4.54.1)

Given:

$$\dot{x} = -(1+u)x^3$$

Find: Investigate Input-to-State stability

Solution

Input-to-State stability Theorem:

Consider

$$\dot{x} = f(t, x, u)$$

where $f: [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is piecewise continuous in t and Locally Lipschitz (LL) in (x, u). The input functions u are piecewise continuous and bounded $\forall\, t \geq 0$ If

1) There exists a continuously differentiable (CD) function

$$V: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$$

2) There exists functions $\alpha_1, \alpha_2 \in K_\infty$ and $\rho \in K$

3) There exists a continuous positive definite (PD) function (1)

$$W_3: \mathbb{R}^n \rightarrow \mathbb{R}$$

Such that

4) $\forall (t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$

5) $\forall t \in [0, \infty)$, $\forall (x, u): \|x\| \geq \rho(\|u\|) > 0$

$$\dot{V}(t, x, u) \leq -W_3(x)$$

Then the system is input-to-state stable.

Where

Definition:

A function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class K_∞ ($\alpha \in K_\infty$) if $\alpha \in K$ and $\lim_{r \to \infty} \alpha(r) = \infty$. (2)

Definition

A function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class K ($\alpha \in K$) if it is CD, strictly increasing, and $\alpha(0) = 0$ (3)

Definition:

A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to belong to class KL if it is continuous and (4)

1) $\forall s \in [0, \infty)$, $\beta(\cdot, s) \in K$

2) $\forall r \in [0, \infty)$, $\beta(r, \cdot)$ is decreasing and

$$\lim_{s \to \infty} \beta(r, \cdot) = 0$$

Lemma:

Suppose $f(t, x, u)$ is CD & GL in (x, u), uniformly in t. If the unforced system has a GAS equilibrium point at the origin $x = 0$, then the system is input-to-state-stable.

Lets begin by assuming $V(x) = \frac{x^2}{2} = \alpha_1(x) = \alpha_2(x)$

$$\dot{V} = x\dot{x} = x(-(1+u)x^3) = -(1+u)x^4 = -x^4 - x^4 u$$

This will be negative definite if $\underbrace{u > 1}_{\rho}$

However, ρ is not class K.

Therefore, based on this choice of V, α_1, and α_2 the system is not input-to-state-stable by (1)

4.54.2)

Given:

$$\dot{x} = -(1+u)x^3 - x^5$$

Find:

see (4.54.1)

Solution:

Suppose $V(x) = \frac{x^2}{2} = \alpha_1(x) = \alpha_2(x)$

$$\dot{V}(x) = x\dot{x} = x(-(1+u)x^3 - x^5)$$
$$= -x^4 - ux^4 - x^6$$

If $u > -\frac{x^2}{x^4} = -x^2$

$\Rightarrow$ $\left.\sqrt{u} < |x|\right\} \rho$

Then $\dot{V}(x)$ will be N.D., and if $-W_3(x) = -x^4$

$$\dot{V}(x) < -W_3(x)$$

Therefore the system is input-to-state-stable by (1)

4.55.1)

Given:

$\dot{x}_1 = -x_1 + x_1^2 x_2 \qquad \dot{x}_2 = -x_1^3 - x_2 + u$

Find:

See (4.54.1)

Solution:

Suppose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1 + x_1^2 x_2) + x_2(-x_1^3 - x_2 + u)$$
$$= -x_1^2 + \cancel{x_1^3 x_2} - \cancel{x_1^3 x_2} - x_2^2 + x_2 u$$
$$= -x_1^2 - x_2^2 + x_2 u$$

If $\rho(\|u\|) < \|x_2\|$ and $-W_3 = -x_1^2$

Then the system is input-to-stable by (1)

Checks:

$\rho \in K_\infty \in K \qquad W_3 > 0$

13.6)

Given:

$$J\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u_1$$

$$J\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u_2$$

$$J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + u_3$$

Find:

Show that the state equation is feedback linearizeable

Solution:

