Answers to questions in Lab 1: Filtering operations

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Instructions: Complete the lab according to the instructions in the notes and respond to the questions stated below. Keep the answers short and focus on what is essential. Illustrate with figures only when explicitly requested.

Good luck!

Question 1: Repeat this exercise with the coordinates p and q set to (5, 9), (9, 5), (17, 9), (17, 121), (5, 1) and (125, 1) respectively. What do you observe?

Answers: We observe that the amplitude is the same for all values (within rounding error) in the matrix, that the wavelength is inversely proportional to the distance from origo to the closest "periodic" point, that the frequency is proportional to the distance from origo to the closest point, and that the angle of the waves is the same as the angle from origo to the closest point

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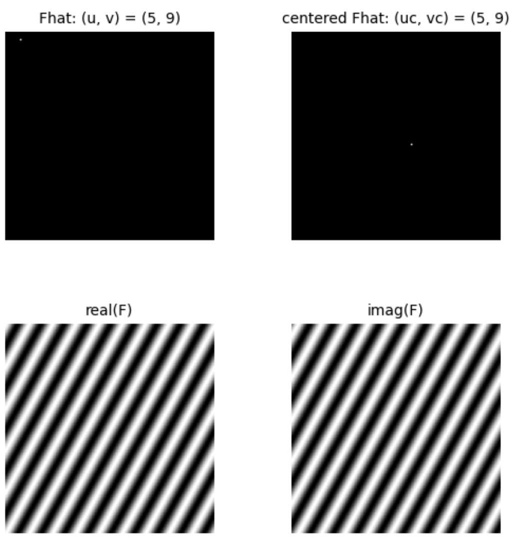
Question 2: Explain how a position (p, q) in the Fourier domain will be projected as a sine wave in the spatial domain. Illustrate with a Matlab figure.

Answers: Fhat(u) will be 1 for one u-vector and 0 for all others, so the value of F(x) will be equal to 1/N * the exponential term where we take the dot product of u and x, and the final value corresponds to the combination of cos and sin by the Euler identity. Changing the x-vector in F(x) increases the euler exponential term and thus changes the phase shift.



Question 3: How large is the amplitude? Write down the expression derived from Equation (4) in the notes. Complement the code (variable amplitude) accordingly.

Answers:

$$F(x) = \mathcal{F}_D^{-1}(\hat{F})(x) = \frac{1}{N} \sum_{u \in [0..N-1]^2} \hat{F}(u)\, e^{\frac{+2\pi i u^T x}{N}}. \tag{4}$$

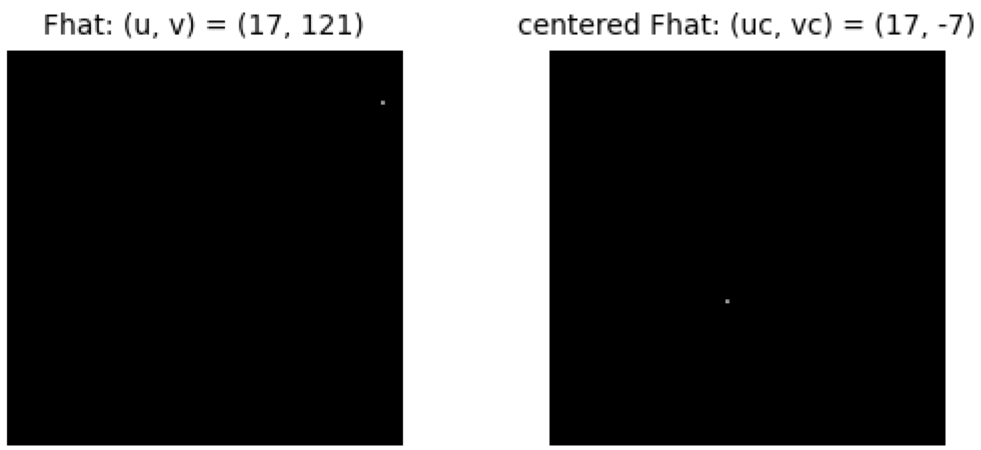
That is 1 for a single pixel and the amplitude will be 1 for the exponential since the cos^2+sin^2=1, thus we get: amplitude = 1/N (for all x)
The position does not matter since all amplitudes are the same when only one point is 1, and the amplitude here would be 1/128. The actual amplitude in the code is 1/(128*128), likely because they only normalize the inverse with N^2 instead of normalizing both ways with N.

Question 4: How does the direction and length of the sine wave depend on p and q? Write down the explicit expression that can be found in the lecture notes. Complement the code (variable wavelength) accordingly.

Answers: Direction depends on the direction from origo to the closest "periodic" point, i.e. uc and vc. Wavelength is inversely proportional to the distance from uc and vc to origo. wavelength = sz / np.sqrt(uc**2 + vc**2), expressed in amounts of pixels.

Question 5: What happens when we pass the point in the center and either p or q exceeds half the image size? Explain and illustrate graphically with Matlab!

Answers: Then that point is no longer the closest to origo, since the “period” or “repeated” point to the left/up from it, since the image is repeated infinitely in all directions.



