A Position Allocation Approach to the Scheduling of Battery Electric Bus ~~Scheduling~~ Charging

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***Abstract*—[To do: Copied] A major challenge to adopting battery electric buses into bus fleets is the scheduling of the battery charging while considering route timing constraints, battery charge, and battery health. This work develops a scheduling framework to balance the use of slow and fast chargers assuming the bus routes and charger locations are fixed. Slow chargers are utilized when possible for sake of battery health and fast chargers are used when needed to accommodate timing constraints and ensure a sufficient charge for route execution. A directed, acyclic graph is used to model the available charge times for buses that periodically return to a depot for charging. A constrained network flow Mixed-Integer Linear Program (MILP) problem is formulated to optimize the scheduling of chargers as well as to determine the number of chargers required to meet charging thresholds. Results are presented using a randomly generated bus schedule for thirty buses and demonstrate the ability of the planner to consider peak time charging costs while planning with fixed and variable numbers of chargers.**

***Index Terms*—**

Will this # hurt us?

I. Introduction

The public transportation system is crucial in any urban area; however, the increase awareness and concern of environmental impacts of petroleum based public transportation has driven an effort to reduce the pollutant footprint [1]–[4]. Particularly, the electrification of public bus transportation via battery power, ~~known as~~ battery electric buses (BEBs), has received significant attention [4]. Although the technology provides benefits beyond reduction in emissions such as lower driving costs, lower maintenance costs, and reduced vehicle noise; battery powered systems introduce new challenges such as larger upfront costs, and potentially several hour long "refueling" periods [2], [4]. Furthermore, the problem is exacerbated by the constraints of the transit schedule the fleet must adhere to, the limited amount of chargers available, as well as the adverse affects in the health of the battery due to fast charging [5]. This paper presents a continuous scheduling framework for a BEB fleet that shares limited fast and slow chargers. This framework takes into consideration linear charging dynamics and a fixed bus schedule while meeting a certain battery percentage threshold while remaining.

comma

Are we doing different charger types?

[To do: Copied] Recent research seeks to enable BEB fleet deployment by solving two problems: providing BEB scheduling (when to charge, at which charger) and determining BEB infrastructure (charger placement, route design). Much attention has been given to solving both problems simultaneously [6]–[9]. Additional variations in addressing the infrastructure include determining which existing buses should be replaced by a BEB [10], assignments of buses to routes [11], and determining locations of fast wireless chargers along the routes [12], [13]. The added complexity of considering both the BEB charge scheduling and the infrastructure problems necessitates simplifications for sake of computation.

Shorten and combine this and the next paragraph and add paragraph about PAP and BAP.

[To do: Copied] Two such simplifications are common. First, only fast chargers are utilized in planning [6], [7], [9]–[14]. Second, significant simplifications to the charging models are made. Some approaches assume full charge [6], [9], [10], [13]. Others have assumed that the charge received is proportional to the time spent on the charger [11], [12], which can be a valid assumption when the battery state-of-charge (SOC) is below 80% charge [11]. ~~Day to day operations require higher fidelity charging models to ensure buses have sufficient charge and to better incorporate the monetary cost of charging~~. In [7], [14], high-fidelity models are used at the price of requiring computationally intensive searches; [7] uses a genetic algorithm and [14] uses an exhaustive search strategy.

We are only doing a proportional model

The contribution of this work is a Mixed Integer Linear Program (MILP) scheduling framework that considers bus schedules, charging battery dynamics, charge limits, and availability of slow and fast chargers. The bus schedules are assumed to be fixed for the duration of the time horizon. The linear program is formed, and extends upon, the Position Allocation Problem (PAP) [15]. Linear charging dynamics are assumed, costs are assumed to be constant for the duration of the time horizon, and the amount of chargers is assumed to be constant. The MILD MILP? framework allows the addition and replacing of constraints. As such, battery dynamic constraints may be replaced with first-order dynamic modeling and costs can be made dynamic. The solution of the problem provides the arrival time, selected charger (fast or slow), initial charge time, final charge time, and departure time from the station.

The remainder of the paper proceeds as follows.

II. Preliminaries

The BEB charge scheduling formulation in this work builds upon the PAP.

