

Final

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1

Given

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= x_1 - x_2^3\end{aligned}$$

Find

- Determine the equilibria for the given system
- Determine their qualitative nature (saddle, node, etc)
- Plot the vector field

Solution

Begin by finding equilibrium points x_{eq} (See appendix for worksheet)

$$\dot{x} = \begin{bmatrix} -x_1^3 + x_2 \\ x_1 - x_2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Where the equilibrium points are found to be $x_{eq} = \{(0, 0), (1, 1), (-1, -1)\}$. Taking the Jacobian as $J = \frac{\partial f}{\partial x}$:

$$J = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix}$$

Solving for x_1 and x_2 we find: $x_2 = x_1 = \pm\sqrt{1/3}$. Therefore $x_{eq} = \{(\sqrt{1/3}, \sqrt{1/3}), (-\sqrt{1/3}, \sqrt{1/3}), (\sqrt{1/3}, -\sqrt{1/3})\}$
 $(0, 0)$

$$J = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$eig(J) = -1, 1$: Saddle

$(1, 1)$

$$J = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$eig(J) = -2, -4$: Stable node

$$(-1, -1)$$

$$J = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{eig}(J) = -2, -4: \text{ Stable node}$$

2

Given

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^3 - x_2\end{aligned}$$

Find

Investigate (by Lyapunov Analysis) stability of the origin for the following system. Make the strongest statement possible.

Solution

Theorem: Let $V : D \rightarrow \mathbb{R}$ be a continuously differentiable function such that

1) $V(0) = 0$

2) $\forall x \in D \setminus \{0\}, V(x) > 0$

S) If $\forall x \in D, \dot{V}(x) \leq 0$ then $x = 0$ is stable

AS) If $\forall x \in D, \dot{V}(x) < 0$ then $x = 0$ is asymptotically stable

Attempt 1

Let $V = 1/2 x^T x$, then $\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$

$$\begin{aligned}x_1 x_2 + x_2(-x_1^3 - x_2) \\ = x_2(x_1 - x_1^3 - x_2)\end{aligned}$$

Can't really say anything from this, but we are close to a form that we want.

Attempt 2

Let $V = 1/4 x_1^4 + 1/2 x_2^2$, then $\dot{V} = x_1^3 \dot{x}_1 + x_2 \dot{x}_2$

$$\begin{aligned}x_1^3 x_2 - x_2 * x_1^3 - x_2^2 &= \\ -x_2^2 &< 0\end{aligned}$$

Therefore by the theorem above, the system is asymptotically stable.

3

Given

$$\begin{aligned}\dot{x}_1 &= x_1^3 - x_2 \\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

Find

Investigate (by whatever means) stability of the origin for the following system. Make the strongest statement possible.

Solution

4

Given

$$\dot{x} = -h(x) + u^2$$

$$\exists a > 0, \forall y \in \mathbb{R} \ y h(y) \geq ay^2$$

Find

Investigate input-to-state stability of the following system.

Solution

Theorem: Consider the system

$$\dot{x} = f(t, x, u)$$

where $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in (x, u) . The input functions u are piecewise continuous and bounded for all $t \geq 0$. If

1. There exists a continuously differentiable $V : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$
2. There exists continuously differentiable functions $\alpha_1, \alpha_2 \in K_\infty$ and $\rho \in K$
3. There exists a continuous positive definite function $w_3 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that
4. $\forall (t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \ \alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$
5. $\forall t \in [0, \infty), \forall (x, u) : \|x\| \geq \rho(\|u\|) > 0 \ \dot{V}(t, x, u) \leq -w_3(x)$

then the system is input-to-state stable.

If $h(x) \geq ax$ then suppose $\dot{x} = -h(x) + u^2 \leq -ax + u^2$

Let $\alpha_1 = \alpha_2 = V(x) = 1/2x^T x$, therefore

$$\dot{V}(x) = x\dot{x} = x(-ax + u) = -ax^2 + ux$$

Let $\|x\| \geq |u| = \rho(|u|)$. Then $|xu| \leq x^2$

$$\dot{V}(x) = -ax^2 - ux \leq$$

$$-ax^2 + x^2 =$$

$$x^2(1 - a)$$

Given that $w_3 = ax^2$, then the system is input-to-state stable.

