

4.22)

Given:

$$PA + A^T P = -C^T C \quad \text{where } (A, C) \text{ is observable}$$

Find:

- Show  $A$  is Hurwitz iff  $P = P^T > 0$  that satisfies the equation
- Show that if  $A$  is Hurwitz, the Lyapunov function will have a unique solution.

Solution:

**Hint:** Apply LaSalle's Theorem and recall that for an observable pair  $(A, C)$ , the vector  $C \exp(At)x = 0 \iff x = 0$ .

**LaSalle's Theorem:** Let  $\Omega \subset D$  be a compact set that is positively invariant with respect to

$$\dot{x} = f(x)$$

Let  $V: D \rightarrow \mathbb{R}$  be a continuously differentiable function s.t.  $\dot{V}(x) \leq 0$  in  $\Omega$ . Let  $E$  be the set of all points in  $\Omega$  where  $\dot{V}(x) = 0$ . Let  $M$  be the largest invariant set in  $E$ . Then every solution starting in  $\Omega$  approaches  $M$  as  $t \rightarrow \infty$ .

Where: Positively Invariant:  $x(0) \in M \Rightarrow x(t) \in M \quad \forall t \geq 0$

Invariant Set:  $x(0) \in M \Rightarrow x(t) \in M \quad \forall t \in \mathbb{R}$

**Theorem:** A matrix is Hurwitz

$$\text{i)} \quad (\forall \lambda_i, \operatorname{Re}(\lambda_i) < 0)$$

iff

$$\text{ii)} \quad \exists Q = Q^T > 0 \quad \exists P = P^T > 0 \quad \text{s.t.}$$

$$PA + A^T P = -Q$$

(2)

Corollary 4.2: Let  $x = 0$  be an equilibrium point for  
(Invariance principle)  $\dot{x} = f(x)$

(3)

Let  $V: \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuously differentiable, radially unbounded, positive definite function s.t.  $\dot{V}(x) \leq 0 \forall x \in \mathbb{R}^n$ . Let  $S = \{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\}$  and suppose that no solution can stay identically on  $S$ , other than the trivial solution  $x(t) \equiv 0$ . Then the origin is A.S.

Lets use a similar approach used to prove Hurwitz in class (pg 80)

$$\text{i) Suppose } V(x) = x^T P x$$

$$\Rightarrow \dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = \underbrace{x^T (P A + A^T P) x}_{\dot{x} = Ax}$$

$$\text{Now substitute } PA + A^T P = -C^T C$$

$$\Rightarrow \dot{V}(x) = x^T (-C^T C) x = -x^T C^T C x$$

By invariance principle: if  $C^T C x \equiv 0 \Rightarrow \dot{V}(x) = 0$

ii) Now suppose A is Hurwitz (all eigs  $< 0$ )

Define

$$P = \int_0^\infty e^{At} Q e^{At} dt$$

Now to show P is positive definite, suppose that it is not

$$\exists x \in \mathbb{R}^n \setminus \{0\} \text{ s.t. } x^T P x = 0$$

$$\Rightarrow \int_0^\infty e^{At} Q e^{At} x dt = 0$$

$$\Rightarrow e^{At} x = 0 \Rightarrow x = 0 \quad \text{which contradicts what was stated above. i.e. } P \text{ is positive definite.}$$

$\uparrow$   
This can be said because  $(A, C)$  is observable.

Now to show that it is a solution of the Lyapunov Equation  
we use direct substitution

$$\begin{aligned} PA + A^T P &= \int_0^\infty e^{At} C^T C e^{At} dt + \int_0^\infty A e^{At} C^T C e^{At} dt \\ &= \int_0^\infty \frac{d}{dt} (e^{At} C^T C e^{At}) dt \\ &= e^{At} C^T C e^{At} \Big|_0^\infty = \underline{\underline{CC}} \end{aligned}$$

Which is a unique solution

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4.27)

Given:

$$\dot{x}_1 = -x_2 x_3 + 1$$

$$\dot{x}_2 = x_1 x_3 - x_2$$

$$\dot{x}_3 = x_3^2 (1 - x_3)$$

Find:

a) Show that the system has a unique equilibrium point

b) Using linearization, show that that the equilibrium point  
is asymptotically stable. Is it G.A.S.?