The PAP is a rectangle packing problem where a set of rectangles $\mathbb{R}$ are attempted to be optimally placed in a larger

TABLE I: Notation used throughout the paper

Variable	Description	Variable	Description
Input values			
A	Number of buses in use	I	Final index
M	An arbitrary very large upper bound value	N	Number of total visits
Q	Number of chargers	T	Time Horizon
Ξ	$N(N-1)$		
Input variables			
Γ_i	Array of visit id's		
α_i	Initial charge time for visit i	β_i	Final charge for bus i at the end of the work day
ϵ_q	Cost of using charger q per unit time	γ_i	Array of values indicating the next index visit i will arrive
κ_i	Battery capacity for bus i	λ_i	Discharge of visit over route i
ν	Minimum charge allowed on depatrue of visit i	τ_i	Time visit i must leave the station
υ	Minimum final percentage charge at and of time horizon T	a_i	Arrival time of visit i
m_q	Cost of a visit being assigned to charger q	r_q	Charge rate of charger q per unit time
Decision Variables			
δ_{ij}	$v_i < v_j = 1$ or $i \neq j = 0$	η_i	Initial charge for visit i
σ_{ij}	$u_i < u_j = 1$ or $i \neq j = 0$	c_i	detach time from charger for visit i
g_i	Linearization term for bilinear terms $g_i := p_i w_{iq}$	p_i	Amount of time spent on charger for visit i
u_i	Initial charge time of visit i	v_i	Assigned queue for visit i
w_{iq}	Vector representation of queue assignment		

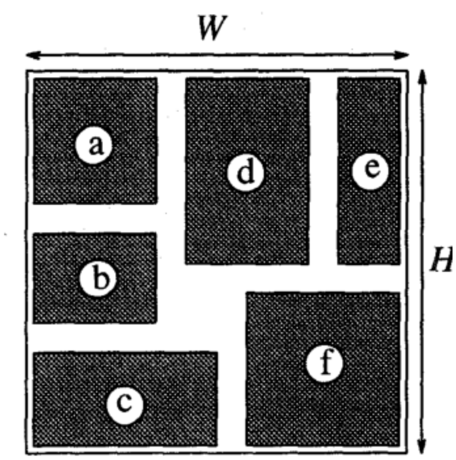


Fig. 1: Example of rectangle packing problem To do: Update Image

Fig. 2: Example of berth allocation. Vessels flow from top to bottom.To do: Update Image

These two paragraphs are great, but we need a transition/connection to our problem.

rectangle R as shown in Fig 1. Both the set $\mathbb{R}$ and rectangle R's width and height are used to represent quantifiable values. The rectangle packing problem is a subset of the packing problem and can be used to describe many real life problems [16]. ~~One of these problems is the Berth Allocation Problem (BAP).~~

The BAP is a form of the rectangle packing problem where the set of rectangles describe incoming vessels to a be docked on a berth to be serviced as shown in Fig 2 [17]. The BAP is commonly formed as a Mixed Integer Linear Program (MILP) [18], [19]. This formulation utilizes the BAP to help determine which queue the bus should be placed on to be charged without violating time or space constraints. Notation is summarized in Table I.

A. The Rectangle Packing Problem

Rectangle packing can be shown to be a NP-hard problem [20]. As shown in Fig 2, there is static sized rectangle R for the set of small rectangles $\mathbb{R}$ to be packed into where the width ~~and height are used to represent measurable values~~ provided ~~by the problem~~. In some problems, the dimensions of $\mathbb{R}$ are held constant such as in the problem of packing modules on a chip, where the widths and height of the rectangles represent the physical width and heights of the modules [20]. Other problems, such as the BAP, allow one or both sides to be decision variables ~~(i.e. the dimension is variable)~~ [19].

B. The Berth Allocation Problem

Avoid "In order to"

The BAP solves the problem of optimally assigning incoming vessels to berth positions ~~in order~~ to be serviced (Fig 2). The width and height of R represent the berth length S and time horizon T, respectively. Similarly, the width and height for $\mathbb{R}$ represent the time spent to service vessel i and the space taken by docking vessel i, respectively. The vessel characteristics (length of the vessel, arrival time, handling time, desired departure time) are assumed to be known for all N vessels being serviced. A representation of a BAP solution is shown in Fig 3.

avoid 1st person

The BAP has characteristics that are both convenient and undesirable in reference of the formulation we wish to develop; to explore these, consider the following MILP [15]:

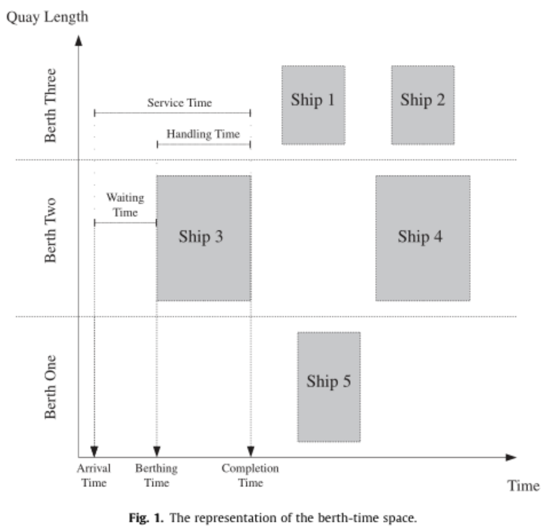


Fig. 3: The representation of the berth-time space To do: Update Image

This isn't the traditional BAP. You need to provide context that this is the BAP applied to vehicle charging.