5

Given

$$\begin{aligned}\dot{x}_1 &= x_2^3 - 1 + (1 - x_1)^3 \\ \dot{x}_2 &= -x_1 + u\end{aligned}$$

Find

- Develop a globally stabilizing state feedback control for the following system using backstepping
- Simulate

Solution

Definition: A nonlinear system

$$\dot{x} = f(x) + G(x)u$$

with $f : \mathbb{R}^n \supset D \rightarrow \mathbb{R}^n$ and $G : \mathbb{R}^n \supset D \rightarrow \mathbb{R}^n$ “sufficiently smooth” is feedback linearizable iff

1. there exists a diffeomorphism $T : D \rightarrow \mathbb{R}^n$ such that $0 \in D_z := T(D) = \{\sigma : \sigma = T(x), x \in D\}$
2. there exists a function $\gamma : D \rightarrow \mathbb{R}^{m \times m}$ such that $\forall x \in D, \gamma(x)$ is non-singular
3. There exists a function $\alpha : D \rightarrow \mathbb{R}^m$
4. There exists a controllable pair (A, B)
5. with $z = T(x), \dot{z} = Az + B\gamma(x)[u - \alpha(x)]$

Theorem: The single input system

$$\dot{x} = f(x) + g(x)u$$

is feedback linearizable iff there exists a domain $D_0 \subset D$

1. the matrix $G(x) = [g(x), ad_f g(x), \dots, ad_f^{n-1} g(x)]$ has rank n
2. The distribution $\Gamma = \text{span}\{g(x), ad_f g(x), \dots, ad_f^{n-1} g(x)\}$ is involutive in D_0

Begin by writing

$$\dot{x} = \begin{bmatrix} x_2 + 1 + (1 - x_1)^3 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where the rank can be seen to be $n = 2$. Let

$$\begin{aligned}ad_f^0 g(x) &= g(x) \\ ad_f^1 g(x) &= \nabla g(x)f(x) + \nabla f(x)g(x)\end{aligned}$$

Where we know $\nabla g(x) = 0$ and

$$\nabla f(x) = \begin{bmatrix} -3x_1^2 + 6x_1 - 3 & 1 \\ -1 & 0 \end{bmatrix}$$

Therefore

$$ad_f^1 g(x) = \begin{bmatrix} -3x_1^2 + 6x_1 - 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

From this, $\text{rank}(G(x)) = 2$ and the distribution is involutive. To find a satisfactory h , we must solve

$$\begin{aligned} L_g L_f^0 h(x) &= \nabla h(x) g(x) = 0 \\ L_g L_f^1 g(x) &= \nabla(\nabla h(x) f(x)) g(x) \neq 0 \\ h(0) &= 0 \end{aligned}$$

The first equation gives

$$\begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0]_{2 \times 2}$$

Therefore $\frac{\partial h}{\partial x_2} = 0$ which implies that h is independent of x_2 . In a similar fashion,

$$\frac{\partial h}{\partial x_1} \begin{bmatrix} -3x_1^2 + 6x_1 - 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial h}{\partial x_1} \neq 0$$

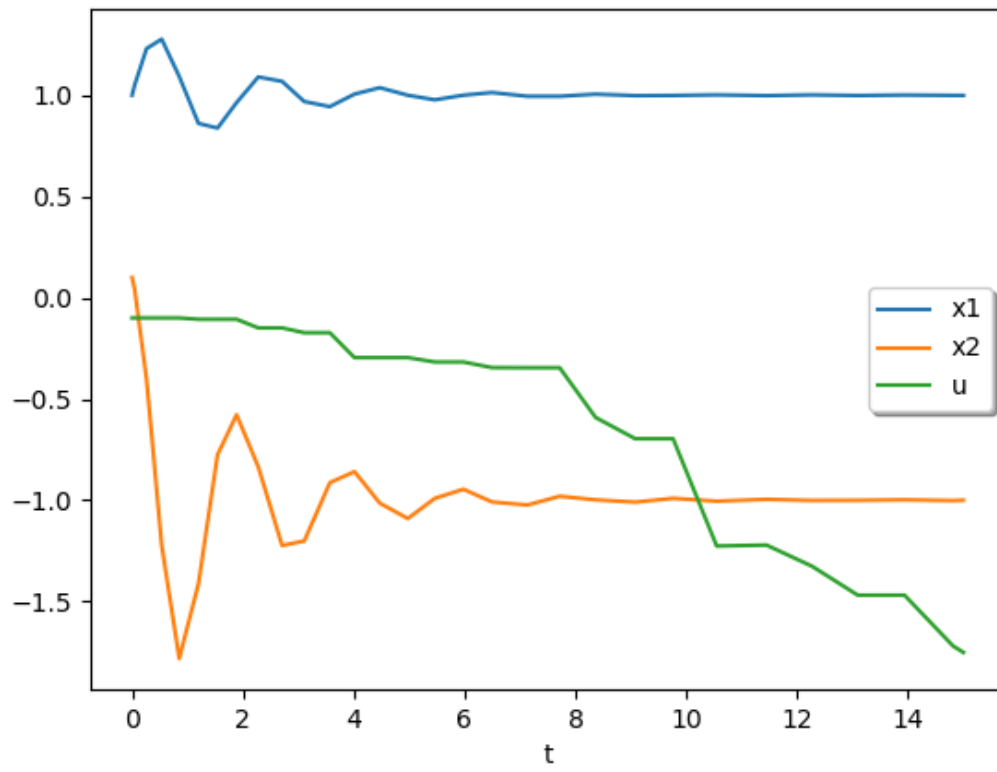
From which we state that h is dependent on x_1 . The first simple choice is then to allow $h(x) = x_1$ (which satisfies $h(0) = 0$)

