

2.2 f.3)

Given:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \mu - x_2 - x_1^2 - 2x_1x_2$$

Find:

Find and classify bifurcations that occur as  $\mu$  varies

Solution:

Begin by finding the equilibrium points ( $\dot{x} = 0$ )

$$x_2 = 0$$

$$0 = \mu - 0 - x_1^2 - 0 \Rightarrow x_1^2 = \mu$$

$$\cdot x_{eq} = [\pm\sqrt{\mu} \quad 0]^T$$

$$J = \frac{df}{dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2x_1 - 2x_2 & -1 - 2x_1 \end{bmatrix} \Big|_{x=x_{eq}}$$

$$\Rightarrow J_{eq} = \begin{bmatrix} 0 & 1 \\ -2\sqrt{\mu} & -2\sqrt{\mu} - 1 \end{bmatrix} \quad \text{For } \mu < 0 \text{ there are no equilibria.}$$

$$-\sqrt{\mu}: -$$

$$J = \begin{bmatrix} 0 & 1 \\ 2\sqrt{\mu} & 2\sqrt{\mu} - 1 \end{bmatrix}$$

$$\text{eig}(J) = -1, 2\sqrt{\mu}$$

Because real parts have opposite sign the equilib. point is a saddle

$$\sqrt{\mu}:$$

$$J = \begin{bmatrix} 0 & 1 \\ -2\sqrt{\mu} & -2\sqrt{\mu} - 1 \end{bmatrix}$$

Saddle-node bifurcation

$$\text{eig}(J) = -1, -2\sqrt{\mu}$$

Both real parts are negative ∴ the equilib. point is a stable node for  $\mu < 0$

2.27.6)

Given:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \mu(x_1 + x_2) - x_2 - x_1^2 - 2x_1x_2$$

Find:

Find and classify bifurcations as  $\mu$  varies

Solution:

Begin by finding equilibrium point

$$\underline{x_2 = 0}$$

$$\begin{aligned} 0 &= \mu(x_1 + 0) - 0 - x_1^2 - 0 & x_{eq} &= [0], [\mu] \\ 0 &= x_1(\mu - x_1) & & [0], [0] \end{aligned}$$

$$\Rightarrow J = \begin{bmatrix} 0 & 1 \\ \mu - 2x_1 - 2x_2 & \mu - 1 - 2x_1 \end{bmatrix}$$

$$x_{eq} = [0, 0]^T$$

$$\Rightarrow J|_{x_{eq}} = \begin{bmatrix} 0 & 1 \\ \mu & \mu - 1 \end{bmatrix} \Rightarrow \text{eig}(J) = -1, \mu$$

If  $\mu < 0 \rightarrow$  stable point

$\mu > 0 \rightarrow$  saddle

$$x_{eq} = [\mu, 0]^T$$

$$J|_{x_{eq}} = \begin{bmatrix} 0 & 1 \\ \mu - 2\mu & \mu - 1 - 2\mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -\mu & -\mu - 1 \end{bmatrix} \Rightarrow \text{eig}(J) = -1, -\mu$$

Because the equilib. points persist  
but properties change this is  
a transcritical bifurcation

$\mu < 0 \rightarrow$  saddle  
 $\mu > 0 \rightarrow$  stable point

4.3.1)

Given:

$$\dot{x}_1 = -x_1 + x_1 x_2$$

$$\dot{x}_2 = -x_2$$

Find:

Use a quadratic Lyapunov function to show the origin is asymptotically stable.

Solution

Definition:

An equilibrium point is A.S. if  $\forall \epsilon > 0 \exists \delta > 0$   
s.t.

$$\forall t \geq 0, \|x(0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon$$

and

$$\lim_{t \rightarrow \infty} x(t) = 0$$

Theorem:

Let  $V: D \rightarrow \mathbb{R}$  be a continuously differentiable function s.t.

$$1) V(0) = 0$$

$$1) " , "$$

$$2) \forall x \in D \setminus \{0\}, V(x) > 0 \quad 2) " . "$$

$$3) \text{ If } \forall x \in D, \dot{V}(x) \leq 0$$

$$3) \|x\| \rightarrow \infty \quad V(x) \rightarrow \infty$$

$$\forall x \in D \setminus \{0\}, \dot{V}(x) < 0 \quad 4) \forall x \in \mathbb{R}^n \setminus \{0\},$$

then  $x=0$  is A.S.

