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R. R. K. Sharma
Indian Institute of Technology Kanpur

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Theory and Methodology

A new dual based procedure for the transportation problem

R.R.K. Sharma *, K.D. Sharma

Department of Industrial and Management Engineering, Indian Institute of Technology, Kanpur 208016, India

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Abstract

We pose the transportation problem in a slightly different form to obtain a dual problem that has a special structure. We exploit the structure of the dual problem to give a new heuristic that runs in $O(cn^2)$ time (n is number of nodes in the network and c is a constant) to improve the dual solution. Our heuristic obtained the optimal solution to several small sized transportation problems that we attempted; and for large sized problems it obtained solutions that were within 82% of the optimal solution on average. Our approach gives good starting solutions for dual based approaches used for solving transportation problems and thus is expected to enhance their computational performance.

Keywords: Transportation; Distribution; Transportation problem

1. Introduction

Transportation problem is as follows. Given are I plants and K markets with cost of shipping unit goods from any plant i to any market k being known. Each plant has known supply and each market has known demand such that sum of all supplies is equal to sum of all market demands. Problem is to prepare a minimal cost shipment plan from plants to markets such that demand at all market points are met without exceeding the supply available at any plant. Mathematically the transportation problem is as given below. In this paper we use the index i for plants/warehouses and index k for the markets.

1.1. Constants of the problem

The number of markets is K and the number of plants is I. D_k is the demand at market k, and d_k is the demand at market point k as a fraction of total market demand. Hence, $d_k = D_k/[\sum_{k=1}^{K} D_k]$, and obviously we have $\sum_{k=1}^{K} d_k = 1$. C_{ik} is the cost of shipping $[\sum_{k=1}^{K} D_k]$ units of goods from plant number i to market number k. B_i is the supply available at plant i, and b_i is the supply available at plant i as a fraction of total market demand, i.e., $b_i = B_i/[\sum_{k=1}^{K} D_k]$, and we assume $\sum_{i=1}^{I} b_i = 1$.

* Corresponding author. Tel.: +91 512 25 7376; fax: +91 512 25 0260.

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In this paper we are considering transportation problems in which total supply from all plants is equal to the total demand at all markets.

1.2. Decision variables

X_{ik} is the quantity received as a fraction of total market demand at market k from plant i.

1.3. Conventional formulation of the transportation problem

Problem Q:

$$\min \quad \sum_k \sum_i X_{ik} C_{ik}$$

s.t.

$$\sum_k X_{ik} = b_i \quad \text{for all } i = 1, \ldots, I, \tag{1}$$

$$-\sum_i X_{ik} = -d_k \quad \text{for all } k = 1, \ldots, K, \tag{2}$$

$$X_{ik} \geqslant 0 \quad \text{for all, } i = 1, \ldots, I \text{ and } k = 1, \ldots, k. \tag{3}$$

The problem Q is a well known transportation problem and we give a brief literature review in Section 2; and we find that well known primal approaches solve the problem Q in $O(n^3 \log(n))$ time. In Section 3 we give a different formulation of the transportation problem (problem P). In Section 4 we examine the special structure of its dual (problem DP) to develop a $O(cn^2)$ procedure for improving the solution of problem DP. In Section 5 we solve three small sized transportation problems to illustrate the solution procedure given in Section 4. In Section 6 we give our computational experience. In Section 7 we give a method to obtain a feasible solution to the dual of problem Q that is as close to the optimal solution as the solution obtained for problem DP. In Appendix A we give ten small sized transportation problems for which the optimal solution was obtained by the methods described in this paper.

Thus in this paper we give a computationally efficient $O(cn^2)$ procedure to obtain good solution for the dual of the problem Q which can be improved further to reach optimal solution by well known dual based approaches given in Ali et al. [2].

2. Literature review

Recent works on primal simplex algorithm for min-cost flow problems have used advanced data structures, see Ref. [1]. Uncapacitated transportation problem (problem Q) is a special class of minimum cost flow network problem. Ahuja et al. [1] have documented that the uncapacitated minimum cost flow problem can be solved by enhanced capacity scaling algorithm in $O(n \log(n)) S(n, m)$ time (pp. 393, Theorem 10.34), where n is number of nodes and m is number of arcs in the network. For the transportation problem m is $O(n^2)$ and $S(n, m)$ is the running time for solving the shortest path problem with n nodes and m arcs and this is $O(m + n \log(n))$. Thus enhanced capacity scaling algorithm runs in $O(n^3 \log(n))$ time. Ahuja et al. [1] have also shown that the uncapacitated minimum cost flow problem can be solved by repeated capacity scaling algorithm in $O(n^2 \log(n) S(n, m))$ time (pp. 386, Theorem 10.24). Thus transportation problem is solved by the enhanced capacity scaling algorithm in $O(n^4 \log(n))$ running time. Ahuja et al. [1] have shown that the cost scaling algorithm for minimum cost flow problem runs in $O(n^3 \log(n))$ time. Thus the best primal based approach solves the uncapacitated transportation problem in $O(n^3 \log(n))$ time.

Now we briefly review dual based approaches to solve the minimum cost flow problems. Orlin [4] and Plotkin and Tardos [5] have developed polynomial time dual network simplex algorithms. Algorithm due to Plotkin and Tardos [5] runs in $O(m^3 \log(n))$ time (n is number of nodes and m is number of arcs in the network) and is considered more efficient. Ali et al. [2] have noted that dual based approaches take lesser number of pivots to reach optimal solution than the primal based approaches; but computational effort required per pivot is higher for dual based approaches. Hence, Ali et al. [2] have presented efficient implementations of the dual network simplex algorithm for the network flow problem which have resulted in superior performance. They have also given a reoptimization procedure where previous bases are used again to obtain optimal solution for a redefined problem with small changes in parameters such as capacity or cost. Ali et al. [2] have conducted an extensive experimental study that compared the running times of primal and dual based approaches for the minimum cost network flow problems, and found that dual based approaches perform significantly better. Bertsekas and Tseng [3] have developed a primal–dual algorithm that has excellent computational performance, but Ahuja et al. [1] have noted that this has poor worst case computational complexity.