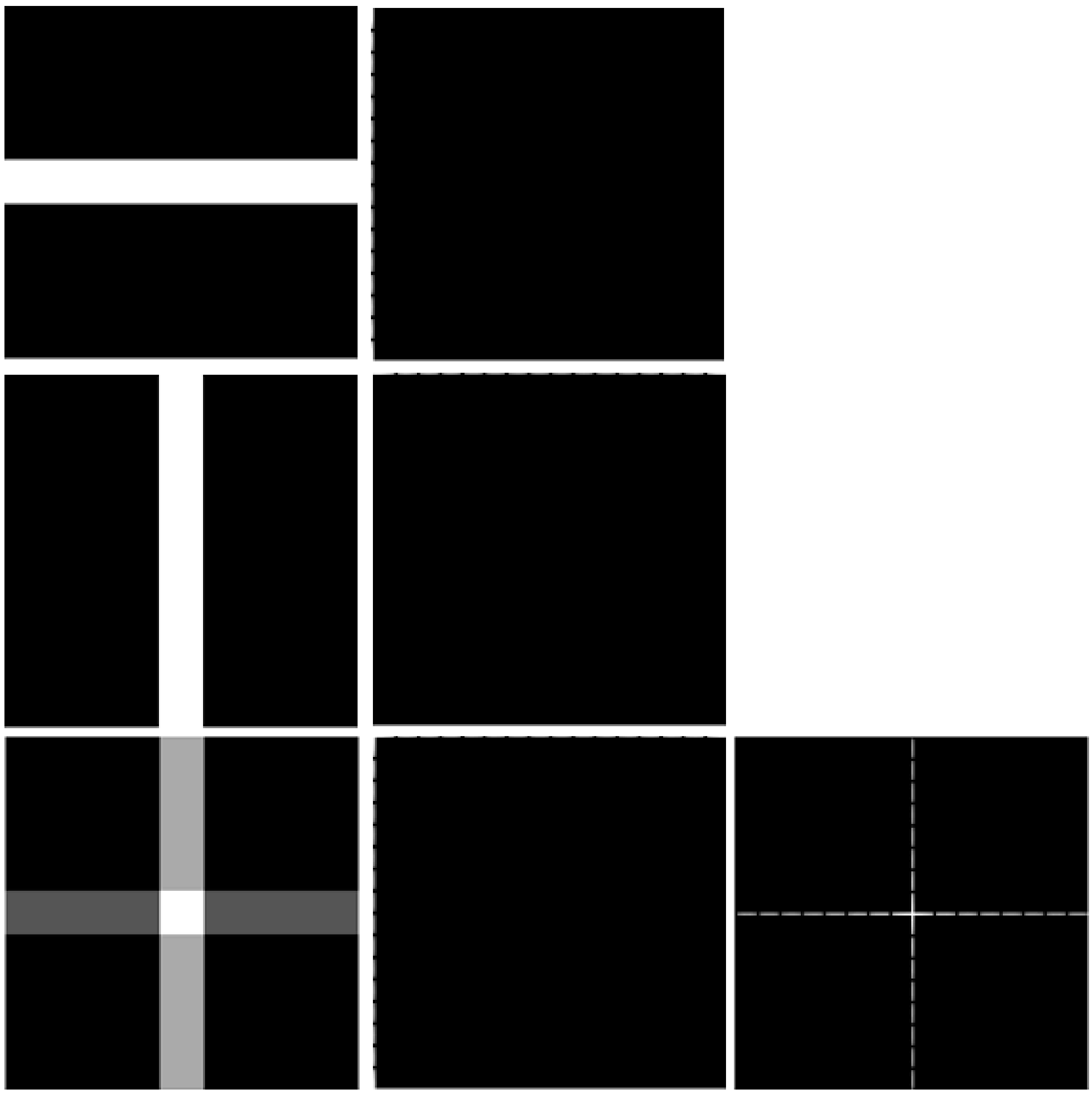
Here, origo is shifted toward the center of the right-side image so we can clearly see that the closest point is down to the left.

Question 6: What is the purpose of the instructions following the question *What is done by these instructions?* in the code?

Answers: To change the input vector to correspond to the closest vector from origo.

Question 7: Why are these Fourier spectra concentrated to the borders of the images? Can you give a mathematical interpretation? Hint: think of the frequencies in the source image and consider the resulting image as a Fourier transform applied to a 2D function. It might be easier to analyze each dimension separately!

Answers: Horizontal lines in the spatial domain are represented by values along the left border in the Fourier domain and vertical lines are represented by values along the top border. This is because many complex numbers are added onto each other with the same angle, so they are amplified instead of canceling out. The point (i, 0) in Fourier space is amplified by all points (i, 0..127) where the Euler exponents generally have the same angle, assuming N=128. Therefore, a horizontal line in the spatial domain becomes the left-side vertical border in the Fourier domain.



Question 8: Why is the logarithm function applied?

Answers: To bring out low values which are still relevant and non-zero due to a logarithmic decrease.