$$min \sum_{i=1}^{N} (c_i - a_i) \tag{1}$$

Also need to state that we provide in entirety as our work builds upon this formulation

Subject to the following constraints:

queueing constraint

$$u_j - u_i - p_i - (\sigma_{ij} - 1)T \geq 0 \tag{2a}$$
$$v_j - v_i - s_i - (\delta_{ij} - 1)S \geq 0 \tag{2b}$$
$$\sigma_{ij} + \sigma_{ji} + \delta_{ij} + \delta_{ji} \geq 1 \tag{2c}$$
$$\sigma_{ij} + \sigma_{ji} \leq 1 \tag{2d}$$
$$\delta_{ij} + \delta_{ji} \leq 1 \tag{2e}$$
$$p_i + u_i = c_i \tag{2f}$$
$$a_i \leq u_i \leq (T - p_i) \tag{2g}$$
$$\sigma_{ij} \in \{0, 1\}, \ \delta_{ij} \in \{0, 1\} \tag{2h}$$

\no indent

Where ~~this problem assumes~~ the following are ~~known~~ constants

Don't indent one-liners

- S : berth length
- T : time horizon
- N : number of incoming vessels
- p_i : charging time for vessel i ;$\forall 1 \leq i \leq N$
- s_i : size of vehicle i ;$\forall 1 \leq i \leq N$
- a_i : arrival time of vessel i ;$\forall 1 \leq i \leq N$

\no indent

and the following are decision variables

- u_i : starting time of service for vessel i ;$\forall 1 \leq i \leq N$
- v_i : berth position i ;$\forall 1 \leq i \leq N$
- c_i : departure time for vessel i ;$\forall 1 \leq i \leq N$

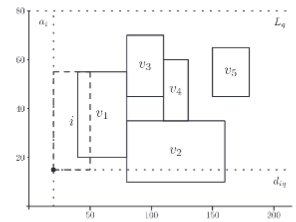
Don't use "left of" or "ahead" of.

- σ_{ij} : 1 if ~~vessel i is of vessel j in time (i.e. i is to the left of j in Fig 3)~~, 0 otherwise $u_j > u_i$ guaranteed and 0 otherwise
- δ_{ij} : 1 if ~~vessel i is head of vessel j in berth space (i.e. i is to the below j in Fig 3), 0 otherwise~~ $v_j > v_i$ guaranteed

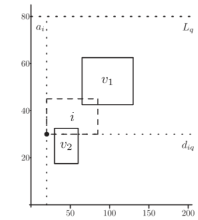
not undesirable, just not applicable to BEB

An undesirable property is that each vessel i ;$\forall 1 \leq i \leq N$ is assumed to be a new (i.e. unique) visit. This becomes a

Describe BAP formulation and then describe how we are going to change it.



(a) Example horizontal overlap in time in time-space plotTo do: Update Image



(b) Example of overlap in space in time-space plotTo do: Update Image

Fig. 4: Visualization of vessel relative positions

problem later when accounting for battery dynamics where each bus visit must be associated with a specific bus ID. However, this formulation allows S and T to be continuous or discrete. Because of this flexibility, the berth can either be left continuous, or discretized to accommodate Q chargers as shown in Fig 3. Time will be kept continuous. [19]. Focus is now directed toward discussing the objective function and constraint properties.

what is [19] referencing

The objective function (1) minimizes the time spent to service each vessel by minimizing over the difference between the departure time (c) and arrival time (a). Constraints 2a-2e are used to prevent overlapping over both space and time as shown in Fig 4. Refer to 2a-2e as "queueing constraint"

Constraint 2a states that the starting service time (u) for vessel j must be greater than the starting time of vessel i plus its service time (p). The last term utilizes the big M method to ~~determine the relevance~~ activate or deactivate of the constraint. ~~As stated before, if j begins being serviced after i is complete $\sigma_{ij} = 1$, otherwise there is some overlap in time and $\sigma_{ij} = 0$ as shown in Fig 4a. If $\sigma = 1$ then $(\sigma_{ij} - 1)T = 0$ leaving $u_i - u_j - p_j \geq 1$ implying that the constraint is now "active"~~. If $\sigma = 0$ then the constraint is of the form ~~$u_i - u_j - p_j - T$ where~~ $T + u_j + p_j > u_i$, rendering the constraint "inactive" because u_i cannot be larger than $T + u_j + p_j$.