Our approach to uncapacitated transportation problem is different from the approaches given above. We pose the uncapacitated transportation problem differently (problem P) and find that its dual (problem DP) has a special structure. We exploit this special structure to develop a computationally attractive $O(cn^2)$ dual based procedure to improve the solution of problem DP which is given in Section 4. Our procedure does not guarantee the optimal solution but empirical investigation revealed that it produced a very good solution. Well known dual based approaches ([3,5,2]) can use our good solution to get an advanced start while solving the simple transportation problem.

Although in Sections 3 and 4 we work on a different formulation of transportation problem, we give necessary transformations to obtain the advanced start solution for the conventional formulation of the transportation problem in Section 7.

3. A different formulation of the transportation problem

The transportation problem can also be set up as given below.

Problem P:

$$\min \quad \sum_k \sum_i X_{ik} C_{ik}$$

s.t.

$$\sum_k \sum_i X_{ik} = 1, \tag{4}$$

$$-\sum_i X_{ik} \geqslant -d_k \quad \text{for all } k = 1, \dots, K, \tag{5}$$

$$-\sum_k X_{ik} \geqslant -b_i \quad \text{for all } i = 1, \dots, I. \tag{6}$$

and Eq. (3)

Eq. (4) ensures that adequate quantities are shipped from the plants so that the entire market demand is met. Eq. (5) ensure that the quantity received at markets is not more than the demand at that point. It is to be noted that these equations will be satisfied as equality constraints due to Eq. (4). Eq. (6) ensure that shipments out of a plant is within its capacity. Eq. (3) are the non-negativity restrictions on X_{ik}'s. Objective function of problem P seeks to minimize the shipment cost over all feasible solutions to constraints (4), (5) and (6).

It is to be noted that constraints of in problem P are put differently. But the assumptions such as $\sum d_k = 1$ and $\sum b_i = 1$ ensure that problem P is a valid formulation of the transportation problem.

In Section 4 given below we show that dual of problem P has a special structure which can be exploited to develop a new solution procedure.

4. Solution method

We associate v_0, v_k and z_i as dual variables with constraints (4), (5) and (6) respectively, and write the dual of problem P as follows.

Problem DP:

$$\max \quad v_0 - \sum_{k=1}^{K} d_k v_k - \sum_{i=1}^{I} b_i z_i$$

s.t.

$$v_0 - v_k - z_i \leqslant C_{ik} \quad \text{for all } i = 1, \dots, I \text{ and } k = 1, \dots, K, \tag{7}$$

$$v_k \geqslant 0 \quad \text{for all } k = 1, \dots, K, \tag{8}$$

$$z_i \geqslant 0 \quad \text{for } i = 1, \dots, I, \tag{9}$$

$$v_0 \text{ unrestricted in sign.} \tag{10}$$

Our objective is to improve the solution to problem DP as far as possible. In problem DP first we set all z_i's equal to zero to obtain problem DP1. The problem DP1 is a special class of linear programming problem which has been solved by Sharma [7] in a different context. The solution procedure for problem DP1 is reproduced in Section 4.1. Later in problem DP we allow only one z_i to increase in value to obtain problem DP2. The solution procedure to solve problem DP2 is given in Section 4.2. In Sections 4.3 and 4.4 we give a solution procedure which attempts to improve the objective function value of problem DP by increasing values of more than one z_i at a time. In Section 4.5 we determine the extent of increase in values of these z_i's that increase in value. It is to be noted that all these procedures run in $O(n^2)$ time. Finally, in Section 4.6 we combine all the heuristics given in Sections 4.1–4.4 to obtain a procedure which seeks to improve the objective function value of problem DP in $O(cn^2)$ time where c is a constant.

Now we set $z_i = 0\ \forall i$ in problem DP and obtain problem DP1 as given below.

4.1. Optimal solution to problem DP with all $z_i = 0$, $i = 1, \ldots, I$

If we let $z_i = 0$, for $i = 1, \ldots, I$, then the problem DP has the following form.

Problem DP1:

$$\begin{aligned} \max \quad & v_0 - \sum_{k=1}^{K} d_k v_k \\ \text{s.t.} \quad & v_0 - v_k \leqslant C_{ik} \quad \text{for all } i = 1, \ldots, I, \\ & \qquad\qquad\qquad k = 1, \ldots, K, \\ & \text{Eqs. (8) and (10).} \end{aligned} \tag{11}$$

Sharma [7] has given an efficient algorithm to optimally solve problem DP1 and it is reproduced here for the sake of completeness.

4.1.1. Algorithm for solving the problem DP1

Algorithm A1

Step 1. Now find $C_k^* = \min_i(C_{ik})$ for all $k = 1, \ldots, K$ and remove all the redundant constraints (of Eq. (11)) in problem DP1. In case of a tie, that is $C_{i1\,k} = C_{i2\,k}$, only one equation is retained and all others are eliminated. Then the problem DP1 becomes the following:

Problem DP11:

$$\begin{aligned} \max \quad & v_0 - \sum_{k} d_k v_k \\ \text{s.t.} \quad & v_0 - v_k \leqslant C_k^* \quad \text{for all } k = 1, \ldots, K. \end{aligned} \tag{12}$$

Step 2. We sort the values of C_k^* in an increasing order and reindex such that $C_r^* \leqslant C_{r+1}^*$ for all $r = 1, \ldots, K-1$.

Step 3. Since $\sum_{k=1}^{K} d_k = 1$, we set $v_0 = C_K^*$, $v_k = v_0 - C_k^*$ for all k.

We show in Theorem 1 below that Algorithm A1 produces an optimal solution to problem DP1.

Theorem 1. *The algorithm A1 produces an optimal solution to problem DP1.*

Proof. At step 3 of the algorithm A1, we have $\sum_{k=1}^{K} d_k = 1$, then we let $v_0 = C_K^*,\ v_k = v_0 - C_k^*$ for all k. Now if v_0 is increased by an amount δ, then v_k also goes up by δ for $k = 1, \ldots, K$, and it does not improve the objective function value of problem DP1 as $(1 - \sum_{k=1}^{K} d_k)\delta = 0$.

When v_0 is negative it is optimal to set all values of v_k at zero. As value of v_0 increases in the range $(0, C_k^*)$ objective function value of problem DP1 steadily increases.

Result 1. *The optimal solution to problem DP1 is given by* $\left(\sum_{k=1}^{K} C_k^* d_k\right)$.

Proof. It is easy to see.

The algorithm described in previous section optimally solved the problem DP when all z_i's are set to zero. In problem DP we now allow exactly one z_i to increase in value. We analyze the properties of the problem DP with these restrictions in Section 4.2 given below.