Question 9: What conclusions can be drawn regarding linearity? From your observations can you derive a mathematical expression in the general case?

Answers: The Fourier transform is linear.

$$\hat{F}_1(u) = \frac{1}{N} \sum_{x \in [0..N-1]^2} F_1(x)\, e^{\frac{-2\pi i u^T x}{N}} \qquad \hat{F}_2(u) = \frac{1}{N} \sum_{x \in [0..N-1]^2} F_2(x)\, e^{\frac{-2\pi i u^T x}{N}}$$

$$F_3(x) = F_1(x) + F_2(x)$$

$$\hat{F}_3(u) = \frac{1}{N} \sum_{x \in [0..N-1]^2} F_3(x)\, e^{\frac{-2\pi i u^T x}{N}}$$

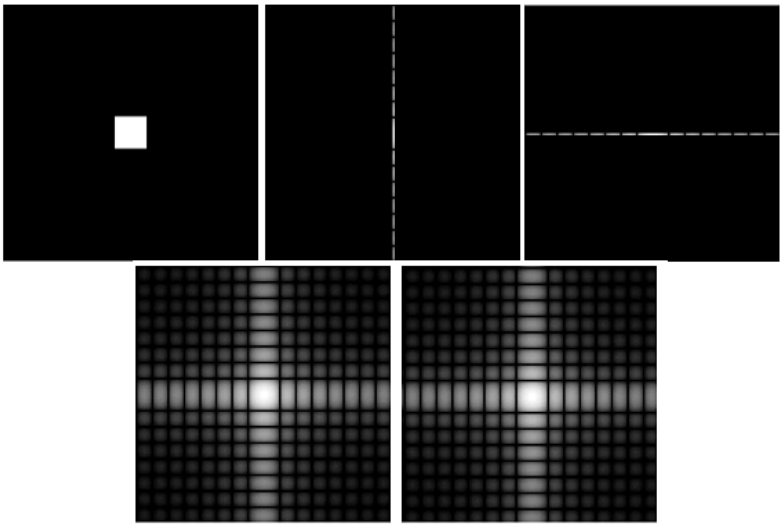
$$\hat{F}_3(u) = \frac{1}{N} \sum_{x \in [0..N-1]^2} \Big(F_1(x) + F_2(x)\Big) \cdot e^{\frac{-2\pi i u^T x}{N}}$$

$$\hat{F}_3(u) = \frac{1}{N} \sum_{x \in [0..N-1]^2} F_1(x)\, e^{\frac{-2\pi i u^T x}{N}} + \frac{1}{N} \sum_{x \in [0..N-1]^2} F_2(x)\, e^{\frac{-2\pi i u^T x}{N}}$$

$$\hat{F}_3(u) = \hat{F}_1(u) + \hat{F}_2(u)$$

Question 10: Are there any other ways to compute the last image? Remember what multiplication in Fourier domain equals to in the spatial domain! Perform these alternative computations in practice.

Answers: We can compute it by doing the convolution between fft2(F) and fft2(G), since multiplication in the spatial domain is the same as convolution in the Fourier domain.



$$\hat{F}_1(u) = \frac{1}{N}\sum_{x\in[0..N-1]^2} F_1(x)\, e^{\frac{-2\pi i u^T x}{N}} \qquad \hat{F}_2(u) = \frac{1}{N}\sum_{x\in[0..N-1]^2} F_2(x)\, e^{\frac{-2\pi i u^T x}{N}}$$

$$F_1(x) = \frac{1}{N}\sum_{u\in[0..N-1]^2} \hat{F}_1(u)\, e^{\frac{+2\pi i u^T x}{N}} \qquad F_2(x) = \frac{1}{N}\sum_{u\in[0..N-1]^2} \hat{F}_2(u)\, e^{\frac{+2\pi i u^T x}{N}}$$

$$F_3(u) = F_1(u) \cdot F_2(u)$$

$$\hat{F}_3(u) = \frac{1}{N}\sum_{x\in[0..N-1]^2} F_3(x)\, e^{\frac{-2\pi i u^T x}{N}}$$

$$\hat{F}_3(u) = \frac{1}{N}\sum_{x\in[0..N-1]^2} F_1(x)\cdot F_2(x)\cdot e^{\frac{-2\pi i u^T x}{N}}$$

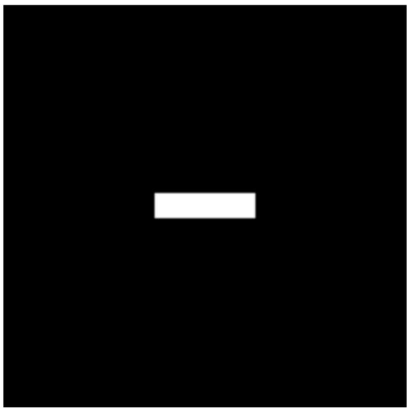
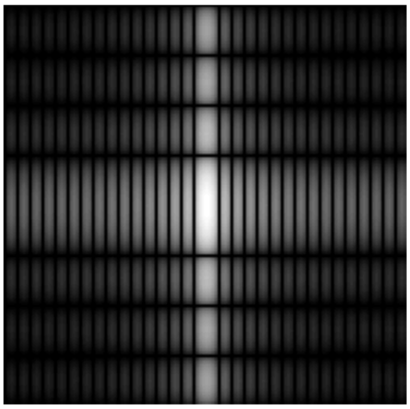
$$\hat{F}_3(u) = \frac{1}{N}\sum_{x\in[0..N-1]^2} \left(\frac{1}{N}\sum_{v\in[0..N-1]^2} \hat{F}_1(v)\, e^{\frac{+2\pi i v^T x}{N}}\right)\cdot F_2(x)\cdot e^{\frac{-2\pi i u^T x}{N}}$$

