Chapter 4.3

Properties of Functions and Set Cardinality

Chapter 4.4

Properties of Relations

...

Written by Rachel J. Morris, last updated 4/10/17

Properties of Functions

Onto:

A function *f* is *onto* if everything in the codomain really is an output of *f*. That is, for every element *y* in the codomain, there must be (at least one) x in the domain where f(x) = y.

One-to-one:

A function *f* is *one-to-one* if nothing in the codomain is an output via two different inputs. That is, for every choice of different elements x_1 and x_2 in the domain, $f\left(x_1\right)$ and $f\left(x_2\right)$ must be different.

Properties of Functions

Invertible:

A function *f* is a *one-to-one correspondence* if it is both *one-to-one* and *onto*. This is equivalent to saying that *f is invertible*.

Another way of stating this is:
The function $f: A \rightarrow B$ is *invertible* if there is a function $f^{-1}: B \rightarrow A$ such that $f(x) = y$ if and only if $f^{-1}(y) = x$.

The notation is read as "*f* inverse" and the symmetry of the definition means that $(f^{-1})^{-1} = f$.

Properties of Functions

In diagramming terms...

- A function is *onto* if every point in the codomain has an arrow ending at that point.
- A function is *one-to-one* if no point in the codomain has two or more arrows ending at a point.
- A function is a *one-to-one correspondence (invertible)* if every point in the codomain has *exactly one arrow* ending at that point.

Properties of Functions

Definition of an invertible function…
A function $f : A \rightarrow B$ is invertible if there is a function $f^{-1} : B \rightarrow A$ such that $f(x) = y$, if and only if $f^{-1}(y) = x$.

And from definition of a function…
The function f associates with each input in A one and only one output in B.

Properties of Functions

Definition of an invertible function…
A function $f : A \to B$ is invertible if there is a function $f^{-1} : B \to A$ such that $f(x) = y$, if and only if $f^{-1}(y) = x$.

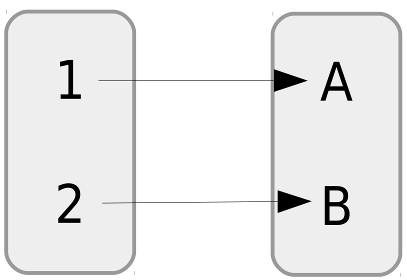
And from definition of a function…
The function f associates with each input in A one and only one output in B.

Properties of Functions

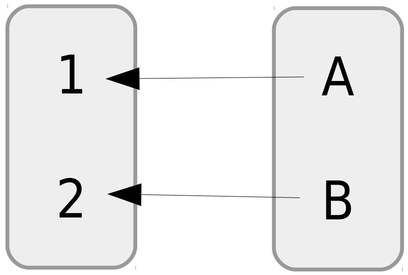
If a function is invertible (i.e., it is both one-to-one *and* onto), then its inverse is also a valid function.

Properties of Functions

$$f : A \to B$$



$$f^{-1} : B \to A$$



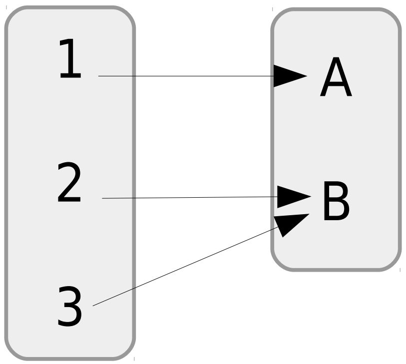
Onto?
Yes, every point in the codomain has an arrow leading to it.

One-to-one?
Yes, there is no point in the codomain that is pointed to from two or more sources in the domain.

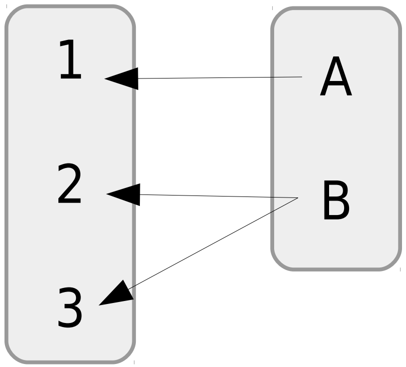
Invertible?
Yes, as it is both onto and one-to-one.

