Final

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1

Given

$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = x_1 - x_2^3$$

Find

- Determine the equilibria for the given system
- Determine their qualitative nature (saddle, node, etc)
- Plot the vector field

Solution

Begin by finding equilibrium points x_{eq} (See appendix for worksheet)

$$\dot{x} = \begin{bmatrix} -x_1^3 + x_2 \\ x_1 - x_2^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Where the equilibrium points are found to be $x_{eq} = \{(0,0), (1,1), (-1,-1)\}$. Taking the Jacobian as $J = \frac{\partial f}{\partial x}$:

$$J = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix}$$

Solving for x_1 and x_2 we find: $x_2 = x_1 = \pm\sqrt{1/3}$. Therefore $x_{eq} = \{(\sqrt{1/3}, \sqrt{1/3}), (-\sqrt{1/3}, \sqrt{1/3}), (\sqrt{1/3}, -\sqrt{1/3})\}$

$(0,0)$

$$J = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$eig(J) = -1, 1$: Saddle

$(1,1)$

$$J = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$eig(J) = -2, -4$: Stable node

$(-1, -1)$

$$J = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$eig(J) = -2, -4$: Stable node

2

Given

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1^3 - x_2$$

Find

Investigate (by Lyapunov Analysis) stability of the origin for the following system. Make the strongest statement possible.

Solution

Theorem: Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

1) $V(0) = 0$

2) $\forall x \in D \backslash \{0\},\ V(x) > 0$

S) If $\forall x \in D$, $\dot{V}(x) \leq 0$ then $x = 0$ is stable

AS) If $\forall x \in D$, $\dot{V}(x) < 0$ then $x = 0$ is asymptotically stable

Attempt 1

Let $V = 1/2x^Tx$, then $\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$

$$x_1x_2 + x_2(-x_1^3 - x_2)$$
$$= x_2(x_1 - x_1^3 - x_2)$$

Can't really say anything from this, but we are close to a form that we want.

Attempt 2

Let $V = 1/4x_1^4 + 1/2x_2^2$, then $\dot{V} = x_1^3\dot{x}_1 + x_2\dot{x}_2$

$$x1^3x_2 - x2 * x_1^3 - x_2^2 =$$
$$-x_2^2 < 0$$

Therefore by the theorem above, the system is asymptotically stable.

3

Given

$$\dot{x}_1 = x_1^3 - x_2$$
$$\dot{x}_2 = x_1 - x_2$$

Find

Investigate (by whatever means) stability of the origin for the following system. Make the strongest statement possible.

Solution

Attempt 1

Let $V(x) = 1/4x_1^4 + 1/2x_2^2$, therefore

$$\begin{gathered}\dot{V}(x) = x_1^3\dot{x}_1 + x_2\dot{x}_2 = \\ x_1^6 - x_1^3x_2 + x_1x_2 - x_2^2\end{gathered}$$

From this, nothing can be said.

Attempt 2

Let $V(x) = 1/2x_1^2 + 1/2x_2^2$, therefore

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1^4 - x_2^2$$

From this, nothing can be said.

Attempt 3

Let $V(x) = 1/2x_2^2$, therefore

$$\dot{V}(x) = x_2\dot{x}_2 = (x_1 - x_2)x_2 = x_1x_2 = x_2^2$$

From this, nothing can be said.

Attempt 4

Can't attempt LaSelle's Invariance Principle because we cannot get a negative semi-definite $\dot{V}(x)$.

Attempt 5

Lets just try linearizing and see what happens locally (See appendix for worksheet). Solving for equilibrium points gives

