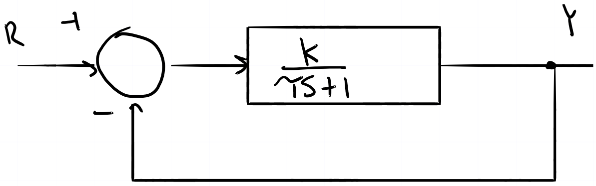
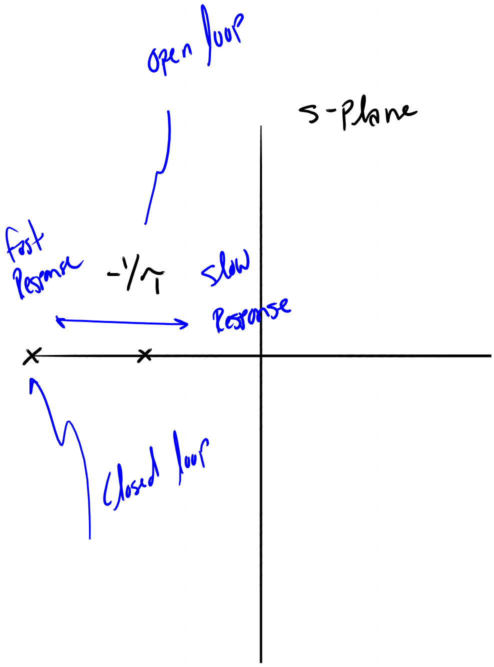
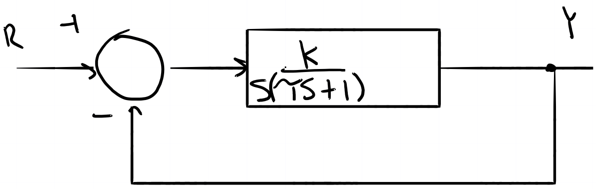
Root Locus



$$\frac{Y(s)}{R(s)} = \frac{K}{\tau s+1+K} = \frac{K/\tau}{s+1/\tau(1+K)}$$

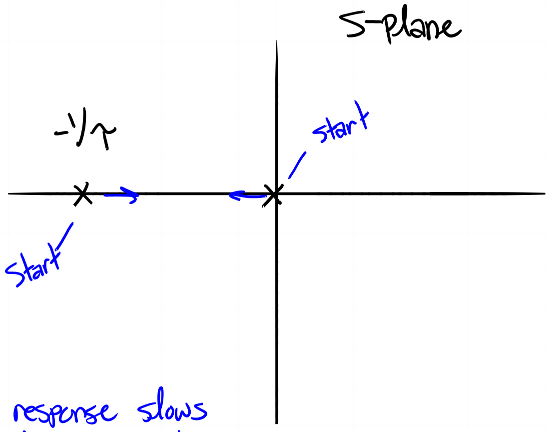


* As we increase K, the pole keeps moving to the left.



$$\frac{Y}{R} = \frac{K}{s(\tau s+1)+K} = \frac{K}{\tau s^2+s+K}$$

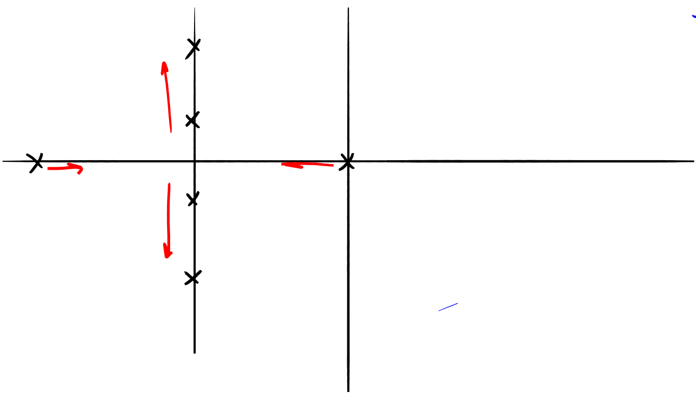
$$\frac{-1 \pm \sqrt{1-4\tau K}}{2\tau}$$



* As K increases, they will converge.

* However, when we make K larger, the response slows down. The root on the right is the dominant root.

If is is large enough, the roots will diverge off of the real axis as shown below.



The further we are from the axis represents more and more overshooting the system will do.