Solution:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_2 \\ \vdots \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_2 x_3 + 1 \\ \vdots \\ x_1 x_3 - x_2 \\ \vdots \\ x_3^2 (1 - x_3) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$x_3^2 (1 - x_3) \Rightarrow x = 0, 1$$

$$\underline{x_3 = 0};$$

$$-x_2(0) + 1 = 0$$

$$x_1(0) - x_2 = 0$$

$$\underline{1 \neq 0}$$

$$\underline{x_3 = 1};$$

$$-x_2(1) + 1 = 0$$

$$x_1(1) - x_2 = 0$$

$$-x_2 = -1$$

$$\underline{x_1 = x_2 = 1}$$

$$\underline{x_2 = 1}$$

The only equilibrium point is  $\mathbf{x}^* = [1 \ 1 \ 1]^T$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -x_3 & -x_2 \\ x_3 & -1 & x_1 \\ 0 & 0 & 2x_3 - 3x_3^3 \end{bmatrix} \mid x = x^{-1}$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} := A$$

$$\text{eig}(A) = -\frac{1 \pm i\sqrt{3}}{2} \text{ or } -1$$

Because  $\text{Re}(\lambda_i) \geq 0$ , then we can say it is A.S.  
 However for  $x_3 = 0$ ,  $x_1$  grows unbounded, i.e. the system is not G.A.S.

4.32.1)

Given:

$$\dot{x}_1 = -x_1 + x_1^2 \quad \dot{x}_2 = -x_2 + x_2^2 \quad \dot{x}_3 = x_3 - x_1^2$$

Find:

Investigate whether the origin is stable, asymptotically stable, or unstable

Solution:

To determine stability of the origin we can try to linearize about it to see if any information can be extracted.

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \\ \vdots & \ddots & \end{bmatrix} = \begin{bmatrix} 2x_1 - 1 & 0 & 0 \\ 0 & -1 & 2x_3 \\ -2x_1 & 0 & 1 \end{bmatrix} \quad |_{x=0}$$

$$\Rightarrow J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{eig}(J) = -1, -1, 1$$

The origin is unstable because:

Theorem: Let  $x=0$  be an equilibrium point for

$$\dot{x} = f(x)$$

where  $f: D \rightarrow \mathbb{R}^n$  is continuously differentiable and  $D$  is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(0)$$

Let  $\lambda_i$  denote an eigenvalue of  $A$

- i) If  $\forall \lambda_i \operatorname{Re}(\lambda_i) < 0$ , then the origin is A.S.
- ii) If  $\exists \lambda_i \text{ s.t. } \operatorname{Re}(\lambda_i) > 0$  then the origin is not stable.

4.32.2)

Given:

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = -5x_3 + x_1 [-2x_3 - \text{sat}(y)]^2 \quad \dot{x}_3 = -2x_3 - \text{sat}(y)$$

Find: where  $y = -2x_1 - 5x_2 + 2x_3$

See (4.32.1)

Solution:

If we assume  $y(0) = 0$  we get

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ -2x_3^2 & 0 & -\cos x_3 - 4x_3 x_1 \\ 0 & 0 & -2 \end{bmatrix} \Big|_{x=0}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow \text{eig}(J) = \text{No good}$$

Let's try assuming that  $y$  is not saturated near the origin.

$$\text{sat}(y) = y = -2x_1 - 5x_2 + 2x_3$$

$\Rightarrow$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -5x_3 + x_1 [-2x_3 - (-2x_1 - 5x_2 + 2x_3)]^2$$

$$\dot{x}_3 = -2x_3 - (-2x_1 - 5x_2 + 2x_3)$$

$$J = \begin{bmatrix} 0 & 1 & 0 \\ (2x_1 + 5x_2 - 4x_3)^2 + x_1(8x_1 + 20x_2 - 16x_3) & -\cos x_3 - x_1(16x_1 - 40x_2 - 32x_3) & -4 \\ 2 & 5 & -4 \end{bmatrix}$$

$$\mathcal{J}|_{x=[0]} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 5 & -4 \end{bmatrix} \Rightarrow \text{eig}(\mathcal{J}) = -0.3116 \pm 0i \\ \underline{2.1573 \pm 1.205i}$$

Which implies origin is unstable

4.32.3)

Given:

$$\dot{x}_1 = -2x_1 + x_1^3 \quad \dot{x}_2 = -x_2 + x_1^2 \quad \dot{x}_3 = -x_3$$

Find:

$\text{See } (4.3.2.1)$

Solution:

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 3x_1^2 - 2 & 0 & 0 \\ 2x_1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Big|_{x=[0]} \quad |_{x=[0]}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \text{eig}(J) = -2, \underline{-1}, \underline{-1}$$

By (4), the system is A.S.