Definition

A nonlinear system $\dot{x} = f(x) + G(x)u$ where $f: D \to \mathbb{R}^n$ and $G: D \to \mathbb{R}^{n \times p}$ are sufficiently smooth on a domain $D \subset \mathbb{R}^n$, is said to be feedback linearizeable (or input state linearizeable) if there exists a diffeomorphism $T: D \to \mathbb{R}^n$ s.t. $D_z = T(D)$ contains the origin and the change of variables $z = T(x)$ transforms

$$\dot{x} = f(x) + G(x)u$$

into the form

$$\dot{z} = Az + B\gamma(x)[u - \alpha(x)]$$

with (A, B) controllable and $\gamma(x)$ nonsingular $\forall x \in D$. (1)

Since we just want to show that the system is feedback linearizeable, then we just have to satisfy (1).

If $x_1 = \omega_1$, $x_2 = \omega_2$, and $x_3 = \omega_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

If we were to take $(J_i - J_j)x_i x_j := \alpha(x)$, i.e., assume it is the nonlinear part of the dynamics. Then:

$$\xi = u - \alpha(x)$$

where

$$\alpha(x) := \begin{bmatrix} (J_2 - J_3)x_2x_3 \\ (J_3 - J_1)x_3x_1 \\ (J_1 - J_2)x_1x_2 \end{bmatrix}$$

Furthermore; $B = J^{-1}$; $\gamma(x) = 1$; $A = 0_{3\times3}$

$\Rightarrow$ $$\dot{z} = 0_{3\times3}\, z + I_{3\times3}(1)(\xi).$$

Now we need to show (A, B) is controllable.

$\text{rank}([B \;\; AB \;\; A^2B]) = \underline{3 = n}$ controllable

$\gamma(x)$ is nonsingular.

$\therefore$ The system is feedback linearizable

13.7)

Given:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

q: m-dim coordinates of joint positions

u: m-dim torque input

$M(q)$: Symmetric inertia matrix (positive definite)

$C(q,\dot{q})$: Centrifugal and Coriolis forces

$D\dot{q}$: Viscous damping ($D > 0$)

$g(q)$: Gravity forces $g(q) = \left[\frac{\partial P}{\partial q}(q)\right]^T$

$P(q)$: Total potential energy of links due to gravity.

Find:

See (13.6)

Solution:

We want to put this in the form

$$\dot{z} = Az + B\gamma(x)[u - \alpha(x)]$$

To start:

$$\ddot{q} = \frac{u}{M(q)} - \frac{C(q,\dot{q})\dot{q} - D\dot{q} - g(q)}{M(q)}$$

Let

$$x_1 = q \quad x_2 = \dot{q}$$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} \Rightarrow \dot{x}_2 = (u - C(q,\dot{q})\dot{q} - D\dot{q} - g(q))M(q)$$

$$\gamma(x) = M(x) \quad \alpha(x) = +C(q,\dot{q})\dot{q} + D\dot{q} + g(q)$$

Assuming an n dimensional system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{B} M(x) \left[u - \left(C(q,\dot{q})\dot{q} + D\dot{q} + g(q) \right) \right]$$

Check if Controllable

$$\text{rank}\left(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \right) = n \Rightarrow \underline{\text{Controllable}}$$

$\gamma(x)$ is positive definite which implies nonsingularity

$\therefore$ The system is feedback Linearizable

13.13)

Given:

$$\dot{x}_1 = \tan(x_3)$$

$$\dot{x}_2 = -\frac{\tan(x_2)}{a\cos(x_3)} + \frac{1}{b\cos(x_2)\cos(x_3)}\tan(u)$$

$$\dot{x}_3 = \frac{\tan(x_2)}{a\cos(x_3)}$$

$a, b \in \mathbb{R} \quad a, b > 0$

Find:

a) Show system is feedback linearizeable

b) Find the domain of validity of the exact linear model

c) W/ $a = b = 1$ design a state-feedback controller to stabilize the origin and simulate.

Solution:

We have a single input system therefore we can apply

Theorem

The single input system

$$\dot{x} = f(x) + g(x)u$$

is feedback linearizeable iff $\exists$ a domain $D_0 \subset D$ s.t.