Root Locus Method

$$\frac{Y(s)}{C(s)} = \frac{G}{1+GH}$$

$\Rightarrow \quad 1 + GH = 0$ We can rewrite this as a ratio of polynomials

$$1 + \frac{K\,n(s)}{d(s)} = 0$$

$\Rightarrow \quad d(s) + K\,n(s) = 0$

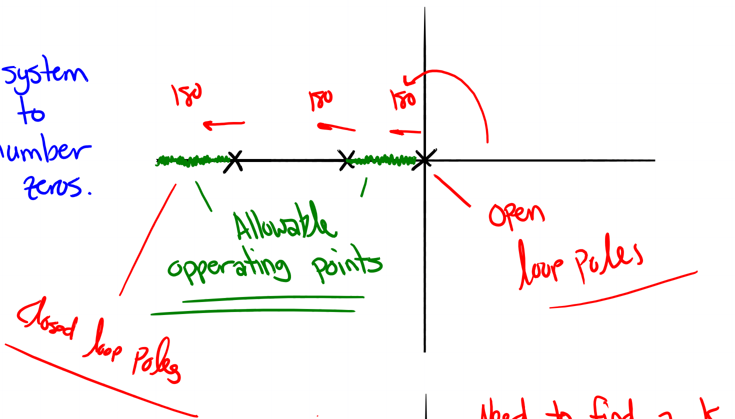
if $d(s) = 0$ → this means the locus starts at the open loop poles.

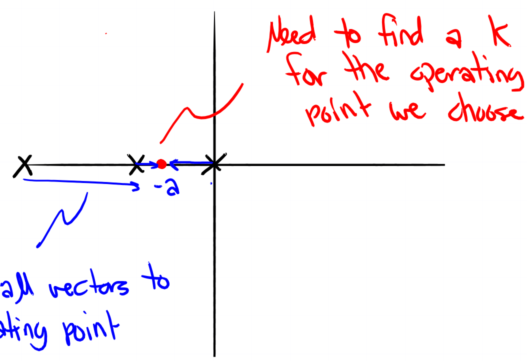
Now if we rewrite $|1| \angle 180°$ in polar form

$$\frac{K\,n(s)}{d(s)} = -1 = -1 + j0$$

This shows that we have an angle criteria of 180°.

For a negative feedback system real axis locus exists to the left of an odd number of real axis poles or zeros.



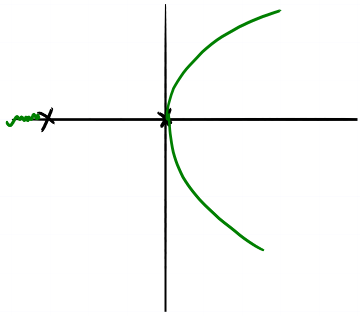


$$|K| = \frac{|d(s)|}{|n(s)|}$$

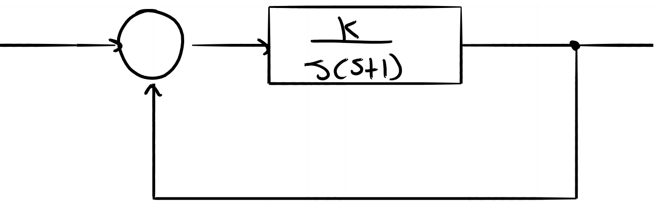
$d(s) = r_1 \cdot r_2 \cdot r_3$ magnitudes

If we have no zeros, we divide by 1. $n(s) = 1$.

If you have a double root such as 2 poles at 0, we would get



Example



$$\frac{Y}{R} = \frac{K}{s^2+s+K}$$

After simplifying

Open loop poles are found using G.

$$s^2+s+K=0 \Rightarrow \frac{-1 \pm \sqrt{1-4K}}{2}$$

In this case it just happens that G & GH both have poles at 0 & -1 because we have unity feedback.

if $K=0 \rightarrow$ the poles are 0 & -1

if $K=1/4 \rightarrow$ poles are at $-1/2$ for both

Closed loop poles converge

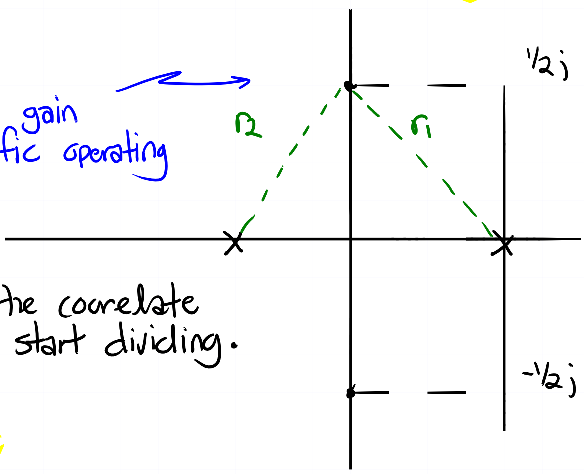
↓

Poles that are found using GH

The parameter K is typically k, but it does not have to be.