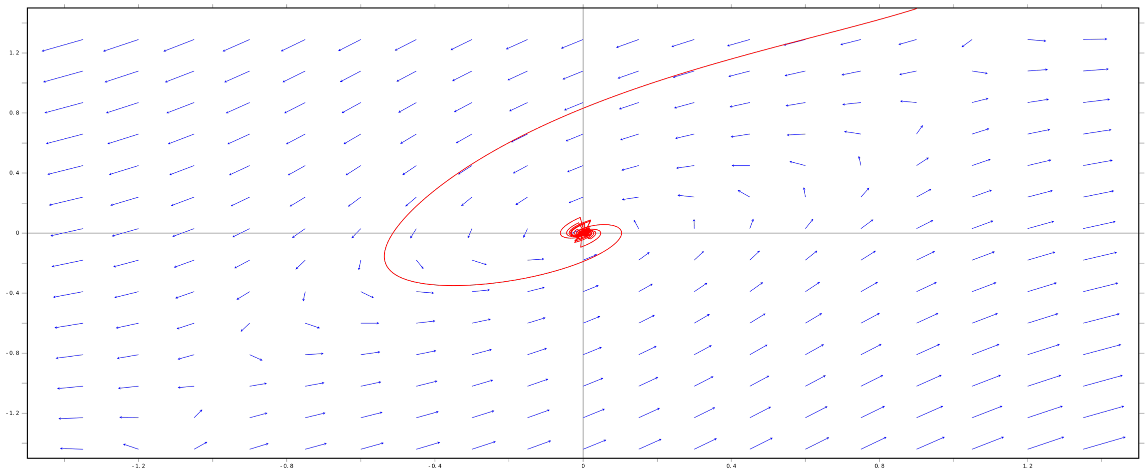
$$x_{eq} = \{(0,0), (1,1), (-1,-1)\}$$

The Jacobian of the system is:

$$J = \begin{bmatrix} 3x_1^2 & -1 \\ 1 & -1 \end{bmatrix}$$

Where the given eigenvalues are

- $(0,0)$: $[-0.5 * 1.732 \pm i]$ - Stable focus
- $(1,1)$: $[-0.7325, 2.73]$ - Saddle
- $(-1,-1)$: $[-0.7325, 2.73]$ - Saddle



4

Given

$$\dot{x} = -h(x) + u^2$$
$$\exists a > 0,\ \forall y \in \mathbb{R}\ yh(y) \geq ay^2$$

Find

Investigate input-to-state stability of the following system.

Solution

Theorem: Consider the system

$$\dot{x} = f(t, x, u)$$

where $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in (x, u). The input functions u are piecewise continuous and bounded for all $t \geq 0$. If

1. There exists a continuously differentiable $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$
2. There exists continuously differentiable functions $\alpha_1, \alpha_2 \in K_\infty$ and $\rho \in K$
3. There exists a continuous positive definite function $w_3 : \mathbb{R}^n \to \mathbb{R}$ such that
4. $\forall (t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m\ \alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||)$
5. $\forall t \in [0, \infty),\ \forall (x, u) : ||x|| \geq \rho(||u||) > 0\ \dot{V}(t, x, u) \leq -w_3(x)$

then the system is input-to-state stable.

If $h(x) \geq ay$ then suppose $\dot{(x)} = -h(x) + u^2 \leq -ax + u^2$

Let $\alpha_1 = \alpha_2 = V(x) = 1/2x^T x$, therefore

$$\dot{V}(x) = x\dot{x} = x(-ax + u) =$$
$$-ax^2 + ux$$

Let $||x|| \geq |u| = \rho(|u|)$. Then $|xu| \leq x^2$

$$\dot{V}(x) = -ax^2 - ux \leq$$
$$-ax^2 + x^2 =$$
$$x^2(1 - a)$$

Given that $w_3 = ax^2$, then the system is input-to-state stable.

5

Given

$$\dot{x}_1 = x_2^3 - 1 + (1 - x_1)^3$$
$$\dot{x}_2 = -x_1 + u$$

Find

- Develop a globally stabilizing state feedback control for the following system using backstepping
- Simulate

Solution

Definition: A nonlinear system

$$\dot{x} = f(x) + G(x)u$$

with $f : \mathbb{R}^n \supset D \to \mathbb{R}^n$ and $G : \mathbb{R} \supset D \to \mathbb{R}^n$ "sufficiently smooth" is feedback linearizeable iff

1. there exists a diffeomorphism $T : D \to \mathbb{R}^n$ such that $0 \in D_z := T(T) = \{\sigma : sigma = T(x), x \in D\}$
2. there exists a function $\gamma : D \to \mathbb{R}^{m \times m}$ such that $\forall x \in D$, $\gamma(x)$ is non-singular
3. There exists a function $\alpha : D \to \mathbb{R}^m$
4. There exists a controllable pair (A, B)
5. with $z = T(x), \dot{z} = Az + B\gamma(x)[u - \alpha(x)]$

Theorem: The single input system

$$\dot{x} = f(x) + g(x)u$$

is feedback lineariazeable iff there exists a domain $D_0 \subset D$

1. x D_0$ the matrix $G(x) = [g(x), ad_f g(x), ..., ad_f^{n-1} g(x)]$ has rank n
2. The distribution $\Gamma = \text{span}\{g(x), ad_f g(x), ..., ad_f^{n-2} g(x)\}$ is involute in D_0

Begin by writing

$$\dot{x} = \begin{bmatrix} x_2 + 1 + (1 - x_1)^3 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

where the rank can be see to be $n = 2$. Let

$$ad_f^0 g(x) = g(x)$$
$$ad_f^1 g(x) = \nabla g(x) f(x) + \nabla f(x) g(x)$$

Where we know $\nabla g(x) = 0$ and

$$\nabla f(x) = \begin{bmatrix} -3x_1^2 + 6x_1 - 3 & 1 \\ -1 & 0 \end{bmatrix}$$

Therefore

$$ad_f^1 g(x) = \begin{bmatrix} -3x_1^2 + 6x_1 - 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

