ECE/MAE 5310
The Basis of Root Locus

Translated mostly from I. M. Horowitz

1 A Brief History

The root locus method was developed by Walter Evans in the late forties early fifties as one of the first open loop frequency domain tools available to engineers. The method amounts to a graphical technique for factoring polynomials with most of the locus being obtained from the open loop transfer function instead of the more complex and often difficult to factor characteristic equation. Root locus is a graphical technique meaning two things: (1) It produces rapid results with great insights into system performance and (2) It is looked down upon by mathematical purists. Note that we are **<u>NOT</u>** mathematicians. "What's a 'locus'?" you ask. A locus is a collection of points that forms a curve. In our case this curve will be on the s-plane and it represents how the roots of the closed-loop characteristic equation change as a single parameter of interest changes between zero and infinity.

Evans also made a tool in the form of a mechanical calculating device called the Spirule. The Spirule made plotting the root locus easy since it allowed the users to locate points on the locus. Evans went on to form a company building the Spirule to sell at a high price and even higher profit. One of the few complaints about the Spirule was that the clutch slipped. Were our complaints about our computers so simple.

2 The Critical Criteria

The root locus plot is based on an analysis of the characteristic equation of the closed loop system. The analysis makes use of the complex nature of the characteristic equation and uses the fact that this equation has both magnitude and phase.

We are now familiar with the form of the overall or closed loop transfer function

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{1}$$

where $G(S)$ is everything in the forward loop (everything from the error to the system output) and $H(s)$ is everything in the feedback loop (everything from the output to the summing junction that calculates the error). The characteristic equation is simply the denominator of $T(s)$. The poles of the closed loop function are simply the roots of the characteristic equation.

Root locus allows us to look at how the closed loop system would vary with a single system parameter varying from zero to infinity. The plot starts on the open-loop poles and terminates on the open-loop zeros, which is in and of itself amazing. Most of the information

regarding the locus in the range between $k = 0$ and $k \to \infty$ is found by manipulating the characteristic equation in the complex s-plane. The 12 steps outlined below guide the construction of a root locus plot and, hopefully, explain the mathematics behind the method itself.
I consider the derivation of the root locus method to be one of practical mathematical brilliance. We open ourselves to the possibility of doing similar things when we see and attempt to understand such brilliance.

2.1 The Angle Criterion

We can manipulate the characteristic equation into the form

$$1 + k\frac{n(s)}{d(s)} = 0, \tag{2}$$

which is manipulated into

$$k\frac{n(s)}{d(s)} = -1. \tag{3}$$

At first glance this equation may not seem to mean much but we have to remember that it is complex. -1 has a lot more meaning in the complex domain than it does in the real domain. It has both magnitude and phase. In fact it can be written in a polar form as

$$-1 = |-1| \angle 180^\circ \tag{4}$$

This means that in order for a point to be the root locus to be on the locus the sum of all the angles of the poles and zeros must be $180^\circ \pm n360^\circ, n = 0, 1, 2, 3....$ What is an angle of a pole or zero? We can evaluate a polynomial transfer function of s at a point s_0 simply by substituting s_0 for s. For example, let

$$\begin{aligned} \frac{n(s)}{d(s)} &= \frac{1}{(s+1)(s+2)} \\ &= \frac{1}{(s-(-1))(s-(-2))} \end{aligned} \tag{5}$$

where -1 and -2 are the pole locations.
Then, if $s = s_0 = -1.5 + j0.5$,

$$\begin{aligned} \frac{n(s_0)}{d(s_0)} &= \frac{1}{(-1.5 + j0.5 - (-1))(-1.5 + j0.5 - (2))} \\ &= \frac{1}{-0.5} \\ &= = -2 \\ &= = 2\angle 180^\circ \end{aligned} \tag{6}$$

with the graphical (vector) equivalent of this is shown in Figure (1).

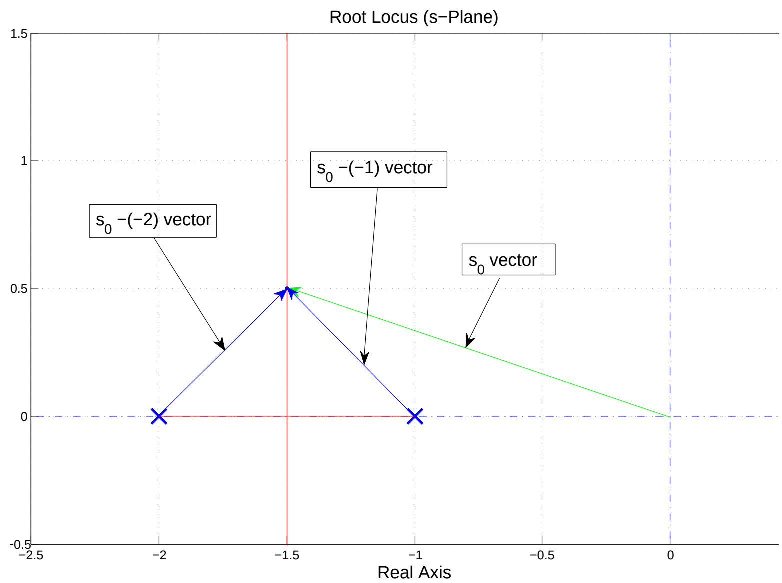

Figure 1: A plot showing the vector nature of poles.

Note that after the subtractions are done we are left with a new complex number. For example

$$s_0 - (-1) = -0.5 + j0.5 \tag{7}$$

and this complex number has a polar representation of

$$-0.5 + j0.5 = .707e^{j135^\circ}, \tag{8}$$

which has the angle of the vector running from $s = -1 \rightarrow s_0$ in Figure (1) with respect to the positive real axis.
To complete the example,

$$s_0 - (-2) = 0.5 + j0.5 \tag{9}$$

and this complex number has a polar representation of

$$0.5 + j0.5 = .707e^{j45^\circ}, \tag{10}$$

which has the angle of the vector running from $s = -2 \to s_0$ in Figure (1) with respect to the positive real axis.