1) $\forall x \in D_0$ the matrix (a)

$$G(x) = [g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)]$$

has rank n,

2) The distribution

$$\Delta = \text{span}\{g, ad_f g, \ldots, ad_f^{n-2} g\}$$

is involute in D_0.

where $ad_f^n g(x)$ are Lie brackets defined as

$$ad_f^0 g(x) = g(x)$$

$$ad_f^1 g(x) = [f, g](x)$$

$$ad_f^k g(x) = [f, ad_f^{k-1} g](x) \quad k \geq 1$$

and $[f, g] := \nabla g(x) f(x) - \nabla f(x) g(x)$

Definition

A distribution Δ is involutive if

$$g_1 \in \Delta \wedge g_2 \in \Delta \Rightarrow [g_1, g_2] \in \Delta$$

(3)

$$\dot{x} = \underbrace{\begin{bmatrix} \tan(x_3) \\ -\dfrac{\tan(x_2)}{a\cos(x_3)} \\ \dfrac{\tan(x_2)}{a\cos(x_3)} \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ \dfrac{\tan(u)}{b\cos(x_2)\cos(x_3)} \\ 0 \end{bmatrix}}_{\text{Almost in the form we want.}}$$

Let $\xi := \dfrac{\tan(u)}{b\cos(x_2)\cos(x_3)}$ then

$$x = f(x) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xi$$

Which is in the form we want

n = 3

$$ad_f^0 g(x) = g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$ad_f g(x) = [f, g](x) = \cancel{\nabla g(x) f(x)}^{\,0} - \nabla f(x) g(x)$$

$$\begin{array}{cccc} & & & \partial f/\partial x_3 \end{array}$$

$$= [0] - \begin{bmatrix} 0 & 0 & \\ 0 & \frac{-(\tan(x_2)^2+1)}{a\cos x_3} & \cancel{\frac{-\sin x_3 \cdot \tan x_2}{a\cos(x_3)^2}} \\ 0 & \frac{\tan(x_2)^2+1)}{a\cos x_3} & \frac{\sin x_3 \cdot \tan x_2}{a\cos(x_3)^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

F

$$\frac{\partial f}{\partial x_2} = \frac{-(\tan(x_2)^2+1)}{a\cos x_3}$$

$-\pi/2 < x_3 < \pi/2$ and $-\pi/2 < x_2 < \pi/2$ } s.t. model is valid

$$ad_f g(x) = [0 \;\; \partial f/\partial x_2 \;\; -\partial f/\partial x_2]^T$$

$$ad_f^2 g(x) = \nabla ad_f g(x) f(x) - \nabla f(x)\, ad_f g(x)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \\ 0 & \frac{-\partial f_2}{\partial x_2} & \frac{-\partial f_3}{\partial x_3} \end{bmatrix} f(x) - F\, ad_f g(x)$$

Didn't want to write it all out.

We can rationalize that each element will be non-zero.

$\therefore$ $\underline{\text{rank}(G(x)) = n}$

We also stated that

$$D \in -\pi/2 < x_1, x_2 < \pi/2$$

To check that Δ is involutive:

$$\Delta = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -(\tan(x_2)^2+1)/a\cos x_3 \\ (\tan(x_2)^2+1)/a\cos x_3 \end{bmatrix} \right\}$$

We need to check

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \Delta \quad \begin{bmatrix} 0 \\ -(\tan(x_2)^2+1)/a\cos x_3 \\ (\tan(x_2)^2+1)/a\cos x_3 \end{bmatrix} \in \Delta$$

and

$$[g(x), \, ad_f g(x)] \in \Delta$$

$$\Rightarrow \quad 0 - \nabla ad_f g(x) g(x) = \frac{\partial a_f g(x)}{\partial x_2}$$

which is linearly independent of $g(x)$ and $ad_f g(x)$ therefore the distribution is involute.