$$\hat{F}_3(u) = \frac{1}{N^2}\sum_{v\in[0..N-1]^2} \hat{F}_1(v) \sum_{x\in[0..N-1]^2} F_2(x)\cdot e^{\frac{-2\pi i (u-v)^T x}{N}}$$

$$\hat{F}_3(u) = \frac{1}{N^2}\sum_{v\in[0..N-1]^2} \hat{F}_1(v)\cdot N\cdot \hat{F}_2(u-v) = \frac{1}{N}\sum_{v\in[0..N-1]^2} \hat{F}_1(v)\, \hat{F}_2(u-v)$$

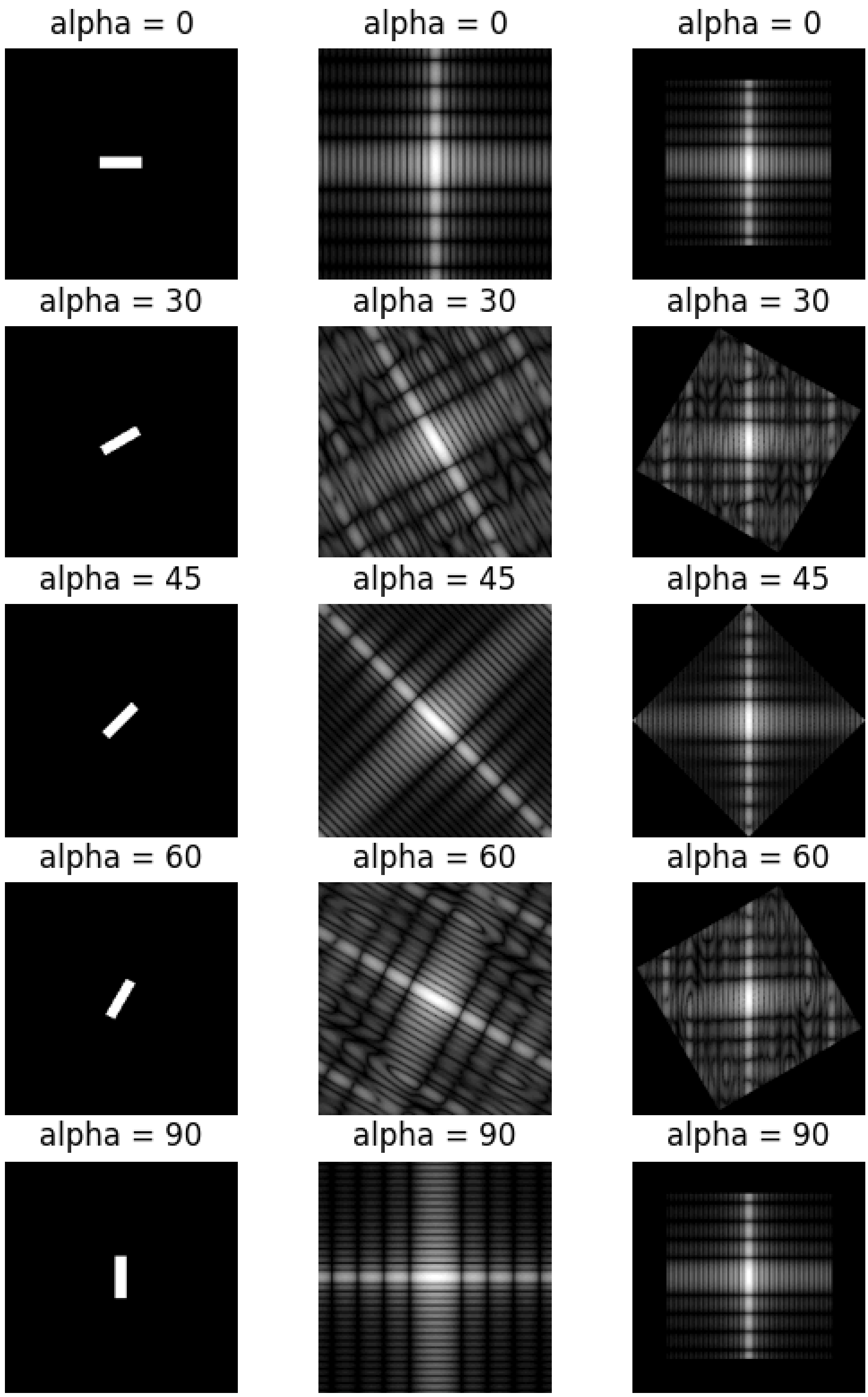
Question 11: What conclusions can be drawn from comparing the results with those in the previous exercise? See how the source images have changed and analyze the effects of scaling.