4.32.4)

Given:

$$\dot{x}_1 = -x_1 \quad \dot{x}_2 = -x_1 - x_2 - x_3 - x_1 x_3 \quad \dot{x}_3 = (x_1 + 1)x_2$$

Find:

$\Sigma x (4.32.1)$

Solution:

$$\Sigma = \frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 0 & 0 \\ -x_3 - 1 & -1 & -x_1 - 1 \\ x_2 & x_1 + 1 & 0 \end{bmatrix} \Big|_{x=[0]}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \text{eig}(\Sigma) = -0.5 \pm 0.826i, -1$$

By (4) the system is A.S.

2.1.1)

Given:

$$\dot{x}_1 = -x_1 + 2x_1^3 + x_2 \quad x_2 = -x_1 - x_2$$

Find:

- Determine type of isolated equilibrium
- Plot vector field

Solution:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_1^3 + x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{x_1 = -x_2}$$

$$-x_1 + 2x_1^3 + x_2 = 0$$

$$-(-x_2) + 2(-x_2)^3 + x_2 = 0$$

$$x_2 - 2x_2^3 + x_2 = 0$$

$$2x_2 - 2x_2^3 = 0$$

$$x_2(1 - x_2^2) = 0$$

$$\underline{x_2 = 0, \pm 1.}$$

$$x^{eq} = [0, 0] ; [1, -1] ; [-1, 1]$$

$$\underline{J = \frac{\partial f}{\partial x} = \begin{bmatrix} 6x_1^2 - 1 & 1 \\ -1 & -1 \end{bmatrix}}$$

i)  $[0, 0]$

$$J|_{[0,0]} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \text{eig}(J) = \underline{-1 \pm i}$$

Stable Focus

ii)  $[1, -1]$

$$J|_{[1,-1]} = \begin{bmatrix} 6-1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \text{eig}(J) = \underline{4.82 \atop -0.82}$$

Saddle ( $\pm$  eigen value)

iii)  $[-1, 1]$

$$J|_{[-1,1]} = \begin{bmatrix} 6-1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \text{eig}(J) = \underline{4.82 \atop -0.82}$$

↑ Same as before

Saddle

2.1.2)

Given:

$$\dot{x}_1 = x_1 + x_1 x_2 \quad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$$

Find:

&lt;see (2.1.1)

Solution:

$$\dot{\mathbf{x}} = \begin{bmatrix} x_1 + x_1 x_2 \\ -x_2 + x_2^2 + x_1 x_2 - x_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(x_1 + x_1 x_2) = x_1(1 + x_2) = 0 \Rightarrow \underline{x_1 = 0 \text{ or } x_2 = -1}$$

$$x_1 = 0$$

$$-x_2 + x_2^2 + (0)x_2 - 0^3 = 0$$

$$-x_2 + x_2^2 = 0$$

$$x_2(-1 + x_2) = 0$$

$$\underline{x_2 = 0, 1}$$

$$x_2 = -1$$

$$-(-1) + (-1)^2 - x_1 - x_1^3 = 0$$

$$2 - x_1 - x_1^3 = 0$$

$$\underline{x_1 = 1}$$

$$\Rightarrow x_{eq} = [0, 0]; [0, 1]; [1, -1]$$

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 1+x_2 & x_1 \\ x_2 - 3x_1^2 & -1+2x_2+x_1 \end{bmatrix}$$

i)  $[0, 0]$ 

$$J|_{[0,0]} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{eig}(J) = \pm 1$$

Saddleii)  $[0, 1]$ 

$$J|_{[0,1]} = \begin{bmatrix} 1+1 & 0 \\ 1-0 & -1+2+0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \text{eig}(J) = 1, 2 \quad \text{Unstable node}$$

iii)  $[1, -1]$ 

$$J|_{[1,-1]} = \begin{bmatrix} 1+(-1) & 1 \\ -1-3(1) & -1+2(-1)+1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}$$

$$\Rightarrow \text{eig}(A) = -1 \pm 1.7321i \quad \text{Stable focus}$$

Given:

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = -0.8x_1 - 10x_1^2 x_2 + u$$

Find:

Develop an approximate, higher order linearization of the following system.