Given that $n(s) = 1$, its angular contribution is zero. Writing $n(s)/p(s)$ in polar form results in

$$\frac{n(s)}{p(s)} = \frac{1}{.707e^{j135^\circ}.707e^{j45^\circ}} \tag{11}$$

and the angle of $n(s)/p(s)$ is

$$\begin{aligned} \angle\frac{n(s)}{p(s)} &= -(135^\circ + 45^\circ) \\ &= -180^\circ \\ &= 180^\circ - 360^\circ \end{aligned} \tag{12}$$

The point s_0 is on the locus since it meets the angle criterion.

As a general rule use the rectangular form of a complex number to add and subtract the numbers and the polar form for multiplication and division.

2.2 The Magnitude Criterion

The magnitude criteria is used to determine the value of the parameter of interest at a point on the locus. From the equation

$$k\frac{n(s)}{d(s)} = -1. \tag{13}$$

or

$$k = \left|\frac{d(s)}{n(s)}\right| \tag{14}$$

Using our example of the previous section

$$\frac{n(s)}{p(s)} = \frac{1}{.707e^{j135^\circ}.707e^{j45^\circ}} \tag{15}$$

and

$$k = \left|.707e^{j135^\circ}.707e^{j45^\circ}\right| = .707^2 = 0.5 \tag{16}$$

So the point $s_0 = -0.5 + j0.5$ is on the axis and the parameter of interest at this point is 0.5. We can check this by expanding

$$1 + k\frac{n(s)}{d(s)} = 0 \tag{17}$$

when $k = 0.5$ resulting in

$$\begin{aligned} (s+1)(s+2) + 0.5 &= 0 \\ s^2 + 3s + 2.5 &= 0 \end{aligned} \tag{18}$$

and $s = -1.5 \pm j0.5$.
We can determine the parameter of interest k at any point on the locus simply by finding the lengths of the vectors from all poles and zeros (you can use a ruler) and then applying the formula

$$k = \frac{\prod \text{lengths of the pole vectors}}{\prod \text{lengths of the zero vectors}} \tag{19}$$

This is Step 12 of the process so more on that later.

3 The Twelve Steps

3.1 Step 1-The Characteristic Equation and the Parameter of Interest

This first step simply involves manipulating the characteristic equation of the system into the following form

$$1 + k\frac{n(s)}{d(s)} = 0. \tag{20}$$

where k is the parameter of interest. k is commonly a gain in the system or more specifically a gain in the controller. However, k can be any system parameter such as a pole or zero location. The migration of poles in the s-plane beginning with $k = 0$ and ending as $k \to \infty$ constitutes a locus of points that define curves on the s-plane. Looking at the extremes in k provides some insight into system behavior.
When $k = 0$ we can't simply multiply out and arrive at the startling conclusion that $1 = 0$. We have to take into account the effect of $d(s)$. Multiplying (20) by $d(s)$ results in

$$d(s) + kn(s) = 0 \tag{21}$$

and setting $k = 0$ results in

$$d(s) = 0. \tag{22}$$

Therefore the locus starts (when $k = 0$) on the zeros of $d(s)$, which are the poles of the open loop transfer function when k is simply a gain.
To investigate what happens as $k \to \infty$ we write the following relationship

$$k = -\frac{d(s)}{n(s)} \tag{23}$$

Taking the limit as $k \to \infty$ results in

$$\lim_{k\to\infty} -\frac{d(s)}{n(s)} = \infty \tag{24}$$

There are two ways that (24) can actually be infinite: if $n(s)$ is zero or if $d(s)$ is infinite. Either and both cases apply. From this we conclude that the poles of $1 + k\frac{n(s)}{d(s)}$ terminate on the zeros of $n(s)$ plus the zeros at infinity caused by $d(s)$.
Let an open loop system be defined by the following transfer function

$$G(s) = \frac{k_0}{s(s+a)}. \tag{25}$$

The characteristic equation is

$$1 + G(s) = 1 + \frac{k_0}{s(s+a)} = 0 \tag{26}$$

and, if k_0 is the parameter of interest, it is already in the proper form with

$$n(s) = 1 \tag{27}$$
$$d(s) = s(s+a). \tag{28}$$

The root locus starts on the open loop poles when $k_0 = 0$ and, in the limit, terminates on the open loop zeros when $k_0 \to \infty$. In this case the poles where the locus starts and the zeros where the locus terminates are simply the open loop poles and zeros of the plant as shown in Figure(2, shown for $a = 1$. However, things change if I want to look at how the system changes with respect to another parameter.
In my example let the parameter of interest be a. In this case I need to expand the equation (26) in order to get it into the proper form. A valid question at this point is, "Why?". Let's answer this question before we move on. Recall at the beginning of the explanation of this step that we found where the locus started and ended. The derivation was for the very specific form of (20) and in order to apply our findings we must place the equation in the same form.
Expanding the characteristic equation of (26) results in

