Systems of Linear Equations and Matrices

Rules of Matrix Arithmetic

1) $A \pm B = B \pm A$
2) $(A \pm B) \pm C = A \pm (B \pm C)$
3) $A(BC) = (AB)C$
4) $A(B \pm C) = (AB \pm AC)$
5) $(B \pm C)A = BA \pm CA$
6) $a(A \pm B) = aA \pm aB$
7) $ab(C) = a(bC)$
8) $a(BC) = (aB)C = B(aC)$

   A product of invertible matrices is always invertible, and the inverse of the product is the product of the inverse in reverse order.

Elementary Matrices and Method for Finding A^{-1}

Definition > An nxn matrix is called an *elementary matrix* if it can be obtained from the nxn identity matrix by performing a single elementary row operation.

Theorem > If the elementary matrix E results from performing a certain row operation on I_m and if A is an mxn matrix, the product EA is the matrix that results when this same row operation is performed on A.

Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results in adding 3 times the first row of I_3 to the third row. This product EA is then

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

Remark: Theorem 8 is primarily of theoretical interest and will be used for developing some results about matrices of linear equations.

Theorem > Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Matrices that can be obtained from one another by a finite sequence of elementary row operations are said to be *row equivalent*.

The inverse of a matrix can be found by multiplying it by a sequence of elementary matrices until the identity matrix is created.

$$A^{-1} = E_1 \cdots E_n I_n$$

Further Results on Systems of Equations and Invertibility

If we have a square, invertible matrix and an equation

$$AX = B$$

then we can say

$$X = A^-1B$$

Determinants

The Determinant Function

Elementary Product

> Any product of n entries from a matrix A, no two of which come from the same column.

Determinant Function

> Let a be a square matrix. The determinant function is denoted by det and we define $det(A)$ to be the sum of all signed elementary products from A.

Evaluating Determinants by Row Reduction

Theorem 1

> If A is any square matrix that contains a row of zeros, then $det(A) = 0$

Theorem 2

> If A is an nxn triangular matrix, then $det(A)$ is the product of the entries on the main diagonal; that is; $det(A) = a_{11}a_{22} \cdots a_{nn}$

Properties of the Determinant Function

Theorem 4

> If A is any square matrix, then $det(A) = det(A^T)$

In general $det(A + B) \neq det(A) + det(B)$, however;

Theorem 5

> If A and B are square matrices of the same size, then $det(AB) = det(A)det(B)$

Theorem 6

> A matrix is invertible iff $det(A) \neq 0$

Cofactor Expansion; Cramer's Rule

Definition: minor entry and cofactor

> If A is a square matrix, then the minor entry a_{ij} is denoted by M_{ij} and is defined to be the determent few the submatrix that remains after the i_{th} row and j_{th} column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij}

$$det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

Definition

> Any nxn matrix made by the cofactors of a_{ij} is called the matrix of cofactors from A. The transpose of this matrix is called the **adjoint** of A and is denoted $adj(A)$.

Theorem 8

> If A in an invertible matrix then

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

Determinants can be computed by multiplying the entries in the first column of A by their cofactors and adding the resulting products. This method is called **cofactor expansion**.

Vector Spaces

Basis and Dissension

You can show a system of linear equations are linearly independent and span R^n by showing the matrix has a non-zero determinant.

Inner Product Spaces

Theorem 15

If u and v are vectors in an inner product space V, then

$$< u, v > \leq < u, u > < v, v >$$

Length and Angle In Inner Product Spaces

Norm: $||u|| =< u, u >^{1/2}$

Distance: $d(u, v) = ||u - v||$

Angle: $cos(\theta) = \frac{<u,v>}{||u||\,||v||}$ $\ and\ $ $0 \leq \theta \leq \pi$

Theorem 17 (Generalized Theorem of Pythagoras): If u and v are orthogonal vectors in an inner product space, then

$$||u + v||^2 = ||u||^2 + ||v||^2$$

Orthonormal Bases; Gram-Schmidt Process

Definition

A set of vectors in an inner product is called an *orthogonal set* if any two distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is called *orthonormal*.

Gram-Schmidt Process

In order to find a orthonormal basis for $\mathbb{R}^n$ consider the following

1. $v_1 = \frac{u_1}{||u_1||}$
2. $v_2 = \frac{u_2 - <u_2, v_1> v_1}{||u_2 - <u_2, v_1> v_1||}$
3. $v_3 = \frac{u_3 - <u_3, v_1> v_1 - <u_3, v_2> v_2}{||u_3 - <u_3, v_1> v_1 - <u_3, v_2> v_2||}$
4. $v_4 = \frac{u_4 - <u_4, v_1> v_1 - <u_4, v_2> v_2 - <u_4, v_3> v_3}{||u_4 - <u_4, v_1> v_1 - <u_4, v_2> v_2|| - <u_4, v_3> v_3}$
5. $\vdots$

Theorem 23 (Best Approximation Theorem):

If W is a finite dimensional subspace of an inner product space V, and if u is a vector in V, then $proj_w u$ is the best approximation to u from w in the sense that

$$||u - proj_W u|| < ||u - w||$$

for every vector w in W different from $proj_W u$.

Coordinates; Change of Basis

Theorem 24: If $S = v_1, v_2, \cdots, v_n$ is a basis for a vector space V, then every vector in v in V can be expressed in the form $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ in exactly one way.

The **coordinate vector** of v relative to S is denoted by $(v)_S$ and is the vector in $\mathbb{R}^n$ defined by:

$$(v)_S = (c_1, c_2, \cdots, c_n)$$

The **coordinate matrix** of v relative to S is denoted by $[v]_S$ and is the $nx1$ matrix defined by:

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

If we change the basis for a vector space V from some old basis B to some new basis B' then the old coordinate matrix $[v]_B$ of a vector v is related to the new coordinate matrix $[v]_{B'}$ by the equation

$$[v]_B = P[v]_{B'}$$

where the columns of P are the coordinate matrices of the old basis vectors relative to the new basis. P is formally known as the **transition matrix** from B to B'.

Theorem 26:

If P is the transition matrix from basis B to basis B' then:

1. P is invertible
2. P^{-1} is the transition matrix from B' to B

Theorem 27:

If P is the transition matrix from one orthonormal basis to another for an inner product space then

$$P^{-1} = P^T$$

Definition:

A square matrix with the property

$$A^{-1} = A^T$$

is said to be an orthogonal matrix.

Theorem 28:

1. A is orthogonal
2. The row vectors of A form an orthonormal set in R^n with the Euclidean inner product
3. The column vectors of A for an orthonormal set in R^n with the Euclidean inner product

References

- Elementary Linear Algebra - Howard Anton