Bus Charging Schedule Simulated Annealing with MILP Constraints

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Contents

1 Preliminaries

1.1 Mixed Integer Linear Program

A mixed integer linear programming (MILP) problem is a class of constrained optimization in which one seeks to find a set of continuous or integer values that maximizes or minimizes an objective function while satisfying a set of constraints [?]. Given an objective function J, decision variables (i.e. variables of optimization) $x_j \in \mathbb{R}$ and $y_k \in \mathbb{Z}^+$, and input parameters $c_j, d_k, a_{ij}, g_{ik}, b_i \in \mathbb{R}$, a MILP has the mathematical structure represented in Equation 1 [?].

$$\begin{array}{lll} \text{Maximize} & J = \sum_j c_j x_j + \sum_k d_k y_k & \\ \text{subject to} & \sum_j a_{ij} x_j + \sum_k g_{ik} y_k \leq b_i & (i = 1, 2, ..., m) \\ & x_j \geq 0 & (j = 1, 2, ..., n) \\ & y_k \in \mathbb{Z}^+ 0 & (k = 1, 2, ..., n) \end{array} \tag{1}$$

This formulation of the MILP is also referred to as "crisp". By this it is meant that each variable in the formulation acts as an injective mapping to its number representation. In other words, no values on the formulation are fuzzy [?].

1.2 Fuzzy Sets Theory

This section introduces the notion of fuzzy numbers and some basic definitions. Concepts from this section are pulled from [?, ?, ?, ?].

1.2.1 Fuzzy Sets

Let's begin with what a fuzzy number is not. A classical (crisp) set is defined as a collection of elements $x \in X$. Crisp sets are binary, either an element either belongs in the set, or it does not [?]. For a fuzzy set, what is known as the characteristic functions applies various degrees of membership for elements of a given set.

[?].

Definition 1.1 *Let X be a collection of objects (often called the universe of discourse [?]). If X is denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs as shown in Equation 2.*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\} \tag{2}$$

$\mu_{\tilde{A}}$ is called the membership function where $\mu_{\tilde{A}}$ is the mapping $\mu_{\tilde{A}} : X \to [0,1]$; which assigns a real number to the interval $[0,1]$. The value of $\mu_{\tilde{A}}$ represents the degree of membership of x in $\tilde{A}$.

This paper will use fuzzy sets defined on the real numbers $\mathbb{R}$. The membership function describes the shape of the fuzzy number. As an example, consider the following definition.

Definition 1.2 *A fuzzy number that is represented by $\tilde{A} = (a, b, c)$ is said to be triangular if its membership function is defined as Equation 3.*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)} & a \le x \le b \\ \frac{(d-x)}{(d-b)} & c \le x \le d \\ 0 & otherwise \end{cases} \tag{3}$$

Definition 1.3 *The fuzzy set tildeA in $\mathbb{R}$ is normal if $max_x \mu_{\tilde{A}}(x) = 1$.*

Definition 1.4 *A fuzzy set $\tilde{A}$ in $\mathbb{R}$ is convex if and only if the membership function of $\tilde{A}$ satisfies the inequality*

$$\mu_{\tilde{A}}[\beta x_1 + (1-\beta)x_2] \ge min[\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)] \ \forall x_1, x_2 \in \mathbb{R} \ \beta \in [0,1]$$

Definition 1.5 *A fuzzy number is a normal convex fuzzy set in $\mathbb{R}$.*

Definition 1.6 *The triangular fuzzy number $\tilde{A}$ is nonnegative $\iff$ $a \ge 0$.*

A more general definition of fuzzy numbers is known as LR fuzzy numbers [?, ?].

Definition 1.7 *A function $L : [0, \infty] \to [0,1]$ (or $R : [0, \infty] \to [0,1]$) is said to be reference a function of the fuzzy number if and only if*

1. *$L(0) = 1$ (or $R(0) = 1$)*
2. *L (or R) is non-increasing on $[0, \infty)$*

Definition 1.8 *A fuzzy number A defined on the let of real numbers, $\mathbb{R}$, denoted as $(m, n, \alpha, \beta)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{tildeA}(x)$ is defined as*

$$\mu_{tildeA}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & x \le m, \alpha > 0 \\ R(\frac{m-n}{\beta}) & x \ge m, \beta > 0 \\ 1 & m \le x \le n \end{cases} \tag{4}$$

Definition 1.9 *An LR flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be a non-negative LR flat fuzzy number if and only if $m - \alpha \ge 0$ and is said to be non-positive LR flat fuzzy number if and only if $m - \alpha \le 0$ is a real number.*