$$s(s+a) + k_0 = s^2 + as + k_0 = 0, \tag{29}$$

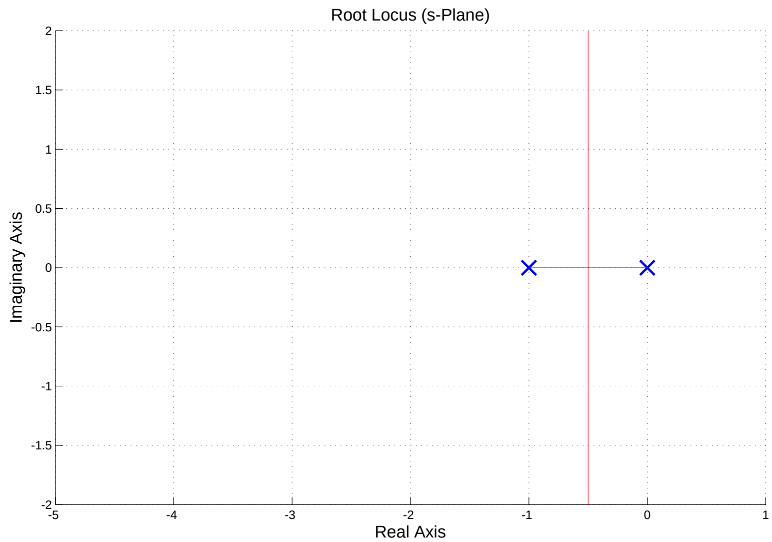

Figure 2: A simple locus plot obtained by varying gain.

which can be reduced to the proper form by dividing through by $s^2 + k_0$ resulting in

$$1 + \frac{as}{s^2 + k_0} = 0 \tag{30}$$

Now, for this case,

$$n(s) = s \tag{31}$$
$$d(s) = s^2 + k_0 \tag{32}$$

In Figure below when k_0 is allowed to vary $a = 1$ and when a is allowed to vary $k_0 = 1$. Are the two plots related? The answer is yes, under the condition that a and k_0 are at their respective fixed values. The intersection of the two loci is pointed out on the plot.
Why are the plots so different? The primary reason is that k_0 does not change pole or zero locations in the plant while a is a pole location and it is being allowed to vary from 0 to ∞. With a varying we want to know how the closed loop poles of the system move. When $a = 0$ the closed loop poles are at $+j\sqrt{k_0}$ and $-j\sqrt{k_0}$ as shown on the plot. Given that the example system is second order any confusion that remains can be cleared up by factoring the characteristic equation as a varies with k_0 fixed and observing that the locus does indeed follow the path shown.

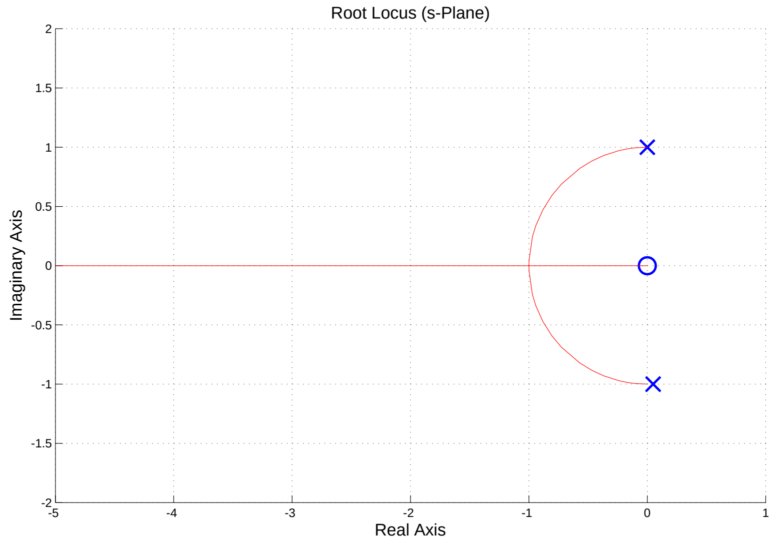

Figure 3: Locus that results from varying a.

3.2 Step 2-Factoring

We must factor the open loop system (sans the varying parameter) so that we can start the root locus plot. Note that $n(s)$ and $d(s)$ have constant coefficients only and you can use a convenient solver to find these roots.

3.3 Step 3-Plotting on the S-Plane

Plotting the open loop poles and zeros (or generally the poles and zeros of $n(s)/d(s)$)is a simple process. On the s-plane, place an 'X' at every pole location and an 'O' at every zero location.

3.4 Step 4-Locating Real Axis Locus

This step is the first of the seemingly strange steps, but it is really easy to understand. Locus (the collection of points as the parameter of interest varies) exists on the real axis for

$k > 0$: to the left of an odd number of real axis poles and zeros (poles and zeros are also called singularities) starting at the right most singularity.

$k < 0$: on all of the real axis that was not covered for $k > 0$ and none of the real axis that was covered.

