

4.54.1)

Given:

$$\dot{x} = -(1+u)x^3$$

Find: Investigate Input-to-State Stability
Solution

Input-to-state stability theorem:

Consider

$$\dot{x} = f(t, x, u)$$

where $f: [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz (LL) in (x, u) . The input functions u are piecewise continuous and bounded $\forall t \geq 0$. If

- 1) There exists a continuously differentiable (CD) function

$$\gamma: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$$

- 2) There exists functions $\alpha_1, \alpha_2 \in K_{\infty}$ and $p \in K$

- 3) There exists a continuous positive definite (PD) function

$$w_3: \mathbb{R}^n \rightarrow \mathbb{R}$$

such that

$$4) f(t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$$

$$\alpha_1(\|x\|) \leq \gamma(t, x) \leq \alpha_2(\|x\|)$$

$$5) \forall t \in [0, \infty), \quad f(x, u): \|x\| \geq p(\|u\|) > 0$$

$$\dot{\gamma}(t, x, u) \leq -w_3(x)$$

Then the system is input-to-state stable.

(1)

Where

Definition:

A function $\alpha : [0, \infty) \rightarrow [0, \infty)$ is said to belong to class K_∞ ($\alpha \in K_\infty$) if $\alpha \in K$ and $\lim_{r \rightarrow \infty} \alpha(r) = \infty$. (2)

Definition

A function $\alpha : [0, \infty) \rightarrow [0, \infty)$ is said to belong to class K ($\alpha \in K$) if it is CD, strictly increasing, and $\alpha(0) = 0$. (3)

Definition:

A function $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is said to belong to class KL if it is continuous and

$$1) \forall s \in [0, \infty), \beta(\cdot, s) \in K$$

$$2) \forall r \in [0, \infty), \beta(r, \cdot) \text{ is decreasing and}$$

$$\lim_{s \rightarrow \infty} \beta(r, \cdot) = 0$$

Lemma:

Suppose $f(t, x, u)$ is CD & GL in (x, u) , uniformly in t . If the unforced system has a GAS equilibrium point at the origin $x=0$, then the system is input-to-state-stable.

Let's begin by assuming $V(x) = \frac{x^2}{2} = \alpha_1(x) = \alpha_2(x)$

$$\dot{V} = x \dot{x} = x(-\underbrace{(1+u)x^3)}_{P}) = -(1+u)x^4 = -x^4 - x^4 u$$

This will be negative definite if $\underbrace{u > 1}_{P}$

However, P is not class K .

Therefore, based on this choice of V , α_1 , and α_2 the system is not input-to-state-stable by (1)

4.54.2)

Given:

$$\dot{x} = -(1+u)x^3 - x^5$$

Find:

See (4.54.1)

Solution:

$$\text{Suppose } V(x) = \frac{x^2}{2} = \alpha_1(x) = \alpha_0(x)$$

$$\begin{aligned}\dot{V}(x) &= x\dot{x} = x(- (1+u)x^3 - x^5) \\ &= -x^7 - ux^4 - x^6\end{aligned}$$

$$\text{If } u = -\frac{x^6}{x^4} = -x^2$$

$$\Rightarrow \sqrt{u} < |x| \quad \left. \right\} \ell$$

Then $\dot{V}(x)$ will be N.D., and if $-W_3(x) = -x^4$

$$\dot{V}(x) < -W_3(x)$$

Therefore the system is input-to-state-stable by (1)

4.55.1)

Given:

$$\dot{x}_1 = -x_1 + x_1^2 x_2 \quad \dot{x}_2 = -x_1^3 - x_2 + u$$

Find:

See (4.54.1)

Solution:

$$\text{Suppose } V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\begin{aligned} \dot{V}(x) &= x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1(-x_1 + x_1^2 x_2) + x_2(-x_1^3 - x_2 + u) \\ &= -x_1^2 + \cancel{x_1^3 x_2} - \cancel{x_1^2 x_2} - x_2^2 + x_2 u \\ &= -x_1^2 - x_2^2 + x_2 u \end{aligned}$$