$\therefore$ The system is feedback linearizeable

To find a h, we must solve

$$L_g L_f h(x) = 0 \Rightarrow \nabla h(x) g(x) = 0$$

$$L_g L_f h(x) \neq 0 \Rightarrow \nabla(\nabla h(x) f(x)) g(x) \neq 0$$

$$h(0) = 0$$

$$\begin{bmatrix} \frac{\partial h}{\partial x_1}, & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{\partial h}{\partial x_2} = 0$$

$$L_g L_f = \nabla h(x) f(x) = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & 0 & \frac{\partial h}{\partial x_2} \end{bmatrix} \begin{bmatrix} \tan(x_3) \\ -\tan(x_2)/a\cos(x_3) \\ \tan(x_2)/a\cos(x_3) \end{bmatrix}$$

$$\Rightarrow \quad L_g L_f = \begin{bmatrix} \tan(x_3)\, \partial h/\partial x_1 \\ 0 \\ \tan(x_2)/a\cos(x_3)(\partial h/\partial x_3) \end{bmatrix}$$

$$L_g L_f h(x) = \nabla \begin{bmatrix} \tan(x_3)\, \partial h/\partial x_1 \\ 0 \\ \tan(x_2)/a\cos(x_3)\, \partial h/\partial x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The only thing that has x_2 is $\frac{\tan(x_2)}{a\cos x_3} \frac{\partial h}{\partial x_3}$

$$\Rightarrow \frac{\partial}{\partial x_2}\left(\frac{\tan(x_2)}{a\cos(x_3)} \frac{\partial h}{\partial x_3} \right) = \frac{\sec^2 x_2}{a\cos x_3} \frac{\partial h}{\partial x_3}$$

Therefore we gather

$$\frac{\partial h}{\partial x_2} = 0 \; ; \; \frac{\partial h}{\partial x_3} \neq 0$$

Try

$$z_1 = h(x) = \sin(x_3) \qquad \Rightarrow z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T(x) = \begin{bmatrix} x_1 \\ \frac{\tan x_2}{a} \end{bmatrix}$$

$$z_2 = \frac{\tan x_2}{a}$$

$$\Rightarrow \quad x = T^{-1}(x) = \begin{bmatrix} z_1 \\ \tan^{-1}(a z_2) \end{bmatrix}$$

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \frac{\tan(x_2)}{a} \\ \frac{\sin x_2}{a^2 \cos^3 x_2 \cos x_3} - \frac{\tan(u)}{ab \cos^2 x_2 \cos x_3} \end{bmatrix}$$

$$\gamma(x) = \frac{1}{\cos^3(x_2)\cos(x_3)} \qquad \alpha(x) = \sin(x_2)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Canvas Problem

Given

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} cos(\phi+\theta) \\ sin(\phi+\theta) \\ sin(\theta) \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

And the Lie Brackets $q_1,\ q_2,\ [q_1, q_2],\ [q_1, [q_1, q_2]]$ spans $\mathbb{R}^4$ making the system STLC from any x

Find

Show that the system is STLC

Solution

```
>>> /* Create grad function */
grad(f,a,b,c,d) := [diff(f,a), diff(f,b), diff(f,c), diff(f,d)]$

/* Create Lie Bracket function */
LB(g1,g2) := block([dg1: grad(g1,x1,x2,phi,theta), dg2: grad(g2,x1,x2,phi,theta)], return(dg2.g1 -
dg1.g2))$

/* Define variables */
g1: matrix([cos(phi+theta)], [sin(phi+theta)], [sin(theta)], [0])$
g2: matrix([0],[0],[0],[1])$

/* Calculate Lie Brackets */
lb1: LB(g1,g2)$
lb2: LB(g1,lb1)$

M: mat_fullunblocker(matrix([g1,g2,lb1,lb2]))$
columnspace(M);
```

$$\operatorname{span}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \cos(\vartheta+\varphi) \\ \sin(\vartheta+\varphi) \\ \sin\vartheta \\ 0 \end{pmatrix}, \begin{pmatrix} \sin(\vartheta+\varphi) \\ -\cos(\vartheta+\varphi) \\ -\cos\vartheta \\ 0 \end{pmatrix}, \begin{pmatrix} \sin\vartheta\cos(\vartheta+\varphi) - \cos\vartheta\sin(\vartheta+\varphi) \\ \sin\vartheta\sin(\vartheta+\varphi) + \cos\vartheta\cos(\vartheta+\varphi) \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\operatorname{notequal}(\sin\vartheta, 0) \wedge \operatorname{notequal}(\sin\vartheta, 0) \wedge \operatorname{notequal}(\sin\vartheta\sin(\vartheta+\varphi) + \cos\vartheta\cos(\vartheta+\varphi), 0) \wedge \operatorname{notequal}$$

It can easily be seen that the span is R^4 .

