Quantitative Macro Final Project

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Simple Variant of the Krusell-Smith Algorithm

1.

Proving Proposition 3, Harenberg and Ludwig (2015) use a guess and verify method. HL guess the form

$$a_{2,t+1} = s(1-\tau)w_t$$

which arises easily taking into account that all households are identical ex ante. Using the FOC of the firms problem and the market clearing condition for assets we arrive at

$$K_{t+1} = a_{2,t+1} = s(1-\tau)(1-\alpha)\Upsilon_t \zeta_t k_t^{\alpha}$$

Or written per efficient unit of labour and using $\Upsilon_{t+1} = (1+g)\Upsilon_t$

$$k_{t+1} = \frac{K_{t+1}}{\Upsilon_{t+1}(1-\lambda)} = \frac{s(1-\tau)(1-\alpha)\zeta_t k_t^{\alpha}}{(1+g)(1+\lambda)}$$

Starting to verify the guess we can simply plug in the guess into the budget constraints. The first one yields

$$c_{1t} = (1-s)(1-\tau)(1-\alpha)\Upsilon_t \zeta_t k_t^{\alpha}$$

and for the second we can additionally use the second FOC of the firms problem $(1+r_t) = \alpha k_t^{\alpha-1}\zeta_t \varrho_t$ and the social security budget constraint $b_t = \lambda w_t \frac{1+\lambda}{1-\lambda}$.

The budget constraint then becomes:

$$c_{i,2,t+1} = s(1-\tau)(1-\alpha)\Upsilon_t\zeta_t k_t^{\alpha}\alpha k_{t+1}^{\alpha-1}\zeta_{t+1}\varrho_{t+1} + \tau(1-\alpha)\Upsilon_{t+1}\varrho_{t+1}k_{t+1}^{\alpha}$$
$$((1+\lambda)+\lambda\eta_{i,2,t+1}) + \lambda\eta_{i,2,t+1}(1-\alpha)\Upsilon_{t+1}\varrho_{t+1}k_{t+1}^{\alpha}$$

$$c_{i,2,t+1} = s(1-\tau)(1-\alpha)\frac{\Upsilon_{t+1}}{1+g}\zeta_t k_t^{\alpha}\alpha k_{t+1}^{\alpha-1}\zeta_{t+1}\varrho_{t+1}$$
$$+ (1-\alpha)\Upsilon_{t+1}\varrho_{t+1}k_{t+1}^{\alpha}(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1})))$$

$$c_{i,2,t+1} = \Upsilon_{t+1}\varrho_{t+1}k_{t+1}^{\alpha}(s(1-\tau)(1-\alpha)\frac{\Upsilon_1}{1+g}\zeta_t k_t^{\alpha}\alpha k_{t+1}^{-1}\varrho_{t+1}$$
$$+ (1-\alpha)(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))))$$

This expression nicely simplifies when we use $k_{t+1} = \dfrac{K_{t+1}}{\Upsilon_{t+1}(1-\lambda)} = \dfrac{s(1-\tau)(1-\alpha)\zeta_t k_t^{\alpha}}{(1+g)(1+\lambda)}$ to

$$c_{i,2,t+1} = \Upsilon_{t+1}\varrho_{t+1}k_{t+1}^{\alpha}(\alpha\frac{\varrho_{t+1}}{1+\lambda} + (1+\alpha)(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))))$$

We can now plug those to expressions into the Euler Equation, which yields

$$1 = \beta\mathbb{E}_t\left[\frac{(1-s)}{s(1+\frac{(1-\alpha)}{\alpha(1+\lambda)\varrho_{t+1}}(\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1}))))}\right]$$

This verifies the guess and proofs the proposition as we can define

$$\Phi = \mathbb{E}_t\left[\frac{1}{1+\frac{(1-\alpha)}{\alpha(1+\lambda)\varrho_{t+1}}((\lambda\eta_{i,2,t+1} + \tau(1+\lambda(1-\eta_{i,2,t+1})))}\right]$$

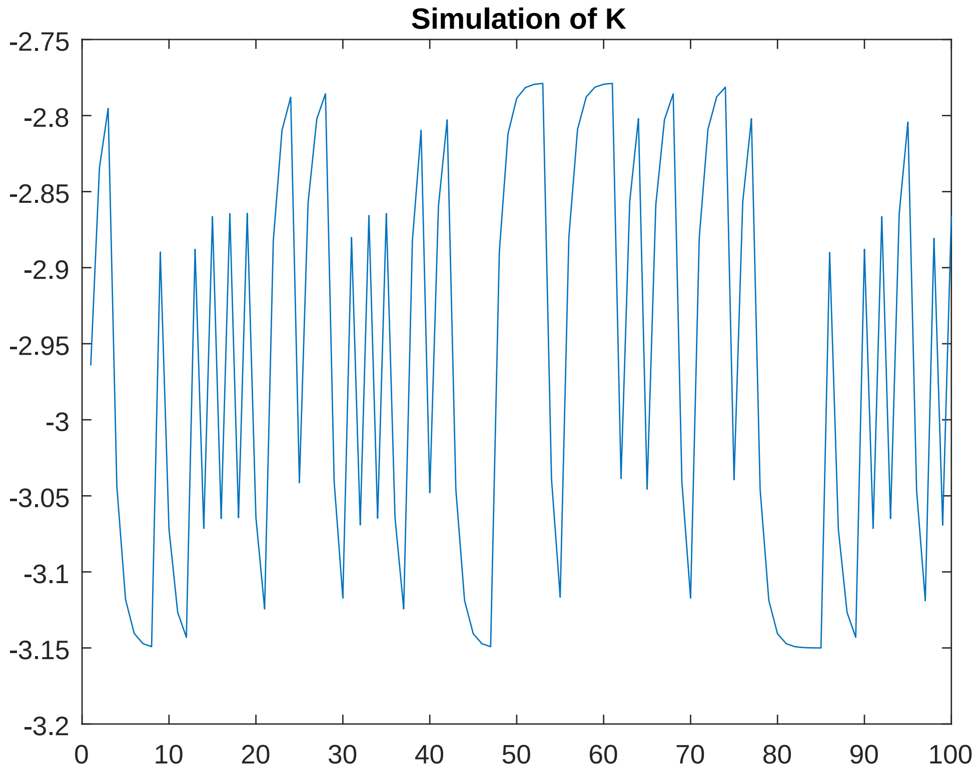
and thus

$$1 = \beta \frac{1-s}{s} \Phi$$

$$s = \frac{\beta \Phi}{1 + \beta \Phi}$$

2.