Let $h(x) = x_1 = z_1$, and $z_2 = \dot{z}_1 = \dot{x}_1 = x_2 + 1 + (1 - x_1)^3$

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} x_2 + 1 + (1 - x_1)^3 \\ \dot{x}_2 - 3x_1^3 \dot{x}_1 + 6x_1 \dot{x}_1 - 3\dot{x}_1 \end{bmatrix}$$

Let $u = x_1 - 3z_2(x_1^2 - 3x_1 + 1) + v$, choose $v = -10x_1^2 - 10x_2^2$.

5



6

Given

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_2 \\ \dot{x}_2 &= u\end{aligned}$$

Find

- Develop a globally stabilizing state feedback control for the following system using feedback linearization
- Simulate

Solution

Let $V(x) = 1/2x_1^2$

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 = \\ &= x_1(x_1^2 + x_2) = x_1^3 + x_1x_2\end{aligned}$$

Let $\phi(x) = x_2 = -x_1^2 - x_1$, therefore

$$\dot{V}(x) = x_1^3 - x_1^3 - x_1^2 \leq 0$$

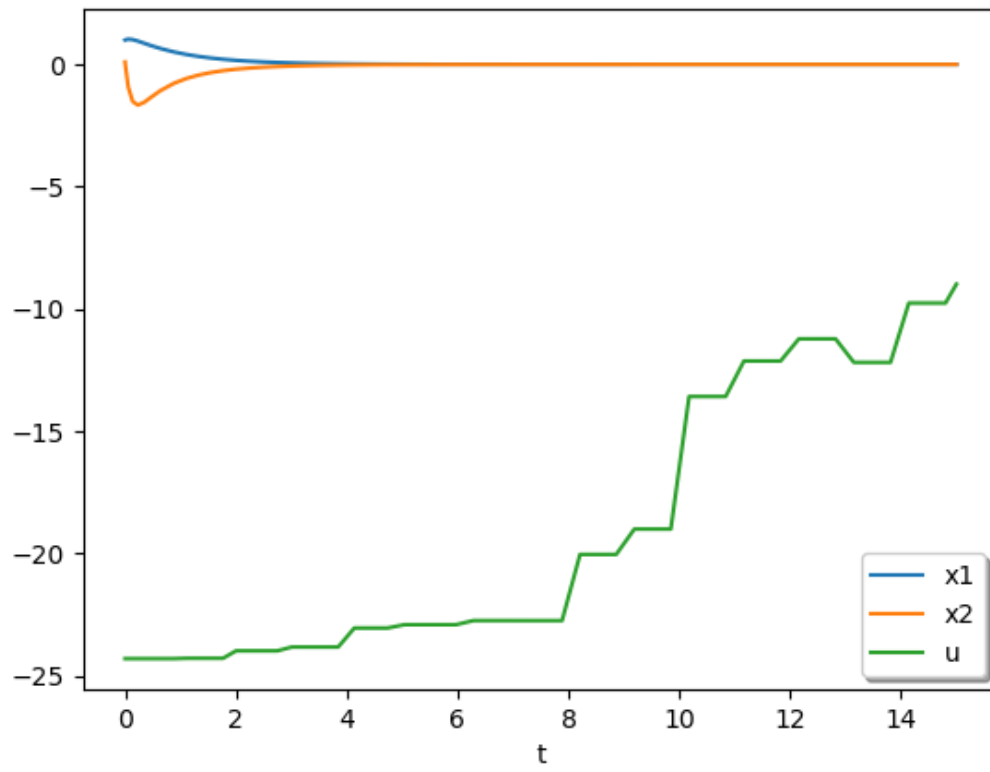
Pick $z = x_2 - \phi(x_1)$, $\dot{z} = \dot{x}_2 - \dot{\phi}(x_1) = u - \dot{\phi}(x_1)$ where $\dot{\phi}(x_1) = -2x_1\dot{x}_1 - \dot{x}_1$