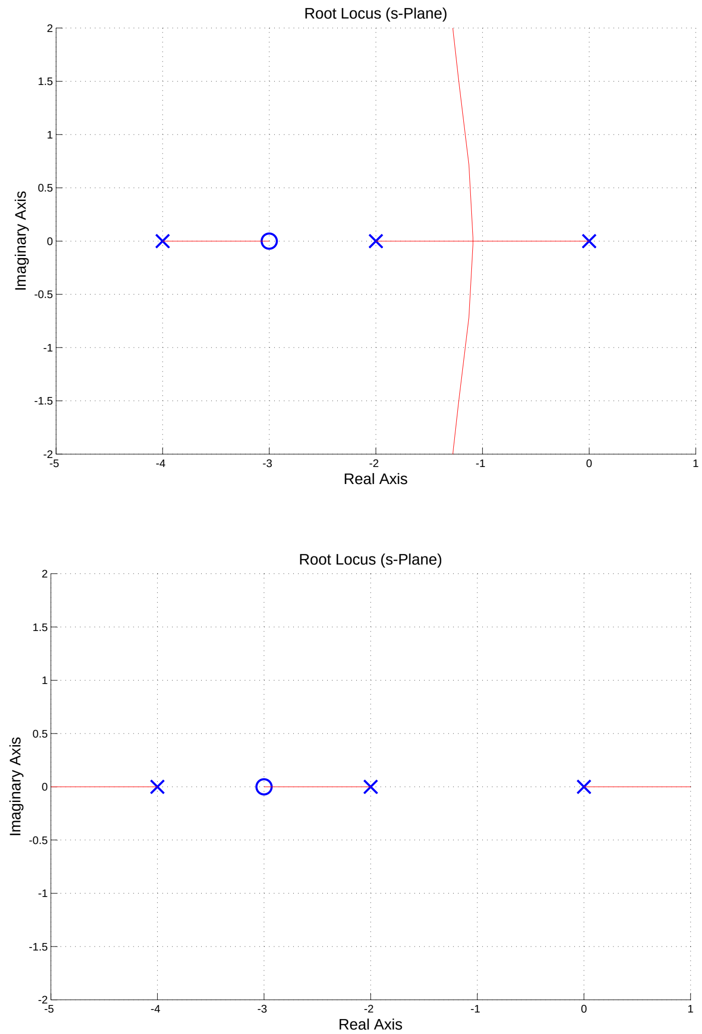

Figure 4: Positive Gain and Negative Gain Real Axis Locus

Note that $k < 0$ indicates positive feedback. For example, let

$$\frac{n(s)}{d(s)} = \frac{s+3}{s(s+2)(s+4)} \tag{33}$$

then the root locus for $k > 0$ and $k < 0$ are plotted in Figure(4)
While this step may appear mysterious it is readily understood by considering the characteristic equation in polar form.
The characteristic equation

$$1 + k\frac{n(s)}{d(s)} = 0 \tag{34}$$

can be rewritten as

$$k\frac{n(s)}{d(s)} = -1 \tag{35}$$

for $k > 0$
Since $kn(s)/d(s) = -1$ is complex -1 can be written in polar form as

$$k\frac{n(s)}{d(s)}) = 1\angle 180^\circ \pm 360^\circ n, n = 0, 1, 2, 3, ... \tag{36}$$

$$\angle zeros - \angle poles = \angle 180^\circ \pm 360^\circ n, n = 0, 1, 2, 3, ... \tag{37}$$

First, given $k > 0$, any singularity (pole or zero) to the right of an arbitrary point on the axis will contribute $\pm 180^\circ$ and any singularity to the left of this point will contribute zero degrees. Thus a single pole or zero contributes $\pm 180^\circ$, a pole and a zero contribute 0°, two poles contribute -360°, and two zeros contribute 360°. An even number of singularities contributes nothing to the angle so locus cannot exist to the left of an even number of singularities. For a single singularity the contributed angle correct and this extends to an odd number of singularities since an odd number can be written as $2n + 1, n = 0, 1, 2, 3...$ the even poles cancel leaving only a single singularity to contribute to the angle. This means that real axis locus exists to the left of an odd number of real axis singularities. A similar argument for $k < 0$ shows that the locus exists on the real axis everywhere it doesn't for $k > 0$.

3.5 Step 5-Determining the Number of Separate Loci

The number of separate loci helps us determine if we have drawn a correct plot. We will have one locus for every each pole. This is because, for a proper system, each open pole must terminate on an open loop zero including those zeros at infinity. Proper being defined as more poles than zeros.

3.6 Step 6-Recognizing that the Loci are Symmetric with the Real Axis

This step is more of an observation than a process step. The locus is symmetric about the horizontal axis because complex poles occur in conjugate pairs and the locus represents poles. This property means that we only have to make calculations for the upper half of the s-plane. The lower half is identical. Use this 'property' to determine if your plot is correct.

3.7 Step 7-Finding the Origin and Angle of Asymptotes

The angle and origin of the asymptotes that the poles approach as the parameter of interest tends to infinity are obtained by a bit of subtle manipulation in the s-plane. If the characteristic equation is given as

$$1 + k\frac{n(s)}{d(s)} = 0 \tag{38}$$

then as k and s tend to infinity

$$\lim_{k,s\to\infty} k\frac{n(s)}{d(s)} \to \frac{k}{s^{\epsilon}} \tag{39}$$

where ϵ is the excess number of poles to zeros. This can be seen by repeatably applying L'Hospital's Rule to the ratio of polynomials.
Then

$$1 + k\frac{n(s)}{d(s)}) = 0 \Rightarrow s^{\epsilon} + k \to 0 \tag{40}$$

Normalize the above equation by defining

$$w \equiv \frac{s}{|k|^{\frac{1}{\epsilon}}}, \tag{41}$$

and (40) reduces to

$$\begin{aligned} &w^{\epsilon} + \operatorname{sgn}(k) \to 0 \\ &w \to -\operatorname{sgn}(k)^{\frac{1}{\epsilon}} \end{aligned} \tag{42}$$

Then as $s \to \infty$

$$s = w\,|k|^{\frac{1}{\epsilon}} \rightarrow |k|^{\frac{1}{\epsilon}}\,(-\mathrm{sgn}(k))^{\frac{1}{\epsilon}} \tag{43}$$

and for $k > 0$

$$s \rightarrow \sqrt[1/\epsilon]{-k} = \sqrt[1/\epsilon]{k}\,\sqrt[1/\epsilon]{-1} \tag{44}$$

or, in words, as s tends to infinity it approaches the ϵ^{th} root of -1 multiplied by a number also approaching infinity.
Thus s at infinity has infinite magnitude and angle(s) of the ϵ^{th} root of -1 where

$$-1^{\frac{1}{\epsilon}} = \left(e^{j\pi}\right)^{\frac{1}{\epsilon}} = e^{j(\pi+2k\pi)/\epsilon} \quad \text{for} \quad k = 0, \pm 1, \pm 2, ... \tag{45}$$

The angle of the asymptotes at infinity is

$$\phi_A = \frac{180°}{\epsilon}, \frac{540°}{\epsilon}, ..., \frac{(2\epsilon-1)360°}{2\epsilon} \tag{46}$$