In actuality, 2a need only be active if vehicles i and j are visiting the same charger ($v_i = v_j$).

A value of $\sigma_{ij} = 1$ will activate the constraint to ensure that i is finished before j

In a similar fashion, δ_{ij} effectively determines whether the vehicles will be located in the same queue. If $\delta_{ij} = 1$ then 2b is rendered active and vehicles i and j must be in different queues.

~~Similarly, constraint 2b states that the berth location for vessel i is greater than the starting berth position of vessel j plus its length (no overlap in physical space on the berth) as in Fig 4b. Again, the big M method is utilized to determine the relevance of the constraint. If $\delta_{ij} = 1$ then the big M term goes to zero rendering the constraint active. If $\delta_{ij} = 0$ then the constraint is inactive.~~

Constraints 2c-2e are used to ~~determine the relative position of each vessel from one another~~ establish the queueing by providing a relationship between queueing variables (u and v). ~~A useful way of describing σ_{ij} and δ_{ij} are in terms of relative position of vessels on the time-space diagram. $\delta_{ij} = 1$ implies i is "to the left of" j and $\sigma_{ij} = 1$ implies i "is below" j. If vessel i is to the left of j then i cannot intersect j vertically ($\sigma_{ij} = 1$ and $\delta_{ij} = 0$), see Fig 4a. Similarly, if i is below j then i is unable to overlap j horizontally ($\sigma_{ij} = 0$ and $\delta_{ij} = 1$), see Fig 4b. Constraint 2c states that vessel i must be to the left of or below vessel j, or vessel j must be to the left of or below i. However,~~

Constraint 2c states that one of the following is true: $u_i < u_j$ ($\sigma_{ij} = 1$), $u_j < u_i$ ($\sigma_{ji} = 1$), $v_i < v_j$ ($\delta_{ij} = 1$), or $v_j < v_i$ ($\delta_{ji} = 1$). Constraints 2d and 2e enforce consistency, i.e. $u_i < u_j$ and $u_j < u_i$ cannot both be true. This enforces a relationship between vehicles:

either one is before the other temporally or they are in different queues.

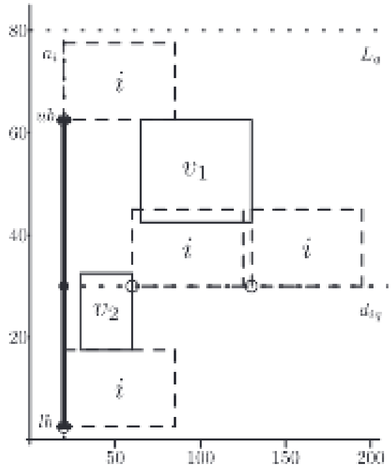


Fig. 5: Example assigning vessel to multiple locations To do: Update Image

~~this constraint does not restrict i being to left and right of j simultaneously (similarly in the vertical sense) as shown in Fig 5. Constraints 2d and 2e add these restrictions by allowing either $_{ij} = 1$ or $_{ji} = 1$ but not both.~~

The last constraints enforce continuity for each ~~vessel.~~ vehicle Constraint 2f states that the service start time (u) plus the time to service vessel i (p) must equal the departure time (c). Constraint 2g enforces the arrival time (a) must be less than or equal to the service start time (u) which must also be less than or equal to the latest time the ~~vessel~~ vehicle may begin to be serviced to stay within the time horizon. Constraint 2h defines the set of values σ and δ.

Describe at high level what we will do differently

III. Problem Formulation

- Buses have multiple visits to charger in day
- Visits are not for a set duration
- Track charge

The MILP is formulated with two sets of constraints: rectangle packing constraints and battery dynamics constraints. This section will build off the previous formulation ((1) and (2)) and progressively construct the battery dynamic constraints to create the Position Allocation MILP.

Need more understanding & the exact problem being solved prior to stating the problem

Sounds like you are using the same MILP, but you are not

A. Rectangle Packing Approach

Similarly to before, we begin with following MILP:

$$\sum_{i=1}^{N}\sum_{q=1}^{Q}\left(w_i m_q + g_i \epsilon_q\right) \tag{3}$$

Need to describe new variables

with the following constraints:

$$u_i - u_j - p_j - (\sigma_{ij} - 1)T \geq 0 \tag{4a}$$
$$v_i - v_j - s_j - (\delta_{ij} - 1)S \geq 0 \tag{4b}$$
$$\sigma_{ij} + \sigma_{ji} + \delta_{ij} + \delta_{ji} \geq 1 \tag{4c}$$
$$\sigma_{ij} + \sigma_{ji} \leq 1 \tag{4d}$$
$$\delta_{ij} + \delta_{ji} \leq 1 \tag{4e}$$
$$p_i + u_i = c_i \tag{4f}$$
$$a_i \leq u_i \leq (T - p_i) \tag{4g}$$
$$c_i \leq \tau_i \tag{4h}$$

Missing $\sigma_{ij} \in \{0,1\}$, $\delta_{ij} \in \{0,1\}$ $\forall i \in \{1,2,\dots,N\}$

The objective function (3) ?