$$K = \frac{|d(s)|}{|n(s)|} = r_1 r_2$$

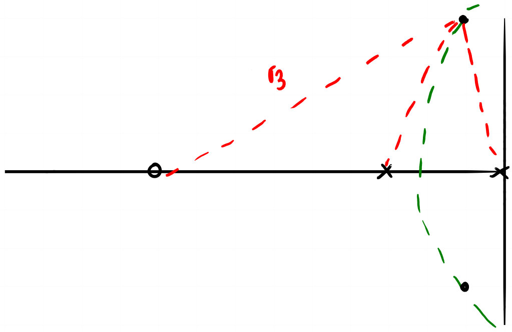
How to find gain for a specific operating point.



if we have s values in numerator, the correlate with n(s) terms. That is when we start dividing.

$$\frac{n(s)}{d(s)} = \frac{\text{open loop poles}}{\text{open loop zeroes}}$$

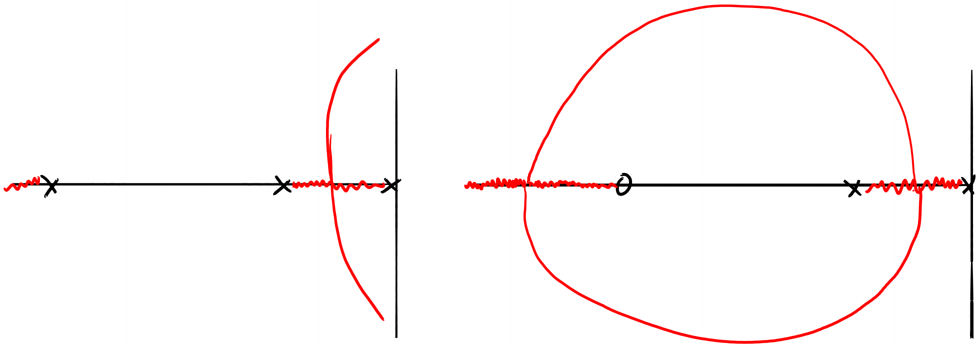
$\frac{k+2}{s(s+1)}$



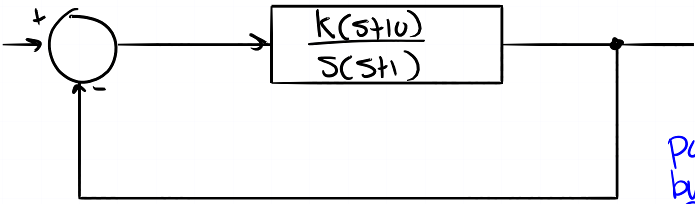
$\Rightarrow k = \frac{d(s)}{n(s)} = \frac{r_1 r_2}{r_3}$

Dominant Poles

Poles tend to repel locus, zeroes act like sinks & attract locus



Example



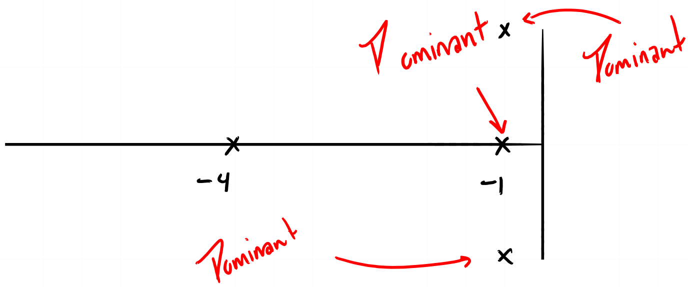
$\frac{Y}{S} = \frac{K(s+10)}{s(s+1)+K(s+10)}$

Poles go all over the place but zeroes are always fixed.

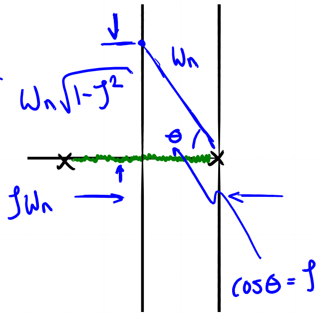
On the s plane, values far from the jw axis act quick, those close move slow. Therefore poles close to the axis are dominant because they act longer

↳ if you have a pole 4 to 5 times away, you can say the pole is dominant.

This holds true for complex pole pairs as well.



