Homework 1

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1.1

Given

A mathematical model that describes a wide variety of physical nonlinear systems is the nth-order differential equation

$$y^n = g(t, y, \dot{y}, ..., y^{n-1}, u)$$

where u and y are scalar variables. With u as input and y as output, find a sate model.

Solution

Suppose that $y = x_1$, $\dot{y} = x_2$, etc. Continuing to differentiate gives:

$$\begin{aligned}
&\dot{x_1} = x_2 = \dot{y} \\
&\dot{x_2} = x_3 \\
&... \\
&\dot{x}_{n-1} = x_n \\
&\dot{x}_n = g(t, y, \dot{y}, ..., y^{n-1}, u)
\end{aligned}$$

1.2

Given

Consider a single-input-single-output system described by the nth-order differential equation

$$y^n = g_1(t, y, \dot{y}, ..., y^{n-1}, u) + g_2(t, y, \dot{y}, ..., y^{n-1})\dot{u}$$

where g_2 is a differentiable function of its arguments. With u as input and y as output, find a state model.

Hint: Take $x_n = y^{n-1} - g_2(t, y, \dot{y}, ..., \dot{y}^{n-1})u$

Solution

Lets begin by defining $x_1 = y$, $x_2 = y^{(1)}$, ..., $x_{n-1} = y^{(n-2)}$, $x_n = y^{(n-1)} - g_2(t, y, \dot{y}, ..., y^{(n-2)})u$. Therefore

$$
\begin{aligned}
&\dot{x}_1 = y^{(1)} = x_2 \\
&\dot{x}_2 = x_3 \\
&... \\
&\dot{x}_{n-1} = y^{(n-1)} = x_n + g_2(t, x_1, x_2, ..., x_{n-1})u \\
&\dot{x}_n = y^{(n)} - \dot{g}_2(t, x_1, x_2, ..., x_{n-q})u - g_2(t, x_1, x_2, ..., x_{n-q})\dot{u}
\end{aligned}
$$

Where $\dot{x}_{n-1}$ is found by solving $x_n = y^{n-1} - g_2(t, y, \dot{y}, ..., \dot{y}^{n-1})u$ for $y^{(n-1)}$ and $\dot{x}_n = y^{(n)} - \dot{g}_2(t, x_1, x_2, ..., x_{n-q})u - g_2(t, x_1, x_2, ..., x_{n-q})\dot{u}$ is found by using the chain rule and product rule.

Chain Rule: $h'(x) = f'(g(x))g'(x)$

Product Rule: $(u \cdot v)' = u' \cdot v + u' \cdot v$

$$
\begin{aligned}
&\dot{x}_n = y^{(n)} - \dot{g}_2(t, x_1, x_2, ..., x_{n-q})u - g_2(t, x_1, x_2, ..., x_{n-q})\dot{u} = \\
&g_1(t, x_1, x_2, ..., x_n + g_2(t, x_2, x_3, ..., x_{n-2})u, u) - \\
&(\tfrac{\partial g_2}{\partial t} + \tfrac{\partial g_2}{\partial x_1}\dot{x}_1 + \tfrac{\partial g_2}{\partial x_2}\dot{x}_2 + ... + \tfrac{\partial g_2}{\partial x_{n-1}}\dot{x}_{n-1})u - g_2(t, x_1, x_2, ..., x_{n-q})\dot{u}
\end{aligned}
$$

1.5

Given

The nonlinear dynamic equations for a single-link manipulator with flexible joints, damping ignored, is given by

$$
\begin{aligned}
&I\ddot{q}_1 + MgLsin(q_1) + k(q_1 - q_2) = 0 \\
&J\ddot{q}_2 - k(q_1 - q_2) = u
\end{aligned}
$$

Solution

Begin by defining $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$:

$$
\begin{aligned}
&\dot{x}_1 = \dot{q}_1 \\
&\dot{x}_2 = \ddot{q}_1 = -\tfrac{MgL}{I} sinx_1 - \tfrac{k(x_1 - x_3)}{I} \\
&\dot{x}_3 = \dot{q}_2 \\
&\dot{x}_4 = \ddot{q}_2 = \tfrac{k(x_1 - x_3)}{J} + \tfrac{u}{J}
\end{aligned}
$$

3.

Given

For each of the functions find whether

- Continuously differentiable
- Locally Lipschitz
- Continuous
- Globally Lipschitz

1. $f(x) = x^2 + |x|$
2. $f(x) = x + sgn(x)$
3. $f(x) = sin(x)sgn(x)$
4. $f(x) = -x + asin(x)$
5. $f(x) = -x + 2\,|x|$
6. $f(x) = tan(x)$
7. $f(x) = \begin{bmatrix} ax_1 + tanh(bx_1) - tanh(bx_2) \\ ax_2 + tanh(bx_1) + tanh(bx_2) \end{bmatrix}$
8. $f(x) = \begin{bmatrix} -x_1 + a|x_2| \\ -(a+b)x_1 + bx_1^2 - x_1x_2 \end{bmatrix}$

Solution

1.