Answers: "Compression (scale down) in spatial domain is same as expansion (scale up) in Fourier domain (and vice versa)." (lecture 3)

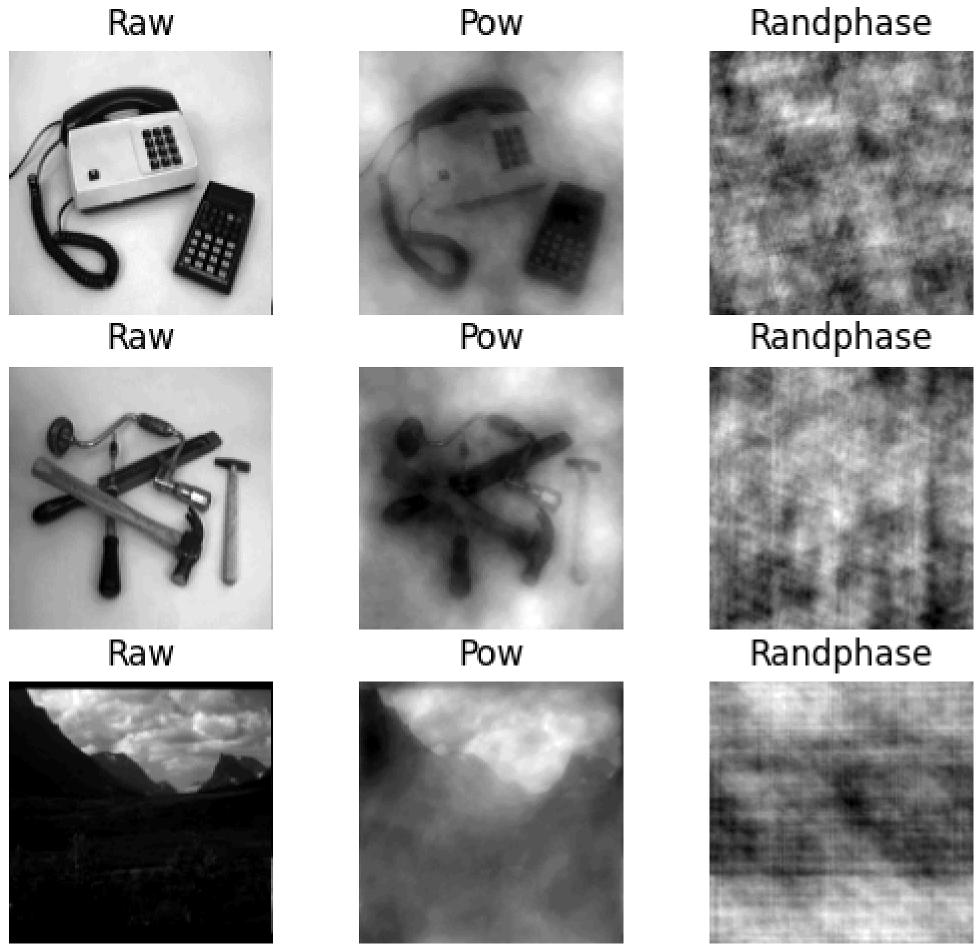
Question 12: What can be said about possible similarities and differences? Hint: think of the frequencies and how they are affected by the rotation.

Answers: Rotation in the spatial domain is represented in the Fourier domain, as we can see when rotating back the Fourier images and observe that the images look the same, minus the loss of resolution. This is quite natural since we need to rotate all sinus waves by the same amount in order to represent a rotated image.



Question 13: What information is contained in the phase and in the magnitude of the Fourier transform?

Answers: We see that with pow2image we can still sort of tell what the original image was, despite now being more “foggy”. However, we cannot tell what the original image was with randphase. With pow2image, we lose the intensity and contrast of the image, which the magnitude is responsible for but we still have information about general edges, which the phase is responsible for. With altered phase, as in randphase, we lose any idea of what the image is supposed to be.



Question 14: Show the impulse response and variance for the above-mentioned t-values. What are the variances of your discretized Gaussian kernel for t = 0.1, 0.3, 1.0, 10.0 and 100.0?

