Lead Compensation Design

$$G_c = k \frac{(s/z + 1)}{(s/p + 1)}$$

Time Constant Form. Tends to be easier to design with when in the frequency domain.

Adds negative phase (lag)

Adds positive phase (lead)

Adjusts the zero dB crossing point.

Requirements:

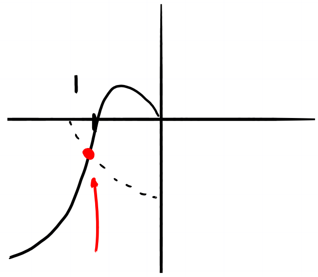
1) Adds no gain at the zero dB crossover frequency
2) Adds phase lead (positive phase) at the crossover frequency
3) Can be analytically designed

$$\angle G_c = \angle k \frac{(j\omega/z + 1)}{(j\omega/p + 1)} = \angle \frac{|\,|e^{j\theta_z}}{|\,|e^{j\theta_p}} = \theta_z - \theta_p$$

$$\Rightarrow \angle G_c = \tan^{-1} \frac{\omega/z}{1} - \tan^{-1} \frac{\omega/p}{1}$$

It is important to divide by one to remember that this is quadrant sensitive.

We want to add a phase 'hump' at the point where the curve on a nyquist plot crosses $r=1$.



$$\frac{d\angle G_c}{d\omega} = \frac{1/z}{1+\omega^2/z^2} - \frac{1/p}{1+\omega^2/p^2} = 0$$

$$\Rightarrow \frac{d\angle G_c}{d\omega} = \frac{1/z\,(1+\omega^2/p^2) - 1/p\,(1+\omega^2/z^2)}{(1+\omega^2/z^2)(1+\omega^2/p^2)} = 0$$

$$\Rightarrow \frac{d\angle G_c}{d\omega} = \frac{1}{z}(1+\omega^2/p^2) - \frac{1}{p}(1+\omega^2/p^2) = 0$$

$$\Rightarrow \omega^2\left[\frac{1}{zp^2} - \frac{1}{pz^2}\right] = \frac{1}{p} - \frac{1}{z}$$

$$\Rightarrow \omega^2\left[\frac{z-p}{z^2p^2}\right] = \frac{z-p}{zp}$$

$$\Rightarrow \omega^2 = \frac{z-p}{zp}\ \frac{z^2p^2}{z-p} = zp$$

$\Rightarrow \omega = \sqrt{zp}$ let this $\omega = \omega_c$ $\rightarrow$ $\omega_c = \sqrt{zp}$

$$\angle G_c(j\omega_c) = \tan^{-1}\frac{\omega_c}{z} - \tan^{-1}\frac{\omega_c}{p} = \tan^{-1}\frac{\sqrt{zp}}{z} - \tan^{-1}\frac{\sqrt{zp}}{p}$$

$$= \tan^{-1}\sqrt{\frac{p}{z}} - \tan^{-1}\sqrt{\frac{z}{p}}$$

Using tables: $\tan^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right) + 90°$

$\Rightarrow \angle G_c(j\omega_c) = 90° - 2\tan^{-1}\sqrt{\frac{z}{p}}$ (in degrees)

Example

$$G_c = \frac{\frac{j\omega}{z}+1}{\frac{j\omega}{p}+1} \qquad |G_c| = k\frac{\sqrt{1+\omega^2/z^2}}{\sqrt{1+\omega^2/p^2}} \qquad k = \frac{1}{|G_c(j\omega)|}$$

$$\Rightarrow k = \frac{\sqrt{1+z/p}}{\sqrt{1+p/z}} = \sqrt{\frac{z}{p}}$$

Procedure

1) Plot open loop Bode Plot of the uncompensated system & determine gain & phase margins.

2) Check Gain margin. If it is less than specified reduce overall system by an K_{gm} until the specification is met. Replot new system $K_{gm}G$.

3) Determine the 0 dB crossover frequency ω_c for either G or $K_{gm}G$ as necessary.

4) Determine the additional phase margin Θ_m needed to meet the specification.

$$\Theta_m = \Theta_{DESIRED} - \phi_m$$

Existing uncompensated phase margin

$$\Theta_m = 90 - 2\tan^{-1}\sqrt{\frac{z}{p}}$$

5) $\omega_c = \sqrt{zp}$

6) Solve for z & p

7) Solve for k

8) The controller is $K_{gm}\,k\left(\frac{s/z+1}{s/p+1}\right)$

Example

Given $G = \dfrac{200}{s(s+2)(s+20)}$

Design a controller such that

$\phi_m = 60°$

$g_m > 10$ dB

steady-state in response to a step input is zero.

Based off Bode plots:

$$\sqrt{\frac{z}{p}} = \tan^{-1}\left[\frac{90° - 33°}{2}\right] = 0.543$$

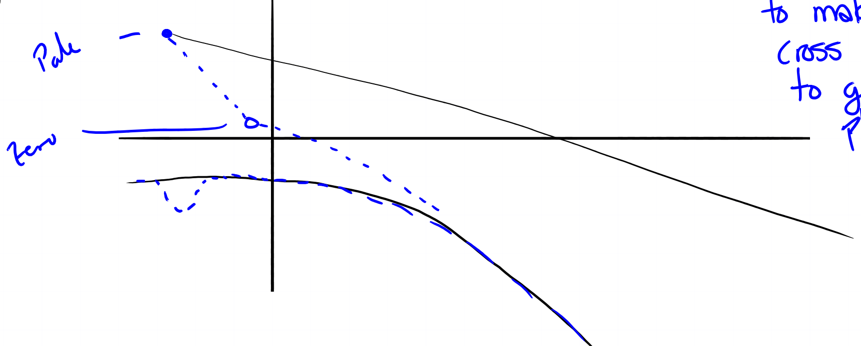
$\omega_c = 0.285$

$z_p = 8.12$ $\qquad \dfrac{z}{p} = 0.295 \quad \Rightarrow \quad \underline{p = 5.25 \quad z = 1.55}$

$$K = \sqrt{\frac{z}{p}} = 0.543$$

$$G_c = 0.543\,\frac{s/1.55 + 1}{s/5.25 + 1}$$

Lag Compensation



Add pole & zero to make magnitude cross x-axis sooner to get desired phase margins.