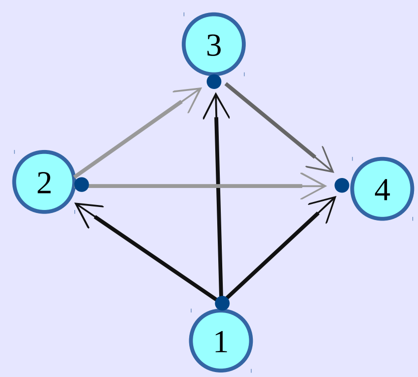
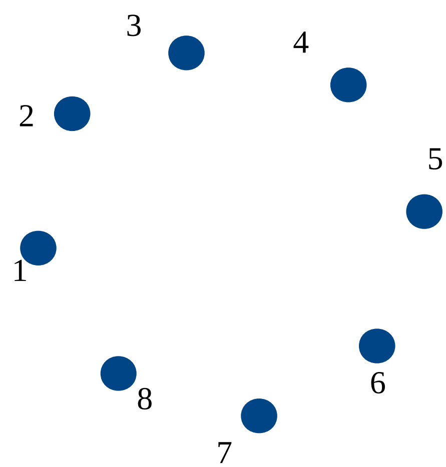
Section 1: Binary Relations

Binary Relations on a set A. reflexive, antisymmetric, & transitive	Let R be a binary relation on a set A. 1. R is said to be **reflexive** if $(a,a) \in R$ for all $a \in A$. In terms of the arrow diagram, this means that every node has a loop. 2. A relation R is called *antisymmetric* if for all $a, b \in A$, if $a \neq b$ and $(a,b) \in R$, then $(b,a) \notin R$. In terms of the arrow diagram, this means that arrow only goes in one direction. 3. A relation R is called *transitive* if whenever $(a,b) \in R$ and $(b,c) \in R$, it must also be the case that $(a,c) \in R$. In terms of the arrow diagram, this means that whenever you can follow two arrows to get from node *a* to node *c*, you can also get there along a single arrow.
Example relation R_1 on the set {1, 2, 3, 4} with the rule "$(x,y) \in R_1 \text{if } x \leq y$"	
Irreflexive	A relation R on A is *irreflexive* if for all $a \in A$, $(a,a) \notin R$. On an arrow diagram, this means no loops. A *strict partial ordering* on the set A is a relation R on A that is transitive, antisymmetric, and irreflexive.

1. Complete the arrow diagram for each of the relations on A = { 1, 2, 3, 4, 5, 6, 7, 8 }, and decide if it has any <u>reflexive</u>, <u>antisymmetric</u>, or <u>transitive</u> properties. For each property that a relation does not have, illustrate this failure with a specific example.

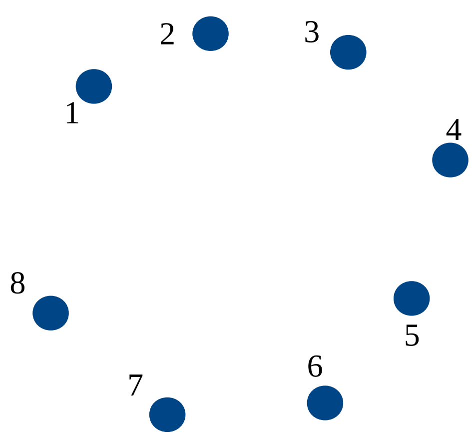
a. R_1 **= { (1, 1), (1, 2), (1, 4), (1, 8), (2, 2), (2, 4), (2, 8), (3, 3), (3, 6), (4, 4), (4, 8), (5, 5), (6, 6), (7, 7), (8, 8) }** **(___/1)**

Is this transitive, antisymmetric, and/or reflexive?



c. R_3 **= { (1, 1), (1, 3), (1, 5), (1, 7), (2, 2), (2, 4), (2, 8), (3, 3), (3, 5), (3, 7), (4, 2), (4, 4), (4, 8), (5, 3), (5, 7), (6, 6), (6, 8), (8, 2), (8, 4), (8, 8) }** **(___/1)**

Is this transitive, antisymmetric, and/or reflexive?



Recap

- **Reflexive:** $(a,a) \in R$ **for all** $a \in A$
- **Irreflexive:** $(a,a) \notin R$ **for all** $a \in A$
- **Antisymmetric: for all** $a, b \in A$ **, if** $a \neq b$ **and** $(a,b) \in R$ **then** $(b,a) \notin R$
- **Transitive: If** $(a,b) \in R$ **and** $(b,c) \in R$ **then** $(a,c) \in R$

2. For the following relation on $\mathbb{Z}$ $R_1 = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a+b \text{ is even}\}$

a. decide if the relation is reflexive or irreflexive. If it does not have one (or both) of these properties, give a specific example to illustrate this. **(___/1)**

Hint: *a* and *b* are both in the set of integers. We are checking to see if the result of (a, a) is always in the relation R, so if you plug *(a, a)* into the, is what you get out still “is even”?

b. decide if the relation is antisymmetric. If it is not, give a specific example to illustrate this. **(___/1)**

Hint: Find some *(a, b)* and *(b, a)* that are both in the relation. Remember that *a* and *b* are both in the set of integers.

3. Let *P* be the set of people who have ever lived. For each of the following relations on *P*, decide if it is reflexive, irreflexive, transitive, or antisymmetric – each can satisfy more than one of these properties. Give explanations on how you decided each of these.

a. $R_1 = \{(\alpha, \beta) \in P \times P : \alpha \text{ is a child of } \beta\}$ **(___/1)**

Reflexive: Is $(a, a) \in P$ valid?

Irreflexive: Is $(a, a) \notin P$ valid?

Transitive: Is there some $(a, b) \in P$ and $(b, c) \in P$?

Antisymmetric: Is $(a, b) \in P$ and $(b, a) \notin P$ valid?

b. $R_2 = \{(\alpha, \beta) \in P \times P : \alpha \text{ is a descendant of } \beta\}$ **(___/1)**

Reflexive: Is $(a, a) \in P$ valid?

Irreflexive: Is $(a, a) \notin P$ valid?

Transitive: Is there some $(a, b) \in P$ and $(b, c) \in P$?

Antisymmetric: Is $(a, b) \in P$ and $(b, a) \notin P$ valid?