Where the objective function (3) is the summation over the cost of assignment of bus visit i to charger q and the usage of charger q. (4a) and (4b) are big M constraints to ensure bus visit i is not overlapping another bus j in either time or space. (4c) is similar to (4a) and (4b) in the sense that it verifies that the bus visit i is not overlapping bus visit j in either time or space, but it also enforces that at least one of the states must be true. (4d) and (4e) are set in place to prevent bus visit i from being assigned to multiple positions in time or space, respectively. In other words, (4a), (4b), (4c), (4d), and (4e) are used together to ensure the bus visit is placed in a single valid position in both time (not encroaching on the bus in front or behind of it in the queue) and space (not allowing more than one bus to reside in the same physical space).

Could state that the are the same queuing constraints as before.

Constraints (4f), (4g), and (4h) are used to enforce time constraints. (4f) states that the initial charge time plus the time on the charger is the detach time. (4g) states that the arrival time is less than the initial charge time and that the initial charge time is sufficient for the bus to be on for the allotted time. (4h) enforces that the detach time of bus visit i is before (or at the same as) the departure time. Constraint (5j) is used to enforce that only a single charger may be chosen for bus visit i.

B. Battery Dynamics Constraints

State what you are trying to do first (i.e. give the charge dynamics and relationship to previous visit)

The set of constraints ((5b), (5c), and (5a)) are the linear battery dynamic constraints. (5b) does not allow bus visit i to over charge, (5c) does not allow the bus to be undercharged as to ensure that the bus can complete its route, and (5a) is the linking item that sets the initial charge for bus visit i's next visit.

The final set of constraints((5f), (5g), (5h), and (5i)), are used to linearize the bilinear term $p_i * w_{iq}$ by using big M constraints.

It is confusing to separate (3), (4), & (5), although I do like them numbered differently

$$\eta_i + \sum_{q=1}^{Q} g_{iq} r_q - \lambda_i = \eta_{\gamma_i} \tag{5a}$$

$$\eta_i + \sum_{q=1}^{Q} g_{iq} r_q \le \kappa \tag{5b}$$

$$\eta_i + \sum_{q=1}^{Q} g_{iq} r_q - \lambda_i \ge \nu * \kappa \tag{5c}$$

asterisks represent convolution. Use \cdot for multiplication

$$\eta_i + \sum_{q=1}^{Q} g_{iq} r_q - \lambda_i \ge \beta * \kappa \tag{5d}$$

$$\eta_i \ge \upsilon * \beta \tag{5e}$$

$$p_i \ge g_{iq} \tag{5f}$$

$$p_i \le g_{iq} - (1 - w_{iq})M \tag{5g}$$

$$M w_{iq} \ge g_{iq} \tag{5h}$$

$$0 \le g_{iq} \tag{5i}$$

$$\sum_{q=1}^{Q} i * w_{iq} = v_i \tag{5j}$$

$$\sum_{q=1}^{Q} w_{iq} = 1 \tag{5k}$$

C. Matrix Notation

There are a few things to note:

- We want to convert this problem to standard LP, for our problem we will mainly be concerned with
  - Inequality of $\ge$ form
- We will be formulating the equation in the form $Ax = b$ and $Ax \ge b$ where
  - A is a $n \times m$ matrix
  - x is a $m \times 1$ vector
  - b is a $n \times 1$ vector

D. Position Allocation Problem

E. Charging Dynamics

IV. Example

V. Conclusion

1) Matrix Deconstruction: The constraint matrix A will be broken down into two parts: A_{eq} for all the equality constraints and A_{ineq} for all the inequality constraints. Both A_{eq} and A_{ineq} formulated with two sub-matrices A_{pack} and $A_{dynamics}$ to represent the portion of the matrix that is utilized for the box packing constraints and the battery dynamics constraints, respectively. For example, A_{eq} will be represented in the following manner

$$A_{eq} = \begin{bmatrix} A_{\text{pack}} \\ A_{\text{dynamics}} \end{bmatrix}_{eq}$$