4.2. Increasing the objective function value problem DP by increasing the value of a single z_i

If exactly one z_r is to assume positive value in problem DP it results in problem DP2 given below.

Problem DP2:

$$\max \quad v_0 - \sum_{k=1}^{K} d_k v_k - z_r b_r$$

s.t.

$$\begin{aligned} & v_0 - v_k \leqslant C_{ik} \quad \text{for } I = 1, \ldots, r-1 \\ & \quad \text{and } r+1, \ldots, I \text{ and for } k = 1, \ldots, K, \end{aligned} \tag{13}$$

$$v_0 - v_k - z_r \leqslant C_{rk} \quad \text{for } k = 1, \ldots, K, \tag{14}$$

$$z_r \geqslant 0, \tag{15}$$

Eq. (8).

For all $k = 1, \ldots, K$ we define $C1_k = \min_i(C_{ik})$, and $l_k = \{i{:}\, C1_k = C_{ik}\}$. We also define $C2_k = \min_{i\in\{1,\ldots,I\}-lk}(C_{ik})$. It may be noted that $C1_k = C^*_k$.

$C1_k$ denotes the least per unit cost of supplying goods to market k. l_k is the set of plants that have least per unit cost of supplying goods to market k. $C2_k$ denotes the second minimum per unit cost of supplying goods to market k. Below we prove a few results which help us to determine the optimal positive value of a given z_r.

Theorem 2. *z_r has the value zero in the optimal solution to the problem DP2 if any of the following condition holds.*

(i) *If for some $k1$, l_{k1} is not singleton and $r \in l_{k1}$ and for all $k \neq k1$, $r \notin l_k$.*

(ii) *For all k, $r \notin l_k$.*

(iii) *For all k, l_k is not singleton.*

Proof. We may set z_r at any positive value in problem DP2 which then gets reduced to problem DP1. Optimal solution to problem DP1 depends on the values C^*_k, $k = 1, \ldots, K$. For some k, if condition (i) holds, then for any positive value of z_r the value of C^*_k is unaffected even if $r \in l_k$. Similarly the value of C^*_k is unaffected if (ii) holds. Thus if conditions (i) or (ii) is true for all $k = 1, \ldots, K$, then values of C^*_k are unaffected for all $k = 1, \ldots, K$; and if z_r is set a positive value then the objective function value of problem DP2 decreases by b_r z_r for any positive value of z_r. Similar argument holds if (iii) is true.

Theorem 2 helps us to identify those z_r's which are unable to increase the objective function value of problem DP2 by attaining a positive value.

We now explore the case when a single z_r can be set at a positive value in problem DP2. We define $\mathrm{ziv}_{rk} = C2_k - C1_k$ if $r \in l_k$, and l_k is singleton; and $\mathrm{ziv}_{rk} = 0$ otherwise. Theorem 3 establishes the range in which the optimal value of z_r lies in the problem DP2.

Theorem 3. *Optimal value of z_r for the problem DP2 occurs in the range* $(0, \max_{k\in(1,\ldots,K)}\mathrm{ziv}_{rk})$

Proof. It is clear that objective value of problem DP1 increases with increases in values of C^*_k, $k = 1, \ldots, K$. For the constraint set $v_0 - v_k - z_r \leqslant (C_{ik})$, $i = 1, \ldots, I$, if $l_k = \{r\}$ and $(C2_k - C1_k > 0)$ then the objective function value of problem DP1 increases as z_r increases from zero to $C2_k - C1_k$. Any increase beyond this range does not affect the value of C^*_k for the associated problem DP1. Thus for a constraint set $v_0 - v_k - z_r \leqslant (C_{ik})$, $i = 1, \ldots, I$, and some k, the optimal value of problem DP1 occurs when z_r varies in the range $(0, \mathrm{ziv}_{rk})$. Hence the result follows.

Theorem 3 allows us to identify those z_r's which can be increased to a positive value to improve the objective function value of problem DP. Theorem 4 is useful to determine the exact value of z_r which will result in maximum improvement in objective function value of problem DP.

Theorem 4. *In the range indicated by Theorem 3, with the increase in value of z_r the objective function value of DP2 increases in a piece wise linear fashion but at a decreasing rate in the initial stages and later decreases at an increasing rate.*

Proof. We assume without loss of generality that $\mathrm{ziv}_{rk} > 0$ for all $k = 1, \ldots, K$. We sort the values of ziv_{rk} in an increasing order and without loss of generality reindex such that

$$\mathrm{ziv}_{rk} \leqslant \mathrm{ziv}_{r(k+1)}, \quad \text{for } k = 1, \ldots, K-1. \tag{16}$$

Thus the value of z_r can be increased in a maximum of K sub ranges, i.e., $(0, \mathrm{ziv}_{r1}), (\mathrm{ziv}_{r1}, \mathrm{ziv}_{r2}), \ldots, (\mathrm{ziv}_{rk}, \mathrm{ziv}_{r(k+1)}), \ldots, (\mathrm{ziv}_{r(K-1)}, \mathrm{ziv}_{rK})$. When z_r is set at zero, the C^*_k required for the heuristic are $(C1_1, C1_2, \ldots, C1_m, C1_{(m+1)}, \ldots, C1_{(K-1)}, C1_K)$.

Now if z_r is set at the upper limit of the mth sub range, then the C^*_k required for the Algorithm A1 are $(C1_1 + \mathrm{ziv}_{r1}), (C1_2 + \mathrm{ziv}_{r2}), \ldots, (C1_m + \mathrm{ziv}_{rm})$, $(C1_{(m+1)} + \mathrm{ziv}_{rm})$, $(C1_K + \mathrm{ziv}_{rm})$.