Definition 1.10 *An LR flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be an unrestricted LR flat fuzzy number if and only if $m - \alpha$ is a real number.*

1.2.2 Fuzzy Arithmetic

If two triangular fuzzy numbers $\tilde{a}_1 = \{a_1, a_2, a_3\}$ and $\tilde{b}_1 = \{b_1, b_2, b_3\}$ are nonnegative then the following operations are defined in Equation 5.

$$\begin{aligned} \tilde{a} \oplus \tilde{b} &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ \tilde{a} \ominus \tilde{b} &= (a_1 + b_3, a_2 + b_2, a_3 + b_1) \\ \tilde{a} \otimes \tilde{b} &= (a_1 b_1, a_2 b_2, a_3 b_3) \end{aligned} \tag{5}$$

1.2.3 Comparing Fuzzy Numbers

Fuzzy numbers do not directly provide a method of ordering nor do they always provide an ordered set like real numbers [?]. There are multiple methods for ordering fuzzy numbers, each coming with advantages and disadvantages. Different properties have been applied to justify comparison of fuzzy numbers, such as: preference, rationality, and robustness [?, ?, ?]. These methods are commonly known as ranking functions or ordering functions [?, ?, ?]. Commonly, including in this work, the First index of Yager [?] is used (Equation 6).

$$\Re(\tilde{A}) = \frac{\sum_i a_i}{|\tilde{A}|} \tag{6}$$

where $|\cdot|$ represents the cardinality of the fuzzy number. In words, Equation 6 is merely the average of the values in the fuzzy number.

1.3 Fully Fuzzy Linear Programming

Much like the MILP, fully fuzzy linear programs (FFLP) it is a class of constrained optimization in which one seeks to find a set of continuous variables that either maximizes or minimizes an objective function, J, while satisfying a set of constraints. The key difference in FFLP is that it is designed to accommodate imprecise information [?, ?]. In FFLP, the parameters and decision variables are fuzzy and linear. A general FFLP is represented as shown in Equation 7. The subscripts $\cdot_e$, $\cdot_l$, and $\cdot_g$ indicate to equality, less than, and greater than constraints, respectively.

$$\begin{array}{lll} \text{Maximize} & J = \sum_j \tilde{C}_j \otimes \tilde{X}_j & \\ \text{subject to} & \sum_j \tilde{a}_{ej} \otimes \tilde{x}_j = \tilde{b}_e & \forall e = 1, 2, 3, ... \\ & \sum_j \tilde{a}_{lj} \otimes \tilde{x}_j \leq \tilde{b}_l & \forall l = 1, 2, 3, ... \\ & \sum_j \tilde{a}_{gj} \otimes \tilde{x}_j \geq \tilde{b}_l & \forall g = 1, 2, 3, ... \end{array} \tag{7}$$

There are many methods of solving FFLP [?, ?, ?, ?]; however, most solution methods convert the fuzzy model into a crisp model that can be solved using traditional methods [?]. In [?, ?], the method of converting the FFLP into a crisp MILP is simply by applying the ranking function to the objective function and breaking the constraints down into a set of crisp constraints as shown in Equation 8. The constraints are separated according to the definition of fuzzy set multiplication defined in Equation 5. The fuzzy number index is represented is the exponent rather than the subscript to clearly distinguish between the indexed value in the fuzzy number and the constraint index (i.e. $\tilde{A} = (a^1, a^2, a^3)$). Furthermore it is assumed that the fuzzy numbers are nonnegative.

$$
\begin{array}{lll}
\text{Maximize} & J = \Re\Big(\sum_j (c_j^1, c_j^2, c_j^3)(x_j^1, x_j^2, x_j^3)\Big) & \\
\text{subject to} & \sum_j a_{ej}^1 x_j^1 = b_e^1 & \forall e = 1,2,3,... \\
& \sum_j a_{lj}^1 x_j^1 \leq b_l^1 & \forall l = 1,2,3,... \\
& \sum_j a_{gj}^1 x_j^1 \geq b_l^1 & \forall g = 1,2,3,... \\
& \sum_j a_{ej}^2 x_j^2 = b_e^2 & \forall e = 1,2,3,... \\
& \sum_j a_{lj}^2 x_j^2 \leq b_l^2 & \forall l = 1,2,3,... \\
& \sum_j a_{gj}^2 x_j^2 \geq b_l^2 & \forall g = 1,2,3,... \\
& \sum_j a_{ej}^3 x_j^3 = b_e^3 & \forall e = 1,2,3,... \\
& \sum_j a_{lj}^3 x_j^3 \leq b_l^3 & \forall l = 1,2,3,... \\
& \sum_j a_{gj}^3 x_j^3 \geq b_l^3 & \forall g = 1,2,3,... \\
& x_j^2 - x_j^1 \geq 0 & x_j^3 - x_j^2 \geq 0
\end{array}
\tag{8}
$$

