1)

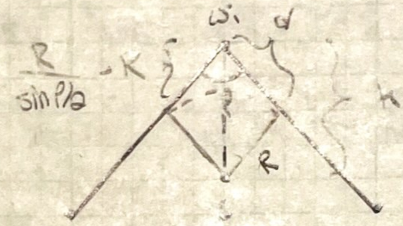
Given:

$$d = \frac{R}{\tan \frac{\rho}{2}}$$

Find:

Prove this

Solution:



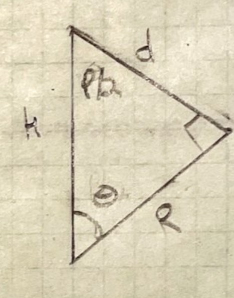
Note: Line is tangent to the circle.

Let $h = W_i - c := R + \frac{R}{\sin \rho/2} \cancel{- R} = \frac{R}{\sin \rho/2}$

Law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ (1)

The triangle we are looking at is



$$180° = \pi = \theta + \frac{\pi}{2} + \frac{\rho}{2}$$

$$\Rightarrow \frac{\pi}{2} = \theta + \frac{\rho}{2} \Rightarrow \theta = \frac{\rho}{2} + \frac{\pi}{2}$$

$$\frac{d}{\sin\theta} = \frac{R}{\sin \rho/2} \Rightarrow d = \frac{R \sin\theta}{\sin \rho/2} = \frac{R \sin(\rho/2 + \pi/2)}{\sin(\rho/2)}$$

$$\Rightarrow d = R\,\frac{\cos(\rho/2)}{\sin(\rho/2)} = \frac{R}{\tan(\rho/2)}$$

2)

Given:

$$k = \frac{R}{\sin \frac{\rho}{2}}$$

Find:

Prove this

Solution:

Using the image from problem (1) and Theorem (1).

$\Rightarrow$ The main objective is to find the distance from w_i to the circle. Lets begin with the triangle

$l := k - R \Rightarrow k = l + R$

$\theta := \rho/2 + \pi/2$

$$\Rightarrow \quad \frac{k}{\sin \pi} = k = \underline{\frac{R}{\sin \rho/2}} = l + R$$

$$\Rightarrow \quad \underline{\underline{l = \frac{R}{\sin \rho/2} - R}}$$

3)

Given:

A singularity happens in a straight-line switching half plane.

Find:

- When does this happen?
- Use math to justify.

Solution:

We know that the half plane is defined by

The unit vector pointing in the direction of $\overline{w_i w_{i+1}}$

$$q = \frac{w_{i+1} - w_i}{\| w_{i+1} - w_i \|}$$

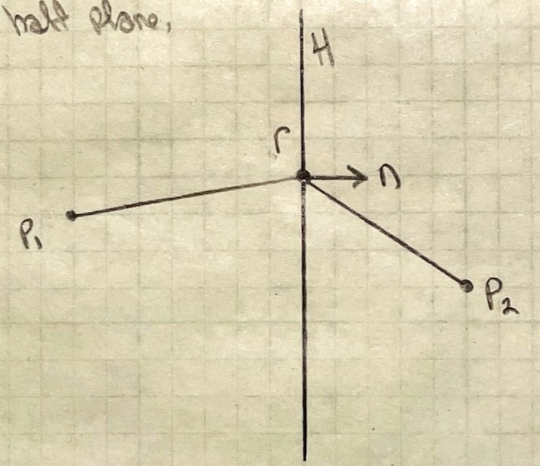
and the normal vector that separates $\overline{w_{i-1} w_i}$ from $\overline{w_i w_{i+1}}$

$$n_i = \frac{q_{i-1} + q_i}{\| q_{i-1} + q_i \|}$$

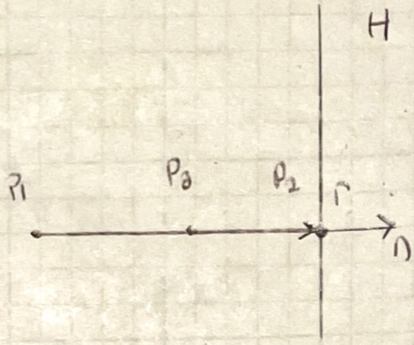
and finally the half plane

$$H(r, n) := \{ p \in \mathbb{R}^3 : (p - r)^T n \geq 0 \}$$

Which states that some vector defined be p and r in $\mathbb{R}^3$ and normal vector $n \in \mathbb{R}^3$ can be used to determine if p has past the half plane.



Now suppose the scenario:



$$n_3 = \frac{p_2 + p_3}{\|p_2 + p_3\|}$$

Note:

$$p_2 = \frac{w_{i+1} - w_i}{\|w_{i+1} - w_i\|}$$ which is a unit vector

$\Rightarrow$ $$n_3 = \frac{-p_3 + p_3}{\|-p_3 + p_3\|}$$ $\Rightarrow$ singularity

This will happen every time one wants to backtrack over the previous route traveled.

4).

Given:

$$c = w_i - \left(\frac{R}{\sin\frac{\rho}{2}}\right)\frac{q_{i-1} - q_i}{\|q_{i-1} - q_i\|}$$

Find:

- When singularites occur
- Defend with math

Solution:

c defines the center of the fillet.

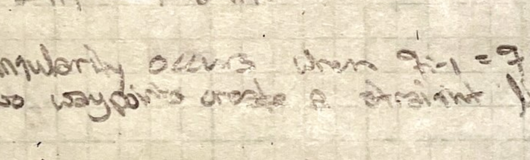
$\frac{R}{\sin\frac{\rho}{2}}$ defines the distance from w_i to c

$\frac{q_{i-1} - q_i}{\|q_{i-1} - q_i\|}$ defines the average direction normal.

If $\sin\frac{\rho}{2} = 0$, meaning that the next waypoint goes back to original direction

↳ $\rho = 0, 2\pi, \ldots, 2\pi n$

The second singularity occurs when $q_{i-1} = q_i$. In other words, the two waypoints create a straight line



Because q is a unit vector pointing in the direction of travel, the magnitude does not matter.

5)

Given:

$$|W_f| = |W| + \sum_{i=2}^{N} \left(R(\pi - \rho_i) - \frac{2R}{\tan\frac{\rho_i}{2}} \right)$$

Find:

Derive this function

Given:

We know

$$|W| := \sum_{i=2}^{N} \|w_i - w_{i-1}\|$$ which defines the length of the waypoint path.

Now to calculate the length of the fillet traversed.

$L = r \cdot \theta$ & $r := R$ $\theta := \pi - \rho$ (derived in problem 1)

$\Rightarrow$ $\underline{L = R(\pi - \rho)}$

Furthermore, the straight line that is removed by the fillet d (derived in (1)) is

$$d = \frac{R}{\tan\frac{\rho}{2}}$$

Note,

$$\Rightarrow |W_f| = |W| + \sum_{i=2}^{N} \left(R(\pi - \rho_i) - \frac{2R}{\tan\frac{\rho_i}{2}} \right)$$

Straight line

Plus arc lengths

Subtract out linear portions that are replaced.