Properties of Functions

$f : A \rightarrow B$



$f^{-1} : B \rightarrow A$



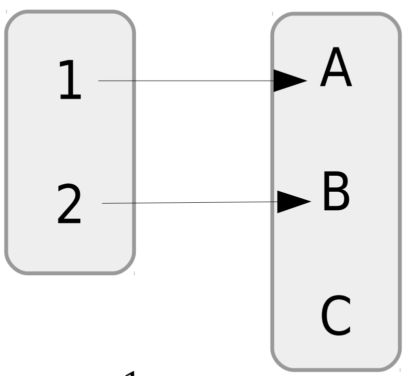
Onto?
Yes, every point in the codomain has an arrow leading to it.

One-to-one?
No, “B” is pointed at from two sources.

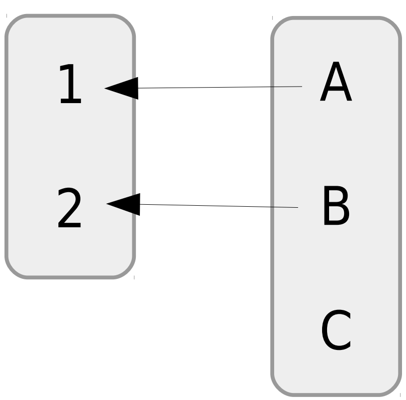
Invertible?
No, the inverse of *f* is not a function.

Properties of Functions

$f : A \to B$



$f^{-1} : B \to A$



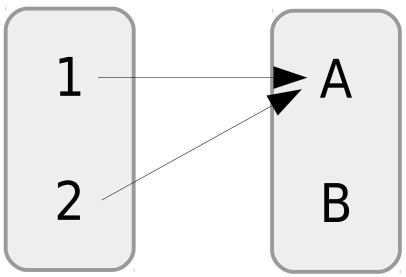
Onto?
No, "C" has nothing pointing to it.

One-to-one?
Yes, there is no point in the codomain that is pointed to from two or more sources in the domain.

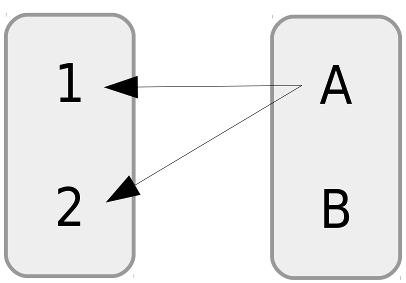
Invertible?
No, the inverse of f is not a function.

Properties of Functions

$f : A \rightarrow B$



$f^{-1} : B \rightarrow A$



Onto?
No, “B” in the codomain does not have an arrow pointing to it.

One-to-one?
No, “A” is being pointed to by two sources.

Invertible?
No, the inverse of *f* is not a function.

Properties of Functions

Without diagrams…

Given the function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \text{ with } f(x) = x^2 + x + 1$$

Why is this function **not onto?** *(for every element y in the codomain, there must be (at least one) x in the domain where f(x) = y.)*

Can we find some *x* that gives us an *f(x)* not in the set of all real numbers?

Properties of Functions

Without diagrams...

Given the function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \text{ with } f(x) = x^2 + x + 1$$

Why is this function **not onto?** *(for every element y in the codomain, there must be (at least one) x in the domain where f(x) = y.)*

Can we find some *x* that gives us an *f(x)* not in the set of all real numbers?

How about f(x) = 0? We can find the roots with the quadratic formula...

$$x = (-1)^{2/3}$$

$$\boxed{x = -\sqrt[3]{-1}}$$

This isn't in the set!

Properties of Functions

Without diagrams...

Given the function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \text{ with } f(x) = x^2 + x + 1$$

Why is this function **not onto?** *(for every element y in the codomain, there must be (at least one) x in the domain where f(x) = y.)*

Therefore, this function is **not onto**, and not invertible.

$$x = (-1)^{2/3}$$

$$x = -\sqrt[3]{-1}$$

Properties of Functions

Without diagrams...

Given the function:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \text{ with } f(x) = x^2$$

Why is this function **not one-to-one?** *(A function f is one-to-one if nothing in the codomain is an output via two different inputs.)*

Can we get the same *f(x)* for two different *x* values?

Properties of Functions

Without diagrams...

Given the function:

$$f:\mathbb{R}\to\mathbb{R}\text{, with } f(x)=x^2$$

Why is this function **not one-to-one?** *(A function f is one-to-one if nothing in the codomain is an output via two different inputs.)*

Can we get the same *f(x)* for two different *x* values?

$$f(2)=4$$

$$f(-2)=4$$

Therefore it is not one-to-one.

Properties of Functions

Without diagrams...