To be more succinct, the FFLP can also equivalently be written as Equation 9.

$$
\begin{array}{lll}
\text{Maximize} & J = \Re\Big(\sum_j (c_j^1, c_j^2, c_j^3) \otimes (x_j^1, x_j^2, x_j^3)\Big) & \\
\text{subject to} & \sum_j a_{ej}^k x_j^k = b_e^k & \forall e = 1,2,3,... \\
& \sum_j a_{lj}^k x_j^k \leq b_l^k & \forall l = 1,2,3,... \\
& \sum_j a_{gj}^k x_j^k \geq b_l^k & \forall g = 1,2,3,... \\
& x_j^2 - x_j^1 \geq 0 & x_j^3 - x_j^2 \geq 0 \\
& \forall k \in \{1,2,...\} &
\end{array}
\tag{9}
$$

Where k has a max value equal to the cardinality to the type of fuzzy number being utilized. This can be further be elaborated on by rewriting the inequality constraints as equality constraints by introducing slack as equality constraints by introducing slack as equality constraints by introducing slack variables. This is useful as it represents the formulation in a standard form [?, ?]. It also has the slightly less useful benefit of (mostly) providing the solver a set of equations called a hyperplane [?].

The given method is called the Kumar and Kaurs method. Generally speaking, it is designed to solve FFLP problems with inequality constraints having LR flat fuzzy numbers. Given the FFLP Equation 7 and assuming that $\tilde{x}_j$ is an LR flat fuzzy number, the problem can be reformulated as Equation 10.

$$
\begin{array}{lll}
\text{Maximize} & J = \sum_j \tilde{C}_j \otimes \tilde{X}_j & \\
\text{subject to} & \sum_j \tilde{a}_{ej} \otimes \tilde{x}_j = \tilde{b}_e & \forall e = 1,2,3,... \\
& \sum_j \tilde{a}_{lj} \otimes \tilde{x}_j \oplus \tilde{S}_l = \tilde{b}_l \oplus \tilde{S'}_l & \forall l = 1,2,3,... \\
& \sum_j \tilde{a}_{gj} \otimes \tilde{x}_j \oplus \tilde{S}_e = \tilde{b}_l \oplus \tilde{S'}_g & \forall g = 1,2,3,... \\
& \Re(\tilde{S}_l) - \Re(\tilde{S}_l') \geq 0 & \forall l = 1,2,3,... \\
& \Re(\tilde{S}_g) - \Re(\tilde{S}_g') \leq 0 & \forall g = 1,2,3,...
\end{array}
\tag{10}
$$

Expanding the set of equation and using the condensed notation in Equation 9 we find Equation 11.

$$
\begin{array}{lll}
\text{Maximize} & J = \Re\Big(\sum_j (c_j^1, c_j^2, c_j^3) \otimes (x_j^1, x_j^2, x_j^3)\Big) & \\
\text{subject to} & \sum_j a_{ej}^k x_j^k = b_e^k & \forall e = 1,2,3,... \\
& \sum_j a_{lj}^k x_j^k s_l^k \leq s_l'^k b_l^k & \forall l = 1,2,3,... \\
& \sum_j a_{gj}^k x_j^k s_g^k \geq s_l'^k b_l^k & \forall g = 1,2,3,... \\
& \Re(\tilde{S}_l) - \Re(\tilde{S}_l') = 0 & \forall l = 1,2,3,... \\
& \Re(\tilde{S}_g) - \Re(\tilde{S}_g') = 0 & \forall g = 1,2,3,... \\
& x_j^2 - x_j^1 \geq 0 & x_j^3 - x_j^2 \geq 0 \\
& s_j^2 - s_j^1 \geq 0 & s_j^3 - s_j^2 \geq 0 \\
& s_j'^2 - s_j'^1 \geq 0 & s_j'^3 - s_j'^2 \geq 0 \\
& \forall k \in \{1,2,...\} &
\end{array}
\tag{11}
$$

2 The Crisp BAP and PAP