Let $V_2(x) = V_1 + 1/2z^2$, then $\dot{V}_2(x) = \dot{V}_1(x) + z\dot{z}$

$$\dot{V}_2(x) = -x_1^2 + z(u - \dot{\phi}(x_1))$$

Let $u = \dot{\phi}(x_1) + kz$

6



7

Given

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + \text{sat}(u)\end{aligned}$$

Find

- Develop a globally stabilizing state feedback control for the following system using passivity-based control
- Simulate

Solution

Theorem: If the system

$$\dot{x} = f(x, u), \quad y = h(x)$$

is

1. passive with radially unbounded positive definite storage function
2. zero-state observable

then the origin $x = 0$ is globally asymptotically stabilizable with $u = -\phi(y)$ where ϕ is any locally Lipschitz function such that

$$\phi(0) = 0 \quad \text{and} \quad \forall y \neq 0, \quad y^T \phi(y) > 0$$

Begin by checking if the system is passive

Let $V(x) = 1/2 x^T x$ therefore

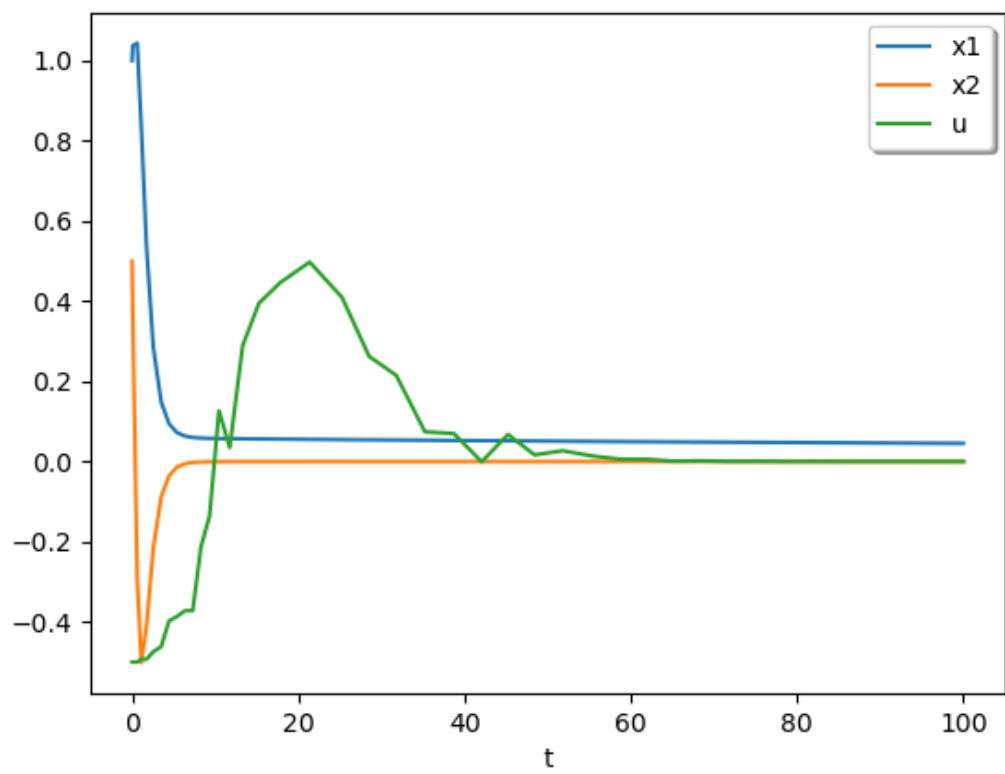
$$\begin{aligned}\dot{V} &= x_1 \dot{x}_1 + x_2 \dot{x}_2 = \\ &= x_1 x_2 - x_1^3 x_2 + x_2 \text{sat}(u)\end{aligned}$$

To check for passivity we write

$$-x_1 x_2 + x_1^3 x_2 = x_2 \text{sat}(u)$$

Let $y = x_2$ then the system is passive. Choose $u = y = x_2$

7



8

Given

$$\begin{aligned}\dot{x}_1 &= x_2 + \sin(x_1) \\ \dot{x}_2 &= \theta x_1 x_2 + u \\ 0 &\leq \theta \leq a\end{aligned}$$