We set v_0 to C^*_m, and obtain the value of v_k as follows:

$$v_k = v_0 - (C1_k + \mathrm{ziv}_{rk}) \quad \text{for } k < m,$$

$$v_k = v_0 - (C1_k + \mathrm{ziv}_{rm}) \quad \text{for } k \geqslant m.$$

We denote objective function value of problem DP2 by OFV_DP2. Then the objective function value of DP2 within $z_r = \text{ziv}_{rm}$ is

$$\text{OFV\_DP2}(z_r = \text{ziv}_{rm})$$
$$= v_0 - \sum_{k=1}^{m-1} d_k[v_0 - (C1_k + \text{ziv}_{rk})]$$
$$- \sum_{k=m}^{K} d_k[v_0 - (C1_k + \text{ziv}_{rm})] - b_r \,\text{ziv}_{rm},$$

Since $\sum_{k=1}^{K} d_k = 1$, it becomes equal to

$$\sum_{k=1}^{K} d_k\, C1_k + \sum_{k=1}^{m-1}(d_k\,\text{ziv}_{rk}) + \sum_{k=m}^{K} d_k\,\text{ziv}_{rm} - b_r\,\text{ziv}_{rm}. \tag{17}$$

Similarly,

$$\text{OFV\_DP2}(z_r = \text{ziv}_{r(m-1)})$$
$$= \sum_{k=1}^{K} d_k\, C1_k + \sum_{k=1}^{m-2}(d_k\,\text{ziv}_{rk})$$
$$+ \sum_{k=m-1}^{K} d_k\,\text{ziv}_{r(m-1)} - b_r\,\text{ziv}_{r(m-1)}.$$

Thus OFV_DP2 $(z_r = \text{ziv}_{rm})$ − OFV_DP2 $(z_r = \text{ziv}_{r(m-1)})$ is equal to

$$\left(\sum_{k=m}^{K} d_k - b_r\right)[\text{ziv}_{rm} - \text{ziv}_{r(m-1)}]. \tag{19}$$

The quantity $[\text{ziv}_{rm} - \text{ziv}_{r(m-1)}]$ denotes the increase in value of z_r and the increase or decrease in objective function value of problem DP2 is determined by the quantity $(\sum_{k=m}^{K} d_k - b_r)$. It may be positive initially when $m = 1$ and later keeps reducing and finally becomes negative. Hence the result folows that the objective function value of problem DP2 increases at a decreasing rate initially and later reduces at an increasing rate with increase in value of any z_r. Similar result holds even if value of z_r increases in subranges less than K.

The results of Theorems 2, 3 and 4 can be combined to develop a simple algorithm for preparing the optimal solution to problem DP2 which is given below in Section 4.2.1.

4.2.1. Algorithm for optimally solving the problem DP2

Algorithm A2

Step 1. For all $k = 1, \ldots, K$, compute $C1_k = \min_{i \in \{1,\ldots,I\}}(C_{ik})$ $\quad l_k = (i{:}\, C1_k = (C_{ik}))$; and $C2_k = \min_{i \in \{1,\ldots,I\} - l_k}(C_{ik})$

Step 2. Compute for all $r = 1, \ldots, I$, $\text{ziv}_{rk} = C2_k - C1_k$ if $r \in l_k$, and I_k is singular; and $\text{ziv}_{rk} = 0$ otherwise. Set $r = 0$.

Step 3. Set $r = r+1$. If $r > I$ then stop, else find p such that $\text{ziv}_{rp} = \min_{k \in \{1,\ldots,K\}}(\text{ziv}_{rk} > 0)$ and then use results of Theorem 3 to determine if it can be set at positive value. If yes then go to step 4. If no then the z_r remains at zero in the optimal solution, go to step 3.

Step 4. Sort ziv_{rk}, $k = 1, \ldots, K$, in an increasing order and use results of Theorem 4 (Eq. (19)) to determine the optimal positive value of z_r. If z_r is set at positive value then stop else go to step 3.

Result 2. *Algorithm A2 runs in* $O(n^2)$ *time* (*where n is the sum of supply and demand points*).

Proof. It can be seen that each of the steps 1 and 4 can be executed in $O(IK)$ number of steps. And steps 2 and 3 are executed in $O(K)$ number of steps. Hence the result follows.

Algorithm A2 improves the objective function value of problem DP by setting some single z_r at a positive value. However it is possible that no single z_r can be increased to improve the objective function of problem DP.

If no single z_i can be increased in value, it does not imply that the optimal solution to problem DP is obtained. It is possible that we may improve dual solution (of problem DP) by increasing a collection of z_i's simultaneously. We now describe a heuristic which helps us to determine such a collection of z_i's.

4.3. Increasing a collection of z_i's to improve solution of problem DP

The essence of heuristic is as follows. We sort l_k in decreasing order of cardinal value and then take l_{k1} which has largest cardinal value. We let all z_i's to increase in value (we call this set as z_for_increase). Thus objective function value of problem DP increases by $d_{k1} - (\sum b_i : i \in l_{k1})$. Later we scan all l_{k2} whose cardinal value is less than or equal to that of l_{k1} and see if we benefit (in terms of increasing value of problem DP) by increasing values z_i's which belong to l_{k2} but had not been increased previously. At the end we compute total increase possible in objective value of problem DP and if it is positive then we are successful. These ideas are incorporated in Heuristic H1 given below.

Heuristic H1

Step 0. Sort sets l_k in decreasing order of cardinal value; combination_found = false; $k1 = 0$.
Step 1. $k1 = k1 + 1$.
if ($k1 <= K$) and not (combination_found) and ($|l_{k1}| > 1$))
then go to step 2 else go to step 5.
Step 2. total_benefit = $d_{k1} - (\sum b_i \quad \forall i: i \in l_{k1})$;
z_for_increase = $\{i: i \in l_{k1}\}$;
$K1 = \{k1\}$;
Step 3. for $k2 := (k1 + 1)$ to K do
begin
benefit_from_k2 = $d_{k2} - (\sum b_i \quad \forall i: i \in l_{k2})$ and $i \notin$ z_for_increase);
if (benefit_from_k2 > 0)
then
begin
total_benefit = total_benefit + benefit_from_k2;
z_for_increase = z_for_increase + $\{i: i \in l_{k2}\}$
$K1 = K1 + \{k2\}$;
end;
end;
Step 4. if (total_benefit > 0)
then combination_found = true and terminate
else Goto Step 1.
Step 5. Combination not found. Terminate.

If we are successful then $z_i: i \in$ z_for_increase in the collection of z_i's that can be increased in value together to improve objective function value of problem DP. We now show in Result 3 that running time complexity of Heuristic H1 is $O(n^2)$.

Result 3. *Running time complexity of Heuristic H1 is* $O(n^2)$.

Proof. Step 0 of H1 takes $O(K \log(K))$ time. Step 1 is repeated for K times and for each execution of step 1, step 3 is executed $O(K)$ times. Hence running time complexity of Heuristic H1 is $O(K^2)$ or $O(n^2)$.