If $\rho(\|u\|) < \|x_2\|$ and $-w_3 = -x_1^2$ Then the system is input-to-stable by (1)

Checks:

$$\rho \in \mathbb{K} \subseteq \mathbb{K} \quad w_3 > 0$$

(B.6)

Given:

$$J\dot{w}_1 = (J_2 - J_3)w_2 w_3 + u_1$$

$$J\dot{w}_2 = (J_3 - J_1)w_3 w_1 + u_2$$

$$J_3 \dot{w}_3 = (J_1 - J_2)w_1 w_2 + u_3$$

Find:

Show that the state equation is feedback linearizable

Solution:

Definition

A nonlinear system $\dot{x} = f(x) + G(x)u$ where $f: D \rightarrow \mathbb{R}^n$ and $G: D \rightarrow \mathbb{R}^{n \times p}$ are sufficiently smooth on a domain $D \subset \mathbb{R}^n$, is said to be feedback linearizable (or input state linearizable) if there exists a diffeomorphism $T: D \rightarrow \mathbb{R}^n$ s.t. $D_T = T(D)$ contains the origin and the change of variables $\dot{z} = T(x)$ transforms

$$\dot{x} = f(x) + G(x)u$$

into the form

$$\dot{z} = Az + Bf(x)[u - \alpha(x)]$$

with (A, B) controllable and $f(x)$ nonsingular $\forall x \in D$.

Since we just want to show that the system is feedback linearizable, then we just have to satisfy (1).

If $x_1 = w_1$, $x_2 = w_2$, and $x_3 = w_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

If we were to take $(J_1 - J_3)x_i x_j := \alpha(x)$, i.e., assume it is the nonlinear part of the dynamics, then:

$$\xi = u - \alpha(x)$$

where

$$\alpha(x) := \begin{bmatrix} (J_2 - J_3)x_2 x_3 \\ (J_3 - J_1)x_3 x_1 \\ (J_1 - J_2)x_1 x_2 \end{bmatrix}$$

$$\text{Furthermore, } B = J^{-1} \Rightarrow \gamma(x) = I \Rightarrow A = O_{3 \times 3}$$

$$\Rightarrow \dot{z} = O_{3 \times 3} z + I_{3 \times 3}(1)(\xi).$$

Now we need to show (A, B) is controllable.

$$\text{rank}([B \ AB \ A^2 B]) = \underline{\underline{3}} = n \quad \text{controllable}$$

$\gamma(x)$ is nonsingular.

\therefore The system is feedback linearizable

13.7)

Given:

$$M(\dot{q})\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u$$

q : m-dim coordinates of joint positions

u : m-dim torque input

$M(q)$: Symmetric inertia matrix (positive definite)

$C(q, \dot{q})$: Centrifugal and Coriolis forces

$D\dot{q}$: Viscous damping ($D > 0$)

$g(q)$: Gravity forces $g(q) = \left[\frac{\partial P(q)}{\partial q} \right]^T$

$P(q)$: Total potential energy of links due to gravity.

Find:

See (13.6)

Solution:

We want to put this in the form

$$\ddot{x} = Ax + B\gamma(x)[u - \alpha(x)]$$

To start:

$$\ddot{q} = \frac{u}{M(q)} - \frac{C(q, \dot{q})\dot{q} - D\dot{q} - g(q)}{M(q)}$$

Let

$$x_1 = q \quad x_2 = \dot{q}$$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} \Rightarrow \dot{x}_2 = (u - C(q, \dot{q})\dot{q} - D\dot{q} - g(q))M(q)$$

$$\gamma(x) = M(x) \quad \alpha(x) = +C(q, \dot{q})\dot{q} + D\dot{q} + g(q)$$

Assuming an n dimensional system

$$\dot{x} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} H(x) [u - ((q_1 \dot{x}_1) + D\dot{x}_2 + g(q))]$$

\curvearrowright \curvearrowright
 \downarrow B

Check if Controllable

$$\text{rank}([B \ AB \ \dots \ A^{n-1}B]) = n \Rightarrow \underline{\text{Controllable}}$$