>>>
```
/* Sanity check, solve the unicycle problem done in class */

/* Create grad function */
grad(f,a,b,c) := [diff(f,a), diff(f,b), diff(f,c)]$

/* Create Lie Bracket function */
LB(g1,g2) := block([dg1: grad(g1,x1,x2,phi), dg2: grad(g2,x1,x2,phi)], return(dg2.g1 - dg1.g2))$

/* Define variables */
g1: matrix([cos(phi)], [sin(phi)], [0])$
g2: matrix([0],[0],[1])$

/* Calculate Lie Brackets */
lb1: LB(g1,g2)$


M: mat_fullunblocker(matrix([g1,g2,lb1]));
columnspace(M);
```

$$\begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ \sin\varphi & 0 & -\cos\varphi \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathrm{notequal}\left(\cos\varphi, 0\right) \wedge \mathrm{notequal}\left(\cos\varphi, 0\right) \wedge \mathrm{notequal}\left(-\sin^2\varphi - \cos^2\varphi, 0\right)$$

$$\mathrm{span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix}\right)$$

13.13)

Given:

$$\dot{x}_1 = tan(x_3)$$
$$\dot{x}_2 = -\frac{tan(x_2)}{acos(x_3)} + \frac{tan(u)}{bcos(x_2)cos(x_3)}$$
$$\dot{x}_3 = \frac{tan(x_2)}{acos(x3)}$$

Find:

1. Show system is feedback linearizeable
2. Find the domain of validity of the exact linear model
3. With $a = b = 1$ design a state feedback controller to stabilize the origin and simulate

13.13)

Given:

$$\dot{x}_1 = tan(x_3)$$
$$\dot{x}_2 = -\frac{tan(x_2)}{acos(x_3)} + \frac{tan(u)}{bcos(x_2)cos(x_3)}$$
$$\dot{x}_3 = \frac{tan(x_2)}{acos(x3)}$$

Find:

1. Show system is feedback linearizeable
2. Find the domain of validity of the exact linear model
3. With $a = b = 1$ design a state feedback controller to stabilize the origin and simulate

---

Solution

Note: This second just covers part (3)

>>>
```
/* Define system */
xd1: tan(x3)$
xd2: -tan(x2)/(a*cos(x3)) + tan(u)/(b*cos(x2)*cos(x3))$
xd3: tan(x2)/(a*cos(x3))$
```

Choose $z1 = h(x) = sin(x3)$

>>>
```
/* Define and calcualte derivatives */
/* z1  = sin(x3)                     */
/* z2  = dz1/dt                      */
/* zd2 = dz2/dt                      */

z1: sin(x3);
z2: trigsimp(xd3*cos(x3));
zd2: trigsimp(sec(x2)^2/a*xd2);
```

$$\sin x3$$

$$-\frac{b\cos u\sin x2 - a\sin u}{a^2 b\cos u\cos^3 x2\cos x3}$$

$$\frac{\sin x2}{a\cos x2}$$

>>>

4.54.2)

Given:

$$\dot{x} = -(1+u)x^3 - x^5$$

Find:

Investigate Input-to-State stability

Solution

Suppose $V(x) = \frac{x^2}{2} = \alpha_1(x) = \alpha_2(x)$

$$\begin{aligned} \dot{V}(x) = x\dot{x} &= x(-(1+u)x^3 - x^5) \\ &= -x^4 - ux^4 - x^6 \end{aligned}$$

If $u > -\frac{x^6}{x^4} = -x^2$

$$\sqrt{u} < |x|$$

Then $\dot{V}(x)$ will be negative definite and if $-W_3(x) = -x^4$

$$\dot{V}(x) < -W_3(x)$$

Therefore the system is input-to-stable by (1)