In order to simulate the evolution of capital using equation (1) I first define a set of shocks and corresponding probabilities. I define both ζ and ϱ to take two states with equal probabilities. The states are each defined to be the mean plus standard deviation and the mean minus standard deviation, so $ln\zeta \in [-0.13, 0.13]$ and $ln\varrho \in [-0.5, 0.5]$. Furthermore η takes 11 states with corresponding probabilities defined by Gaussian Quadrature. Then using the analytical expressions for Φ and s they can be calculate as $\Phi = 0.5502$ and $s = 0.269$. Steady state capital can then be easily calculated by setting $lnk_{t+1} = lnk_t$ and by (1) $lnk^{ss} = -2.964$ and $k^{ss} = 0.0516$. Drawing a random vector of ζ I can then simulate 50000 observations of lnk. Plotting the first 100 clearly shows that k is alternating around the steady state.



3. Krusell-Smith Algorithm

In order to find the solution to the model numerically I first compute theoretical values for $\psi_i(z)$. Using (1) I define everything that is multiplying lnk as $\psi_1(z)$ and everything else as the constant. It follows that

	$z = high$	$z = low$
ψ_0	-1.945	-2.205
ψ_1	α	α

Solving the Household Problem

To solve the model I firstly use the above values as first guesses for $\psi_i(z)$. In addition I create a grid of k with 5 nodes around the steady state value of k. I can then create a grid for k_{t+1} which is defined by the grid of k and equation

(2). It therefore has size 2x5, for each state z and each value in the grid k. I can now solve the household problem. To do so I firstly use the Euler Equation and the two budget constraints. Plugging in yields

$$a_{t+1}(\tau,k,z) = \frac{(1-\tau)w_t - \frac{1}{\beta}\lambda(1-\tau)E\left[\frac{\eta}{R_{t+1}}w_{t+1}\right] - (1-\lambda)E\left[\frac{b}{R_{t+1}\beta}\right]}{1+\frac{1}{\beta}}$$

where $R = 1 + r$

Factor prices are defined by the firm problem as $R_t = \alpha k_t^{\alpha-1}\zeta\varrho$ and $w_t = (1-\alpha)k_t^{\alpha}\zeta$. The pension b is defined by $\tau w\frac{1+\lambda}{1-\lambda}$ and is zero in the first part of the exercise.

To solve this now the interesting part is $E\left[\frac{\eta}{R_{t+1}}w_{t+1}\right]$, because wages and returns share a common shock and are therefore not independent. Including η there are in fact 44=2x2x11 possible values for this expectation expression. Now for each (k,z) this expectation together with the other variables can be calculated and by this $a'(\tau,k,z)$ follows and dividing this by today's income yields $s(\tau,k,z)$. It is furthermore important to add a non negativity constraint for assets, as we cannot take logs of negative savings rates when simulating the model in the next step.

Simulation and Regression

Now I can again simulate an economy for 50000 periods using (1) only this time instead of the theoretical value for s I use the policy function $s(\tau,k,z)$ obtained in the household problem. Again starting in steady state and using the simulation of aggregate shocks from before I calculate a time series of lnk. Finally I regress k' on k seperatly for both states and use the regression coefficients to update the guesses on $\psi_i(z)$.

Results

The code converges in 45 iterations with values for $\psi_i(z)$ close to the theoretical ones from before:

	$z = high$	$z = low$
ψ_0	-2.2257	-2.2944
ψ_1	0.3198	0.3021

The savings rate is close to the theoretical one for the high income state, $s = 2.407$ for k^{ss}, but noticeable smaller for the low income state, i.e. $s = 1.931$ for k^{ss}. The savings rate is furthermore decreasing in k, which makes intuitively sense.

Pension System

Now I introduce a pension system by increasing τ to be equal to 0.1. The algorithm now converges to

	$z = high$	$z = low$
ψ_0	-2.40	-2.367
ψ_1	0.2907	0.2982

Interestingly, the savings rate increases slightly for all states. This is counter intuitive as we should expect some crowding out due to the pension system. In fact when calculating the theoretical value for s like in 2.) it does fall in comparison to no pension.

Welfare Comparison

Having both solution we can look at a welfare comparison. In order to do so I need in addition to the time series of random ζ a time series of random return shocks ϱ. I can then calculate average realized ex ante cohort utilities. For each cohort the life time utility is defined as

$$U_t(\tau) = (1-\tilde{\beta})log(c_t) + \tilde{\beta}E\left[log(c_{t+1})\right]$$

where $\tilde{\beta} = \dfrac{\beta}{1+\beta}$ as in Harenberg and Ludwig (2015). Here c_t and c_{t+1} follow directly from the budget constraints for the current states of k and the shocks in each period of the simulation.

As results I get

	EU
$\tau = 0$	-1.76
$\tau = 0.1$	-1.88

so introducing the pension system actually decreases utility ! In consumption equivalent terms: $g = \dfrac{EU_{tau=0.1} - EU_{tau=0}}{\beta} - 1 = -0.15$. So consumption has to be decreased by 15 percent to make households indifferent between the two policies.

