2.27.3)

Given:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \mu - x_2 - x_1^2 - 2x_1x_2$$

Find:

Find and classify bifurcations that occur as μ varies

Solution:

Begin by finding the equilibrium points ($\dot{x} = 0$)

$$x_2 = 0$$

$$0 = \mu - 0 - x_1^2 - 0 \Rightarrow x_1^2 = \mu$$

$$x_{eq} = [\pm\sqrt{\mu} \quad 0]^T$$

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2x_1 - 2x_2 & -1 - 2x_1 \end{bmatrix}\Bigg|_{x = x_{eq}}$$

$$\Rightarrow J_{eq} = \begin{bmatrix} 0 & 1 \\ -2\sqrt{\mu} & -2\sqrt{\mu} - 1 \end{bmatrix}$$

For $\mu < 0$ there are no equib points.

$-\sqrt{\mu}$:

$$J = \begin{bmatrix} 0 & 1 \\ 2\sqrt{\mu} & 2\sqrt{\mu} - 1 \end{bmatrix}$$

$$\text{eig}(J) = -1, 2\sqrt{\mu}$$

Because real parts have opposite sign the equib. point is a saddle

$\sqrt{\mu}$:

$$J = \begin{bmatrix} 0 & 1 \\ -2\sqrt{\mu} & -2\sqrt{\mu} - 1 \end{bmatrix}$$

Saddle-node bifurcation

$$\text{eig}(J) = -1, -2\sqrt{\mu}$$

Both real parts are negative $\therefore$ the equib. point is a stable node for $\mu < 0$

2.27.c)

Given:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \mu(x_1 + x_2) - x_2 - x_1^2 - 2x_1x_2$$

Find:

Find and classify bifurcations as μ varies

Solution:

Begin by finding equilibrium point

$$\underline{x_2 = 0}$$

$$0 = \mu(x_1 + 0) - 0 - x_1^2 - 0 \qquad x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \mu \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 0 = \underline{x_1(\mu - x_1)}$$

$$J = \begin{bmatrix} 0 & 1 \\ \mu - 2x_1 - 2x_2 & \mu - 1 - 2x_1 \end{bmatrix}$$

$$x_{eq} = [0, 0]^T$$

$$\Rightarrow \quad J|_{x_{eq}} = \begin{bmatrix} 0 & 1 \\ \mu & \mu - 1 \end{bmatrix} \Rightarrow eig(J) = -1, \mu$$

If $\mu < 0 \rightarrow$ stable point

$\mu > 0 \rightarrow$ saddle

Because the equilib. points persist but properties change this is a transcritical bifurcation

$$x_{eq} = [\mu, 0]^T$$

$$J|_{x_{eq}} = \begin{bmatrix} 0 & 1 \\ \mu - 2\mu & \mu - 1 - 2\mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\mu & -\mu - 1 \end{bmatrix} \Rightarrow eig(J) = -1, -\mu$$

$\mu < 0 \rightarrow$ saddle

$\mu > 0 \rightarrow$ Stable point

4.3.1)

Given:

$\dot{x}_1 = -x_1 + x_1 x_2$

$\dot{x}_2 = -x_2$

Find:

Use a quadratic Lyapunov function to show the origin is asymptotically stable.

Solution

Definition: (1)

An equilibrium point is A.S. if $\forall \epsilon > 0 \ \exists \delta > 0$ s.t.

$\forall t \geq 0, \ \|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon$

and

$\lim_{t \to \infty} x(t) = 0$

Theorem: (2)

Let $V: D \to \mathbb{R}$ be a continuously differentiable function s.t.

1) $V(0) = 0$	1) " "
2) $\forall x \in D \setminus \{0\}, \ V(x) > 0$	2) " "
3) If $\forall x \in D, \ \dot{V}(x) \leq 0$	3) $\|x\| \to \infty \ V(x) \to \infty$
$\forall x \in D \setminus \{0\}, \ \dot{V}(x) < 0$	4) $\forall x \in \mathbb{R}^n \setminus \{0\}$, $\dot{V}(x) < 0$
then $x = 0$ is A.S.	Then $x = 0$ is G.A.S

Suppose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

1) $V(0) = 0$ & 2) $\forall x \in D \setminus \{0\}$ $V(x) > 0$

$\dot{V}(x) = \frac{1}{2}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) = x_1\dot{x}_1 + x_2\dot{x}_2$

$\Rightarrow \dot{V}(x) = x_1(-x_1 + x_1 x_2) + x_2(-x_2)$

$\Rightarrow \dot{V}(x) = -x_1^2 + x_1^2 x_2 - x_2^2$

Theorem (3)

Let $V: D \to \mathbb{R}^n$ be C.D., P.D. function on D s.t. $\dot{V}$ is negative semi-definite on D. Let

$$S = \{x \in D \mid \dot{V}(x) = 0\}$$

If the only "trajectory" in S is $x \equiv 0$ then the origin is asymptotically stable.