As referenced in the class notes, x^2 is locally, but not globally, Lipschitz. $|x|$ on the other hand, it not continuous at 0. Furthermore:

$$\forall x, y \in \mathbb{R} \ \ ||x| - |y|| \leq |x - y|$$

By choosing $\delta = \frac{\epsilon}{3}$:

$$\begin{aligned}
&\left|x^2 + |x| - y^2 - |y|\right| \leq L\,|x - y| \\
&\left|x^2 - y^2\right| + ||x| - |y|| \\
&2\,|x - y| + |x - y| \\
&3\,|x - y| \\
&\text{Note that } \ |x - y| = \delta \\
&3\delta = \tfrac{3\epsilon}{3} = \epsilon
\end{aligned}$$

This shows that there $f(x)$ is locally Lipschitz. Therefore, the following can be said about the function

- Not continuously differentiable
- Locally Lipschitz
- Not continuous
- Not globally Lipschitz

2.

Consider Figure 1:

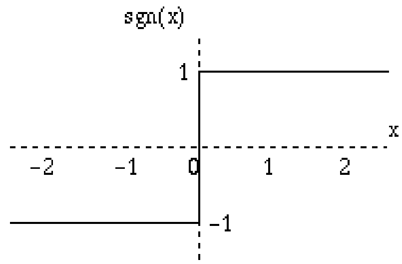

Figure 1: Plot of $sgn(x)$

$sgn(x)$ is not continuous or continuously differentiable. However it is bounded at ± 1 therefore it is globally Lipschitz (implying local as well). x is continuous and continuously differentiable as well as globally Lipschitz. Therefore:

- Continuously differentiable
- Locally Lipschitz
- Continuous
- Globally Lipschitz

3.

Consider the following

$$\frac{\partial f}{\partial x} = cos(x)sgn(x)$$

Furthermore, consider Figures 2 and 3. It can be seen that for $f(x)$, it is piecewise continuous, but not continuously differentiable.

Given the function and its derivative we can conclude:

- Not continuously differentiable
- Locally Lipschitz
- Piecewise continuous
- Globally Lipschitz

4.

$sin(x)$ and x continuous, continuously differentiable, and globally Lipschitz therefore

- Continuously differentiable

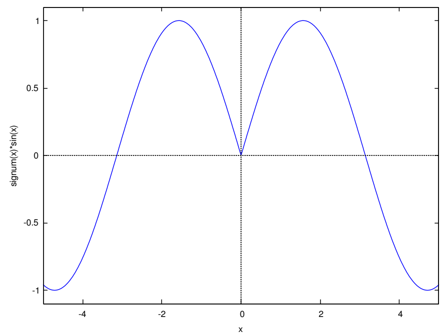

Figure 2: Plot of $sin(x)sgn(x)$

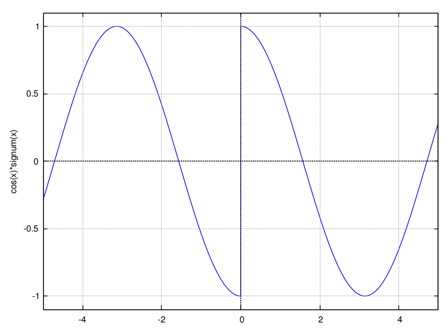

Figure 3: Plot of $cos(x)sgn(x)$

- Locally Lipschitz
- Continuous
- Globally Lipschitz

5.

$-x$ is continuous, continuously differentiable, and globally Lipschitz. However, we have seen the $|x|$ is not continuous or continuously differentiable, but it is locally and globally Lipschitz. Therefore:

$$
\begin{aligned}
&|-x + 2|x| + y - 2|y||\\
&|x - y| + |2x - 2y|\\
&3\,|x - y|
\end{aligned}
$$

Implying L = 3. Note that this holds for $\mathbb{R}$.

- Not continuously differentiable
- Locally Lipschitz
- Not continuous
- Globally Lipschitz

6.