The same derivation can also be done with negative k, which is typical for a positive feedback system.
The origin of asymptotes is a more interesting derivation. When studying the root locus the formula

$$\sigma_A = \frac{\sum poles - \sum zeros}{\#poles - \#zeros} \tag{47}$$

appears out of nowhere never to be questioned. Fortunately, I. M. Horowitz questioned it and I clean up and present his derivation here. The seatbelt sign is on, buckle up, it is going to be a bumpy ride.
We know that the asymptotes must be symmetric about the real axis because complex pole pairs must be conjugate at all points. This implies that the origin of the asymptotes is on the real axis. We know the angle of the asymptotes, but where do these asymptotes intersect the real axis? Do all asymptotes intersect at the same point on the axis? The answer to these questions follows from a clever construct and a limiting procedure; fancy that. Assume that your origin of asymptotes is at a point σ_A and that A is a point somewhere on the asymptote as shown in Figure (5).
The goal is locate σ_A and this is done by letting $\rho \rightarrow \infty$ and observing the result. The point s_A is simply defined by the vector sum

$$s_A = \Delta + \rho e^{j\theta} \tag{48}$$

and it is worth mentioning again that the point A is not on the locus and that s_A does not satisfy the characteristic equation. However, as $\rho \rightarrow \infty$

$$1 + k\frac{n(s_A)}{d(s_A)} \rightarrow 0 \tag{49}$$

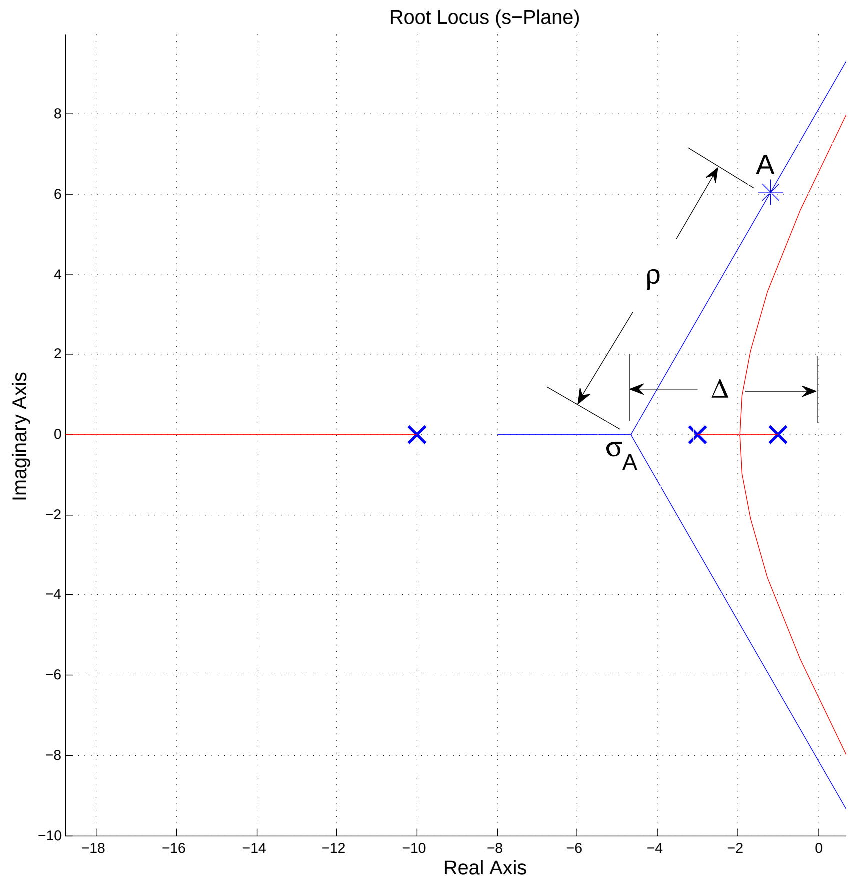

Figure 5: A point on a root locus asymptote.

and in the limit s_A will be on the locus. Therefore it is useful to look at the characteristic equation at $s = s_A$ and see what happens in the limit. Let

$$L\left(s_A\right) = k\frac{n(s_A)}{d(s_A)} = k\frac{\prod_1^m \left(s_A + z_i\right)}{\prod_1^n \left(s_A + p_h\right)} = k\frac{\prod_1^m \left(\Delta + z_i + \rho e^{j\theta}\right)}{\prod_1^n \left(\Delta + p_h + \rho e^{j\theta}\right)} \tag{50}$$

While this equation does not look as if it could reduce to the simple form of (47), it will. The first change that needs to be made to this equation is to change the index bounds on the products so that they are expressed in a common term. Note that since we are dealing with a proper transfer function there are ϵ excess poles over zeros. I want to manipulate this equation into a form where I can take advantage of the fact that ρ is getting large and I can use it to access a binomial series expansion and change the products into sums.

$$L\left(s_A\right) = k\frac{\prod_1^m \left(\Delta + z_i + \rho e^{j\theta}\right)}{\prod_1^{m+\varepsilon} \left(\Delta + p_h + \rho e^{j\theta}\right)} \tag{51}$$