4.4. Another heuristic for increasing a collection of z_i's to improve solution of problem DP

The essence of the heuristic is as follows. We seek to partition set $\{1, \dots, K\}$ into two disjoint set $K1$ and $K2$ such that set $U_{k \in K1} l_k$ and $U_{k \in K2} l_k$ are disjoint and none equal to $\{1, \dots, I\}$. If none of the sets $U_{k \in K1} l_k$ and $U_{k \in K2} l_k$ is equal to $\{1, \dots, I\}$ then we can improve the solution of problem DP by increasing z_i's such that $i \in U_{k \in K2} l_k$ or $i \in U_{k \in K1} l_k$. This is so because either

$$\left(\sum_{k \in K1} d_k - \sum_{i \in lk1} b_i\right) > 0 \text{ or } \left(\sum_{k \in K2} d_k - \sum_{i \in lk2} b_i\right) > 0.$$

The details of the heuristic are as given below.

Heuristic H2

Step 0. Set union_of_l_sets = $\bigcup_{k=1}^{K} l_k$
z_for_increase = l_1
$K1 = \{1\}$
Step 1. For $k1 := 2$ to K do
if (l_{k1} and z_for_increase have common elements)
then
begin z_for_increase = z_for_increase + l_{k1}
$K1 = K1 + \{k1\}$
end
Step 2. if (z_for_increasse = $\{1, \dots, I\}$)
then combination_found = false
else go to step 3.
Step 3. Set benefit = $(\sum_{k \in K1} d_k - \sum_{i \in z\_for\_increase} b_i)$
if (benefit > 0)
then combination_found = true
else if (benefit < 0)
then begin
combination_found = true

z_for_increase = union_of_1_sets − z_for_increase
$K1 = \{1..K\} - K1$
end
else combination_found = false {benefit = 0}
If combination_found = true then we can increase a collection of z_i's: $i \in z$_for_increase.

We now show in Result 4 that running time complexity of Heuristic H2 is $O(n^2)$.

Result 4. *Running time complexity of Heuristic H2 is* $O(n^2)$.

Proof. It may be noted that maximum of K elements may be added in the set $K1$. Hence step 2 is executed maximum of K times. Each time step 2 is executed it takes $O(K)$ time. Hence Heuristic H2 runs in $O(K^2)$ or $O(n^2)$ time.

4.5. *To determine extent of increase for z_i: $i \in z$_for_increase*

Having determined a collection of z_i's that can be increased to improve the objective function value of problem DP, we now give a heuristic procedure to determine extent of increase of z_i's in set z_for_increase. We give such a procedure in Heuristic H3 below, but we define the following first.

$$\text{second\_min}_k = \min_{i \in \{1,\dots,I\} - z\_\text{for\_increase}} \{C_{ik}\} \quad \forall k = 1,\dots,K$$

and
possible_increase$_k$ = (second_min$_k$ − C1$_k$)
Re-index without loss of generality such that,
possible_increase$_k$ < = possible_increase$_{k+1}$
$\forall k = 1,\dots,K-1$.
If extent of increase is strickly less than possible_increase$_1$, then benefit possible is

$$\left(\sum_{k \in K1} d_k - \sum_{i \in z\_\text{for\_increase}} b_i\right),$$

which is greater than zero as determined by Heuristics H1 or H2. If extent of increase is strictly greater than possible_increase$_1$ and strictly less than possible_increase$_2$, then benefit possible is

$$\left(\sum_{k \in K1 - \{1\}} d_k - \sum_{i \in z\_\text{for\_increase}} b_i\right).$$

If extent of increase is strictly greater than possible_increase$_k$, then benefit possible is

$$\left(\sum_{k \in K1 - \{1,\dots,k\}} d_k - \sum_{i \in z\_\text{for\_increase}} b_i\right).$$

We would like to increase z_i's in the set z_for_increase to a quantity such that k is as large as possible to have the following:

$$\left(\sum_{k \in K1 - \{1,\dots,k\}} d_k - \sum_{i \in z\_\text{for\_increase}} b_i\right) > 0.$$

These ideas are condensed into Heuristic H3 which determines maximum extent of increase possible once the set z_for_increase has been identified by Heuristics H1 or H2.

Heuristic H3
Step 0. Determine second_min$_k$ =

$$\min_{i \in \{1,\dots,I\} - z\_\text{for\_increase}} \{C_{ik}\} \quad \forall k = 1,\dots,K$$

and
possible_increase$_k$ = second_min$_k$ − C1$_k$
Reindex, such that $\forall k = 1,\dots,K-1$, we have, possible_increase$_k \leqslant$ possible_increase$_{k+1}$,
Step 1. Find largest k:

$$\left(\sum_{k \in K1 - \{1,\dots,k\}} d_k - \sum_{i \in z\_\text{for\_increase}} b_i\right) > 0$$

and set extent of increase possible = possible_increase$_k$, and stop.

Result 5. *Running time complexity of heuristic H3 is* $O(n^2)$.

Proof. Step 0 takes $O(I^2)$ steps and C_{1k} and second_min$_k$ can be found in $O(I)$ steps and have to be repeated for K steps. Step 1 is executed in

O(K) step. Hence running time complexity of Heuristic H3 is O(K^2), that as O(n^2).

Heuristics H1, H2 and H3 could be combined to yield a heuristic that can improve the solution to problem DP in O(n^2) time that is given below.

4.6. Combining heuristics H1, H2 and H3 into a single heuristic

Heuristic H4

Step 0. Set all $z_i = 0$ in problem DP and obtain optimal solution to by Algorithm A1.

Step 1. Use Algorithm A2 or Heuristics H1 or H2 to see if solution to problem DP can be improved by either increasing a single z_i or a combination of z_i's. If solution can be improved then go to step 2, else stop and print best solution obtained so far.

Step 2. Determine extent of increase by using Heuristic H3 and update C_{ik}'s and go to step 1.

Result 6. *Heuristic H4 runs in* O(cn^2) *time.*

Proof. Algorithm A1 and Heuristics H1, H2 and H3 run in O(n^2) time. At each step of the heuristic we are assured of a strictly positive improvement in values of z_i's (that can be raised in values) and hence solution to problem DP can be improved by a positive quantity which is product of increase in value of z_i's and quantity equal to

$$\min\left(\left|\sum_{k\in T1\subseteq\{1,\ldots,K\}} d_k - \sum_{i\in T2\subseteq\{1,\ldots,I\}} b_i\right| > 0\right).$$

Let this number be denoted by δ. And optimal solution to problem DP is a finite real number bound by sum of all C_{ik}'s. Hence maximum number of iterations required by Heuristic H4 is given by (sum of all C_{ik}'s)/δ which is equal to c.