Where we can define the full equality as:

$$\begin{bmatrix} A_{\text{pack}} \\ A_{\text{dynamics}} \end{bmatrix}_{eq} \begin{bmatrix} x_{\text{pack}} \\ x_{\text{dynamics}} \end{bmatrix}_{eq} = \begin{bmatrix} b_{\text{pack}} \\ b_{\text{dynamics}} \end{bmatrix}_{eq}$$

$$A_{eq} x_{eq} = b_{eq}$$

Similarly for the inequality constraints:

$$\begin{bmatrix} A_{\text{pack}} \\ A_{\text{dynamics}} \end{bmatrix}_{ineq} \begin{bmatrix} x_{\text{pack}} \\ x_{\text{dynamics}} \end{bmatrix}_{ineq} \ge \begin{bmatrix} b_{\text{pack}} \\ b_{\text{dynamics}} \end{bmatrix}_{ineq}$$

Finally, the entire constraint formulation will be written as:

$$\begin{bmatrix} A_{\text{pack}} \\ A_{\text{dynamics}} \end{bmatrix}_{eq} \begin{bmatrix} x_{\text{pack}} \\ x_{\text{dynamics}} \end{bmatrix}_{eq} = \begin{bmatrix} b_{\text{pack}} \\ b_{\text{dynamics}} \end{bmatrix}_{eq} \tag{6a}$$

$$\begin{bmatrix} A_{\text{pack}} \\ A_{\text{dynamics}} \end{bmatrix}_{ineq} \begin{bmatrix} x_{\text{pack}} \\ x_{\text{dynamics}} \end{bmatrix}_{ineq} \ge \begin{bmatrix} b_{\text{pack}} \\ b_{\text{dynamics}} \end{bmatrix}_{ineq} \tag{6b}$$

A. Formulating A_{pack}

1) Formulating $A_{pack_{eq}}$: The components that make up the equality constraints for the box packing problem are:

- $p_i + u_i = c_i$
- $\sum_{q=1}^{Q} w_{iq} = 1$

Placing them together in A_{eq} results in:

$$A_{eq} = \begin{bmatrix} A_{detach_{N\times 2N}} & \mathbb{0}_{N\times NQ} \\ \mathbb{0}_{N\times 2N} & A_{w_{N\times NQ}} \end{bmatrix}_{2N\times(2N+NQ)}$$

$$x_{eq} = \begin{bmatrix} p_{i_{N\times1}} \\ u_{i_{N\times1}} \\ w_{iq_{NQ\times1}} \end{bmatrix}_{2N+NQ} \qquad b_{eq} = \begin{bmatrix} c_{i_{N\times1}} \\ \mathbb{1}_{N\times1} \end{bmatrix}_{2N\times1} \tag{7}$$

$$\begin{bmatrix} A_{detach_{N\times 2N}} & \mathbb{0}_{N\times NQ} \\ \mathbb{0}_{N\times 2N} & A_{w_{N\times NQ}} \\ \mathbb{0}_{N\times 2N} & A_{v_{N\times NQ}} \end{bmatrix} \begin{bmatrix} p_{i_{N\times1}} \\ u_{i_{N\times1}} \\ w_{iq_{NQ\times1}} \end{bmatrix} = \begin{bmatrix} c_{i_{N\times1}} \\ \mathbb{1}_{N\times1} \\ v_{i_{N\times1}} \end{bmatrix}$$

Where

2) Formulating $A_{pack_{ineq}}$: The components that make up the inequality constraints for the box packing problem are

- $u_j - u_i - p_i - (\sigma_{ij} - 1)T \ge 1$
- $v_j - v_i - s_i - (\delta_{ij} - 1)S \ge 1$
- $\sigma_{ij} + \sigma_{ji} + \delta_{ij} + \delta_{ji} \ge 1$
- $\sigma_{ij} + \sigma_{ji} \le 1$
- $\delta_{ij} + \delta_{ji} \le 1$
- $a_i \le c_i \le (T - p_i)$
- $c_i \le \tau_i$
- $p_i \ge g_{iq}$
- $p_i \le g_{iq} - (1 - w_{iq})M$
- $M w_{iq} \ge g_{iq}$
- $0 \le g_{iq}$