Solution:

Lets attempt a simple redefinition of

$$\tilde{x}_1 = x_2; \quad \tilde{x}_2 = -0.8x_1 \quad \tilde{x}_3 = -10x_1^2 x_2$$

Differentiating gives

$$\dot{\tilde{x}}_1 = \dot{x}_2 = -0.8x_1 - 10x_1^2 x_2 + u = \tilde{x}_2 + \tilde{x}_3 + u$$

$$\dot{\tilde{x}}_2 = -0.8\dot{x}_1 = -0.8x_2 = -0.8\tilde{x}_1$$

$$\begin{aligned}\dot{\tilde{x}}_3 &= -10x_1 \dot{x}_2 - 10x_1^2 x_2 \\ &= \frac{20\tilde{x}_2 x_1^2}{0.8} - \frac{10x_2^2 x_1}{0.8}\end{aligned}$$

↓ Continue in PDF

## 2.1.1)

### Given

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2x_2^3 + x_2 \\ \dot{x}_2 &= -x_1 - x_2\end{aligned}$$

### Find

1. Find all equilibrium points
2. Determine the type of each isolated equilibrium
3. Draw vector field plot

### Solution

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#### Find equilibrium points

```
>>> /* Define equations */
      xd1: -x1 + 2*x1^3 + x2;
      xd2: -x1 - x2;

      /* Find roots */
      solve([xd1=0, xd2=0]);

x2 + 2x1^3 - x1
[[x2 = 1, x1 = -1], [x2 = -1, x1 = 1], [x2 = 0, x1 = 0]]
-x2 - x1
```

#### Determine equilibrium point types

```
>>> /* Calculate Jacobian */
      J:jacobian([xd1, xd2], [x1, x2]);

      /* Calculate eigenvalue for each equilibrium point */
      /* The eigenvalue output is of the following format */
      /* [[eigenvalues], [multiplicity]] */

      float(eivals(psubst([x1=-1, x2=1], J)));
      float(eivals(psubst([x1=1, x2=-1], J)));
      float(eivals(psubst([x1=0, x2=0], J)));

[[[-0.8284271247461907, 4.828427124746191], [1.0, 1.0]]
 [[-0.8284271247461907, 4.828427124746191], [1.0, 1.0]]
 [[-1.0i - 1.0, i - 1.0], [1.0, 1.0]]

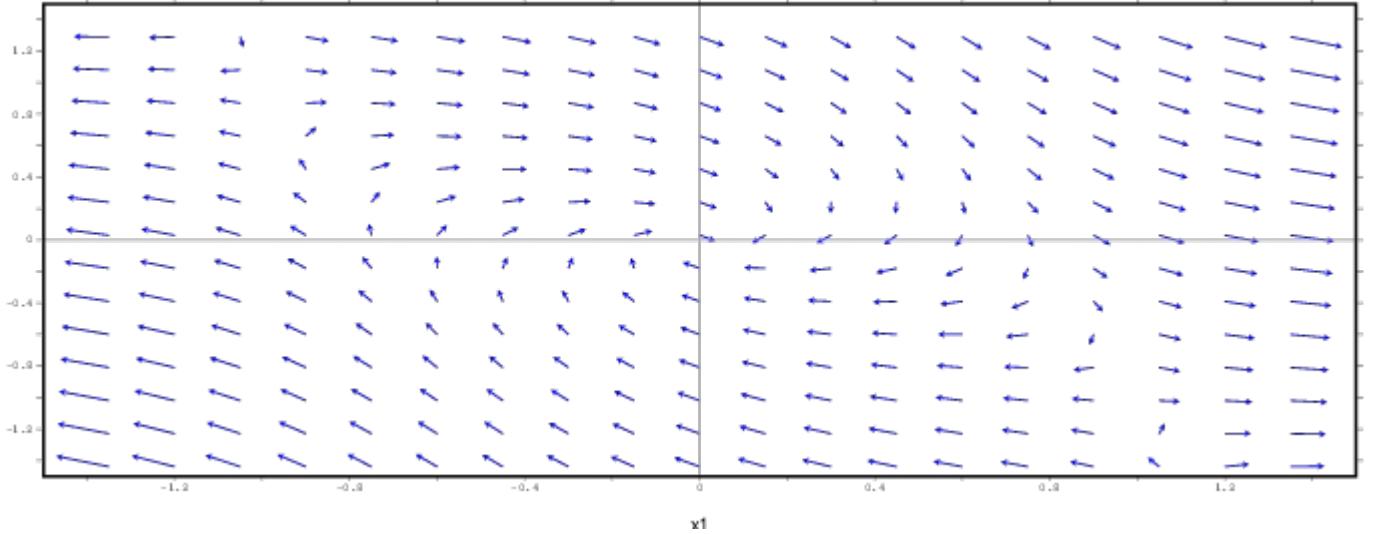
\begin{pmatrix} 6x_1^2 - 1 & 1 \\ -1 & -1 \end{pmatrix}
```