From this, $\text{rank}(G(x)) = 2$ and the distribution is involutive. To find a satisfactory h, we must solve

$$\begin{aligned} &L_g L_f^0 h(x) = \nabla h(x) g(x) = 0 \\ &L_g L_f^1 g(x) = \nabla(\nabla h(x) f(x)) g(x) \neq 0 \\ &h(0) = 0 \end{aligned}$$

The first equation gives

$$\begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0]_{2\times 2}$$

Therefore $\frac{\partial h}{\partial x_2} = 0$ which implies that h is independent of x_2. In a similar fashion,

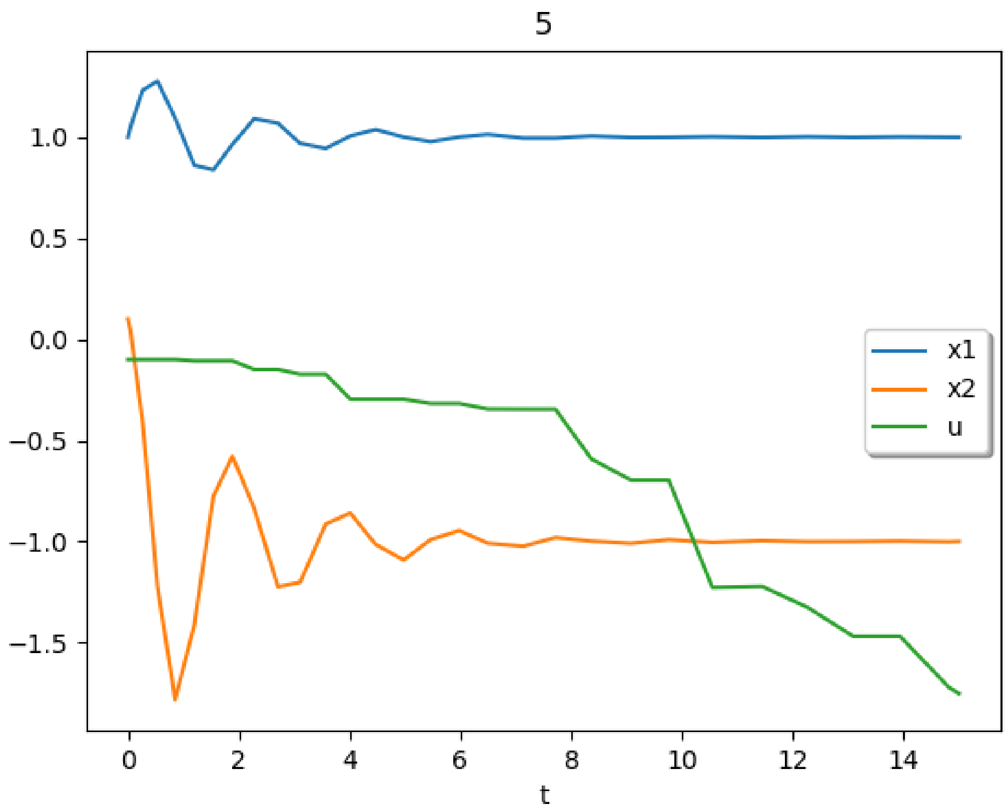
$$\frac{\partial h}{\partial x_1} \begin{bmatrix} -3x_1^2 + 6x_1 - 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial h}{\partial x_1} \neq 0$$

From which we state that h is dependent on x_1. The first simple choice is then to allow $h(x) = x_1$ (which satisfies $h(0) = 0$)

Let $h(x) = x_1 = z_1$, and $z_2 = \dot{z}_1 = \dot{x}_1 = x_2 + 1 + (1 - x_1)^3$

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} x_2 + 1 + (1 - x_1)^3 \\ \dot{x}_2 - 3x_1^3 \dot{x}_1 + 6x_1 \dot{x}_1 - 3\dot{x}_1 \end{bmatrix}$$

Let $u = x_1 - 3z_2(x_1^2 - 3x_1 + 1) + v$, choose $v = -10x_1^2 - 10x_2^2$.