A_{ineq} takes the form of:

$$
A_{ineq} = \begin{bmatrix}
A_{time_{\Xi\times(2\Xi+2N)}} & \mathbb{0}_{\Xi\times(3\Xi+4N+3NQ)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbb{0}_{\Xi\times(2\Xi+2N)} & A_{queue_{\Xi\times(2\Xi+2N)}} & \mathbb{0}_{\Xi\times(\Xi+2N)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbb{0}_{N\times 2N} & A_{\sigma_{N\times\Xi}} & \mathbb{0}_{N\times(2N+3NQ)} & A_{\delta_{N\times\Xi}} & \mathbb{0}_{N\times(2\Xi+2N)} & \cdots & \cdots & \cdots & \cdots \\
\mathbb{0}_{N\times 2N} & -A_{\sigma_{N\times\Xi}} & \mathbb{0}_{N\times(3\Xi+4N+3NQ)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbb{0}_{N\times(2\Xi+2N)} & \mathbb{0}_{N\times(2N)} & \cdots & -A_{\delta_{N\times\Xi}} & \mathbb{0}_{N\times(2N+3NQ)} & \cdots & \cdots & \cdots & \cdots \\
\mathbb{0}_{N\times(4\Xi+4N)} & \cdots & \cdots & \cdots & -A_{a_{N\times N}} & \mathbb{0}_{N\times(N+3NQ)} & \cdots & \cdots & \cdots \\
\mathbb{0}_{N\times(4\Xi+5N)} & \cdots & \cdots & \cdots & \cdots & -A_{c_{N\times N}} & \mathbb{0}_{N\times 3NQ} & \cdots & \cdots \\
\mathbb{0}_{N\times(4\Xi+5N)} & \cdots & \cdots & \cdots & \cdots & -A_{c_{N\times N}} & \mathbb{0}_{N\times 3NQ} & \cdots & \cdots \\
\mathbb{0}_{4N\times(4\Xi+6N)} & \cdots & \cdots & \cdots & \cdots & \cdots & A_{gg_{4N\times NQ}} & A_{gw_{4N\times NQ}} & \mathbb{1}_{4N\times NQ}
\end{bmatrix}_{(2\Xi+10N)\times(4\Xi+6N+3NQ)}
$$

$$
x_{ineq} = \begin{bmatrix}
u_{i_{N\times1}} \\ p_{i_{N\times1}} \\ \sigma_{ij_{\Xi\times1}} \\ \mathbb{1}_{\Xi\times1} \\ v_{i_{N\times1}} \\ s_{i_{N\times1}} \\ \delta_{ij_{\Xi\times1}} \\ \mathbb{1}_{\Xi\times1} \\ a_{i_{N\times1}} \\ c_{i_{N\times1}} \\ q_{iq_{NQ\times1}} \\ w_{iq_{NQ\times1}} \\ \mathbb{1}_{NQ\times1}
\end{bmatrix}_{(4\Xi+6N+3NQ)\times1}
\qquad
b_{ineq} = \begin{bmatrix}
\mathbb{1}_{\Xi\times1} \\ \mathbb{1}_{\Xi\times1} \\ \mathbb{1}_{N\times1} \\ -\mathbb{1}_{N\times1} \\ -\mathbb{1}_{N\times1} \\ -c_{i_{N\times N}} \\ -(T-p_i)_{N\times N} \\ -\tau_{i_{N\times N}} \\ -p_{i_{N\times1}} \\ p_{i_{N\times1}} \\ \mathbb{0}_{2N\times1}
\end{bmatrix}_{(5\Xi+7N)\times1}
\tag{8}
$$

B. Formulating $A_{dynamics}$

1) Formulating $A_{dynamics_{eq}}$: The components that make up the equality constraint for the dynamics problem are

- $\eta_{\gamma_i} = \eta_i + g_{iq}r_q - \lambda_i$

$A_{dynamics}$ takes the form of:

$$
A_{eq} = \begin{bmatrix}
A_{\text{init charge}_{N\times N}} & \mathbb{0}_{N\times(N+NQ)} \\
A_{\text{next charge}_{N\times 2N+NQ}} & \cdots
\end{bmatrix}_{2N\times(2N+NQ)}
$$
$$
x_{eq} = \begin{bmatrix} \eta_{1_{N\times1}} \\ g_{iq_{NQ\times1}} \\ \lambda_{i_{N\times1}} \end{bmatrix}_{(2N+NQ)\times1}
\qquad
b_{eq} = \begin{bmatrix} \eta_{i_{N\times1}} \\ \eta_{\gamma_i\ N\times1} \end{bmatrix}_{2N\times1}
\tag{9}
$$