Based on the eigenvalues found from the Jacobians found at each of the equilibrium points, we can determine that they are:

- (-1, 1): Saddle
- (0,0): Stable focus
- (1, -1): Saddle

#### Draw vector field

```
>>> plotdf([xd1, xd2], [x1, x2], [x1, -1.5, 1.5], [x2, -1.5, 1.5])$
```



## 2.1.2)

### Given

$$\begin{aligned}\dot{x}_1 &= x_1 + x_1 x_2 \\ \dot{x}_2 &= -x_2 + x_2^2 + x_1 x_2 - x_1^3\end{aligned}$$

### Find

1. Find all equilibrium points
2. Determine the type of each isolated equilibrium
3. Draw vector field plot

### Solution

---

#### Find equilibrium points

```
>>> /* Define equations */
      xd1: x1 + x1*x2;
      xd2:-x2 + x2^2 + x1*x2 - x1^3;

      /* Find roots */
      solve([xd1=0, xd2=0]);
```

$\left[ [x_2 = 1, x_1 = 0], [x_2 = 0, x_1 = 0], [x_2 = -1, x_1 = 1], \left[ x_2 = -1, x_1 = -\frac{\sqrt{7}i + 1}{2} \right], \left[ x_2 = -1, x_1 = \frac{\sqrt{7}i - 1}{2} \right] \right]$

$x_1 x_2 + x_1$   
 $x_2^2 + x_1 x_2 - x_2 - x_1^3$

#### Determine equilibrium point types

```

>>> /* Calculate Jacobian */
J:jacobian([xd1, xd2], [x1, x2]);

/* Calculate eigenvalue for each equilibrium point */
/* The eigenvalue output is of the following format */
/* [[eigenvalues], [multiplicity]] */

float(eivals(psubst([x1=0, x2=1], J)));
float(eivals(psubst([x1=0, x2=0], J)));
float(eivals(psubst([x1=1, x2=-1], J)));

[[-1.0, 1.0], [1.0, 1.0]]


$$\begin{pmatrix} x_2 + 1 & x_1 \\ x_2 - 3x_1^2 & 2x_2 + x_1 - 1 \end{pmatrix}$$


[[-1.732050807568877i - 1.0, 1.732050807568877i - 1.0], [1.0, 1.0]]

[[1.0, 2.0], [1.0, 1.0]]

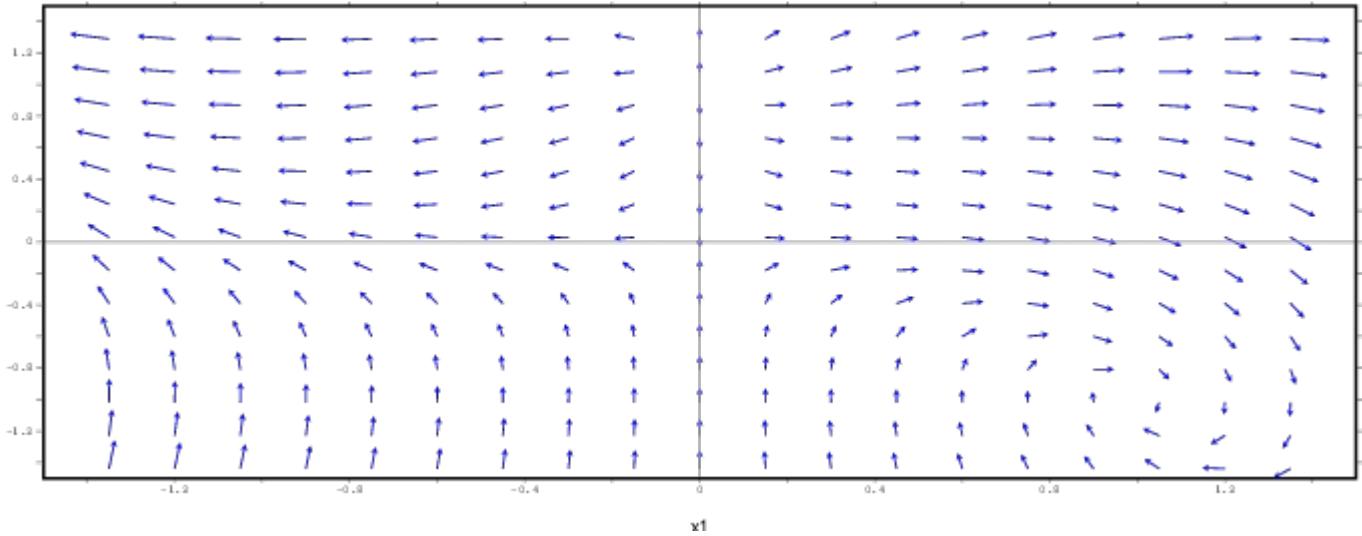
```