$\therefore \dot{V}(x)$ can be rewritten as

$$\dot{V}(x) = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}}_{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Where P is negative definite. $\therefore$ the origin is G.A.S.

↳ See Lecture notes p.g. ~~75~~ 3 (2)

4.3.2)

Given:

$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2)$

$\dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$

Find:

See (4.3.1)

Solution:

Suppose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2$

$= x_1(-x_2 - x_1(1 - x_1^2 - x_2^2)) + x_2(x_1 - x_2(1 - x_1^2 - x_2^2))$

$= -\cancel{x_1 x_2} - x_1^2(1 - x_1^2 - x_2^2) + \cancel{x_1 x_2} - x_2^2(1 - x_1^2 - x_2^2)$

$= -x_1^2 + x_1^4 + x_2^2 x_1 - x_2^2 + x_1^2 x_2^2 + x_2^4$

$= x_2^4 + x_2^2(2x_1^2 - 1) + x_1^4 - x_1^2$

$= x_2^4 + x_1^4 + x_2^2(2x_1^2 - 1) - x_1^2$

$= x_2^4 + x_2^2(2x_1^2 - 1) + x_1^4 - x_1^2$

$= x_2^2(x_2^2 + 2x_1^2 - 1) + x_1^2(x_1^2 - 1)$

$\rightarrow -x_1^2 + x_1^2 x_1^2 + x_2^2 x_1 - x_2^2 + x_1^2 x_2^2 + x_2^2 x_2^2$

$-(\underbrace{x_1^2 + x_2^2}_{2V})(1 \underbrace{- x_1^2 - x_2^2}_{-2V})$

$\Rightarrow \dot{V}(x) = -2V(1 - 2V)$

If $V = 1/2$ then $\dot{V}(x) \rightarrow 0$

If $V < 1/2$ $\dot{V}(x) < 0$

If $V > 1/2$ $\dot{V}(x) > 0$

$\dot{V}(x)$ is N.D. on $V < 1/2$ making the origin A.S.

↳ See (2)

4.3.3)

Given:

$$\dot{x}_1 = x_2(1-x_1^2)$$

$$\dot{x}_2 = -(x_1+x_2)(1-x_1^2)$$

Find:

See (4.3.1)

Solution:

Suppose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\Rightarrow \dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(x_2(1-x_1^2)) + x_2(-(x_1+x_2)(1-x_1^2))$$

$$= x_2^2(x_1^2-1) = x_2^2x_1^2 - x_2^2$$

For terms << 1 (much smaller than 1) $x_1^2x_2^2 << x_2^2$.

$\therefore$ the origin is A.S. for points near the origin.

see (2)

4.3.4)

Given:

$\dot{x}_1 = -x_1 - x_2$; $\dot{x}_2 = 2x_1 - x_2^3$

Find:

See (4.3.1)

Solution:

$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1 - x_2) + x_2(2x_1 - x_2^3)$

$= -x_1^2 - x_1x_2 + 2x_1x_2 - x_2^4$

$= -x_2^4 - x_1^2 + x_1x_2$ If it were $2x_1x_2$, we could cancel out the terms

Suppose $V(x) = x_1^2 + \frac{1}{2}x_2^2$

$\dot{V}(x) = 2x_1\dot{x}_1 + x_2\dot{x}_2 = 2x_1 \cdot (-x_1 - x_2) + x_2(2x_1 - x_2^3)$

$= -2x_1^2 - \cancel{2x_1x_2} + \cancel{2x_1x_2} - x_2^4$

$= -x_2^4 - 2x_1^2$

$\dot{V}(x)$ is Negative Definite $\therefore$ the origin is G.A.S. (See (2))

4.4)

Given:

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u_1$$

$$J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u_2$$

$$J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + u_3 .$$

Find:

a) With $u_1 = u_2 = u_3 = 0$ the origin $\omega = 0$ is stable. Is it A.S.?

b) Suppose $u_i = -k_i\omega_i$ where $k_i > 0$. Show that the closed-loop system is GAS.