Now I can factor a $\rho e^{j\theta}$ out of each product term in the numerator and denominator of (51) resulting in

$$\begin{aligned} L\left(s_A\right) &= k\frac{\rho e^{jm\theta}}{\rho e^{j(m+\varepsilon)\theta}} \frac{\prod_1^m \left(\frac{\Delta+z_i}{\rho e^{j\theta}} + 1\right)}{\prod_1^{m+\varepsilon} \left(\frac{\Delta+p_h}{\rho e^{j\theta}} + \rho e^{j\theta}\right)} && (52) \\ &= \frac{k}{\rho^{\varepsilon} e^{j\varepsilon\theta}} \frac{\prod_1^m \left(\frac{\Delta+z_i}{\rho e^{j\theta}} + 1\right)}{\prod_1^{m+\varepsilon} \left(\frac{\Delta+p_h}{\rho e^{j\theta}} + 1\right)} && (53) \end{aligned}$$

with the fraction of products being expanded using the binomial series expansion to produce

$$\begin{aligned} \frac{\prod_1^m \left(\frac{\Delta+z_i}{\rho e^{j\theta}} + 1\right)}{\prod_1^{m+\varepsilon} \left(\frac{\Delta+p_h}{\rho e^{j\theta}} + 1\right)} \cong & \left(1 + \frac{\Delta + z_1}{\rho e^{j\theta}} + H.O.T\right)\left(1 - \frac{\Delta + p_1}{\rho e^{j\theta}} + H.O.T\right) \\ & \ldots \left(1 + \frac{\Delta + z_m}{\rho e^{j\theta}} + H.O.T\right)\left(1 - \frac{\Delta + p_m}{\rho e^{j\theta}} + H.O.T\right) \\ & \ldots \left(1 - \frac{\Delta + p_n}{\rho e^{j\theta}} + H.O.T\right) \end{aligned} \tag{54}$$

(54) can be expanded and higher order terms neglected resulting in

$$L\left(s_A\right) = \frac{k}{\rho^{\varepsilon} e^{j\varepsilon\theta}} \left[1 + \frac{E}{\rho e^{j\theta}}\right] \tag{55}$$

where E is defined as

$$E = \sum_{i=1}^{m} z_i - \sum_{h=1}^{m+\varepsilon} p_i - \Delta\varepsilon \tag{56}$$

The characteristic equation is now written as

$$\begin{aligned} 1 + L\left(s_A\right) &= 1 + \frac{k}{\rho^{\varepsilon} e^{j\varepsilon\theta}} \left[1 + \frac{E}{\rho e^{j\theta}}\right] \\ &= \rho^{\varepsilon} e^{j\varepsilon\theta} + k\left[1 + \frac{E}{\rho} e^{-j\theta}\right] && (57) \\ & && (58) \end{aligned}$$

and this can be expanded to

$$
\begin{aligned}
1+L\left(s_A\right) &= \rho^{\varepsilon}\left[\cos(\theta\varepsilon)+j\sin(\theta\varepsilon)\right]+k\left[1+\frac{E}{\rho}\left(\cos(\theta)-j\sin(\theta)\right)\right] \\
&= \rho^{\varepsilon}\cos(\theta\varepsilon)+k\left(1+\frac{E}{\rho}\cos(\theta)\right)+j\left(\rho^{\varepsilon}\sin(\theta\varepsilon)-k\left(\frac{E}{\rho}\sin(\theta)\right)\right) \qquad (59)
\end{aligned}
$$

As ρ and k go to infinity the point on the asymptote,S_A, and the root locus converge meaning that the characteristic equation is satisfied at infinity. Therefore both the real and imaginary parts of the equation in (59) must go to zero. We are shooting to show that $E = 0$ so that we can solve for the origin of asymptotes. Working on the real part first

$$
\rho^{\varepsilon}\cos(\theta\varepsilon)+k+\frac{kE}{\rho}\cos(\theta)\rightarrow 0 \qquad (60)
$$

but $\cos(\theta\varepsilon)\rightarrow -1$ for $k>0$ from the angle of asymptotes formula and $\sin(\theta\varepsilon)\rightarrow 0$. Putting (60) over a common denominator and allowing $\rho\rightarrow\infty$ results in

$$
k\cong\rho^{\varepsilon} \qquad (61)
$$

Because $\sin(\theta\varepsilon)\rightarrow 0$ the imaginary part of (59) reduces to

$$
\frac{kE}{\rho}\sin(\theta)\rightarrow 0 \qquad (62)
$$

In (62) $\sin(\theta)$ does not go to zero in fact

$$
\frac{k}{\rho}\sin(\theta)\cong\frac{\rho^{\varepsilon}}{\rho}\sin(\theta)\rightarrow\infty \qquad (63)
$$

The only possible way that part of (62) can go to infinity and the whole quantity go to zero is if the remaining part of the equation is zero. The remaining part of the equation is E and

$$
E = 0 \qquad (64)
$$

Finally, solving (56) for Δ results in

$$
\begin{aligned}
E &= 0=\sum_{i=1}^{m} z_i-\sum_{h=1}^{m+\varepsilon} p_i-\Delta\varepsilon \\
\Delta &= \frac{\sum\limits_{i=1}^{m} z_i-\sum\limits_{h=1}^{m+\varepsilon} p_i}{\varepsilon} \qquad (65)
\end{aligned}
$$

and Δ is the origin of asymptotes.

Note that this formula uses the form $s + z_i$ so, for example, a zero at -2 has $z_i = 2$. If you want to use the actual pole locations you need to reverse the formula and obtain

$$\sigma_A = \frac{\sum poles - \sum zeros}{\#poles - \#zeros}, \tag{66}$$

which is what we were after in the first place.

3.8 Step 8-Determining the Point(s) When/If Locus Crosses the Imaginary Axis

Knowing when the locus crosses the imaginary axis tells us when, or if, a linear system will go unstable. We can determine the gain at this point by using the Routh-Hurwitz criterion. The actual point at which the crossing occurs can be obtained by solving the auxiliary polynomial. For example, assume that

$$G(s) = \frac{1}{s(s+1)(s+3)}. \tag{67}$$

is a plant in negative feedback with a gain of interest k.
Then the characteristic equation for this problem is

$$1 + kG(s) = s^3 + 4s^2 + 3s + k. \tag{68}$$

The Routh array for this problem is

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 4 & k \\ s^1 & \frac{12-k}{4} & 0 \\ s^0 & k & \end{array}$$

The edge of instability (sitting on the $j\omega$-axis) occurs when

$$\frac{12-k}{4} = 0 \tag{69}$$

or when

$$k = 12. \tag{70}$$

The system goes thus reaches the $j\omega$-axis at $k = 12$ and the frequency where this occurs is given by the solution of the auxiliary polynomial

$$\begin{aligned} 4s^2 + 12 &= 0 \qquad (71) \\ s &= \pm j\sqrt{3} \qquad (72) \end{aligned}$$

The locus for this system is plotted in Figure(6) showing the crossing at $\omega = \sqrt{3}$.