5
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t
x1
x2
u

6

Given

$$\dot{x}_1 = x_1^2 - x_2$$
$$\dot{x}_2 = u$$

Find

- Develop a globally stabilizing state feedback control for the following system using feedback linearization
- Simulate

Solution

Let $V(x) = 1/2x_1^2$

$$\dot{V}(x) = x_1\dot{x}_1 =$$
$$x_1(x_1^2 + x_2) = x_1^3 + x_1x_2$$

Let $\phi(x) = x_2 = -x_1^2 - x_1$, therefore

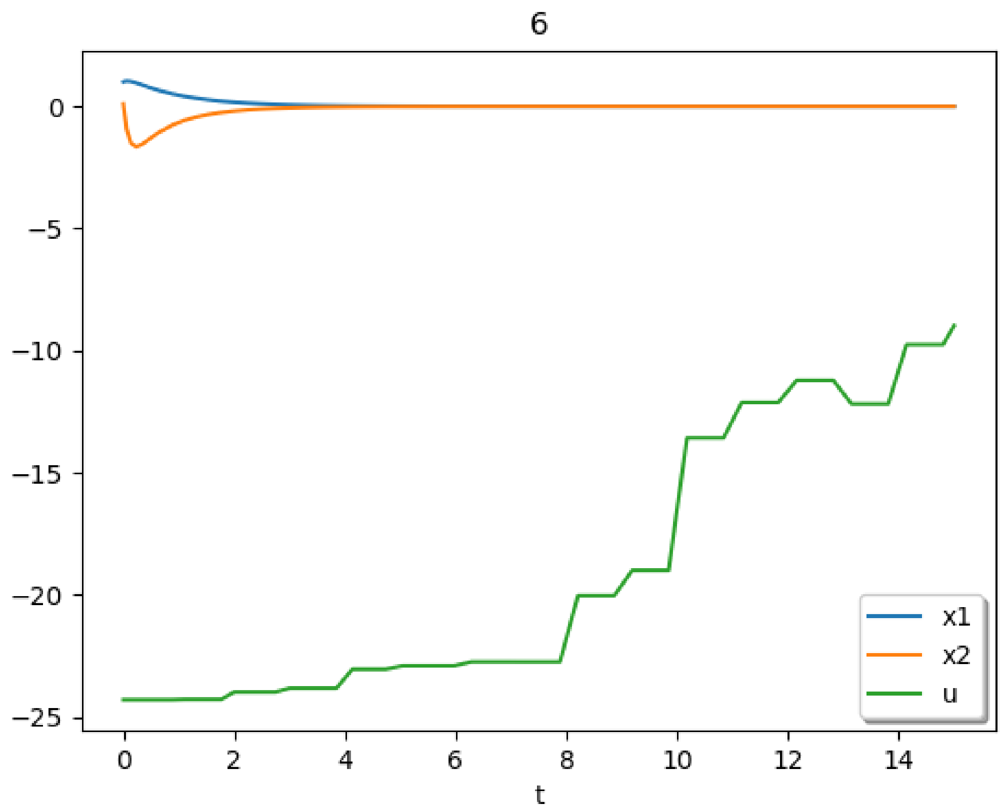
$$\dot{V}(x) = x_1^3 - x_1^3 - x_1^2 \leq 0$$

Pick $z = x_2 - \phi(x_1)$, $\dot{z} = \dot{x}_2 - \dot{\phi}(x_1) = u - \dot{\phi}(x_1)$ where $\dot{\phi}(x_1) = -2x_1\dot{x}_1 - \dot{x}_1$

Let $V_2(x) = V_1 + 1/2z^2$, then $\dot{V}_2(x) = \dot{(V)}_1(x) + z\dot{z}$

$$\dot{V}_2(x) = -x_1^2 + z(u - \dot{\phi}(x_1))$$

Let $u = \dot{\phi}(x_1) + kz$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7

Given

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 + \text{sat}(u)$$

Find

- Develop a globally stabilizing state feedback control for the following system using passivity-based control
- Simulate

Solution

Theorem: If the system

$$\dot{x} = f(x, u),\ y = h(x)$$

is

1. passive with radially unbounded positive definite storage function
2. zero-state observable

then the origin $x = 0$ is globally asymptotically stabilizeable with $u = -\phi(y)$ where ϕ is any locally Lipschitz function such that