Solution:

Suppose $V(\omega) = \frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2)$

$$\dot{V}(\omega) = J_1\omega_1\dot{\omega}_1 + J_2\omega_2\dot{\omega}_2 + J_3\omega_3\dot{\omega}_3$$

$$\dot{V}(\omega) = \cancel{J_1}\omega_1\left(\frac{J_2 - J_3}{\cancel{J_1}}\omega_2\omega_3 + \frac{u_1}{\cancel{J_1}}\right) + \cancel{J_2}\omega_2\left(\frac{J_3 - J_1}{\cancel{J_2}}\omega_3\omega_1 + \frac{u_2}{\cancel{J_2}}\right) + \cancel{J_3}\omega_3\left(\frac{J_1 - J_2}{\cancel{J_3}}\omega_1\omega_2 + \frac{u_3}{\cancel{J_3}}\right)$$

a) $u_1 = u_2 = u_3 = 0$

$$\dot{V}(\omega) = (J_2 - J_3)\omega_1\omega_2\omega_3 + (J_3 - J_1)\omega_1\omega_2\omega_3 + (J_1 - J_2)\omega_1\omega_2\omega_3$$

$\dot{V}(\omega) = 0$

$\therefore$ The origin is stable but is not A.S. because $\dot{V}(\omega) \equiv 0$

↳ See page 71 in class notes

b) $u_i = -k_i \omega_i$

$$\dot{V}(\omega) = (J_2 - J_3)\,\omega_1\omega_2\omega_3 - k_1\omega_1^2 + (J_2 - J_3)\,\omega_1\omega_2\omega_3 - k_2\omega_2^2 + (J_1 - J_2)\,\omega_1\omega_2\omega_3 - k_3\omega_3^2$$

$$= \omega_1\left((J_2 - J_3)\,\omega_2\omega_3 - k_1\omega_1\right) + \omega_2\left((J_2 - J_3)\,\omega_1\omega_3 - k_2\omega_2\right) + \omega_3\left((J_1 - J_2)\,\omega_1\omega_2 - k_3\omega_3\right)$$

$\Rightarrow \dot{V}(\omega) = -k_1\omega_1^2 - k_2\omega_2^2 - k_3\omega_3^2 < 0$

$\therefore$ The origin is G.A.S. as per (2)

4.9)

Given:

$$V(x) = \frac{(x_1+x_2)^2}{1+(x_1+x_2)^2} + (x_1-x_2)^2$$

Find:

a) $V(x) \to \infty$ as $\|x\| \to \infty$ along the lines $x_1 = 0$ or $x_2 = 0$.

b) Show that $V(x)$ is not radially unbounded

Solution:

a)

$\underline{x_1 = 0}$

$$V(x) = \frac{x_2^2}{1+x_2^2} + x_2^2$$

$$\lim_{|x| \to \infty} V(x) \Big|_{x_1 = 0} = \infty$$

$\underline{x_2 = 0}$

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_1^2$$

$$\lim_{|x| \to \infty} V(x) \Big|_{x_2 = 0} = \infty$$

b) (4)

Definition:

A function $f: \mathbb{R}^n \to \mathbb{R}$ for which

$\|x\| \to \infty \Rightarrow f(x) \to \infty$

is said to be radially unbounded.

Suppose $x_1 = x_2$

$$V(x) = \frac{(x_1 + x_1)^2}{1 + (x_1 + x_1)^2} + (x_1 - x_1)^2$$

$$= \frac{2x_1^2}{1 + 2x_1^2} + 0$$ $\to$ Try plugging big numbers in

$\therefore$ as $|x| \to \infty$ $V(x) \to 1$, which is not radially unbounded

4.16)

Given:

$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -x_1^3 - x_2^3$

Find:

The origin is G.A.S.

Solution:

Suppose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1 x_2 + x_2(-x_1^3 - x_2^3)$

$= \underbrace{x_1 x_2 - x_1^3 x_2}_{\text{Want to remove}} - x_2^4$

Try the $V(x)$ used in class

$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^3 \Rightarrow \dot{V}(x) = x_1^3\dot{x}_1 + x_2\dot{x}_2$

$\Rightarrow \dot{V}(x) = \cancel{x_1^3 x_2} - \cancel{x_1^3 x_2} - x_2^4$

$= \underline{-x_2^4}$

If $x_2 \equiv 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow x_1 \equiv 0$

$\therefore$ By (3) the origin is A.S. Furthermore because $V(x)$ is radially unbounded the origin is G.A.S. as per (2)

4.3.3

Given

$$\dot{x}_1 = x_2(1 - x_1^2)$$
$$\dot{x}_2 = -(x_1 + x_2)(1 - x_2^2)$$

Find

See (4.3.1)

Solution

Suppose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_2^2x_1^2 - x_2^2$$

For values << 1 (much smaller than 1) $x_2^2x_1^2 << x_2^2$. Therefore, the origin is asymptotically stable for points near the origin.