So in this case the distortion through taxes means bigger welfare losses then the gain trough insurance. So the greater risk introduced by adding idiosyncratic and aggregate risk in the same model amplifies the distortion to a greater extend then the insurance effect. One reason might be the low risk aversion using log utility.

Complex Variant of Krusell-Smith Algorithm

Set-up

I now augment the general equilibrium OLG model of project 3 by adding aggregate risk. Except this, the model follows exactly project 3. That means I solve an OLG model with heterogeneous agents who live for 80 years and retire after 45 years. Agents are hit by idiosyncratic income shocks that can take two values and follow a Markov process. They can furthermore safe in assets. They earn income from wages while they work and receive a pension when they retire. The amount of the pension is given by a replacement rate that is set by the government and a tax rate that then balances the governments budget. I will compare the states of $rr = 0$ and $rr = 0.6$ as in project 3. On top of this there is now aggregate fluctuations in output as described in the exercise. Households thus solve:

$$V(j, \eta, k, z, x) = max\left\{u(c) + \beta\xi \sum_{\eta'} \sum_{z'} \pi_\eta(\eta'|\eta)\pi_z(z'|z)V(j+1, \eta', k', z', x')\right\}$$

$$s.t.$$

$$a' = s = a(1+r) + \epsilon(\eta w(1 - tau)) + (1 - \epsilon)wrr(1 - \tau) - c$$

$$x = a(1+r) + \epsilon(\eta w(1 - tau)) + (1 - \epsilon)wrr(1 - \tau)$$

Here ξ is the survival risk and $\epsilon = 1$ if a person is in working age an $\epsilon = 0$ if the prson is retired. Also the utility function is a standard CRRA with $\theta = 2$. In the last part of this project I will also use $\theta = 1$, i.e. log utility, which will result in different findings.

Solving the Model

To solve the model I use the Krusell-Smith Algorithm. That is as before I impose that agents approximate the law of motion by $lnK' = \psi_0^z + \psi_1^z K$. In the following description of the solving of the model all numerical values are for the case of $rr = 0.6$ as this is the more interesting case. The model was solved the same for $rr = 0$

1. Step

Here I cannot build on theoretical values for the first guess of ψ. I therefore tried different initial guesses which had no influence on the solution.

2. Step: Solving the Household Problem

To solve the household problem I started by creating a grid for aggregate capital. Here I used 5 nodes centred around the equilibrium value of K from the model without aggregate risk of project 3, i.e. 7.3. For each of the two aggregate shocks and each value in the grid for K a value for K' is generated. The model can then be solved recursively as in the case of no aggregate risk. The only difference is that the interest rate and wages will now not be constant over time. Expectations of tomorrows income always depend additionally on the current K and z. We thus have 5 state variables (j, η, K, z, x). Solving the model yields policy functions $a'(j, \eta, K, z, x)$ and $c(j, \eta, K, z, x)$ as well as value functions $V(j, \eta, K, z, x)$.

3. Step: Simulation

Using those policy functions I can aggregate and simulate. In order to simulate I first generate 100 observations of z. More observations would in the following go beyond the computational power I can currently use. I then start to simulate the economy, where I use as a starting value $K = 7.3$. Given this and the first draw for the aggregate shock I use the policy function on assets and aggregate to obtain:

$$K' = \int a'(j, \eta, K, z, x)\phi(j, \eta, x)$$

Where I iterate on this equation together with the random draw of z to obtain a sequence of 100 Ks, where obviously in each iteration I have to interpolate to fit capital into the grid.

4. Step: Regression

I can then use these simulated values of K to regress K' on K for each of the realizations of z. The regression coefficients are then used to update the guess of ψ using a weighting between the old guess and the new regression coefficients.

Iteration and convergence

In the following I cannot achieve convergence of the coefficients ψ. In general there is no proof for convergence of the Krusell-Smith Algorithm and therefore also no guarantee. In this case I suspect two different reasons for non convergence. First, I am limited in the number of iteration by computational power. After initializing 10 iteration to have better first guesses for ψ I run the iteration process 20 times. The second reason might be the small number of simulations of K. As there are only 100 values, each regression will have in fact $n < 100$ and one of the two regression will have $n < 50$. Obviously regressions with this little amount of observations can be quite problematic, especially because I now cannot disregard the first few hundred observations to avoid dependence on initial conditions.

To be able to still work with the results I first looked at the realizations of ψ for each iteration. In the following I report the coefficients for every fifths iteration:

$m = 5$	$z = high$	$z = low$
ψ_0	1.93	1.79
ψ_1	0.48	0.24
$m = 10$		
ψ_0	2.37	2.0
ψ_1	0.31	0.24
$m = 15$		
ψ_0	2.09	2.34
ψ_1	0.4	0.078
$m = 20$		
ψ_0	1.99	1.94
ψ_1	0.429	0.198

There is no clear direction visible, but all values rather stay in some neighbourhood. Because of this I will assume that $\psi_i(z)$ in fact alternate around the true values, to be able to make some statement about the wealth effects of the introduction of a pension system.