True values: $\theta = 0.1$ and $a = 1$

Find

- Develop a globally stabilizing state feedback control law for the following uncertain system using sliding mode control.
- Simulate

Solution

Begin by choosing manifold to be $s = ax_1 + x_2$, therefore $x_2 = -ax_1$

$$\begin{aligned}\dot{x}_2 &= -ax_1 + \sin(x_1) \\ \text{we choose} \\ V(x) &= 1/2x_1^2 \\ \dot{V}(x) &= x_1\dot{x}_1 = \\ &= x_1(-ax_1 + \sin(x_1))\end{aligned}$$

Therefore $\sin(x_1) - ax_1 < 0$. Now let $\dot{V} = s\dot{x}$

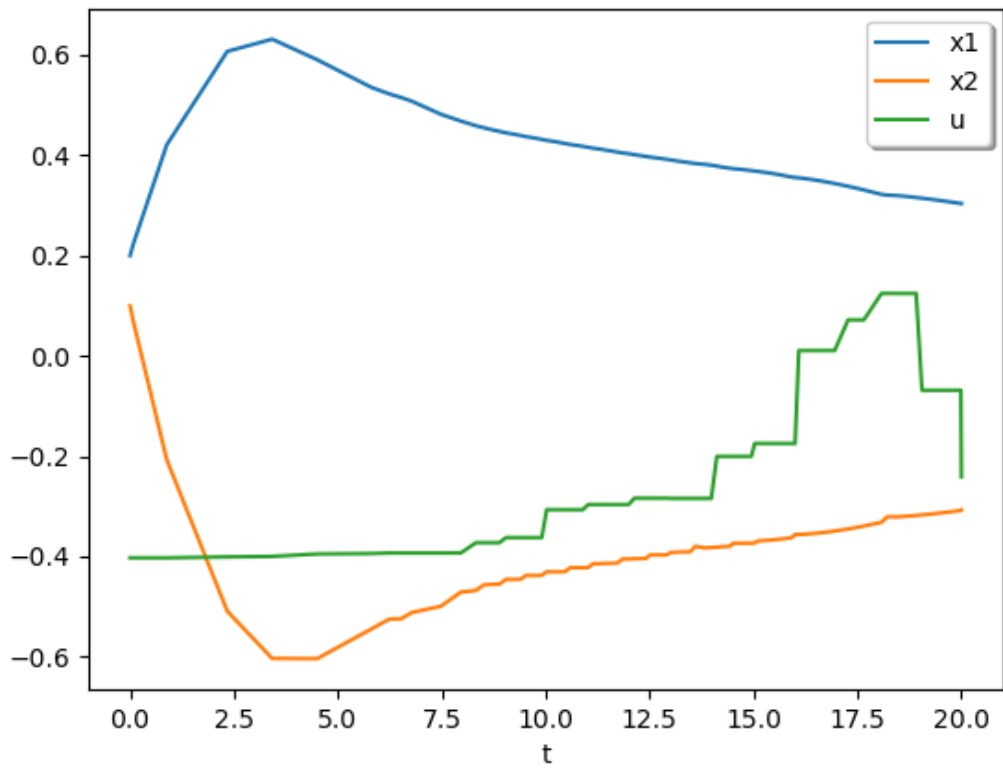
$$\begin{aligned}\dot{V} &= s(ax_1 + \dot{x}_2) = \\ &= s(a(x_2 + \sin(x_1)) + \theta x_1 x_2 + u)\end{aligned}$$

Let $u = -a(x_2 + \sin(x_1)) + v$

$$\dot{V} = s(\theta x_1 x_2 + v)$$

Choose $v = \text{sgn}(s)(-\theta x_1 x_2 - \beta_0)$ where $\beta_0 > 0$

8



9

Given

$$\begin{aligned}\dot{x}_1 &= x_2 + \sin(x_1) \\ \dot{x}_2 &= \theta x_1 x_2 + u \\ 0 &\leq \theta \leq a\end{aligned}$$

True values: $\theta = 0.1$ and $a = 1$

Find

- Develop a globally stabilizing state feedback control for the following system using Lyapunov redesign
- Simulate

Solution

10

Given

$$\dot{x} = ax + u$$

- True values: $a = 1$
- a is unknown to the controller

Find

- Develop an adaptive control law for the scalar system to track the signal $r(t) = \sin(t)$.
- Simulate

Solution