5. Numerical examples

We solve here three sample problems to illustrate working of Algorithm A2 and Heuristic H1 and H2.

5.1. Simple transportation problem SP1

Consider the following simple transportation problem with three plants and four markets and following data:

$$C_{i1} = (\underline{2}, 10, 7),$$
$$C_{i2} = (\underline{2}, 8, 6),$$
$$C_{i3} = (\underline{2}, 5, 7),$$
$$C_{i4} = (\underline{1}, 4, 8),$$
$$d_k = (4/15, 3/15, 4/15, 4/15),$$
$$b_i = (3/15, 7/15, 5/15).$$

5.1.1. Solution procedure: Algorithm A2

Iteration 0:

We let all z_i's to be equal to zero and for the associated problem DP1 we have the following:

$$C_k^* = (2, 2, 2, 1),$$
$$v_0 = 2,$$
$$v_k' \mathrm{s} = (0, 0, 0, 1).$$

Optimal objective value of problem

$$\mathrm{DP1} = \sum_{k=1}^{4} d_k \ * C_k^* = 26/15.$$

Iteration 1

$$l_1 = \{1\}; \quad l_2 = \{1\}; \quad l_3 = \{1\}; \quad l_4 = \{1\}.$$
$$\mathrm{ziv}_{11} = \{5\}; \quad \mathrm{ziv}_{12} = \{4\}; \quad \mathrm{ziv}_{13} = \{3\};$$
$$\mathrm{ziv}_{14} = \{3\}.$$

Loss of objective function value for associated problem DP2 for unit increase in value of $z_1 = 3/15$. In light of Theorem 4, z_1 can be increased to full five units. Now revised C_{ik}'s with five units of increase in z_1 are as follows:

$$C_{i1} = (\underline{7}, 10, \underline{7}),$$
$$C_{i2} = (7, 8, \underline{6}),$$
$$C_{i3} = (7, \underline{5}, 7),$$
$$C_{i4} = (6, \underline{4}, 8),$$

$$\text{Total loss} = 3 \times 5/15 = 15/15.$$

Optimal objective function value of associated problem DP1 with $z_1 = 5$ in problem DP2 is

$$\sum_{k=1}^{4} d_k * C_k^* \equiv (7 \times 4/15 + 6 \times 3/15 + 5 \times 4/15 + 4 \times 4/15) \equiv 82/15.$$

Optimal objective function of problem DP2 with $z_1 = 5 \equiv 82/15 - 15/15 \equiv 67/15$.

Iteration 2:

$$l_1 = \{1,3\}; \quad l_2 = \{3\}; \quad l_3 = \{2\}; \quad l_4 = \{2\}.$$
$$\mathrm{ziv}_{12} = 0; \quad \mathrm{ziv}_{22} = 1; \quad \mathrm{ziv}_{23} = 1; \quad \mathrm{ziv}_{24} = 2.$$

If we increase z_2 by one unit loss of objective function value of problem DP2 goes down by 7/15 while gain is $d_3 + d_4 = 8/15$.

In light of Theorem 4 we find that objective function value of problem DP2 will increase with value of z_2 at one unit. Revised C_{ik}'s are as follows:

$$C_{i1} = (\underline{7}, 11, \underline{7}),$$
$$C_{i2} = (7, 9, \underline{6}),$$
$$C_{i3} = (7, \underline{6}, 7),$$
$$C_{i4} = (6, \underline{5}, 8),$$

loss for current iteration is $7/15 \times 1 = 7/15$,

Total loss $= 15/15 + 7/15 = 22/15$.

Objective function value of associated problem DP1 with $z_2 = 1$ in problem DP2 is

$$\sum_{k=1}^{4} d_k\ C_k = (4/15 \times 7) + (3/15 \times 6) + (4/15 \times 6) + (4/15 \times 5) \equiv 90/15.$$

Objective function value of problem DP2 with $z_2 = 1$ is $90/15 - 22/15 = 68/15$.

Iteration 3:

$$l_1 = \{1,3\}; \quad l_2 = \{3\}; \quad l_3 = \{2\}; \quad l_4 = \{2\}.$$

No further improvement is possible and we have reached the optimal solution. Equivalent transportation problem also has the same solution.

5.2. Simple transportation problem SP2

Consider the following problem with number of plants = 3; and number of markets = 3 and the following data:

$$C_{i1} = (150, \underline{120}, 240),$$
$$C_{i2} = (240, \underline{60}, 270),$$
$$C_{i3} = (300, 90, \underline{30}),$$
$$d_k = (0.16, 0.16, 0.68),$$
$$b_i = (0.33, 0.33, 0.34).$$

5.2.1. Solution procedure: Heuristic H1

Iteration 0:

We let all z_i's to be equal to zero and for associated problem DP1 we have the following:

$$C_k^* = (120, 60, 30).$$

Optimal objective value of problem

$$\mathrm{DP1} = \sum_{k=1}^{3} d_k \ * C_k^* = 49.2.$$

Iteration 1:

$$l_1 = \{2\}; \quad l_2 = \{2\}; \quad l_3 = \{3\}.$$

Now z_2 cannot be increased individually. If only z_2 is increased by one unit, we have loss per unit increase of $z_2 = 0.33$; and benefit per unit increase of z_2 is $d_1 + d_2 = 0.32$; and hence, net benefit is -0.01/unit increase of z_2.

If only z_3 is increased by one unit, we have loss $= -0.34$ and benefit $= 0.68$. Hence, now z_3 can go up.

We compute $\mathrm{ziv}_{13} = 0$; $\mathrm{ziv}_{23} = 0$; $\mathrm{ziv}_{33} = 60$. Hence we let z_3 go up by 60 units. We have (C_{ik}'s + z_3) as follows:

$$C_{i1} = (150, \underline{120}, 300),$$
$$C_{i2} = (240, \underline{60}, 330),$$
$$C_{i3} = (300, \underline{90}, \underline{90}).$$

We have $C_k^* = (120, 60, 90)$.

Optimal objective value of problem DP1 = $\sum_{k=1}^{3} d_k * C_k^* = 90$.