Welfare Analysis

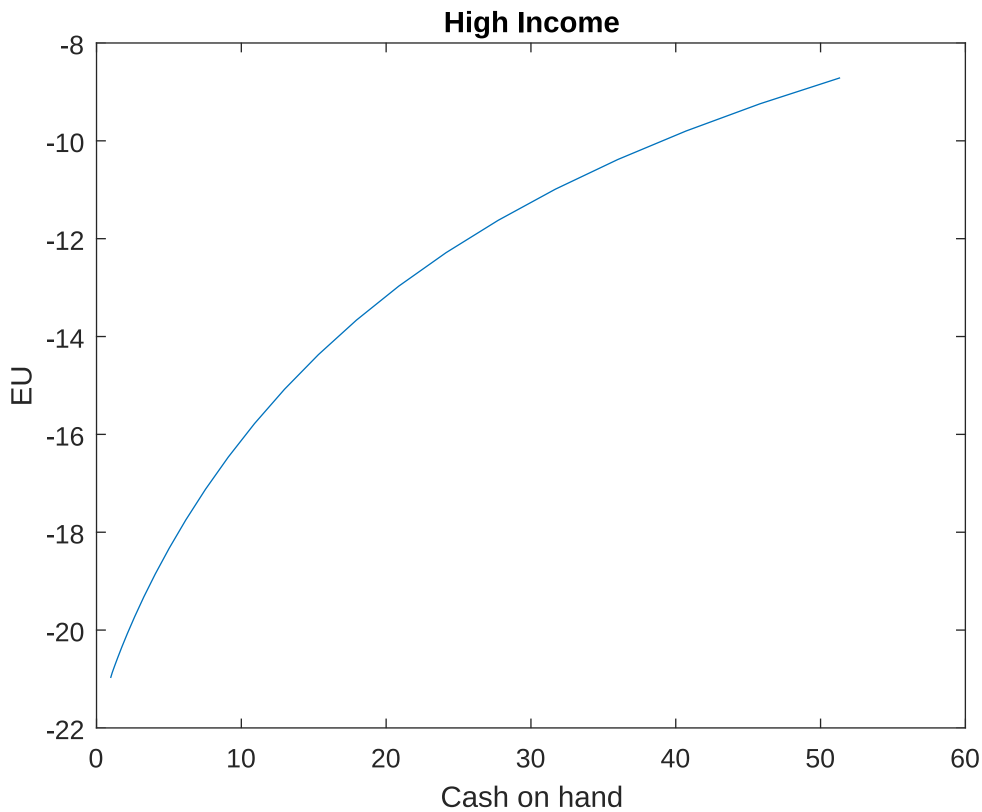
We are again interested in the average life time utility of newborns over our 100 period simulation. This is given by

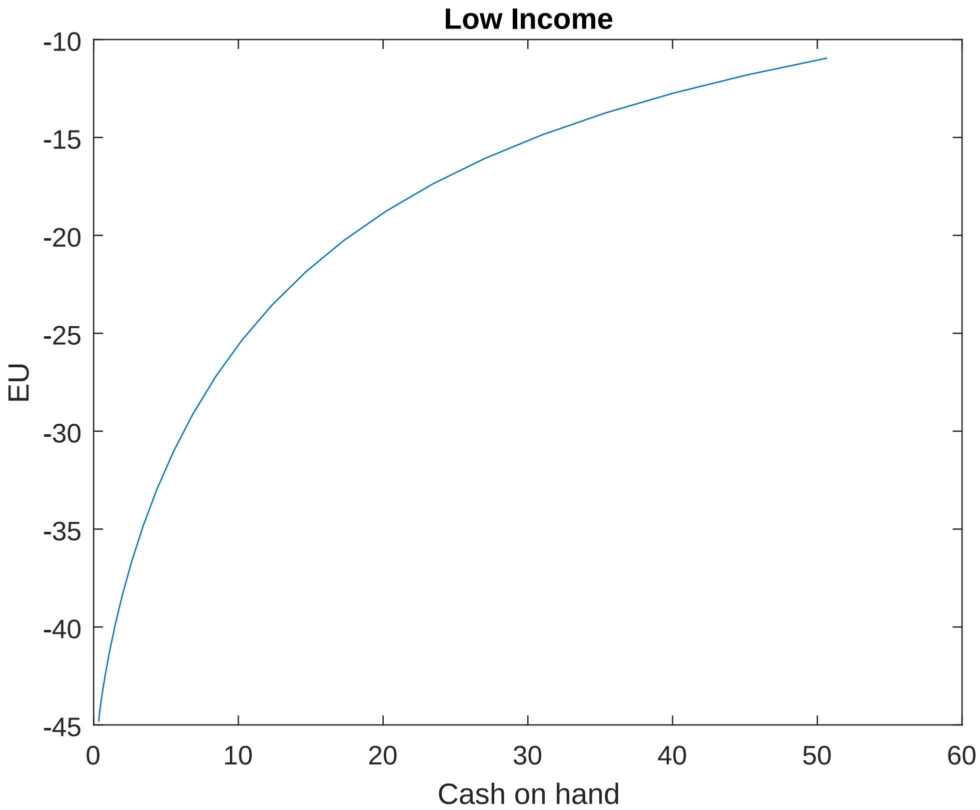
$$EU(\tau, \eta, x) = \frac{1}{100} \sum V(1, \eta, K(t), z(t), x)$$

Since I do not have the true value functions of the model I will approximate $EU(\tau, \eta, x)$.

As explained above I assume that the coefficients in every of the 20 iterations are close to the true ones and at least to some extent alternate around it. As a crude approximation of the true average life time utility I will therefore calculate the average life time utility in every iteration, using the current update of the value function.

In every iteration that means I take the lifetime utility of newborns in every period of the simulation, given the current states K and z for each η and x and take the average. *Important* to note here is, that when summing the value functions we need to interpolate them for every period. Since they are defined for the grid of cash on hand, but the cash on hand grid is endogenous and the value functions therefore live on different grids for each realization of K and z. Now I can take the average of those $EU(\tau, \eta, x)$ for all of the 20 iterations. Again of course interpolating as they are again defined on different grids of cash on hand. This finally yields the following two average life time utilities:



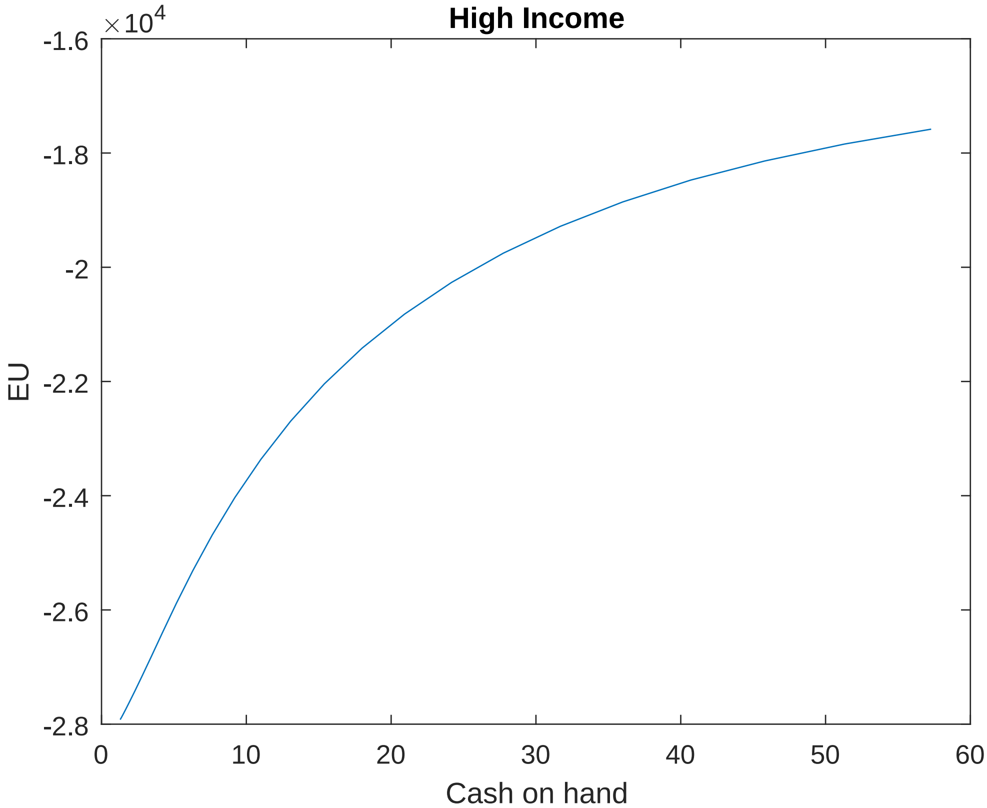


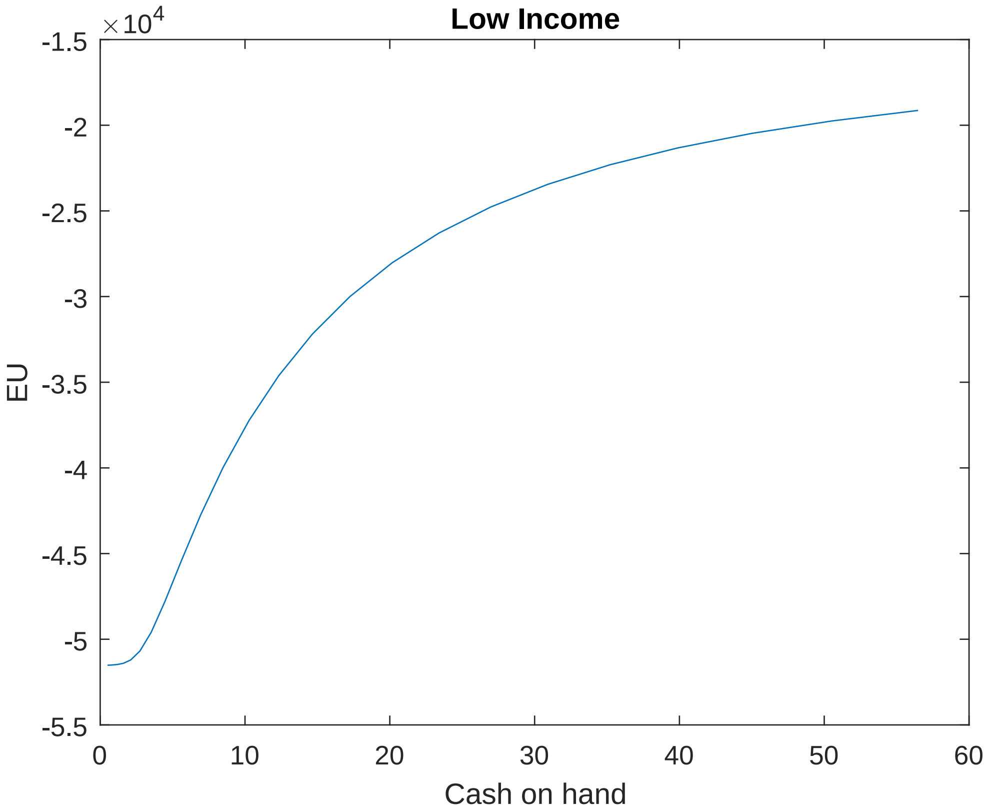
Consumption Equivalent Variation

Redoing the exercise for a replacement rate of zero, i.e. no pension system the algorithm actually converges to following values:

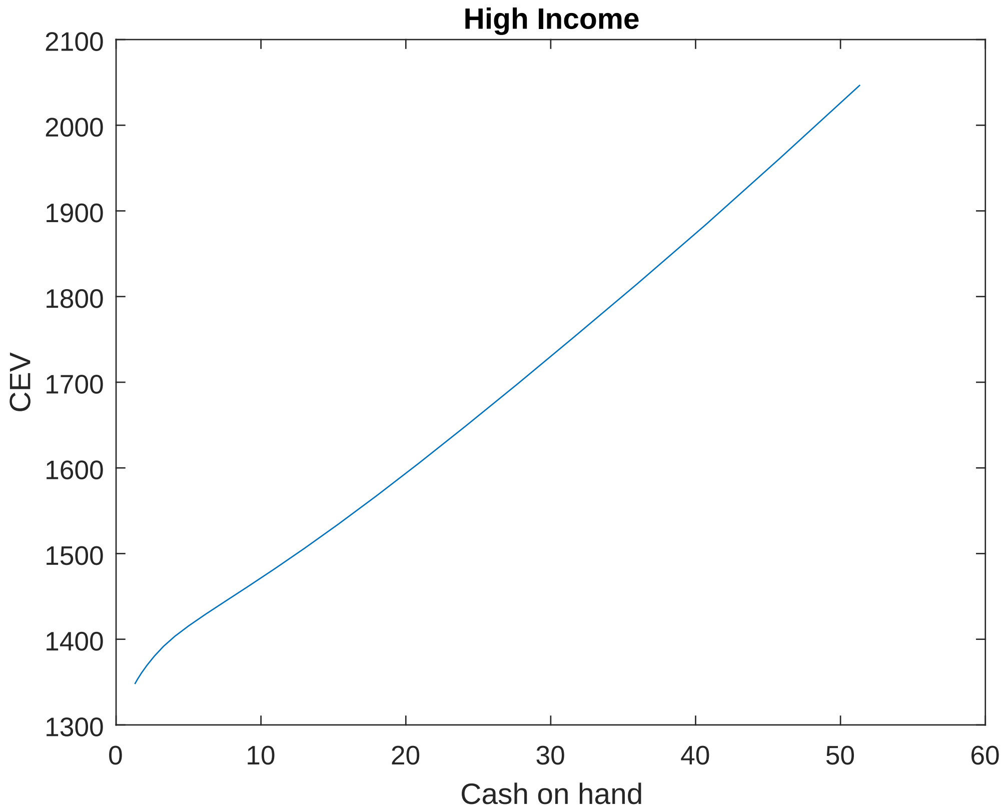
	$z = high$	$z = low$
ψ_0	1.4	12.4
ψ_1	0.8	-0.48

The corresponding EUs are:

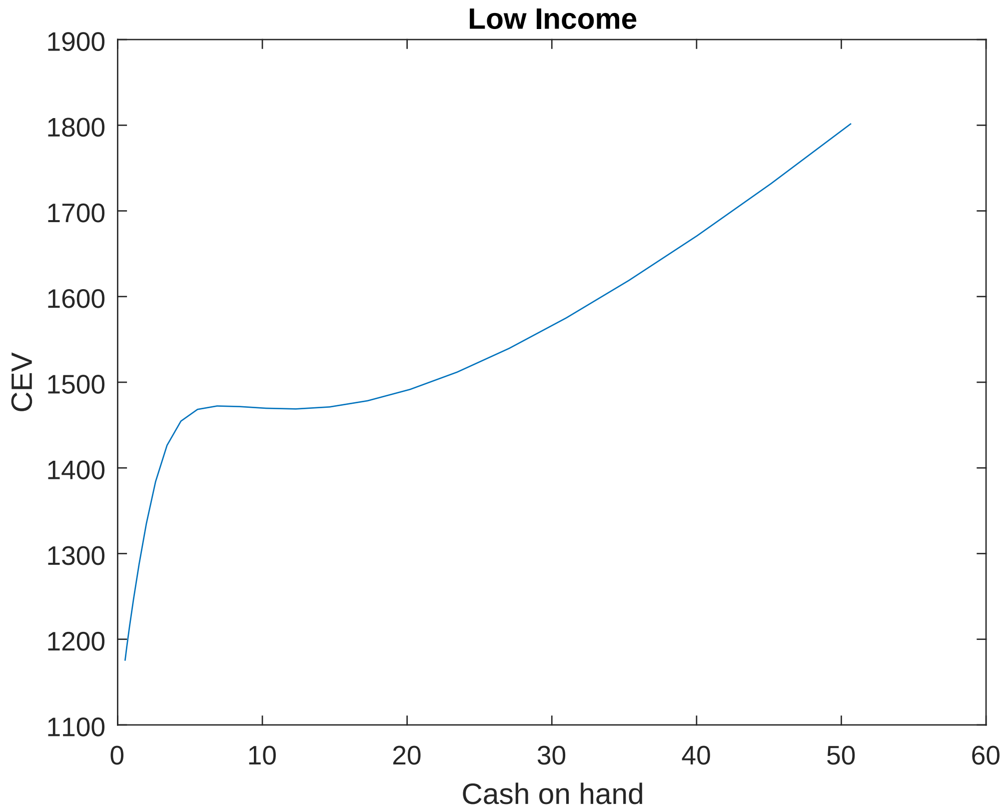




The magnitude alone already makes clear that introducing the pension system will have tremendous gains for the consumer. Formally, I calculate the consumption equivalent variation and plot $g(\eta, x) = \left[\frac{v_T(\eta, x)}{v_0(\eta, x)}\right]^{\frac{1}{1-\theta}}$ for high and low income as follows