Answers:

t=0.1:

1.32967252e-02	1.96231920e-14
1.96231920e-14	1.32967252e-02

t=0.3:

2.81053830e-01	-5.60107516e-14
-5.60107516e-14	2.81053830e-01

t=1.0:

9.99999789e-01	-3.56936702e-14
-3.56936702e-14	9.99999789e-01

t=10:

1.00000000e+01	3.21409566e-14
3.21409566e-14	1.00000000e+01

t=100:

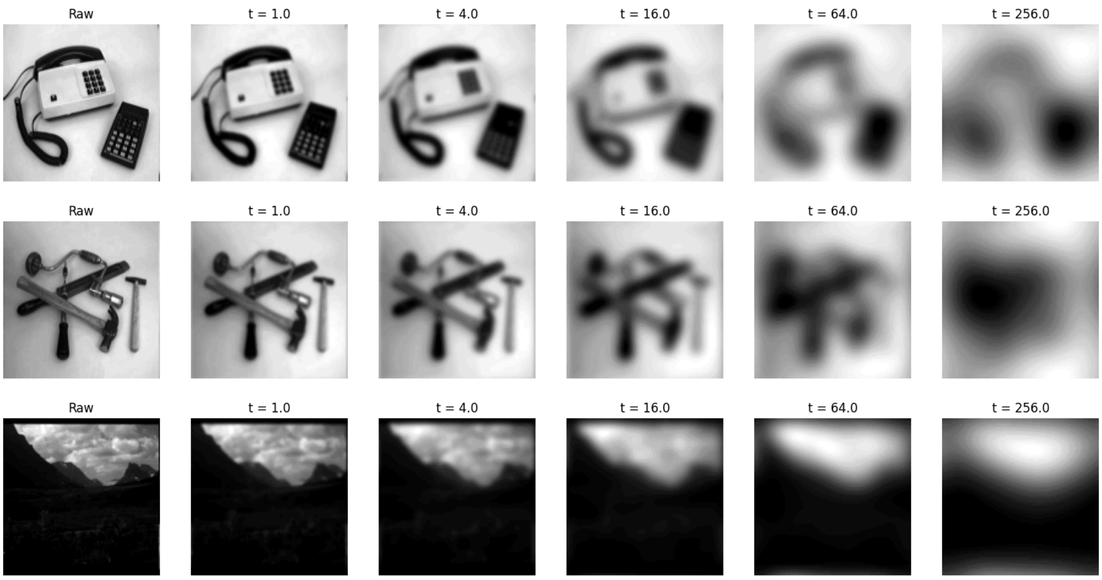
9.99999993e+01	3.77753384e-14
3.77753384e-14	9.99999993e+01

Question 15: Are the results different from or similar to the estimated variance? How does the result correspond to the ideal continuous case? Lead: think of the relation between spatial and Fourier domains for different values of t.

Answers: We see that the obtained results are close to the ideal continuous case, especially for larger t-values and if we consider rounding errors. This is because larger t-values lead to more sampling points for the Gaussian filter which starts to closely correspond to the continuous case. Additionally, if we have fractional t-values, the sample points will land in between pixels which leads to less accurate results.

Question 16: Convolve a couple of images with Gaussian functions of different variances (like t = 1.0, 4.0, 16.0, 64.0 and 256.0) and present your results. What effects can you observe?

Answers: For increased values of t, we see that the image gets more and more blurry, which is expected since more of the surrounding pixels are taken into account.