$\gamma(x)$ is positive definite which implies nonsingularity

i.e. The system is feedback linearizable

13.13)

Given:

$$\dot{x}_1 = \tan(x_3)$$

$$\dot{x}_2 = -\frac{\tan(x_2)}{a \cos(x_3)} + \frac{1}{b \cos(x_2) \cos(x_3)} \tan(u)$$

$$\dot{x}_3 = \frac{\tan(x_2)}{a \cos(x_3)}$$

$$a, b \in \mathbb{R} \quad a, b > 0$$

Find:

- Show system is feedback linearizable
- Find the domain of validity of the exact linear model
- W/ $a=b=1$ design a state-feedback controller to stabilize the origin and simulate.

Solution:

We have a single input system therefore we can apply

Theorem

The single input system

$$\dot{x} = f(x) + g(x)u$$

is feedback linearizable iff \exists a domain $D_0 \subset D$ s.t.

1) $\forall x \in D_0$ the matrix

(2)

$$G(x) = [g(x), \text{ad}_f g(x), \dots, \text{ad}_f^{n-1} g(x)]$$

has rank n .

2) The distribution

$$D = \text{span} \{ g, \text{ad}_f g, \dots, \text{ad}_f^{n-2} g \}$$

is involutive in D_0 .

where $\text{ad}^n g(x)$ are Lie brackets defined by

$$\text{ad } f^0 g(x) = g(x)$$

$$ad(f'g(x)) = [f,g](x)$$

$$ad^k f g(x) = [f, ad^{k-1} g](x) \quad k \geq 1$$

$$\text{and } [f, g] := fg\cos x - f\cos g(x)$$

Definition

A distribution Π is involutive if

$$g_1 \in D \wedge g_2 \in D \Rightarrow [g_1, g_2] \in D$$

$$\dot{x} = \begin{bmatrix} -\tan(x_3) \\ -\frac{\tan(x_2)}{a \cos(x_3)} \\ \frac{\tan(x_2)}{a \cos(x_3)} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\tan(u)}{b \cos(x_2) \cos(x_3)} \\ 0 \end{bmatrix}$$

\sim \sim

$f(x)$ Almost in the form

Almost in the form
we want.

$$\text{Let } g := \frac{\tan(u)}{b \cos(x_2) \cos(x_1)} \quad \text{then}$$

$$x = f(x) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \left\{ \begin{array}{l} \text{Which is in the form} \\ \underline{\text{we want}} \end{array} \right.$$

n = 3

$$\text{ad}f^0 g(x) = g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \text{ad}fg(x) &= [f, g](x) = \cancel{fg(x)} f(x) - \cancel{gf(x)} g(x) \\ &= [0] - \begin{bmatrix} 0 & 0 & \frac{\partial f}{\partial x_3} \\ 0 & \frac{-\tan(x_2)^2 + 1}{\cos x_3} & -\frac{\sin x_3 \cdot \tan x_2}{\cos(x_3)^2} \\ 0 & \frac{\tan(x_2)^2 + 1}{\cos x_3} & \frac{\sin x_3 \cdot \tan x_2}{\cos(x_3)^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \frac{\partial f}{\partial x_2} &= \underline{-\frac{\tan(x_2)^2 + 1}{\cos x_3}} \quad -\pi/2 < x_3 < \pi/2 \quad \left. \begin{array}{l} \text{and} \\ \text{s.t. model} \\ \text{is valid} \end{array} \right\} \\ &\quad -\pi/2 < x_2 < \pi/2 \end{aligned}$$

$$\text{ad}fg(x) = [0 \ \frac{\partial f}{\partial x_2} \ -\frac{\partial f}{\partial x_2}]^\top$$

$$\text{ad}f^2 g(x) = \cancel{\tau(\text{ad}f g(x))} f(x) - \cancel{\delta f(x)} \text{ad}f g(x)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ 0 & -\frac{\partial f}{\partial x_2} & -\frac{\partial f}{\partial x_3} \end{bmatrix} f(x) - F \text{ad}fg(x)$$

Didnt want to write it all out.