Current loss $z_3 \times b_3 = 60 \times (0.34) = 20.4$.

Dual objective function value = $90.0 - 20.4 = 69.6$.

Iteration 2:

$l_1 = \{2\}; \quad l_2 = \{2\}; \quad l_3 = \{2,3\}.$

Now we see that z_2 or z_3 cannot be increased individually. If z_2 is increased by one unit, per unit loss = 0.33, and gain per unit is $d_1 + d_2 = 0.32$ which is less than 0.33. If z_3 is increased by one unit, loss per unit = 0.34; and gain per unit is zero as l_3 is not singular; hence, net gain is negative.

But it is easy to see that z_2 and z_3 can be increased by a common value. If that is done loss per unit increase = $b_2 + b_3 = 0.67$ and gain per unit increase = $d_1 + d_2 + d_3 = 1.00$. We have possible_increase$_1$ = 30; possible_increase$_2$ = 180; possible_increase$_3$ = 210. Hence z_2 and z_3 can go up by a common value of 30 units. Beyond that value our benefit per unit increase in value of z_1 and z_3 is $d_2 + d_3 = 0.84$ which is more than loss per unit = $b_2 + b_3 = 0.67$. Hence now z_2 and z_3 can go up to 180. Beyond that benefit per unit increase in value of z_2 and z_3 is 0.68 and loss = $b_2 + b_3 = 0.67$. Hence, now z_2 and z_3 can go upto 210 units. With z_2 and z_3 at 210 units; we have $C_{1k} + z_i$'s as follows:

$$C_{i1} = (\underline{150}, 330, 510),$$
$$C_{i2} = (\underline{240}, 270, 540),$$
$$C_{i3} = (\underline{300}, \underline{300}, \underline{300}).$$

We have $C_k^* = (150, 240, 300)$.
Optimal objective value of problem DP1 = $\sum_{k=1}^{3} d_k * C_k^* = 266.4$.
Current loss = 210 (0.67) = 140.7.
Cumulative loss = 20.4+140.7 = 161.1.
Objective function value = 266.4 − 161.1 = 105.3.

Iteration 3:

$l_1 = \{1\}; \quad l_2 = \{1\}; \quad l_3 = \{1,2,3\}.$

Now it can be easily seen that none of the z_i's individually or collectively can be increased to improve the dual solution by H1. We can verify that we have reached the optimal solution.

5.3. Simple transportation problem SP3

Consider the following problem with number of plants = 3; and number of markets = 4, and the following data:

$$C_{i1} = (30, 150, 105),$$
$$C_{i2} = (30, 120, 90),$$
$$C_{i3} = (30, 75, 90),$$
$$C_{i4} = (15, 60, 120),$$
$$d_k = (0.2667, 0.2, 0.2667, 0.2666),$$
$$b_i = (0.2, 0.4667, 0.3333).$$

5.3.1. Solution procedure: Heuristic H2

We let z_i's to be equal to zero and for associated problem DP1 we have the following: $C_k^* = (30, 30, 30, 15)$. Optimal objective value of problem DP1 = $\sum_{k=1}^{4} d_k \; C_k^* = 26.0025$

Iteration 1

$l_1 = \{1\}; \quad l_2 = \{1\}; \quad l_3 = \{1\}; \quad l_4 = \{1\}.$

We increase z_1. Benefit per unit increase of $z_1 = d_1 + d_2 + d_3 + d_4 = 1.0$ and loss per unit increase of $z_1 = 0.2$. Hence net benefit = 0.8.

$\text{ziv}_{11} = 75; \quad \text{ziv}_{12} = 60; \quad \text{ziv}_{13} = 45; \quad \text{ziv}_{14} = 45.$

Using Theorem 4 or heuristic H3, z_1 can be increased to 75. Current loss = 75 × 0.2 = 15.0; Cumulative Loss = 15.0. If we increase z_1 by 75, the $C_{ik} + z_i$ becomes equal to

$$C_{i1} = (\underline{105}, 150, \underline{105}),$$
$$C_{i2} = (105, 120, \underline{90}),$$
$$C_{i3} = (105, \underline{75}, 90),$$
$$C_{i4} = (90, \underline{60}, 120).$$

Optimal value of associated problem DP1 is

$$\sum_{k=1}^{4} C_k^* \; d_k = 105 \times 0.2667 + 90 \times 0.2 + 75 \times 0.2667 + 60 \times 0.2666 = 80.002.$$

Hence objective value of problem DP = 82.002 − 15 = 67.002.

Iteration 2:

$l_1 = \{1,3\}; \quad l_2 = \{3\}; \quad l_3 = \{2\}; \quad l_4 = \{2\}.$

For $z\_for\_increase = \{1, 3\}$,
benefit $= d_1 + d_2 = 0.4667$ and loss $= b_1 + b_3 = 0.5333$,
Net benefit 0.4667 – 0.5333 is negative, and hence can not be increased.

Complement of {1,3} is {2} and can be increased as net benefit is = 0.5333 – 0.4667, which is also equal for d_3 + d_4 – (b_2). Now, $ziv_{23} = 15$ and $ziv_{24} = 30$ by using results of Theorem 3, z_2 can be increased only by 15 units. Now (C_{ik} + z_i) become equal to

$$C_{i1} = (\underline{105}, 165, \underline{105}),$$
$$C_{i2} = (105, 135, \underline{90}),$$
$$C_{i3} = (105, \underline{90}, \underline{90}),$$
$$C_{i4} = (90, \underline{75}, 120).$$

Current loss $= 15 \times 0.4667 = 7.0005$.
Cumulative loss $= 15 + 7.005 = 22.005$.

Optimal objective value of problem DP1 is

$$\sum_{k=1}^{4} d_k\ C_k^* = 0.2667 \times 105 + 0.2 \times 90 + 90 \times 0.2267 + 75 \times 0.2666 = 90.0015.$$

Current solution to problem DP = 90.0015 – 22.0005 = 68.0010.

Iteration 3:

$$l_1 = \{1, 3\};\quad l_2 = \{3\};\quad l_3 = \{2, 3\};\quad l_4 = \{2\}.$$

None of the combinations of z_i can now be increased to improve dual solution by H4. We verify that we have the optimal solution.

6. Computational experience

We have solved 50 small sized simple transportation problems taken from various books. All these problems were optimally solved by the Heuristic H4 presented in this paper. The details of these can be found in Sharma and Sharma [8], and the computer program in PASCAL can be found in Sharma [6]. We give details of 10 small sized problems in Appendix A.