Based on the eigenvalues found from the Jacobians found at each of the equilibrium points, we can determine that they are:

- (0,1): Unstable node
- (0,0): Saddle
- (1,-1): Stable focus

### Draw vector field

```
>>> plotdf([xd1, xd2], [x1, x2], [x1, -1.2, 1.5], [x2, -1.5, 3.5])$
```



## Canvas Problem)

### Given

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.8x_1 - 10x_1^2 x_2 + u\end{aligned}$$

### Find

1. Develop an approximate, higher order linearization
2. Plot results

### Solution

```

>>> /* Define original state */
 $xd1: x2$ 
 $xd2: -0.8*x1 - 10*x1^2*x2$ 

/* Find roots */
solve([xd1=0, xd2=0]);

/* Calculate Jacobian */
J:jacobian([xd1, xd2], [x1, x2])$

/* Calculate eigenvalue for each equilibrium point */
/* The eigenvalue output is of the following format */
/* [[eigenvalues], [multiplicity]] */
float(evals(psubst([x1=0, x2=0], J))));

/* Draw vector field */
plotdf([xd1, xd2], [x1, x2], [x1, -1.2, 1.5], [x2, -1.5, 3.5])$
```

**rat:** replaced -0.8 by -4/5 = -0.8  
[[x1 = 0, x2 = 0]]  
[[-0.8944271909999159i, 0.8944271909999159i], [1.0, 1.0]]

### Define new linearized states

```

>>> /* Define state */
xt1: x2(t)$
xt2: -0.8*x1(t)$
xt3: -10*x1(t)^2*x2(t)$

/* Differentiate to get forms of xt# */
diff(xt1);
diff(xt2);
diff(xt3);

/* Define first derivative of new state */
xdt1: x2 + x3$ 
xdt2: -0.8*x1$ 
xdt3: (20*x2*x1^2)/(0.8) - (10*x2^2*x1)/(0.8)$

/* Calculate Jacobian */
J:jacobian([xdt1, xdt2, xdt3], [x1, x2, x3])$

/* Evaluate Jacobian at equilibrium point */
float(evals(psubst([x1=0, x2=0], J))));
```

$-0.8 \left( \frac{d}{dt} x_1(t) \right) dt$   
 $\left( -10x_1(t)^2 \left( \frac{d}{dt} x_2(t) \right) - 20x_1(t)x_2(t) \left( \frac{d}{dt} x_1(t) \right) \right) dt$   
[[-0.8944271909999159i, 0.8944271909999159i, 0.0], [1.0, 1.0, 1.0]]  
 $\frac{d}{dt} x_2(t) dt$

From this we can see that we have retained the same eigenvalues (showing that the higher order state is another realization). Below are some slices of the contour plots.

```

>>> /* Draw vector field slices */
plotdf([xdt1], [x2, x3])$
```

```
>>> plotdf([xdt2, xdt3], [x1, x2])$
```