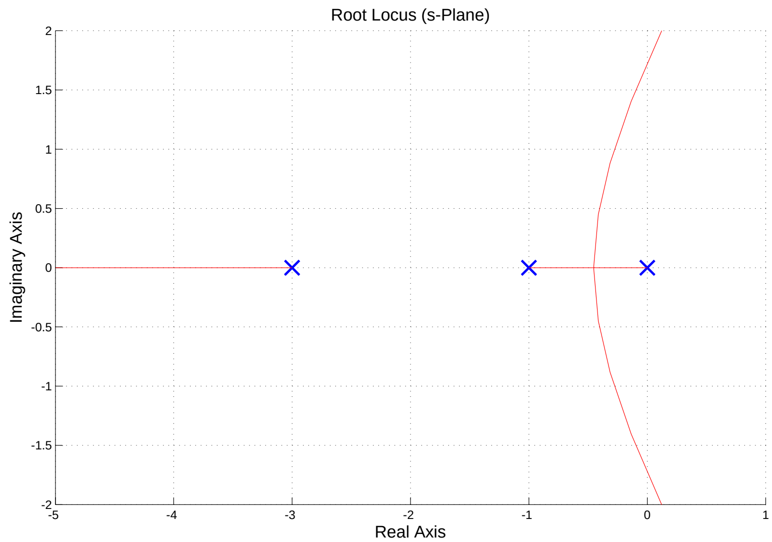

Figure 6: Locus that crosses the imaginary axis.

3.9 Step 9-Determining the Real Axis Breakaway Point

The point or points where the locus leaves the real axis tell us when to start to expect sinusoidal terms in the time response of our system. The point where poles leave the real axis and begin to be complex is called the breakaway point. The point can be located by isolating the gain k, differentiating the equation with respect to s, and finding the roots of the equation. For an example let

$$\frac{n(s)}{d(s)} = \frac{s+1}{s(s+1.5)(s+10)} \tag{73}$$

then the characteristic equation can be reordered to isolate k

$$k = -\frac{s^3 + 11.5s^2 + 15s}{s+1}. \tag{74}$$

Differentiating the above equation with respect to s results in

$$\frac{dk}{ds} = -1/2\,\frac{4\,s^3 + 29\,s^2 + 46\,s + 30}{(s+1)^2} = 0, \tag{75}$$

which has solution $s = -5.3$. This is shown in Figure(7).

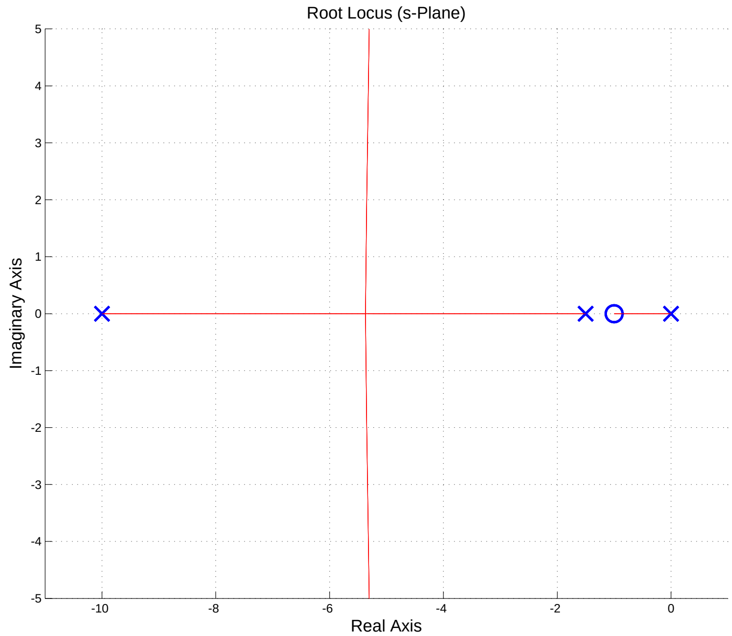

Figure 7: Locus leaving the real axis.

This step can also yield the point where locus reattaches to the real axis, called the coalescence point. If

$$\frac{n(s)}{d(s)} = \frac{s+1}{(s+0.5)(s+0.6)} \tag{76}$$

Then the locus for this system both leaves the real axis and reattaches to the real axis as shown in Figure(8)
After differentiating with respect to s

$$\frac{dk}{ds} = -1/5\,\frac{5\,s^2 + 10\,s + 4}{(s+1)^2} = 0, \tag{77}$$

which has two solutions $s = -.55$ (the breakaway point) and $s = -1.45$ (the coalescence point).