$$\phi(0) = 0 \quad \text{and} \quad \forall y \neq 0,\ y^T\phi(y) > 0$$

Begin by checking if th system is passive

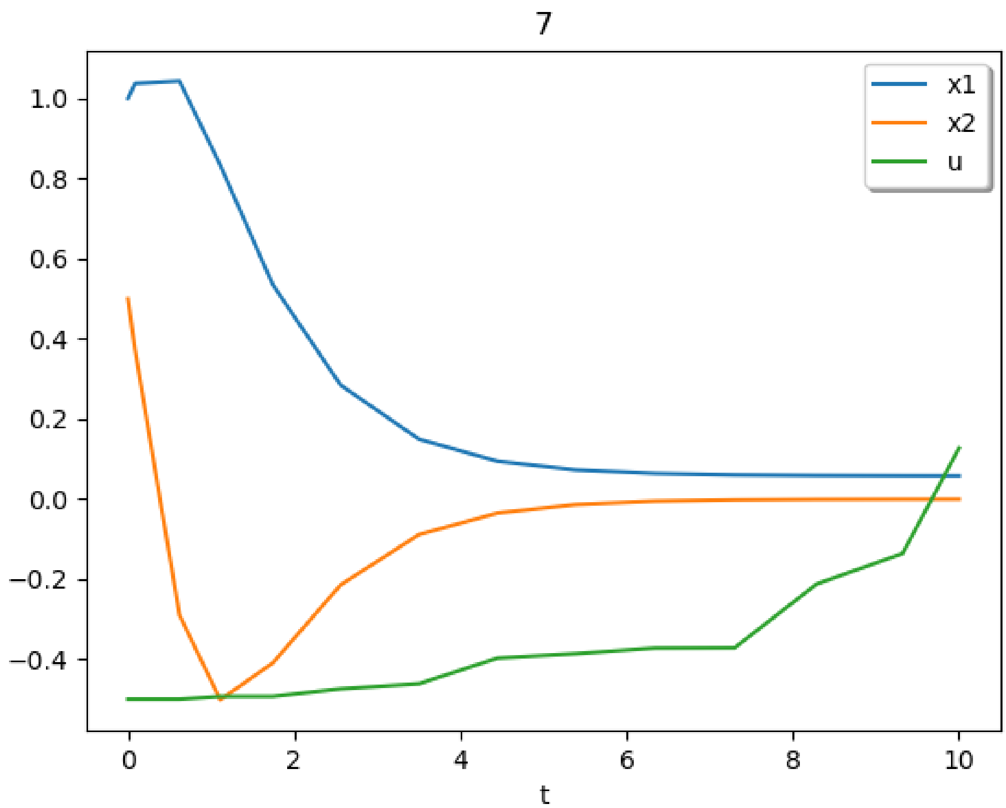
Let $V(x) = 1/2x^Tx$ therefore

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 =$$
$$x_1x_2 - x_1^3x_2 + x_2 sat(u)$$

To check for passivity we write

$$-x_1x_2 + x_1^3x_2 = x_2 sat(u)$$

Let $y = x_2$ then the system is passive. Choose $u = y = x_2$


7
1.0
0.8
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0.0
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−0.4
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t
x1
x2
u

8

Given

$$\dot{x}_1 = x_2 + sin(x_1)$$
$$\dot{x}_2 = \theta x_1 x_2 + u$$
$$0 \leq \theta \leq a$$

True values: $\theta = 0.1$ and $a = 1$

Find

- Develop a globally stabilizing state feedback control law for the following uncertain system using sliding mode control.
- Simulate

Solution

Begin by choosing manifold to be $s = ax_1 + x_2$, therefore $x_2 = -ax_1$

$$\dot{x}_2 = -ax_1 + sin(x_1)$$
we choose
$$V(x) = 1/2x_1^2$$
$$\dot{V}(x) = x_1\dot{x}_1 =$$
$$x_1(-ax_1 + sin(x_1))$$