From Figure 4, we can see that the plot is continuous over $(-\pi/2, \pi/2)$, similarly for Figure 5. However, there is a clear asymptote at $\pm\pi/2$. From this we can say

- Continuously differentiable over $(-\pi/2, \pi/2)$
- Locally Lipschitz
- Continuous over $(-\pi/2, \pi/2)$
- Not globally Lipschitz

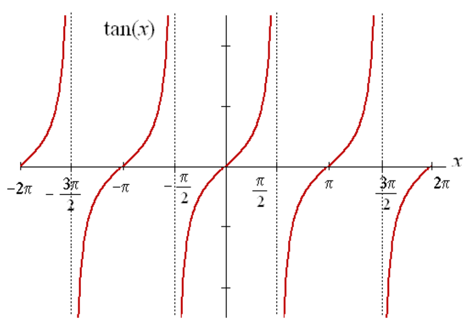

Figure 4: Plot of $tan(x)$

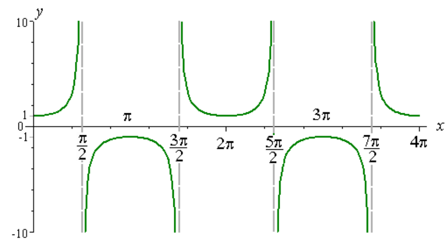

Figure 5: Plot of $sec^2(x)$

7.

From Figure 6, it is easily noticed that the function is continuous, continuously differentiable, and globally Lipschitz. ax_i hold similar properties. Therefore we can say the following

- Continuously differentiable
- Locally Lipschitz
- Continuous
- Globally Lipschitz

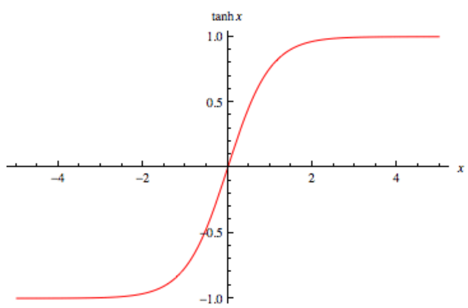

Figure 6: Plot of $tanh(x)$

8.

x_i is continuous, continuously differentiable and globally Lipschitz. However, $|x|$ is not continuous or continuously differentiable, and x^2 is not globally Lipschitz (as shown in the class notes). Therefore

- Not continuously differentiable
- Locally Lipschitz
- Not continuous
- Not globally Lipschitz

3.3

Given

Show that if $f_1 : R \to R$ and $f_2 : R \to R$ are locally Lipschitz, then $f_1 + f_2$, $f_1 f_2$, and $f_1 \circ f_2$ are locally Lipschitz

Solution

Refer to Theorems Lemmas and Definitions for the Definition of locally Lipschitz.

a) $f_1 + f_2$

Suppose $f_3 := f_1 + f_2$. Using the definition of locally Lipschitz

$$\begin{aligned}
&|f_1(x) + f_2(x) - f_1(y) - f_2(y)| = \\
&|f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| = \\
&L_1 |x - y| + L_2 |x - y|
\end{aligned}$$

b) $f_1 f_2$

Suppose $f_3 := f_1 f_2$. Using the definition of locally Lipschitz

$$\begin{aligned}
&|f_1(x)f_2(x) - f_1(y)f_2(y)| = \\
&|f_1(x)f_2(y) + f_1(x)f_2(y) - f_1(x)f_2(y) - f_1(x)f_2(y)| = \\
&|f_1(f_2(x) - f_2(y)) + f_2(y)(f_1(x) - f_1(y))| = \\
&C_2 L_1 |x - y| + C_1 L_2 |x - y|
\end{aligned}$$

c) $f_1 \circ f_2$

Suppose $f_3 := f_1 \circ f_2 = f_2(f_1(x))$. Using the definition of locally Lipschitz

$$\begin{aligned}
&|f_2(f_1(x)) - f_2(f_2(y))| = \\
&L_2 |f_1(x) - f_1(y)| = \\
&L_1 L_2 |x - y|
\end{aligned}$$

3.7

Given

Let $g : R^n \to R^n$ be continuously differentiable for all $x \in R^n$ and define $f(x)$ by

$$f(x) = \frac{g(x)}{1 + g^T(x)g(x)}$$

Show that $\dot{x} = f(x)$, with $x(0) = x_0$, has a unique solution defined for all $t \geq 0$.

Solution

Uniqueness can be shown by showing $f(x)$ is globally Lipschitz (Theorems Lemmas and Definitions). Lets begin by substituting $g(x) := y$.

$$\begin{aligned} f(x) &= \tfrac{y}{1+y^2} \\ \tfrac{\partial f}{\partial y} &= -\tfrac{y^2-1}{y^4+2y^2+1} \end{aligned}$$

which shows that $f(x)$ is continuous and continuously differentiable. Therefore, by definition, section, the function can be identified as locally Lipschitz. Furthermore taking the limit to ∞, $\frac{\partial f}{\partial x} \to 0$ showing that $\frac{\partial f(x)}{\partial x}$ uniformly bounded. Therefore, by definition, section, $f(x)$ can be said to be globally Lipschitz.