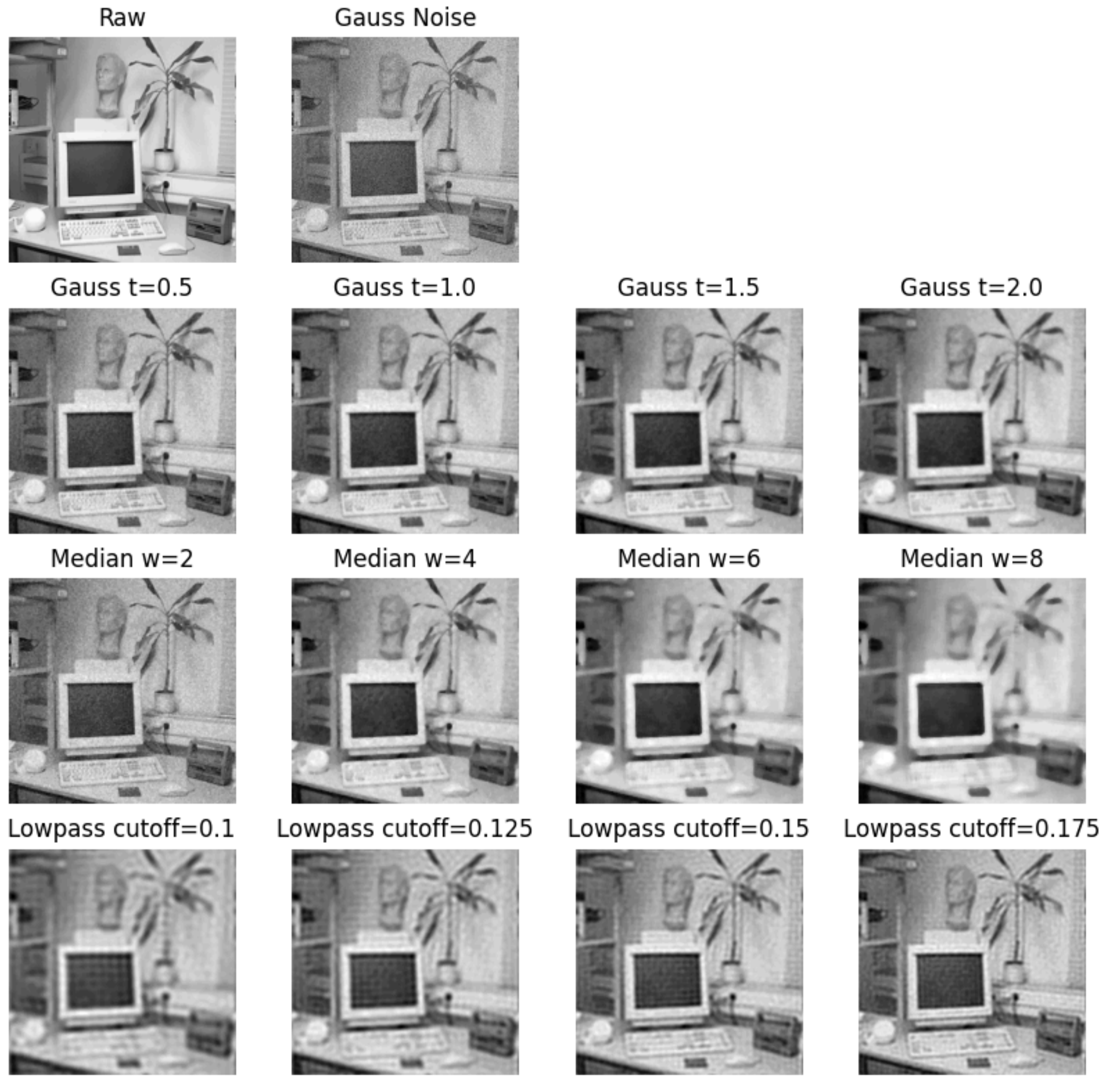


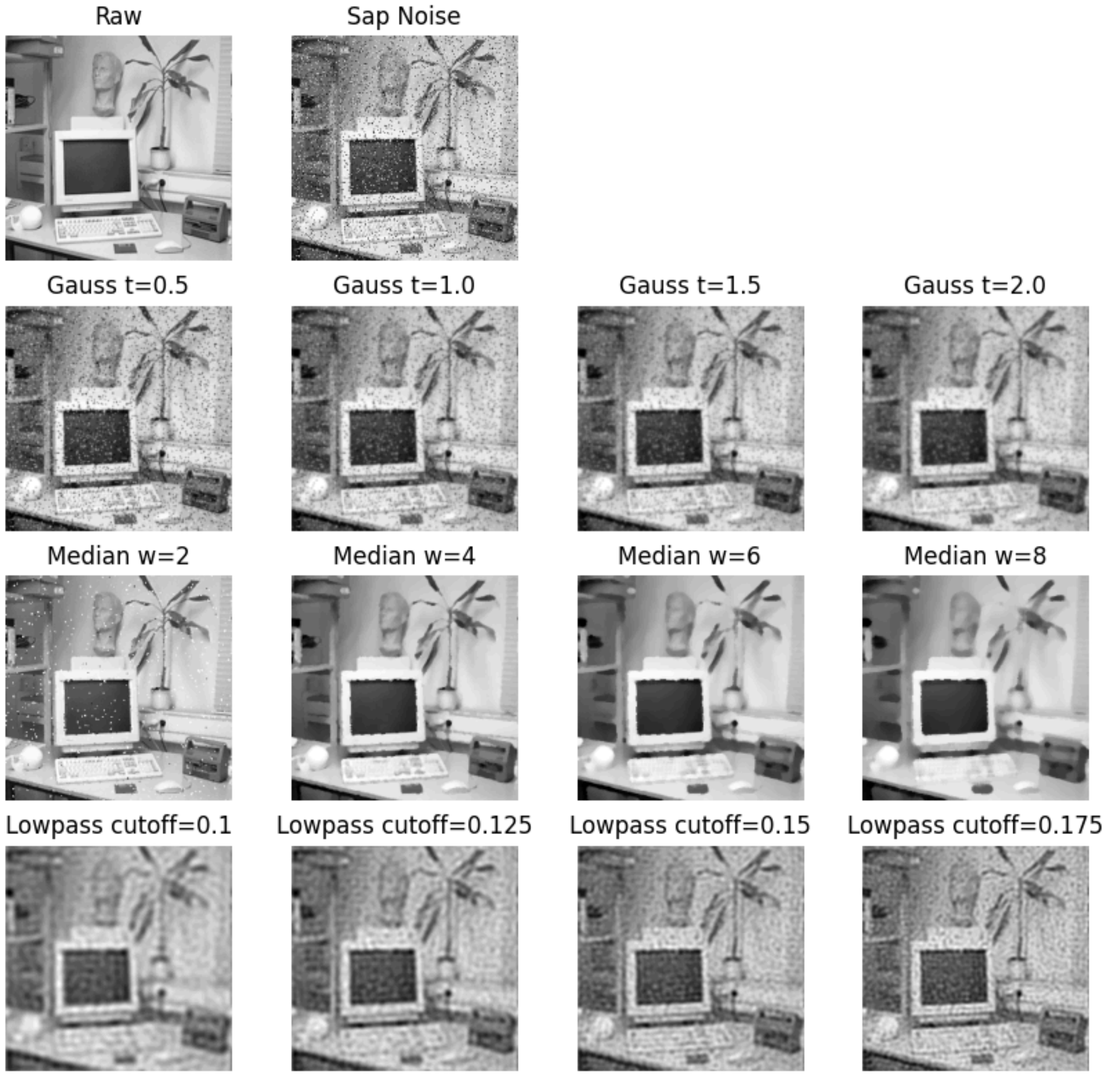
Question 17: What are the positive and negative effects for each type of filter? Describe what you observe and name the effects that you recognize. How do the results depend on the filter parameters? Illustrate with Matlab figure(s).

