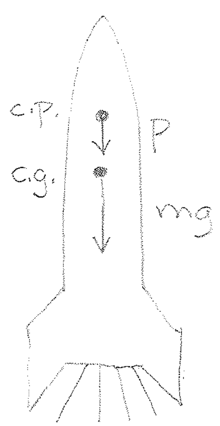
ECE/MAE 5310 Stability More Examples Chap. 5 Material

Stability

We will work from the definition of stability

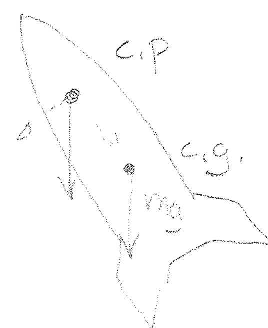
as a bounded input produces a bounded output

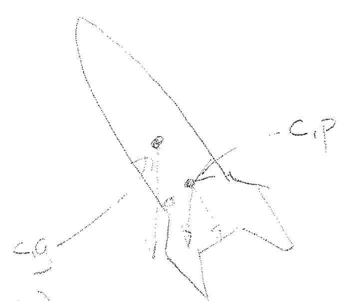
Your book uses a cone to illustrate stability. Cones are boring rockets are interesting. We'll use a rocket, read about the cone.



forces on the body

perturbed rocket





(show rocket demonstrator)

the rocket is unstable if the c.p is in front of the c.g. stable if it is behind the c.g. and neutral if it is co-located.

The rocket illustrates bounded input/bounded output or BIBO stability when the c.g. is in front of the center of pressure.

A linear system is stable only if all the poles of the system (closed loop) have negative real parts

A linear system is marginally stable if it has poles on the $j\omega$-axis (oscillatory)

$$T(s) = \frac{100}{(s+5)(s+20)(s+1)}$$ stable

$$T(s) = \frac{100}{(s-5)(s+20)(s+1)}$$ unstable RHP pole $\underline{s=5}$

$$T(s) = \frac{100}{(s^2+10)}$$ marginally stable $\pm j\sqrt{10}$

The denominator of $T(s)$ is called the characteristic equation of $T(s)$. The closed loop system poles are the zeros of the denominator of $T(s)$

$$T(s) = \frac{100}{(s+5)(s+20)(s+1)}$$

$q(s) = (s+5)(s+20)(s+1)$ is the characteristic equation

this one is factored and it is easy to tell stability

$q(s) = 0 = (s+5)(s+20)(s+1)$ when $s=-5$, $s=-20$, $s=-1$

It is more difficult to know stability when it is not factored. or when one of the coefficient is a variable rather than a number

1877 1964

Routh-Horwitz Criterion

This was such a problem that Horwitz found a solution, realized that Routh had found it 90 yrs. earlier and was adult enough to put his name first.

The criterion requires the manipulation of a special array that allows you to determine system stability

given a general characteristic equation

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

you form the Routh array as follows

$$
\begin{array}{c|cccc}
s^n & \boxed{a_n} & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & \boxed{a_{n-3}} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\
\vdots & \vdots & & & \\
s^0 & h_{n-1} & & &
\end{array}
$$

all the coefficients of $q(s)$ go in these two rows

you fill in the rest of the rows as follows

$$b_{n-1} = \frac{(a_{n-1})(a_{n-2}) - (a_n)(a_{n-3})}{a_{n-1}}$$

$$
\begin{array}{c|cccc}
s^n & \boxed{a_n} & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & \boxed{a_{n-5}} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \\
\vdots & \vdots & & & \\
s^0 & h_{n-1} & & &
\end{array}
$$

$$b_{n-3} = \frac{(a_{n-1})(a_{n-4}) - (a_n)(a_{n-5})}{a_{n-1}}$$

$$
\begin{array}{c|ccc}
s^n & a_n & a_{n-2} & a_{n-4} \\
s^{n-1} & \boxed{a_{n-1}} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_{n-1} & \boxed{b_{n-3}} & b_{n-5} \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \\
\vdots & \vdots & & \\
s^0 & h_{n-1} & &
\end{array}
\qquad
c_{n-1} = \frac{(b_{n-1})(a_{n-3}) - (a_{n-1})(b_{n-3})}{b_{n-1}}
$$

what's c_{n-3}

$$c_{n-3} = \frac{(b_{n-1})(a_{n-5}) - (a_{n-1})(b_{n-5})}{b_{n-1}}$$

Routh-Hurwitz Criterion

"the number of roots of $q(s)$ with positive real parts is equal to the number of sign changes in the first column of the Routh array.