Given the function:

$$h:\{a,b,c\}\rightarrow\{1,2,3\}\text{ with the rule }\{(a,1),(b,2),(c,3)\}$$

Is it onto?

(for every element y in the codomain, there must be (at least one) x in the domain where f(x) = y.)

Properties of Functions

Without diagrams...

Given the function:

$$h:\{a,b,c\}\rightarrow\{1,2,3\}\text{ with the rule }\{(a,1),(b,2),(c,3)\}$$

Is it onto?

(for every element y in the codomain, there must be (at least one) x in the domain where f(x) = y.)

Yes!

Is it one-to-one?

(A function f is one-to-one if nothing in the codomain is an output via two different inputs.)

Properties of Functions

Without diagrams…

Given the function:

$$h:\{a,b,c\}\rightarrow\{1,2,3\}\text{ with the rule }\{(a,1),(b,2),(c,3)\}$$

Is it onto?

(for every element y in the codomain, there must be (at least one) x in the domain where f(x) = y.)

Yes!

Is it one-to-one?

(A function f is one-to-one if nothing in the codomain is an output via two different inputs.)

Yes!

So is it invertible?

Properties of Functions

Without diagrams...

Given the function:

$h:\{a,b,c\} \rightarrow \{1,2,3\}$ with the rule $\{(a,1),(b,2),(c,3)\}$

Onto: Yes!

One-to-one: Yes!

Invertible: Yes!

$h^{-1}:\{1,2,3\} \rightarrow \{a,b,c\}$ with the rule $\{(1,a),(2,b),(3,c)\}$

Properties of Relations

Let R be a binary relation on set *A*.

<u>Reflexive:</u> *R* is reflexive if $(a,a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

<u>Antisymmetric:</u> *R* is antisymmetric if for all $a,b \in A$
If $a \neq b$ and $(a,b) \in R$, then $(b,a) \notin R$
In an arrow diagram – arrows only go in one direction.

<u>Transitive:</u> *R* is transitive if whenever $(a,b) \in R$ and $(b,c) \in R$
It must also be the case that $(a,c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

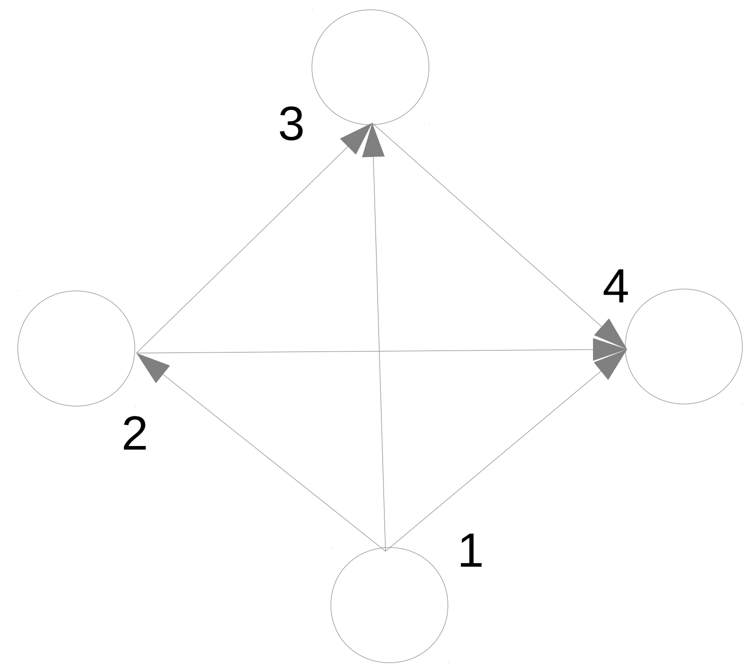
Properties of Relations

Let R be a binary relation on set *A*.

<u>**Irreflexive:**</u> *R* is irreflexive if for all $a \in A, (a, a) \notin R$
In an arrow diagram – there are **no loops**.