Table 1

Problem size	No. of problems randomly generated	% Closeness reached by Heuristic H4	Average no. of iterations taken by H4
10×10	10	100.000	16.5
20×20	10	94.240	42.8
30×30	10	88.854	36.7
40×40	10	87.133	55.2
50×50	10	83.132	83.8
60×60	10	76.280	92.4
70×70	10	74.449	88.3
80×80	10	75.705	91.7
90×90	10	72.465	111.0
100×100	10	75.897	87.4

Later we solved 100 transportation problems of various sizes. Details of the same appear in Sharma and Sharma [9] and condensed information is given in Table 1. We find that our heuristics on average obtained dual solution that was within 82% of the optimal solution. We now show in Section 7 how to obtain feasible dual solution to the conventional formulation of the simple transportation problem (i.e. dual of the problem Q) which is as close to the optimal solution as obtained by heuristic H4 given in this paper to obtain solution for problem DP.

7. Dual solution of the conventional formulation

At the termination of the heuristic H4 let the best possible solution of problem DP be represented by the dual variables v_0^b, z_i^b ($\forall i = 1, \ldots, I$), v_k^b ($\forall k = 1, \ldots, K$). However, the dual based approaches to simple transportation problem use problem Q as the standard formulation. Its dual is problem DQ as given below.

Let the dual variables associated with constraints (1) be $z1_i$ and those with constraints (2) be $v1_k$, then the dual problem Q is formulated as given below.

Problem *DQ*:

$$\max \quad \sum_{i=1}^{I} b_i\ z1_i - \sum_{k=1}^{K} d_k\, v1_k$$

s.t.

$$z1_i - v1_k \leqslant C_{ik} \quad \forall i = 1, \ldots, I\ \ \forall k = 1, \ldots, K, \quad (20)$$

$$v1_k \text{ uis } \forall k = 1, \ldots, K, \quad z1_i \text{ uis } \forall i = 1, \ldots, I. \quad (21)$$

We let

$$v1_k = v_k^b - v_0^b/2 \quad \forall k = 1, \ldots, K$$

and

$$z1_i = -z_i^b + v_o^b/2 \quad \forall i = 1, \ldots, I. \tag{22}$$

It may be noted that problems DQ and DP are different formulations of the same transportation problem and have same optimal objective function value. It can be easily seen that feasible solution obtained by Heuristic H4 for problem DP and feasible solution obtained for problem DQ as given by Eq. (22) have same objective function value, and thus they are as equally close to the optimal solution. Solution procedures due to Ali et al. [2] can now start from good solution obtained for problem DQ.

8. Conclusions

In this paper we have presented a O(cn^2) heuristic for solving the uncapacitated transportation problem. It has optimally solved a collection of small sized problems. For larger sized problems it obtained dual solution that was on average within 82% of the optimal solution. We have empirically shown that the procedure developed in this paper will give good starting solutions to dual based procedures used for solving the uncapacitated transportation problems.

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Appendix A

Problem No. 1:
Number of plants, number of markets 3, 4
b_i's 0.2, 0.4667, 0.3333
d_k's 0.2667, 0.2, 0.2667, 0.2667
C_{ik}'s
30, 150, 105
30, 120, 90
30, 75, 90
15, 60, 120
Optimal solution obtained by us: 67.98

Problem No. 2:
Number of plants, number of markets 3, 4
b_i's 0.2800, 0.4000, 0.3200
d_k's 0.3600, 0.1600, 0.2400, 0.2400
C_{ik}'s
1000, 1125, 1750
750, 1500, 1125
1250, 1625, 2000
1125, 875, 625
Optimal solution obtained by us: 1020

Problem No. 3
Number of plants, number of markets 5, 6
b_i's 0.15, 0.30, 0.15, 0.30, 0.10
d_k's 0.25, 0.10, 0.15, 0.15, 0.20
C_{ik}'s
1500, 1000, 2000, 500, 1000
1000, 4500, 4000, 1000, 800
100, 3500, 4000, 3000, 400
1000, 2500, 4000, 4000, 100
2000, 2500, 4000, 1000, 100
Optimal solution obtained by us: 850

Problem No. 4
Number of plants, number of markets 4, 3
b_i's 0.185185, 0.370370, 0.185185, 0.259259
d_k's 0.296296, 0.333333, 0.370370
C_{ik}'s
20250, 21330, 22950, 0.0
16200, 19710, 20250, 0.0
18630, 18360, 18900, 0.0
Optimal solution obtained by us: 13610.01

Problem No. 5
Number of plants, number of markets 3, 4
b_i's 0.400000, 0.333333, 0.266667
d_k's 0.266667, 0.466667, 0.066667, 0.200000
C_{ik}'s
1800, 2100, 2550
2100, 1950, 2250
2400, 2850, 2700
000, 000, 000
Optimal solution obtained by us: 1580

Problem No. 6
Number of plants, number of markets 3, 4
b_i's 0.166667, 0.333333, 0.500000
d_k's 0.100000, 0.100000, 0.400000, 0.400000
C_{ik}'s
600, 150, 540
330, 270, 210
90, 300, 120
180, 60, 30
Optimal solution obtained by us: 98

Problem No. 7
Number of plants, number of markets 3, 4
b_i's 0.1, 0.4, 0.5
d_k's 0.2, 0.3, 0.3, 0.2
C_{ik}'s
2200, 3100, 2500
3600, 1900, 2500
2400, 3200, 1600
2300, 2600, 2200
Optimal solution obtained by us: 2000

Problem No. 8
Number of plants, number of markets 2, 2
b_i's 0.25, 0.75
d_k's 0.5, 0.5
C_{ik}'s
100, 200
200, 400
Optimal solution obtained by us: 250

Problem No. 9
Number of plants, number of markets 2, 3
b_i's 0.5, 0.5
d_k's 0.125, 0.5, 0.375
C_{ik}'s
160, 200
320, 120
400, 160
Optimal solution obtained by us: 215

Problem No. 10
Number of plants, number of markets 3, 3
b_i's 0.166667, 0.333333, 0.500000
d_k's 0.333333, 0.166667, 0.500000
C_{ik}'s
240, 180, 480
120, 300, 360
360, 540, 60
Optimal solution obtained by us: 110

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