3.10 Step 10-Determining the Departure/Arrival Angles

The departure angle at poles and the arrival angle at zeros is a useful sketching aid and the angle can be easily determined. The characteristic equation can be manipulated so that

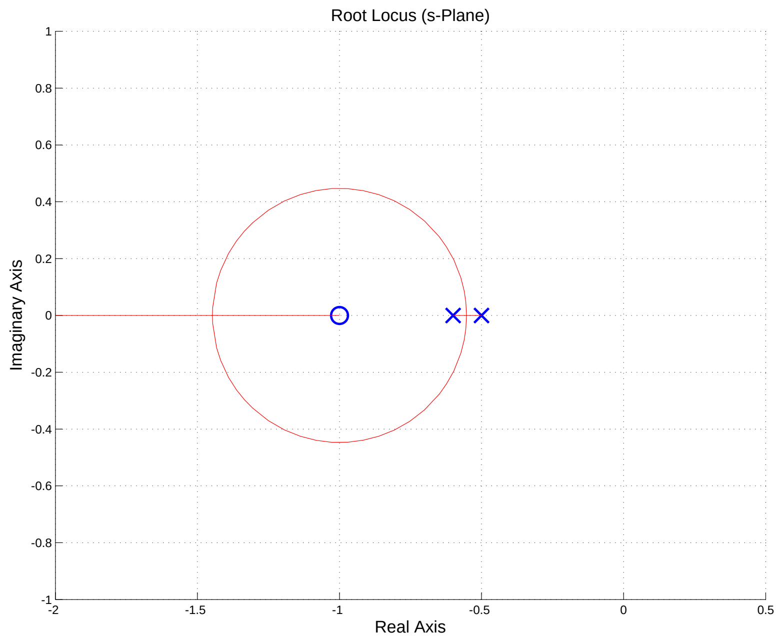

Figure 8: Locus leaving and returning to the real axis.

$$k\frac{n(s)}{d(s)} = -1 = 1\angle 180^\circ \tag{78}$$

Since k is real it contributes no angular component to the left side of the above equation and it is safe to say that all the angular component is contributed by $n(s)/d(s)$. Any point that is on the locus meets the angle criterion so that

$$\angle\frac{n(s_0)}{d(s_0)} = \angle 180^\circ \tag{79}$$

where s_0 is the point under consideration.
When the poles and zeros are complex the locus departs or arrives at an angle that completes the angle criteria for that pole or zero. In equation form the departure angle from a pole is

$$\theta_d = \sum \angle\text{zeros} - \sum \angle\text{other poles} \pm 180^\circ(2n+1), n = 0, 1, 2, ... \tag{80}$$

The arrival angle at a zero is

$$\theta_a = \sum \angle\text{poles} - \sum \angle\text{other zeros} \pm 180^\circ(2n+1), n = 0, 1, 2, ... \tag{81}$$

3.11 Step 11-Using the Angle Criterion to Sketch Points on the Locus

The Spirule was used to accumulate angles to determine if a point really was on the locus. This calculation can also be carried out analytically using the angle criterion.
As stated earlier if a point in the s-plane meets the angle criteria it is on the locus. The easiest way to find out is a point is on the locus is to use a protractor. You measure the angle from each zero of $n(s)/d(s)$ to the point of interest in the plane and add them all together. Then you measure all the angles from the poles to the point and add them together. You then subtract the sum of the angles from the poles from the angles of the zeros and if the resulting angle is

$$\begin{aligned} \angle\frac{n(s)}{p(s)} &= \angle n(s) - \angle d(s) = 180^\circ \\ &= -180^\circ + 360n^\circ, n = 0, 1, 2, ... \end{aligned} \tag{82}$$

then the point is on the locus.

3.12 Step 12-Determining the Parameter of Interest at a Point on the Locus

Once the root locus is plotted we can determine the value of the parameter of interest at any point on the locus. This a simple matter of measuring the length from each pole to the point of interest in the s-plane and multiplying all these lengths together. Then the lengths from each each zero to the point are measured and these lengths multiplied together. k is found by dividing the product of the lengths to the poles to the product of the lengths to zeros or in equation form

$$k = \frac{\prod \text{lengths of the pole vectors}}{\prod \text{lengths of the zero vectors}}. \tag{83}$$

4 The Twelve Steps of Root Locus-Short Reference

1. Step 1-Write the characteristic equation and isolate the parameter of interest $(1+k\frac{n(s)}{d(s)})$.
2. Step 2-Factor $n(s)$ and $d(s)$ to obtain open loop pole and zero locations.
3. Step 3-Plot the poles and zeros from Step 2 in the s-plane.
4. Step 4-Locate the real-axis root locus (to the left of an odd number of real axis poles of zeros starting at the rightmost real-axis pole or zero).
5. Step 5-Determine the number of separate loci (same as the number of open loop poles).
6. Step 6-Remember that the root locus is symmetric about the real-axis.
7. Step 7-Find the origin of asymptotes $(\sigma_A = \frac{\sum poles - \sum zeros}{\# poles - \# zeros})$ and the angle of asymptotes $(\phi_A = \frac{180^\circ}{\epsilon}, \frac{540^\circ}{\epsilon}, ..., \frac{(2\epsilon-1)360^\circ}{2\epsilon})$.
8. Step 8-Determine the imaginary axis crossing point using the Routh Array.
9. Step 9-Determine real-axis breakaway and coalescence points $(\frac{dk}{ds} = -\frac{d}{ds}\frac{d(s)}{n(s)} = 0)$
10. Step 10-Determine the departure angle from a complex pole ($\theta_d = \sum \angle$zeros $- \sum \angle$other poles $\pm\, 180^\circ(2n+1), n = 0, 1, 2, ...$) and the arrival angle for a complex zero ($\theta_a = \sum \angle$poles $- \sum \angle$other zeros $\pm\, 180^\circ(2n+1), n = 0, 1, 2, ...$).
11. Step 11-Determine if a point is on the locus by summing the angles from all system poles and zeros $(\angle\frac{n(s)}{p(s)} = \angle n(s) - \angle d(s) = 180^\circ$ $= -180^\circ + 360n^\circ, n = 0, \pm 1, \pm 2, ...)$
12. Step 12-Determine the parameter of interest at a point on the locus $(k = \frac{\prod \text{lengths of the pole vectors}}{\prod \text{lengths of the zero vectors}})$.