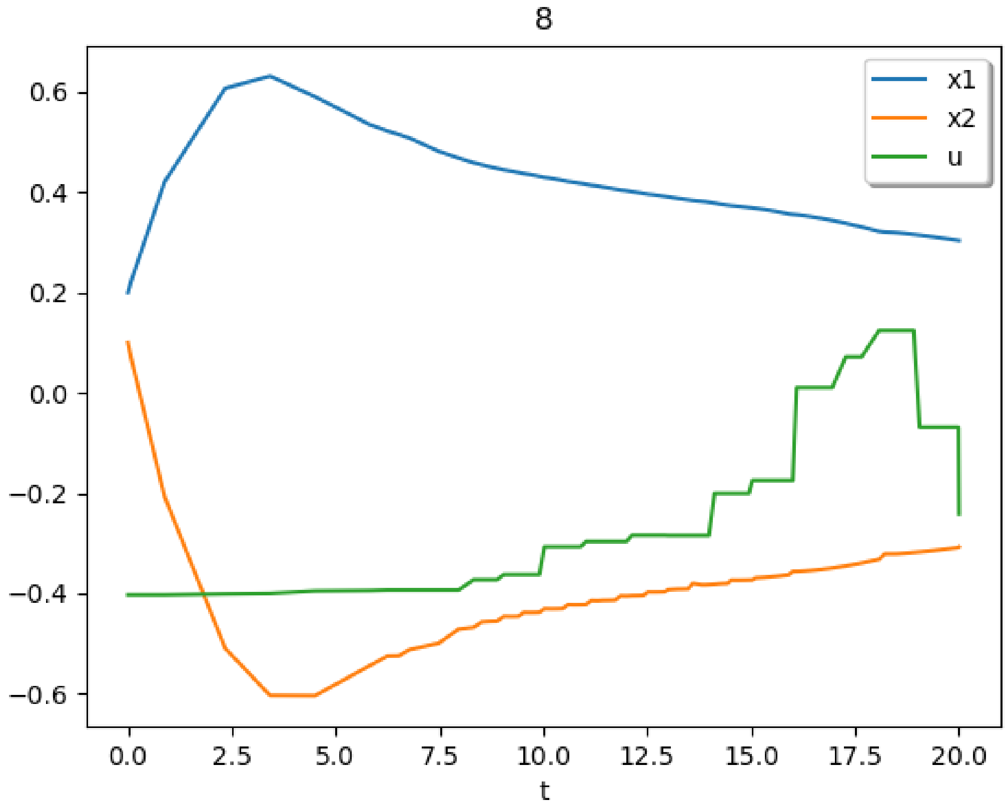
Therefore $sin(x_1) - ax_1 < 0$. Now let $\dot{V} = s\dot{x}$

$$\dot{V} = s(a\dot{x_1} + \dot{x}_2) =$$
$$s(a(x_2 + sin(x_1)) + \theta x_1 x_2 + u)$$

Let $u = -a(x_2 + sin(x_1)) + v$

$$\dot{V} = s(\theta x_1 x_2 + v)$$

Choose $v = sgn(s)(-\theta x_1 x_2 - \beta_0)$ where $\beta_0 > 0$


8
x1
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0.6
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−0.4
−0.6
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
t

9

Given

$$\dot{x}_1 = x_2 + sin(x_1)$$
$$\dot{x}_2 = \theta x_1 x_2 + u$$
$$0 \leq \theta \leq a$$

True values: $\theta = 0.1$ and $a = 1$

Find

- Develop a globally stabilizing state feedback control for the following system using Lyapunov redesign
- Simulate

Solution

Stabilize the nominal system

Begin by writing

$$\begin{bmatrix} x_1 + sin(x1) \\ \theta_0 x_1 x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Let $V(x) = 1/2x^T x$, therefore

$$\dot{V}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 =$$
$$x_1(x_2 sin(x1)) + x2(\theta_0 x_1 x_2 + u) =$$
$$x_1 x_2 sin(x_1) + \theta_0 x_1 x_2^2 + x_2 u$$

We choose $u = x_1 sin(x_1) + \theta_0 x_1 x_2 - kx$, therefore $\dot{V}(x) = -kx$. However, because we want to control a nominal and uncertain system, let the u defined above be redefined as $\psi = x_1 sin(x_1) + \theta_0 x_1 x_2 - kx$. We now let $u = \psi + v$.

Stabilize the actual system

We begin by defining $\delta(t, x, u) = \bar{\theta} x_1 x_2$

$$\bar{\theta} x_1 x_2 \leq |\bar{\theta} x_1 x_2| \leq \rho(x) + k_0 v$$

Where $k_0 = 0$ because there is no uncertainty in the control. Now we define $\eta = \nabla V(x) G(x) = x^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x2$. We then choose v to be

$$v = -(\frac{\rho}{1 - k_0} + \beta_0)\frac{x_1}{|x_2|}$$

Where the total control is $u = \psi + v$.

[](img/9.png

10

Given

$$\dot{x} = ax + u$$

- True values: $a = 1$
- a is unknown to the controller

Find

- Develop an adaptive control law for the scalar system to track the signal $r(t) = sin(t)$.
- Simulate