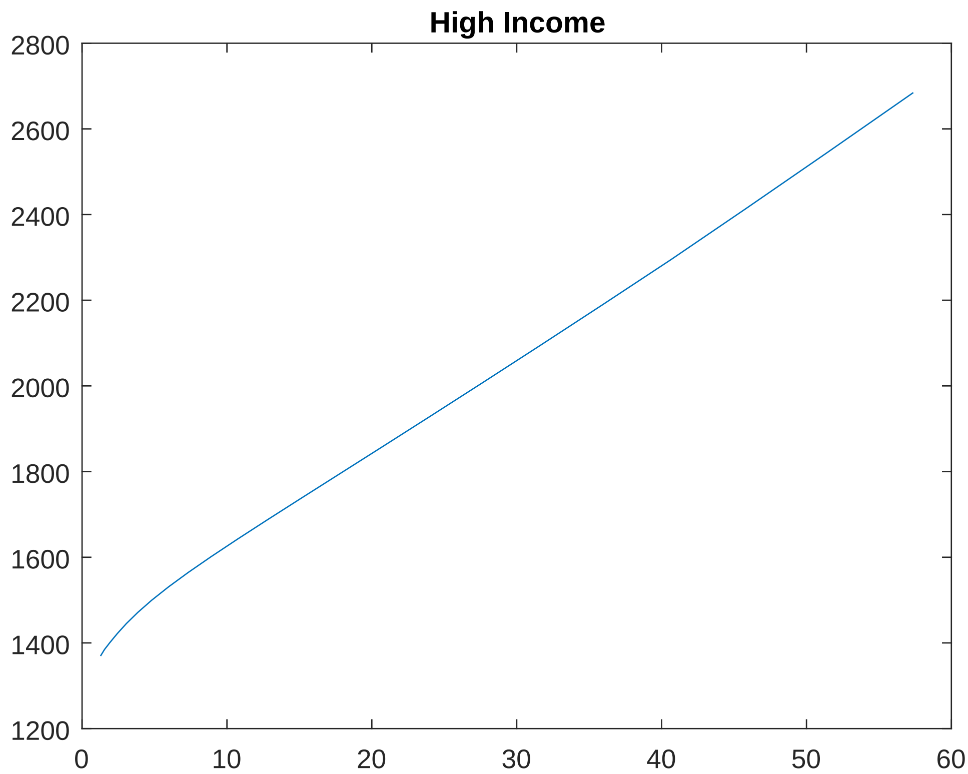
High Income
CEV
2100
2000
1900
1800
1700
1600
1500
1400
1300
0
10
20
30
40
50
60
Cash on hand

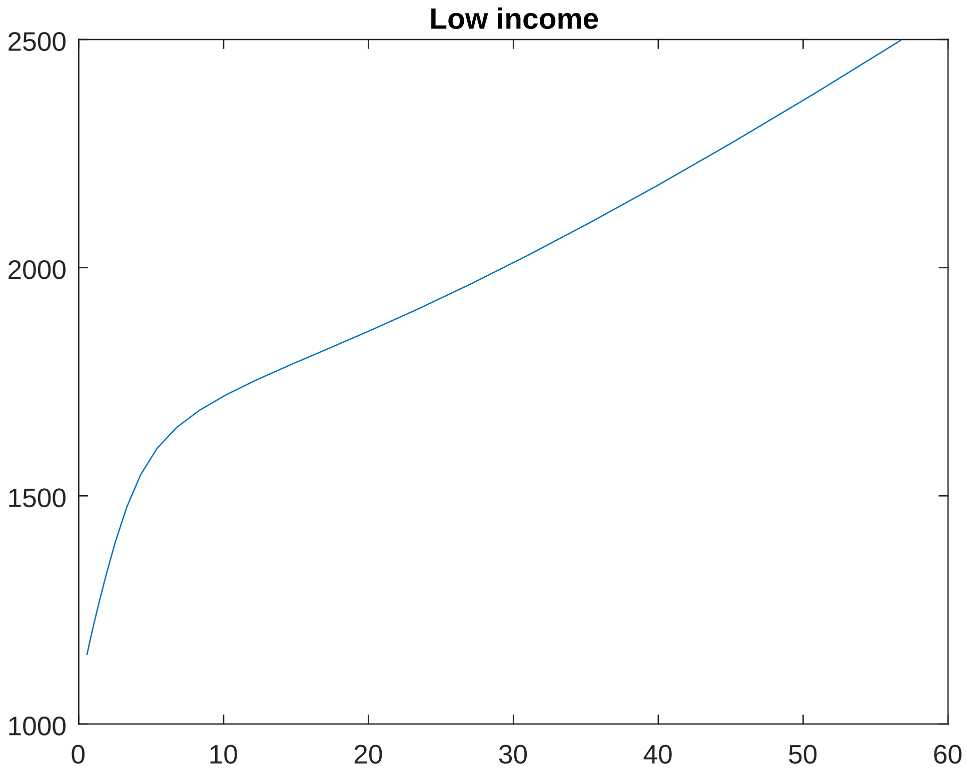


Interpretation and Comparison

Introducing a pension system in this OLG model with CRRA utility and θ increases utility extremely as g is between 1200 and 1800 for low income and 1400 and 2000 for high income. These magnitudes are actually comparable to the ones I found in project 3 without aggregate risk. Similar to there, the gain from the new system is increasing in wealth. As I decomposed in project 3 this is probably again because, while all households gain due to an positive insurance effect, the general equilibrium effect is stronger for richer households. This is because as before, by introducing the pension system, there is less precautionary and life time saving, which is increasing the interest rate, while decreasing wages. Rich households benefit more from the higher interest rate since they generate more of their income through savings.

Now the difference is that households also face aggregate risk, this de-facto increases the risk households face and by that welfare gains from insurance but at the same time welfare losses due to the crowding out of distortionary taxes are amplified as described in Harenberg and Ludwig (2015). The gains obviously still greatly outweigh the losses. Below plotting again the welfare gains I calculated in project 3 without aggregate risk, it however becomes clear, that welfare gains in the current environment are lower over the whole asset distribution. So increase in risk, which arises by the interaction between idiosyncratic and aggregate risk, amplified the losses due to crowding out more than it increased the gains from insurance.





