Chapter 3

Differential Equations

Diff EQ $\xrightarrow[\text{Derived from}]{}$ Physical Law

- In Controls, the diff EQ is identical in both physical & electrical systems.

Modeling dynamic systems (A simple strategy)

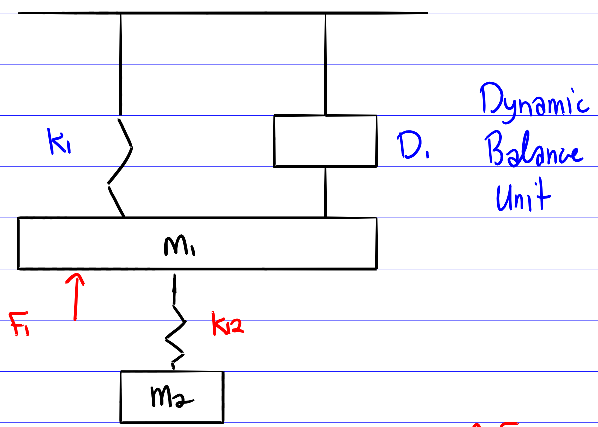
Things that balance in systems: Force, charge, current, torque

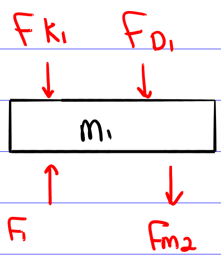
1) Basic physical Laws (Find the ones you need)

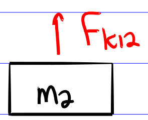
Electrical	Mechanical
a) KVL	a) $F = ma$
b) Ohm's Law	b) $M = I\dot{\omega}$
c) KCL	

2) Write down what you know

Example







At gravitational equilibrium $x_1 = x_2 = 0$

$m_1 a_1 = F_1 - F_{K1} - F_{D1} - F_{K12}$

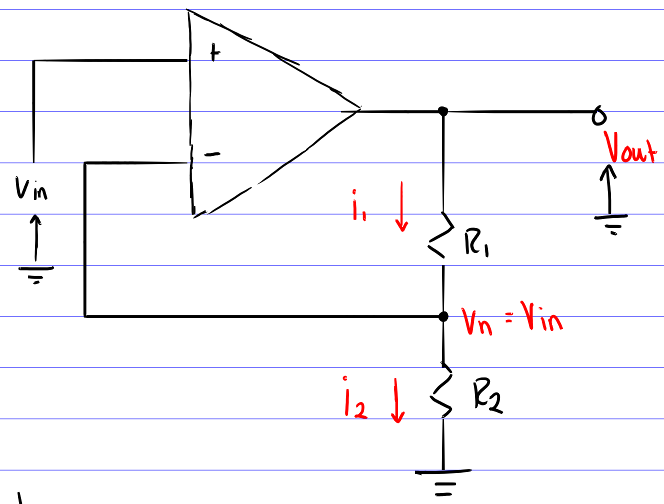
$m_2 a_2 = F_{K21}$

3) Figure out what is unknown

F_{K1}, F_{D1}, F_{K12}

$F_{K1} = k x_1$
$F_{D1} = D_1 \dot{x}_1$
$F_{K12} = K_{12}(x_1 - x_2)$
$F_{K21} = K_{12}(x_2 - x_1)$

Example



1) Ohm's Law
KCL
KVL

2) What do we know

- An omp amp in negative feedback will do whatever it can to keep $V_+ = V_-$.
- No current flows into (+) or (-) terminals.

Because it is difficult to define a loop in this system, lets use KCL.

$$i_1 = \frac{V_{out} - V_{in}}{R_1}$$ This is Ohm's Law

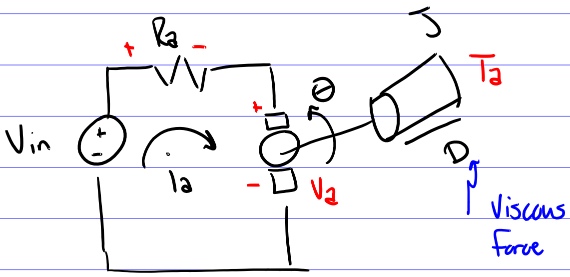
$$i_1 = i_2 = \frac{V_{in}}{R_2}$$

Do some manipulation

$$\frac{V_{out}}{V_{in}} = 1 + \frac{R_1}{R_2}$$

Example

Permanent Magnet DC-Motor (Brushed)



1) KVL (We have a nice loop)

$$J\ddot{\theta} = T$$

KVL

$$-V_{in} + V_R + V_a = 0$$

$V_R \rightarrow i_a R_a$

Rotational

$$J\ddot{\theta} = T = T_a - D\dot{\theta}$$

$T_a \rightarrow$ Applied Torque

3) $V_a = ?$

$T_a = ?$

$$V_a = f(\theta, \dot{\theta}) \quad \& \quad T_a = g(v, i)$$

$V_a = k_0 \Theta$ or $k_0 \dot{\Theta}$
$T_a = k_1 V$ or $k_1 i$
The equations are going to have one of these two forms

$V_a = k_0 \dot{\Theta}$ ← Back emf (Electro-motive Force)
$T_a = k_1 i$

→ Both k values are the same, they are related to the magnetic field.