Solution

Define the error as

$$e(t) = x(t) - r(t)$$
$$\dot{e} = (\hat{a} - a)x + be - \dot{r}$$

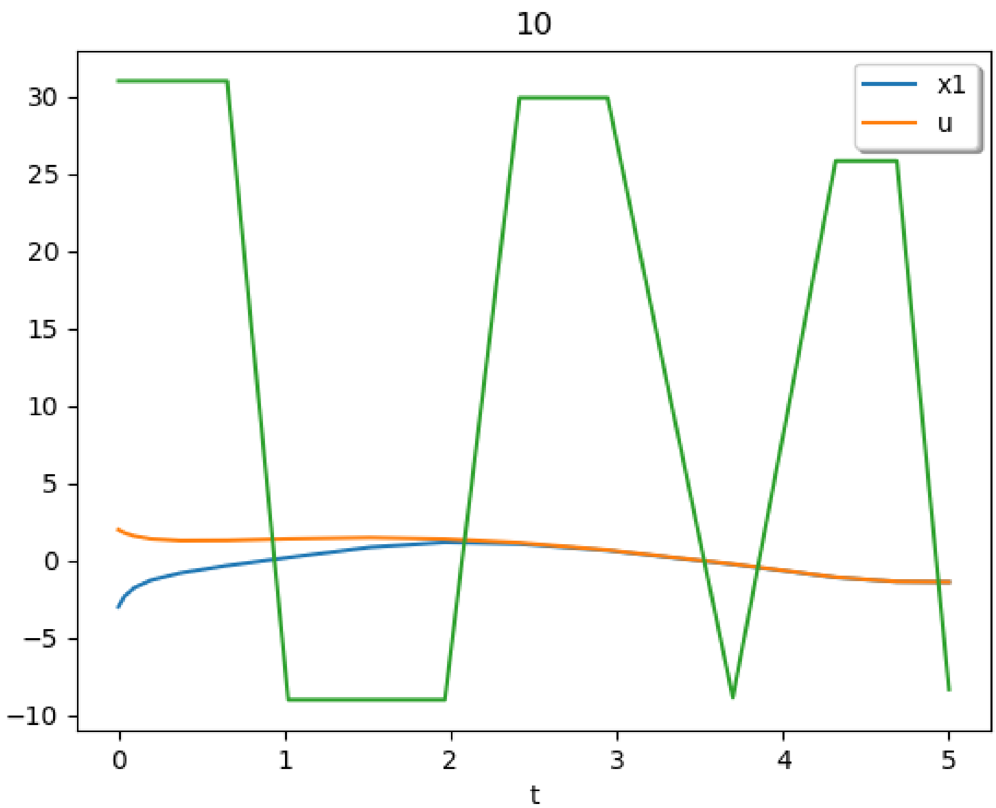
Where $r(t)$ is a reference trajectory. Choose $u = -1[ke + \hat{a}x - \dot{r}]$ such that

$$\dot{e} = -ke$$

where $\hat{a}$ is "adaptive" value of a. Choose $V = \frac{1}{2}e^2 + \frac{1}{2\gamma}\tilde{a}^2$, therefore

$$\dot{V} = e\dot{e} + \frac{1}{\gamma}\tilde{a}\dot{\hat{a}}$$

Choose $\dot{\hat{a}} = \gamma ex$, thus $\dot{V}(x) = -ke^2 < 0$. Therefore, $u = -1(ke - \dot{r} + \widehat{a(t)}x)$


10
x1
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5
0
−5
−10
0
1
2
3
4
5
t

Appendix

1)

Given

$\dot{x}_1 = -x_1^3 + x_2$
$\dot{x}_2 = x_1 - x_2^3$

Find

1. Find all equilibrium points
2. Determine the type of each isolated equilibrium
3. Draw vector field plot

Solution

Find equilibrium points

\>>>

```
/* Define equations */
xd1: -x1^3 + x2;
xd2: x1 - x2^3;

/* Find roots */
solve([xd1=0, xd2=0]);
```

```
x2-x1^3

x1-x2^3

[[x2 = -(-1)^(1/4),x1 = sqrt(-%i)],[x2 = (-1)^(3/4),x1 = (-1)^(1/4)],
        [x2 = %i,x1 = -%i],[x2 = -%i,x1 = %i],[x2 = -1,x1 = -1],
        [x2 = 1,x1 = 1],[x2 = 0,x1 = 0]]
```

Determine equilibrium point types

\>>>

```
/* Calculate Jacobian */
J:jacobian([xd1, xd2], [x1, x2]);

/* Calculate eigenvalue for each equilibrium point  */
/* The eigenvalue output is of the following format */
/* [[eigenvalues], [multiplicity]]                  */

float(eivals(psubst([x1=1, x2=1], J)));
float(eivals(psubst([x1=-1, x2=-1], J)));
float(eivals(psubst([x1=0, x2=0], J)));
```

```
matrix([-3*x1^2,1],[1,-3*x2^2])

[[-2.0,-4.0],[1.0,1.0]]

[[-2.0,-4.0],[1.0,1.0]]

[[-1.0,1.0],[1.0,1.0]]
```

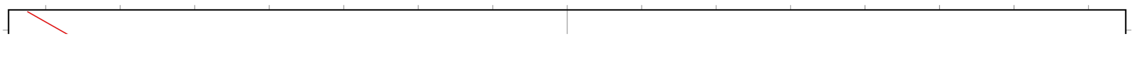
Based on the eigevalues found from the Jacobians found at each of the equilibrium points, we can determine that they are:

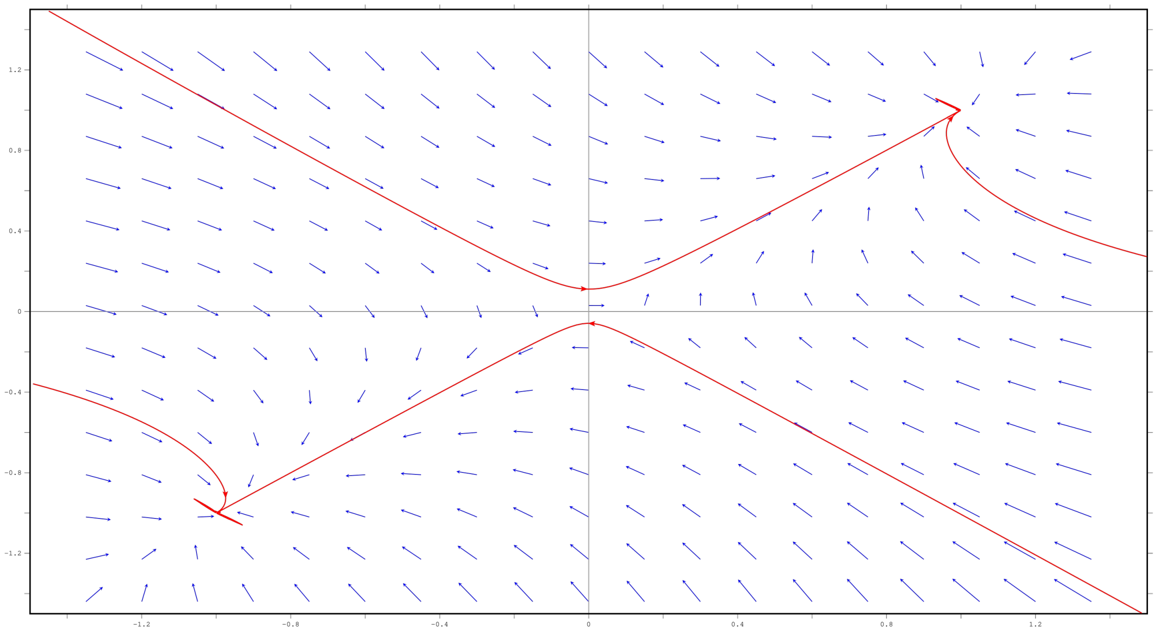
- (1,1): Stable node
- (-1,-1): Stable Node
- (0,0): Saddle

Draw vector field

\>>>

```
plotdf([xd1, xd2], [x1, x2], [x1, -1.5, 1.5], [x2, -1.5, 1.5])$
```





3)

Given

$$\dot{x}_1 = x_1^3 - x_2$$
$$\dot{x}_2 = x_1 - x_2$$

Find

1. Find all equilibrium points
2. Determine the type of each isolated equilibrium
3. Draw vector field plot

Solution

Find equilibrium points

>>>

```
/* Define equations */
xd1: x1^3 - x2;
xd2: x1 - x2;

/* Find roots */
solve([xd1=0, xd2=0]);
```

x1^3-x2

x1-x2

[[x2 = -1,x1 = -1],[x2 = 1,x1 = 1],[x2 = 0,x1 = 0]]

Determine equilibrium point types

>>>

```
/* Calculate Jacobian */
J:jacobian([xd1, xd2], [x1, x2]);

/* Calculate eigenvalue for each equilibrium point  */
/* The eigenvalue output is of the following format */
/* [[eigenvalues], [multiplicity]]                  */

float(eivals(psubst([x1=1, x2=1], J)));
float(eivals(psubst([x1=-1, x2=-1], J)));
float(eivals(psubst([x1=0, x2=0], J)));
```

```
matrix([3*x1^2,-1],[1,-1])

[[-0.7320508075688772,2.732050807568877],[1.0,1.0]]

[[-0.7320508075688772,2.732050807568877],[1.0,1.0]]

[[-0.5*(1.732050807568877*%i+1.0),0.5*(1.732050807568877*%i-1.0)],[1.0,
                                                                     1.0]]
```

Based on the eigevalues found from the Jacobians found at each of the equilibrium points, we can determine that they are:

- (1,1): Saddle
- (-1,-1): Saddle
- (0,0): Saddle

Draw vector field

>>>

```
plotdf([xd1, xd2], [x1, x2], [x1, -1.5, 1.5], [x2, -1.5, 1.5])$
```