3.20

Given

Show that if $f : R^n \to R^n$ is Lipschitz on $W \subset R^n$, then $f(x)$ is uniformly continuous on W.

Solution

Lets begin with the definition:

$$\forall x, y \in \mathbb{R}^n \ ||f(t,x) - f(t,y)|| \leq L||x - y||$$

Therefore, if $W \subset \mathbb{R}^n$ (assuming the set is compact)

$$\forall x, y \in W \ ||f(t,x) - f(t,y)|| \leq \forall x, y \in \mathbb{R}^n \ ||f(t,x) - f(t,y)|| \leq L||x - y||$$

must be true because every subset of $\mathbb{R}^n$ must also satisfy the Lipschitz conditions.

Canvas Problem

Given

Give the definitions and examples of open sets, closed sets, and compact sets in 2D.

Solution

Open Sets

> *Definition*: An open set in a metric space (X, d) is a subset U of X with the following property: for any $x \in U$ there is a real number ϵ such that any point X that is a distance less than ϵ from x is also contained in U.
> For any point $x \in X$, define $B(x, \epsilon)$ to be the open ball of radius ϵ, which is the set of all points in X that are within distance ϵ from x. Then a set U is open iff for each point $x \in U$, there is an $\epsilon > 0$ such that $B(x, \epsilon)$ is completely contained in U

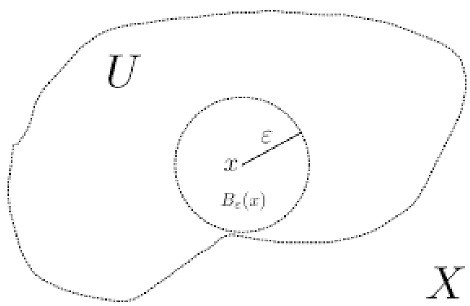

Figure 7: Example of an open set

The open interval $(2, 5)$ is an open set.

Closed Sets

> *Definition*: A closed set in a metric space (X, d) is a subset Z of X with the following property: for any point $x \notin Z$, there is a ball $B(x, \epsilon)$ around x, for some $\epsilon > 0$, which is joint from Z.

The closed set $[2, 5]$ is a closed set.

Compact Sets

> *Definition*: A cover of a set X is a collection of sets whose union includes X as a subset. Formally, if $C = \{U_\alpha : \alpha \in A\}$ is an indexed family of sets U_α, then C is a cover of X if
>
> $$X \subseteq \bigcup_{\alpha \in A} U_\alpha$$
>
> *Definition*: Z is compact if every open cover has a finite subcover.

The closed unit interval $[0, 1]$ is compact.

Theorems Lemmas and Definitions

Locally Lipschitz

Theorems: If $f : I \times D \to \mathbb{R}^n$ is locally Lipschitz at $x_0 \in \mathbb{R}^n$, then $\exists t' \in (t_o, t_1]$ such that the solution of $\dot{x} = f(t,x), x(t_0) = x_0$ exists and is unique on $[t_0, t']$.

Lemma: If f and $\frac{\partial f_i}{\partial x_j}$ are continuous (i.e., f is continuously differentiable) on $I \times x_0$, then f is locally Lipschitz at x_0

Globally Lipschitz

Definition: A function $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz if:

1. $\forall x \in D, f(\cdot, x)$ is piecewise continuous on I
2. $\exists L \in \mathbb{R}^+$ such that
   - $\forall t \in I, \forall x, y \in \mathbb{R}^n$
   - $||f(t,x) - f(t,y)|| \leq L||x - y||$

Theorem: If $f : I \times D \to \mathbb{R}^n$ is globally Lipschitz then the solution of $\dot{x} = f(t,x), x(t_0) = x_0$ exists and is unique on I

Lemma: Suppose f and $\frac{\partial f}{\partial x}$ are continuous on $I \times \mathbb{R}^n$. f is globally Lipschitz iff $\frac{\partial f}{\partial x}$ is uniformly bound on $I \times \mathbb{R}^n$, i.e.

$\exists L \in \mathbb{R}^+$ s.t. $\forall t \in I, \forall x \in \mathbb{R}^n$

$$||\frac{\partial f(t,x)}{\partial x}|| \leq L$$

Theorem: If $I \times R^n \to R^n$ is globally Lipschitz then the solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$ exists and is unique on I.

Sources

- Class notes
- Chain Rule
- Product Rule
- Open Sets
- Closed Sets
- Compact Sets
  - Brilliant
  - Wikipedia