→ Now we have a coupled set of differential equations

$\Rightarrow V_{in} + i_a R_a + k_T \dot{\Theta} = 0$
$J\ddot{\Theta} = k_T i_a - D\dot{\Theta}$

Now we can use Laplace transforms to make the equations easier to solve.

→ "Transforms turn calculus into Algebra"

Laplace Transforms

- For finding tansfer functions we assume all intial conditions are zero.

$k_T \dot{\Theta} + R_a i_a = V_{in}$

→ $k_T s \Theta(s) + R_a I_a(s) = V_{in}(s)$

$J\ddot{\Theta} + D\dot{\Theta} - k_T i_a = 0$

→ $J s^2 \Theta(s) + D s \Theta(s) - k_T I_a(s) = 0$

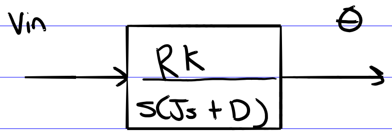
Now we can solve these equations using Algebra

$\Rightarrow (Js^2 + Ds)\Theta(s) - k_T I_a(s) = 0$

Cramer's Rule

$$\Theta(s) = \frac{\begin{vmatrix} C & B \\ 0 & E \end{vmatrix}}{\begin{vmatrix} A & B \\ D & E \end{vmatrix}} \qquad I_a(s) = \frac{\begin{vmatrix} A & C \\ D & 0 \end{vmatrix}}{\begin{vmatrix} A & B \\ D & E \end{vmatrix}}$$

Θ / V_{in}

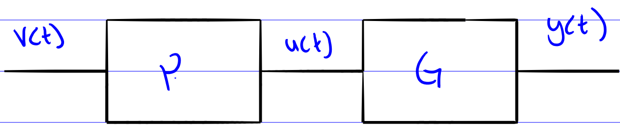


Looks something like that.

Any thing that is real (we can make) can be solved using Laplace transforms.

Laplace Transforms Formally

If you have a system



If we want to solve this in the time domain, we have to use the convolution integral

$$y(t) = \int_{0^-}^{\infty} (rv(t-\tau)) \, d\tau$$

This makes nasty, long equations that are difficult to solve. Therefore to get rid of time we can use Laplace Transforms to move to the frequency domain.

Laplace Transform:

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t) e^{-st} \, dt$$

where s is a complex variable

$s = \sigma + j\omega$

- $u(t)$ unit step function $\mathcal{L}\{u(t)\} = 1/s$
- $tu(t)$ ramp $\mathcal{L}\{t(t)\} = 1/s^2$

$\vdots$

Differential EQ's and Laplace Transforms

$$G: a_n \frac{d^n}{dt^n} c(t) \cdots$$

$$r(t) \rightarrow \boxed{G} \rightarrow c(t)$$

$$= b_m \frac{d^m r(t)}{dt^m} + \cdots$$

$$\Rightarrow G(s) = a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \cdots a_0 C(s)$$

$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \cdots b_0 R(s)$$

Laplace Transform

$$\frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots a_0}$$

General Form of Transfer Function

$G(s)$ is known as the impulse response.

Example

$$r(t) - \boxed{G} - c(t)$$

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

$$sC(s) + 2C(s) = R(s)$$

$$\Rightarrow C(s)[s+2] = R(s)$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{1}{s+2}$$

We can make $R(s)$ be whatever we want it to be now

When we set that equal to 0 then we can say we have a poll at $s = -2$

Let $R(s) = 1/s$

$$\Rightarrow C(s) = \frac{R(s)}{s+2} = \frac{1}{s(s+2)}$$

What is $c(t)$? PG 487, 7th from the top

$$c(t) = \frac{1}{2}\left(1 - e^{-2t}\right)$$ Reverse Laplace Transform

At $t=0$ $c(0) = 0$

At $t=\infty$ $c(\infty) = 1/2$

Poles

$$G(s) = \frac{n(s) \rightarrow \text{Zeros}}{d(s) \rightarrow \text{Poles}}$$

Example

$$C(s) = \frac{1}{s(s+10)(s+1)} = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+1}$$

$$\Rightarrow C(s) = \frac{1}{s} + \frac{1/9}{s+10} + \frac{-10/9}{s+1}$$

Dominant

$$\Rightarrow c(t) = \left[1 + \frac{1}{9}e^{-10t} - \frac{10}{9}e^{-t}\right]u(t)$$