Case I No element in the column is zero

Example 1.1 $\quad q(s) = s^2 + 5s + 6$

$$
\begin{array}{c|cc}
s^2 & 1 & 6 \\
s^1 & 5 & 0 \\
s^0 & \boxed{6} &
\end{array}
\qquad
\boxed{} = \frac{30-0}{5} = 6
$$

stable no sign changes

Example 1.2 General Second Order

$$q(s) = a_2 s^2 + a_1 s^1 + a_0$$

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & a_0 & \end{array} \qquad \longrightarrow \qquad \frac{a_1 a_0 - 0}{a_1} = a_0$$

for a second order system to be stable all the coefficients must be positive (or all negative)

Example 1.3 General Third Order

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & \\ s^0 & c_1 & \end{array}$$

$$b_1 = ? \quad \frac{a_2 a_1 - a_3 a_0}{a_2}$$

$$c_1 = \frac{b_1 a_0 - 0}{b_1} = a_0$$

to be stable a third order system must have all coefficients positive and

$$a_2 a_1 > a_3 a_0$$

zeros in the first column cause trouble because they are used in division so

case 2: There is a zero in the first column but non-zero elements in the row containing the zero.

Example 2.1

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

s^5	1	2	11
s^4	2	4	10
s^3	0	6	
s^2			
s^1			
s^0			

replace w/ ε

s^5	1	2	11
s^4	2	4	10
s^3	ε	6	0
s^2	$\square$	$\triangle$	
s^1	$\bigcirc$		
s^0	10		

$$\square = \frac{4\varepsilon - 12}{\varepsilon} = 4 - \frac{12}{\varepsilon}$$ but ε is very small

$$= -\frac{12}{\varepsilon} \longrightarrow$$ big negative

$$\triangle = 10$$

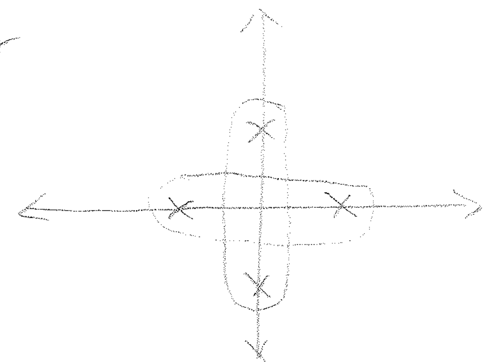
$$\bigcirc = \frac{-\frac{12}{\varepsilon}(6) - 10\varepsilon}{-12/\varepsilon} = \longrightarrow 6$$ back to positive

2 sign changes

Routh cont.)

Case 3 There is a zero in the first column and all the elements in the row are also zero.

Occurs for



Example

$q(s) = s^3 + 2s^2 + 4s + K$

s^3	1	4
s^2	2	K
s^1	$\frac{8-K}{2}$	0
s^0	K	

for $K = 8$ we have a zero row

Then the row above the zero row contains a polynomial whos roots are poles of $q(s)$. It is called the auxillary polynomial. In this example

$$U(s) = 2s^2 + 8 = 0$$

$$s^2 + 4 = 0 \qquad s = \pm j2$$

$(s+j2)(s-j2)$ are roots of $q(s)$ and system poles

check by synthetic division

$$\begin{array}{r|l} & \tfrac{1}{2}s + 1 \\ \hline 2s^2+8 & s^3 + 2s^2 + 4s + 8 \\ & \underline{s^3 + 0s^2 + 4s } \\ & \phantom{s^3+{}} 2s^2 + 8 \end{array}$$

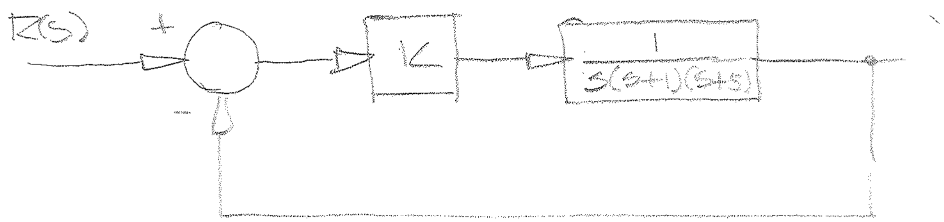
$(s+2)$ is the other root

Case 4 Repeated $j\omega$ roots

Routh does indicate instability
indicates marginal stability

Example

Given a system



Determine the range of K for which the system is stable

$$T(s) = \frac{K}{s(s^2+6s+5)+K} = \frac{K}{s^3+6s^2+5s+K}$$

Forming the Routh array $q(s) = ?$

$$q(s) = s^3 + 6s^2 + 5s + K$$

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 6 & K \\ s^1 & \frac{30-K}{6} & 0 \\ s^0 & K & \end{array}$$

$$\frac{30-K}{6} > 0 \qquad 30 - K > 0$$

$$K < 30$$

$$\underline{K > 0}$$

$$0 < K < 30$$