Log Utility

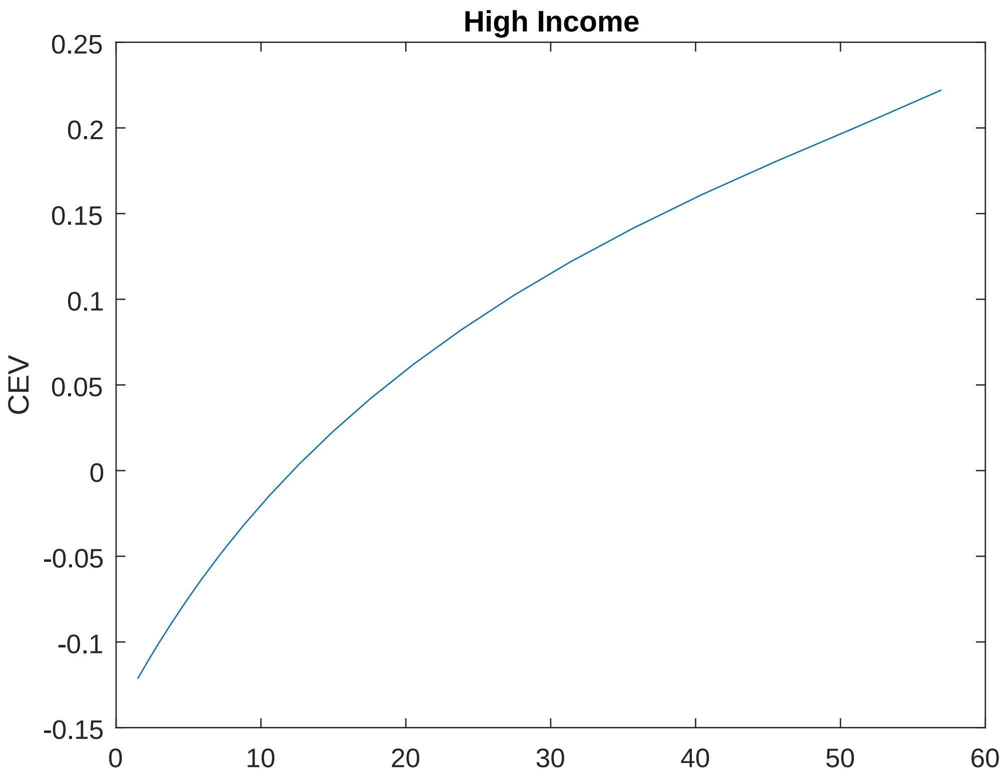
I redo the above exercise for a model with log utility, i.e. $\theta = 1$. After finding the right values for the the grid of aggregate capital the algorithm now converges for both policies. For a replacement rate of zero, i.e. no pension system it converges to

	$z = high$	$z = low$
ψ_0	6.4	9.835
ψ_1	-0.066	-0.9527

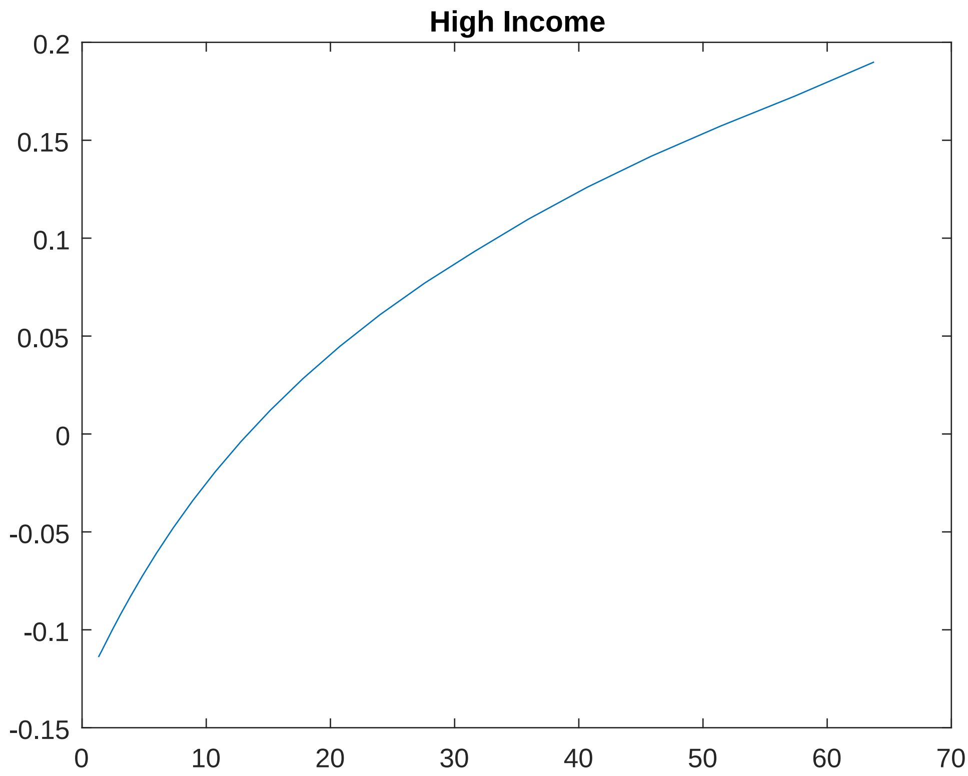
and after introducing the pension system with a replacement rate of 0.6 it converges to

	$z = high$	$z = low$
ψ_0	6.37	5.07
ψ_1	-0.03	-0.1971

Average life time utilities take on the usual concave form for all income states and policies. Looking at the welfare gains now gives an interesting picture. I report the welfare gains in the high state, as the ones in the low state yield the same results.



We can see that rich households gain from the introduction of the pension system. They suffer less from the crowding out effect but rather gain from the still prevailing equilibrium effect of higher interest rates together with the insurance effect. Now poor households actually lose out due to the introduction of the pension system, as the distortions of the taxes and the consequential loss in wage income is not compensated by the insurance effect. Furthermore it is again especially interesting to compare those findings here with the ones from project 3. Below I plotted the consumption equivalence for the high income state and log utility without aggregate risk.



Qualitatively I arrived at the same result, that only rich households benefit from the introduction of the pension system. Quantitatively now both gains and losses are amplified by the interaction of idiosyncratic and aggregate risk. We can see that for poor households losses are bigger with aggregate risk than without while for rich households the gains are also clearly bigger. Without decomposing the effects it is not obvious to explain this result. But it seems to confirm that gains and losses from the pension system are convex functions in risk. Where households already profited from the pension system, adding aggregate risk, exponentially increases those gains, whereas where people suffered from the pension system, the occurred losses also grow with the introduction of aggregate risk.