Properties of Relations

Example relation *R* on the set {1, 2, 3, 4}, with the rule $(x, y) \in R, \text{if } x \leq y$

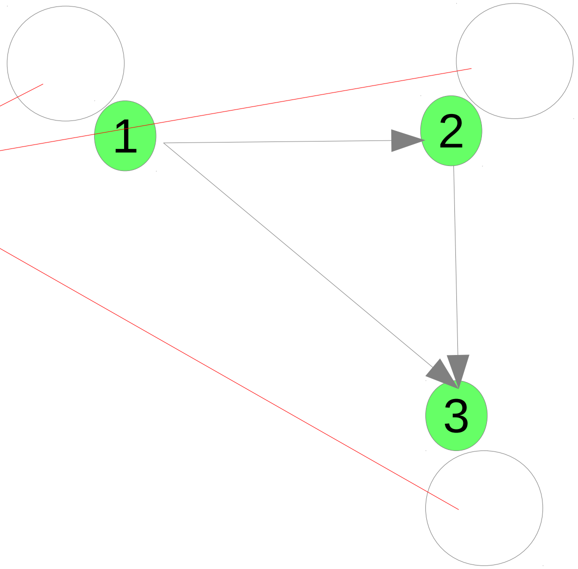


Properties of Relations

Relation: $R_1 = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

<u>Reflexive:</u> *R* is reflexive if every node has a loop to itself.

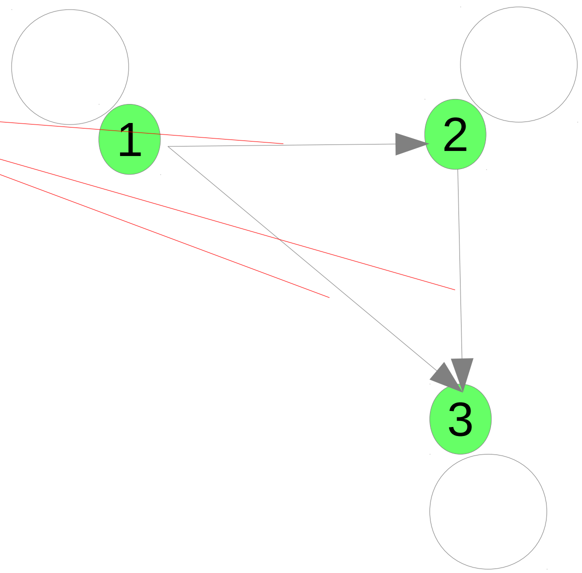
So this relation is reflexive.



Properties of Relations

Relation: $R_1 = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

Antisymmetric: *R* is antisymmetric if arrows only go in one direction.

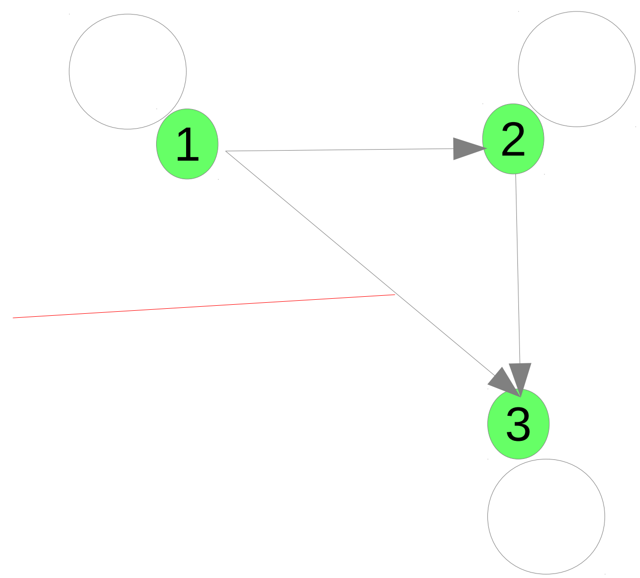


So this relation is antisymmetric.

Properties of Relations

Relation: $R_1 = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$

Transitive: *R* is transitive if you can follow two arrows from ***a*** to ***c***, and you can also get from ***a*** to ***c*** in a single arrow.



So this relation is transitive.

Properties of Relations

Without a diagram...

A = { 1, 2, 3, 4 },
R = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) }

Is this transitive?

Transitive: R is transitive if whenever $(a,b)\in R$ and $(b,c)\in R$
It must also be the case that $(a,c)\in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

Without a diagram...

A = { 1, 2, 3, 4 },
R = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) }

Is this transitive?

(1, 2) exists, (2, 3) exists, and (1, 3) exists.
(1, 2) exists, (2, 4) exists, and (1, 4) exists.
(1, 3) exists, (3, 4) exists, and (1, 4) exists.
… and so on …

Transitive: *R* is transitive if whenever $(a,b)\in R$ and $(b,c)\in R$
It must also be the case that $(a,c)\in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

Without a diagram...

A = { 1, 2, 3, 4 },
R = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) }

Is it reflexive? Or irreflexive?

Irreflexive: *R* is irreflexive if for all $a \in A, (a, a) \notin R$
In an arrow diagram – there are **no loops**.

