\mathsf{sua}}$.

$\textsc{Catt}_{\mathsf{sua}}$ models strictly unital and associative ∞-categories.

$\textsc{Catt}_{\mathsf{sua}}$ are simpler than their $\textsc{Catt}$ or $\textsc{Catt}_{\mathsf{su}}$ equivalents.

Normal forms for $\textsc{Catt}_{\mathsf{sua}}$ are obtained via a reduction system:

This reduction system is strongly terminating and confluent.

Try our interpreter for $\textsc{Catt}_{\mathsf{sua}}$:

`https://github.com/alexarice/catt-strict`

Thank you for listening.