

# Test Flight Problem Set Solutions

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- A1. The statement is false. If  $n \geq 2$ , then for any  $m$ , we have that  $3m + 5n \geq 13$  (since  $3m + 5n \geq 3m + 10 \geq 13$ ). Thus, the only way to find such a solution for  $n$  in the natural numbers would be when  $n = 1$ . Substituting, we have  $3m + 5 \cdot 1 = 3m + 5 = 12$ , or  $3m = 7$ . But since there is no natural number  $m$  satisfying this equation, we have proved the result. QED
- A2. The statement is true. Without loss of generality, we may assume the consecutive five integers may be written in the form:  $n - 2, n - 1, n, n + 1, n + 2$ . If we sum these integers, we have  $5n$ , which is divisible by 5. Hence, we have proved the result. QED
- A3. The statement is true. We may rewrite  $n^2 + n + 1$ , as  $n \cdot (n + 1) + 1$ . If  $n$  is even, then  $n + 1$  is odd. If  $n$  is odd, then  $n + 1$  is even. In either case,  $n \cdot (n + 1)$  is even because the product of an even and odd number is even. Hence, we may write  $n \cdot (n + 1) + 1$  as  $2k + 1$ , which is odd. Hence, we have proved the result. QED
- A4. Recall from the remainder theorem: if  $a, b$  are integers with  $b > 0$ , then there exist unique integers  $q, r$  such that  $a = bq + r$  and  $0 \leq r < b$ . If we let  $b = 4$  (and  $n = q$ ), then we have the statement that  $a = 4n + r$  with  $0 \leq r < 4$ . If  $r = 0$  or  $2$ , then we have  $a = 4n$  or  $a = 4n + 2$ , which are even natural numbers. If  $r = 1$  or  $3$ , we have that  $a = 4n + 1$  or  $a = 4n + 3$ , which are odd natural numbers. Since  $a$  is any odd natural number, satisfying the antecedent, we have that it must be of one of the following forms  $a = 4n + 1$  or  $a = 4n + 3$ . Hence, we have proved the result. QED
- A5. Recall from the remainder theorem: if  $a, b$  are integers with  $b > 0$ , then there exist unique integers  $q, r$  such that  $a = bq + r$  and  $0 \leq r < b$ . If we take  $b = 3$ , then we have the statement that  $a = 3q + r$  with  $0 \leq r < 3$ . Expanding out (and letting  $n = a$ ), we have that  $n = 3q$ , or  $n = 3q + 1$ , or  $n = 3q + 2$ . Let's now write  $n, n + 2$ , and  $n + 4$  in these forms:  $n$  is either  $3q, 3q + 1$ , or  $3q + 2$ .  $n + 2$  is either  $3q + 2, 3q + 3$ , or  $3q + 4$ .  $n + 4$  is either  $3q + 4, 3q + 5$ , or  $3q + 6$ . But we see that in each of the forms, there exists an element which

is divisible by 3 i.e. if  $n$ ,  $3|3q$  and if  $n+2$ ,  $3|(3q+3)$ , and if  $n+4$ ,  $3|(3q+6)$ . Hence, we have proved the result. QED

- A6. We prove this by contradiction i.e. assume there exists  $n > 3$ , such that  $n$ ,  $n+2$ , and  $n+4$  are prime. But from the proof of #5, we have just shown that one of  $n$ ,  $n+2$ ,  $n+4$  must be divisible by 3. And since  $n > 3$ , 3 is not one of the primes. Thus, one of  $n$ ,  $n+2$ ,  $n+4$  is not prime. Hence, we have proved the result. QED
- A7. Let the sum,  $2+2^2+2^3+\dots+2^n$ , be denoted by  $S$ . Multiplying by 2, we have that  $2S = 2^2+2^3+\dots+2^{n+1}$ . Subtracting  $S$  from  $2S$ , we have that  $S = 2^{n+1} - 2$ , which was to be proved. QED
- A8. By the assumption, we have that for any given  $\epsilon > 0$ , there exists an  $n$  where for all  $m \geq n$ ,  $|a_m - L| < \epsilon$ . The statement that  $Ma_n$  tends to  $ML$  as  $n$  tends to infinity is equivalent to saying that for any given  $\epsilon_1 > 0$ , there exists an  $n$  where all  $m \geq n$ ,  $|Ma_m - ML| < \epsilon_1$ . This simplifies to  $|M|a_m - L| \iff M|a_m - L| < \epsilon_1 \iff |a_m - L| < \frac{\epsilon_1}{M}$ . This will be true if we take  $\epsilon_1/M = \epsilon$ , and find such an  $n$ . Since we can always do this, we have proved the result. QED
- A9. Let  $A_n = (0, 1/n)$ . We have that  $A_n$  is a subset of  $A_1$  since  $(0, 1/n)$  is a subset of  $(0, 1)$ . Suppose that  $x$  is an element of  $(0, 1)$ . We can always find a natural number  $m$  such that  $1/m < x$ . But that means that  $x$  is not an element of  $A_m$ . Hence,  $x$  is not an element of the intersection of  $A_n$  where  $n$  is a natural number. Since we can always find this number  $m$ , we must necessarily have that intersection of  $A_n$  is empty. Hence, we have proved the result. QED
- A10. Let  $A_n = [0, 1/n)$ . We may write this set as  $0 \cup B_n$ , where  $B_n = (0, 1/n)$ . The intersection of  $A_n$  for  $n$  in the natural numbers may thus be written as  $0 \cup (\cap B_n)$ . But since we have proved from #9 that intersection of all  $B_n$  is the empty set, we have that  $0 \cup \emptyset = \{0\}$ . Hence, we have proved the result. QED