This is found by solving



$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

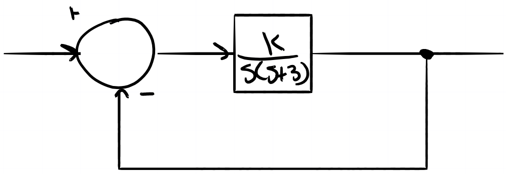
$$\frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

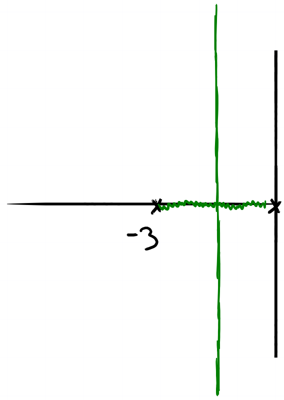
$$\Rightarrow \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2-1)}}{2} = \frac{-2\zeta\omega_n \pm 2\omega_n\left(j\sqrt{1-\zeta^2}\right)}{2}$$

$$\Rightarrow \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Example

$$KG(s) = K\frac{1}{s(s+3)}$$





Specifications

- Steady state error in response to a step ($1/s$) = 0
- Steady state error in response to a ramp < 1
- Percent overshoot < 5%
- Peak time < 2.5 seconds

Percent Overshoot: $P.O. = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \cdot 100 < 5\%$

$$\Rightarrow e^{-(\zeta\pi/\sqrt{1-\zeta^2})} = \frac{5}{100}$$

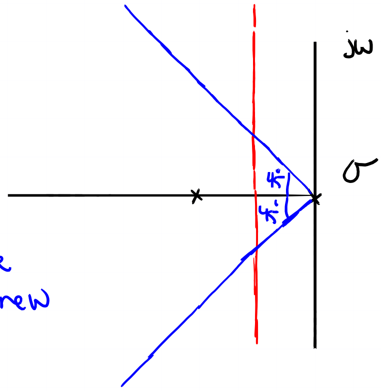
$$\Rightarrow \underline{\zeta = 0.69}$$

$\cos\theta = \zeta \Rightarrow \underline{\theta = 46.4^\circ}$ ~ if we pick a point anywhere below the line this creates, we will meet the P.O. requirement.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad \text{where} \quad \zeta\omega_n = 1.5$$

$$\omega_n = \frac{1.5}{\zeta} = 2.2$$

$\zeta = 0.707$, we are using a new ζ.



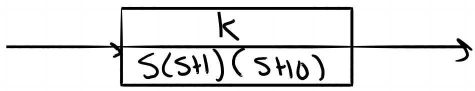
S.S. To a Ramp

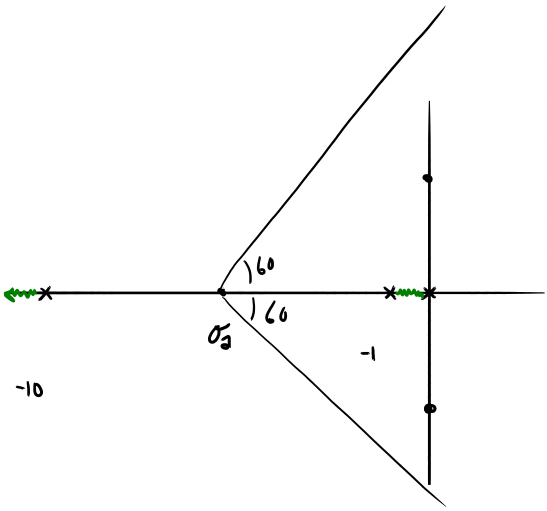
$$\lim_{s\to 0} s\left[\frac{1}{s^2}\cdot\frac{1}{1+KG(s)}\right] = \lim_{s\to 0}\frac{1}{s}\,\frac{s(s+3)}{s^2+3s+k} = \underline{\underline{\frac{3}{K}}}$$

$$\frac{3}{k} < 1 \Rightarrow \underline{\underline{k > 3}} \quad \Big\}\ \text{we know}\ k = \omega_n^2 = \underline{2.12^2 > 3}$$

Example

$$T = \frac{\left(\frac{K}{s(s+1)(s+10)}\right)}{\left(1+\frac{K}{s(s+1)(s+10)}\right)}$$





Step 7

$$\frac{-11-0}{3-0}$$

ϵ = # poles − # zeros

s^3	1	10
s^2	11	K
s^1	$\frac{110-k}{11}$	0
s^0	K	

$k > 110$

if $k = 110 \Rightarrow 11s^2 + 110 = 0$

$$s = \sqrt{-\frac{110}{11}} \approx \pm 3$$

Step 9

$$\frac{k}{s(s+1)(s+10)} = -1$$

$$\Rightarrow \quad k = -\, s(s+1)(s+10)$$

$$\Rightarrow \quad k = -(s^3 + 11s^2 + 10s)$$

$$\Rightarrow \quad \frac{dk}{ds} = -(3s^2 + 22s + 10)$$

Departure Angle Example