Not so dominant

at $t=0$ $c(0)=0$

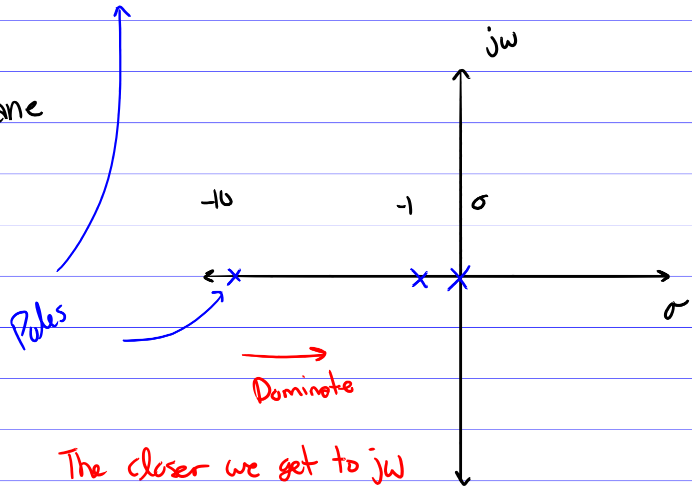
at $t=\infty$ $c(\infty)=1$

$s(s+10)(s+1)=0$

$0, -10, -1$ are the poles

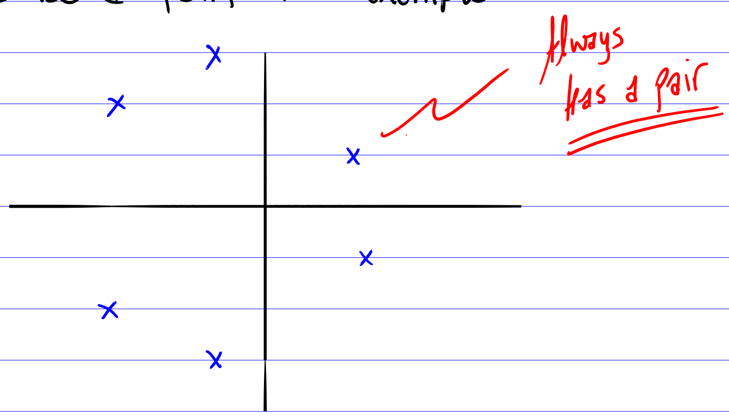
Plotting in s plane

$s = \sigma + j\omega$



The closer we get to jw axis the more dominant the pole is.

- If you have a poll off the axis, there should always be a pair, for example



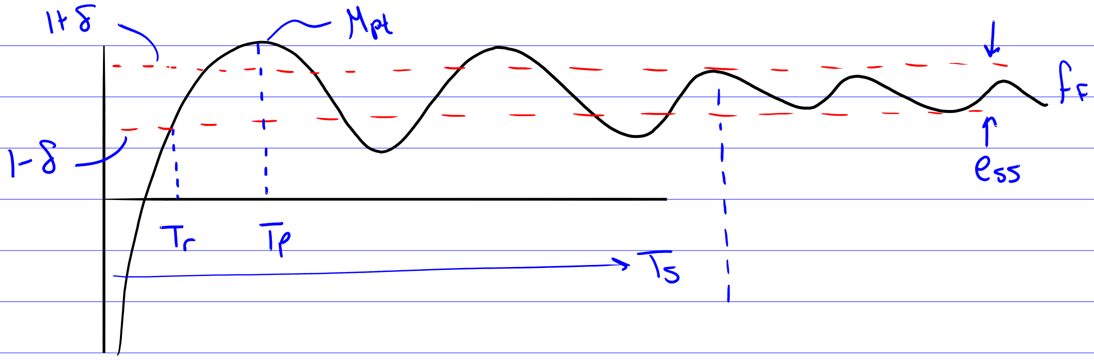
Poles on the axis are exponential values.

Second Order Response

Natural frequency

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

damping coefficient



People usually tend to notice when there is more than 3-5 % overshoot.

T_s = settling time
T_r = rise time
T_p = Peak time
M_{pt} = peak response
P.O. = Percent overshoot
f_v = final value

$$P.O. = \left(\frac{M_{pt} - f_v}{f_v}\right) \cdot 100\ \%$$

Settling Time

$$e^{-\zeta\omega_n t} < 0.2$$ 2% settling

or

$$\zeta\omega_n T_s = -\ln 0.02 = 3.9$$

$$\cancel{T_s \approx \frac{4}{\zeta\omega_n}}$$

Peak Time

$$\frac{d}{dt}y(t) \xrightarrow{\mathcal{L}} sY(s)$$

Assume unit step function input

$$sY(s) = \frac{1}{s}\frac{\omega_n^2 s}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$$\Rightarrow \frac{d}{dt}y(t) = \frac{\omega_n}{\beta}e^{-\zeta\omega_n t}\sin(\omega_n\beta t) \quad \text{for } \zeta < 1$$

$$\sin(\omega_n\beta t) = 0$$

$$\omega_n\beta t = \pi$$

$$\Rightarrow T_p = \frac{\pi}{\omega_n\beta} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$