Answers: Blurring (gauss) is quite effective against gaussian noise, since the noise isn't as intense, but less effective against salt and pepper noise. Here, low values for t achieve decent results.

Conversely, the median filter is most effective against salt and pepper noise, since the noise is intense outliers compared to the normal pixels. With width=4, the sap image is basically back to its original image with little smoothing. It is less effective against gaussian noise, but still decent. For larger values for width, the image becomes excessively smoothed.

With the ideal lowpass filter, we can observe excessive ringing with both gaussian and sap noise. We see a small improvement from the gaussian noise image around cutofff=0.175, but little to no improvement for the sap noise.





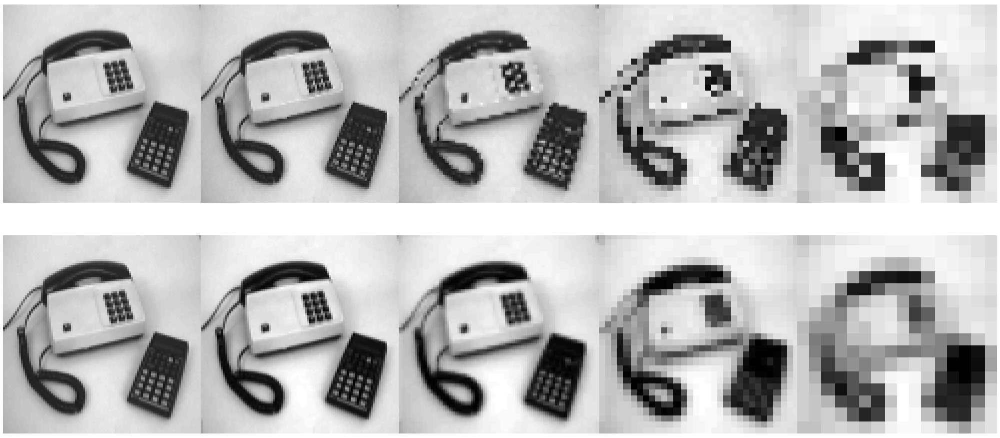
Question 18: What conclusions can you draw from comparing the results of the respective methods?

Answers: We can conclude that gaussian filtering works better for gaussian noise while the median filter works better for salt and pepper noise since that noise is more intense, as previously stated. All three methods still work pretty well against gaussian noise, but pretty much only median filtering works against salt and pepper noise.

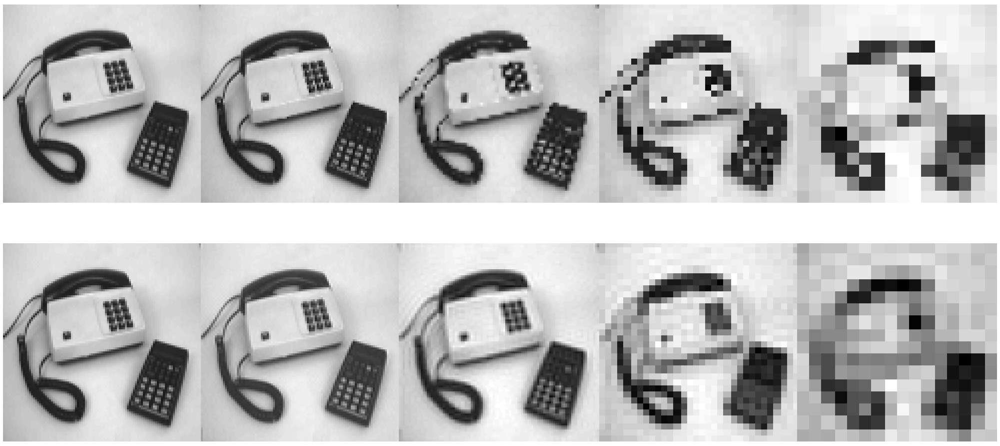
Question 19: What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration i = 4.

Answers: We see that some quality is preserved and that we can still see what the original image was in the fourth iteration.

Gaussian filter (t=0.5):



Ideal lowpass filter (cutoff=0.25):



Question 20: What conclusions can you draw regarding the effects of smoothing when combined with subsampling? Hint: think in terms of frequencies and side effects.

Answers: By smoothing the image before subsampling, we make it so that the subsampled pixels are a decent representation of the pixels that weren't picked, thus making the image more recognizable.