We can rationalize that each element will be non-zero.

$$\therefore \underline{\text{rank}(g(x)) = n}$$

We also stated that

$$D \in -\pi/2 < x_1, x_2 < \pi/2$$

To check that D is involutive:

$$D = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} (\tan(x_2)^2 + 1) / \cos x_3 \\ (\tan(x_2)^2 + 1) / \cos x_3 \\ 0 \end{bmatrix} \right\}$$

We need to check

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in D \quad \begin{bmatrix} (\tan(x_2)^2 + 1) / \cos x_3 \\ (\tan(x_2)^2 + 1) / \cos x_3 \\ 0 \end{bmatrix} \in D$$

and

$$[g(x), \text{ad}fg(x)] \in D$$

$$0 - \nabla(\text{ad}fg(x))g(x) = \frac{\partial \text{ad}fg(x)}{\partial x_2}$$

which is linearly independent of $g(x)$ and $\text{ad}fg(x)$
therefore the distribution is involutive.

∴ The system is feedback linearizable

To find a h , we must solve

$$Lg(L^f h(x)) = 0 \Rightarrow \nabla h(x)g(x) = 0$$

$$Lg(L^f h(x)) \neq 0 \Rightarrow \nabla(h(x)f(x))g(x) \neq 0$$

$$h(0) = 0$$

$$\left[\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}, \frac{\partial h}{\partial x_3} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{\partial h}{\partial x_2} = 0$$

$$\text{Lg L}_f = \nabla h(x) f(x) = \left[\frac{\partial h}{\partial x_1} \ 0 \ \frac{\partial h}{\partial x_3} \right] \begin{bmatrix} \tan(x_3) \\ -\tan(x_2)/a \cos(x_3) \\ \tan(x_2)/a \cos(x_3) \end{bmatrix}$$

$$\Rightarrow \text{Lg L}_f = \begin{bmatrix} \tan(x_3) \frac{\partial h}{\partial x_1} \\ 0 \\ \tan(x_2)/a \cos(x_3) (\frac{\partial h}{\partial x_3}) \end{bmatrix}$$

$$\text{Lg L}_f h(x) = \nabla \begin{bmatrix} \tan(x_3) \frac{\partial h}{\partial x_1} \\ 0 \\ \tan(x_2)/a \cos(x_3) \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The only thing that has x_2 is $\frac{\tan(x_2)}{a \cos(x_3)} \frac{\partial h}{\partial x_3}$

$$\Rightarrow \frac{\partial}{\partial x_2} \left(\frac{\tan(x_2)}{a \cos(x_3)} \frac{\partial h}{\partial x_3} \right) = \frac{\sec^2 x_2}{a \cos x_3} \frac{\partial h}{\partial x_3}$$

Therefore we gather

$$\frac{\partial h}{\partial x_2} = 0; \quad \frac{\partial h}{\partial x_3} \neq 0$$

Try

$$z_1 = h(x) = \sin(x_3) \Rightarrow z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T(x) = \begin{bmatrix} x_1 \\ \frac{\tan x_2}{a} \end{bmatrix}$$

$$z_2 = \frac{\tan x_2}{a}$$

$$\Rightarrow x = T^{-1}(x) = \begin{bmatrix} z_1 \\ \tan^{-1}(az_2) \end{bmatrix}$$

$$\dot{\vec{z}} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -\frac{\tan(\alpha_2)}{a} \\ \frac{\sin(\alpha_2)}{a^2 \cos^2 \alpha_2 \cos \alpha_3} - \frac{\tan(\alpha_1)}{ab \cos^2 \alpha_2 \cos \alpha_3} \end{bmatrix}$$

$$Y(x) = \frac{1}{\cos^3(\alpha_2) \cos(\alpha_3)} \quad \underline{\alpha(x) = \sin(\alpha_2)}$$

$$\underline{A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

$$\underline{B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$