$A_{dyanmics_{ineq}}$ takes the form of:

$$
A_{ineq} = \begin{bmatrix}
-A_{\text{max charge}_{N\times(N+NQ)}} & \mathbb{0}_{N\times N} \\
A_{\text{min charge}_{N\times(2N+NQ)}} & \cdots \\
A_{\text{last charge}_{N\times N}} & \mathbb{0}_{N\times(N+NQ)}
\end{bmatrix}_{3N\times(2N+NQ)}
$$
$$
x_{ineq} = \begin{bmatrix} \eta_{i_{N\times1}} \\ g_{iq_{NQ\times1}} \end{bmatrix}_{(2N+NQ)\times1}
\qquad
b_{ineq} = \begin{bmatrix} -\mathbb{1}_{N+NQ\times1} \\ \mathbb{0}_{2N+NQ\times1} \\ H_{final} * \mathbb{1}_{A\times1} \end{bmatrix}_{3N\times1}
\tag{10}
$$

C. Putting it back together

$$
\begin{bmatrix} A_{\text{pack}} \\ A_{\text{dynamics}} \end{bmatrix}_{eq}
\begin{bmatrix} x_{\text{pack}} \\ x_{\text{dynamics}} \end{bmatrix}_{eq}
=
\begin{bmatrix} b_{\text{pack}} \\ b_{\text{dynamics}} \end{bmatrix}_{eq}
$$
$$
\begin{bmatrix} A_{\text{pack}} \\ A_{\text{dynamics}} \end{bmatrix}_{ineq}
\begin{bmatrix} x_{\text{pack}} \\ x_{\text{dynamics}} \end{bmatrix}_{ineq}
\geq
\begin{bmatrix} b_{\text{pack}} \\ b_{\text{dynamics}} \end{bmatrix}_{ineq}
$$

May also be represented as

$$
A_{eq} = \begin{bmatrix}
\begin{Bmatrix}
A_{detach_{N\times 2N}} & \mathbb{0}_{N\times NQ} \\
\mathbb{0}_{N\times 2N} & A_{w_{N\times NQ}} \\
\mathbb{0}_{N\times 2N} & A_{v_{N\times NQ}}
\end{Bmatrix}_{3N\times(2N+NQ)} & \{\mathbb{0}\}_{3N\times(2N+NQ)} \\
\{\mathbb{0}\}_{NA\times(2N+NQ)} & \{A_{\text{next charge}_{N\times 2N+NQ}}\}_{NA\times(2N+NQ+A)}
\end{bmatrix}_{(3N+NA)\times(4N+NQ+A)}
$$
$$
x_{eq} = \begin{bmatrix}
\begin{Bmatrix} p_{i_{N\times1}} \\ u_{i_{N\times1}} \\ w_{iq_{NQ\times1}} \end{Bmatrix}_{2N+NQ} \\
\begin{Bmatrix} \eta_{\Gamma_i\ N\times1} \\ g_{iq_{NQ\times1}} \\ \lambda_{i_{N\times1}} \end{Bmatrix}_{(2N+NQ+A)\times1}
\end{bmatrix}_{(4N+2NQ+A)\times1}
\qquad
b_{eq} = \begin{bmatrix}
\begin{Bmatrix} c_{i_{N\times1}} \\ \mathbb{1}_{N\times1} \\ v_{i_{N\times1}} \end{Bmatrix}_{3N\times1} \\
\{\eta_{\gamma_i\ N\times1}\}_{N\times1}
\end{bmatrix}_{4N\times1}
$$
$$
\begin{bmatrix}
\begin{Bmatrix}
A_{detach_{N\times 2N}} & \mathbb{0}_{N\times NQ} \\
\mathbb{0}_{N\times 2N} & A_{w_{N\times NQ}} \\
\mathbb{0}_{N\times 2N} & A_{v_{N\times NQ}}
\end{Bmatrix}_{3N\times(2N+NQ)} & \{\mathbb{0}\}_{3N\times(2N+NQ)} \\
\{\mathbb{0}\}_{NA\times(2N+NQ)} & \{A_{\text{next charge}_{N\times 2N+NQ}}\}_{NA\times(2N+NQ+A)}
\end{bmatrix}
\begin{bmatrix}
\begin{Bmatrix} p_{i_{N\times1}} \\ u_{i_{N\times1}} \\ w_{iq_{NQ\times1}} \end{Bmatrix}_{2N+NQ} \\
\begin{Bmatrix} \eta_{\Gamma_i\ N\times1} \\ g_{iq_{NQ\times1}} \\ \lambda_{i_{N\times1}} \end{Bmatrix}_{(2N+NQ+A)\times1}
\end{bmatrix}
$$
$$
= \begin{bmatrix}
\begin{Bmatrix} c_{i_{N\times1}} \\ \mathbb{1}_{N\times1} \\ v_{i_{N\times1}} \end{Bmatrix}_{3N\times1} \\
\{\eta_{\gamma_i\ N\times1}\}
\end{bmatrix}_{4N\times1}
\tag{11}
$$

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