Reflexive: *R* is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Properties of Relations

Without a diagram...

A = { 1, 2, 3, 4 },
R = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) }

Is it reflexive? Or irreflexive?

None of these nodes point back to themselves, so it is **irreflexive**.

Irreflexive: *R* is irreflexive if for all $a \in A, (a, a) \notin R$
In an arrow diagram – there are **no loops**.

Reflexive: *R* is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Properties of Relations

Without a diagram...

A = { 1, 2, 3, 4 },
R = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) }

Is it antisymmetric?

Antisymmetric: *R* is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Properties of Relations

Without a diagram...

A = { 1, 2, 3, 4 },
R = { (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) }

Is it antisymmetric?

Each node only goes one way: (1, 2) but no (2, 1).
So it is antisymmetric.

Antisymmetric: *R* is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Properties of Relations

For the following relation on the set of integers:

$$R_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a + b \text{ is odd}\}$$

Is this reflexive or irreflexive?

Reflexive: R is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: R is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: R is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$
It must also be the case that $(a, c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

For the following relation on the set of integers:

$$R_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a + b \text{ is odd}\}$$

Is this reflexive or irreflexive?

If we plug in *a* as *(a, a)*?
a + a = 2a, which is **not** odd. A node cannot loop back on itself, so it is irreflexive.

Reflexive: *R* is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: *R* is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: *R* is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$
It must also be the case that $(a, c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

For the following relation on the set of integers:

$$R_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a + b \text{ is odd}\}$$

Is this antisymmetric? Can we find some *a, b* and *b, a* that result in the same values?

Reflexive: *R* is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: *R* is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: *R* is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$
It must also be the case that $(a, c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

For the following relation on the set of integers:

$$R_1 = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a + b \text{ is odd}\}$$

Is this antisymmetric? Can we find some *a, b* and *b, a* that result in the same values?

(2, 3) = 5, and (3, 2) = 5, so it is **not** antisymmetric.

Reflexive: *R* is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: *R* is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: *R* is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$
It must also be the case that $(a, c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

For the following relation:

$$R_1 = \{(a, b) \in C \times C : \text{There is a direct flight from } a \text{ to } b\}$$

Is this reflexive?

Reflexive: R is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: R is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: R is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$
It must also be the case that $(a, c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

For the following relation:

$$R_1 = \{(a,b) \in C \times C : \text{There is a direct flight from } a \text{ to } b\}$$

Is this reflexive?
No, you generally cannot get a flight from location "a" back to location "a" (at least not without another stop).

Reflexive: R is reflexive if $(a,a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: R is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a,b) \in R$, then $(b,a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: R is transitive if whenever $(a,b) \in R$ and $(b,c) \in R$
It must also be the case that $(a,c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

For the following relation:

$$R_1 = \{(a, b) \in C \times C : \text{There is a direct flight from } a \text{ to } b\}$$

Is this antisymmetric?

Reflexive: R is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: R is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: R is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$
It must also be the case that $(a, c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

For the following relation:

$$R_1 = \{(a, b) \in C \times C : \text{There is a direct flight from } a \text{ to } b\}$$

Is this antisymmetric?
No, you can get a flight from “a” to “b” and from “b” to “a”.

Reflexive: R is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: R is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: R is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$
It must also be the case that $(a, c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

For the following relation:

$$R_1 = \{(a, b) \in C \times C : \text{There is a direct flight from } a \text{ to } b\}$$

Is this transitive?
For this set, you can have a direct flight from “a” to “b”, and “b” to “c”, but not for “a” to “c”, as it is not defined in the set.

Reflexive: R is reflexive if $(a, a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: R is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a, b) \in R$, then $(b, a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: R is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$
It must also be the case that $(a, c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.

Properties of Relations

For the following relation:

$$R_1 = \{(a,b) \in C \times C : \text{There is a direct flight from } a \text{ to } b\}$$

Is this antisymmetric?

Reflexive: R is reflexive if $(a,a) \in R$ for all $a \in A$
In an arrow diagram – every node has a loop to itself.

Antisymmetric: R is antisymmetric if for all $a, b \in A$
If $a \neq b$ and $(a,b) \in R$, then $(b,a) \notin R$
In an arrow diagram – arrows only go in one direction.

Transitive: R is transitive if whenever $(a,b) \in R$ and $(b,c) \in R$
It must also be the case that $(a,c) \in R$
In an arrow diagram – If you can follow two arrows from ***a*** to ***c***, you can also get from ***a*** to ***c*** in a single arrow.