$$\dot{V}(x) \neq 0$$

Then  $x=0$  is G.A.S

$$\text{Suppose } V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

$$1) V(0) = 0 \quad 2) \forall x \in D \setminus \{0\} \quad V(x) > 0$$

$$\dot{V}(x) = \frac{1}{2} (2x_1\dot{x}_1 + 2x_2\dot{x}_2) = x_1\dot{x}_1 + x_2\dot{x}_2$$

$$\dot{V}(x) = x_1(-x_1 + x_1x_2) + x_2(-x_2)$$

$$\dot{V}(x) = -x_1^2 + x_1^2x_2 - x_2^2$$

### Theorem

Let  $V: D \rightarrow \mathbb{R}$  be C.D., P.D. function on  $D$  s.t.

$\dot{V}$  is negative semi-definite on  $D$ . Let

$$S = \{x \in D \mid \dot{V}(x) = 0\}$$

If the only "trajectory" in  $S$  is  $x = 0$  then the origin is asymptotically stable.

$\dot{V}(x)$  can be rewritten as

$$\dot{V}(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{P}$$

where  $P$  is negative definite.  $\therefore$  the origin is G.A.S.

↳ See Lecture notes p.g. 3 (2)  
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4.3.2)

Given:

$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2)$$

$$\dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$$

Find:

See (4.3.1)

Solution:

$$\text{Suppose } V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\dot{V}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2$$

$$= x_1(-x_2 - x_1(1 - x_1^2 - x_2^2)) + x_2(x_1 - x_2(1 - x_1^2 - x_2^2))$$

$$= -\cancel{x_1}x_2 - \cancel{x_1^2}(1 - x_1^2 - x_2^2) + \cancel{x_1}x_2 - x_2^2(1 - x_1^2 - x_2^2)$$

$$= -x_1^2 + x_1^4 + \cancel{x_2^2}x_1 - x_2^2 + \cancel{x_1^2}x_2^2 + \cancel{x_2^4}$$

$$= x_2^4 + x_2^2(2x_1^2 - 1) + x_1^4 - x_1^2$$

$$= x_2^4 + x_2^2(2x_1^2 - 1) - x_1^2$$

$$= x_2^4 + x_2^2(2x_1^2 - 1) + x_1^4 - x_1^2$$

$$= x_2^2(x_2^2 + 2x_1^2 - 1) + x_1^2(x_1^2 - 1)$$

$$\rightarrow -x_1^2 + x_1^2 x_1^2 + x_2^2 x_1 - x_2^2 + x_1^2 x_2^2 + x_2^2 x_2^2$$

$$- (\underbrace{x_1^2 + x_2^2}_{2V}) (\underbrace{1 - x_1^2 - x_2^2}_{-2V})$$

$\therefore \dot{V}(x) = -2V(1 - 2V)$  If  $V = V_2$  then  $\dot{V}(x) \rightarrow 0$

If  $V < V_2$   $\dot{V}(x) < 0$

If  $V > V_2$   $\dot{V}(x) > 0$

$\dot{V}(x)$  is N.D. on

$V < V_2$  making the origin A.S.

$\hookrightarrow$  See (2)

4.3.3)

Given:

$$\dot{x}_1 = x_2(1-x_1^2)$$

$$\dot{x}_2 = -(x_1+x_2)(1-x_1^2)$$

Find:

See (4.3.1)

Solution:

Suppose  $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\begin{aligned} \Rightarrow \dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(x_2(1-x_1^2)) \\ &\quad + x_2(-(x_1+x_2)(1-x_1^2)) \\ &= x_2^2(1-x_1^2) = x_2^2 x_1^2 - x_2^2 \end{aligned}$$

For terms  $\ll 1$  (much smaller than 1)  $x_1^2 x_2^2 \ll x_2^2$ .  
 $\therefore$  the origin is A.S. for points near the origin.

↳ See (2.)

4.3.4)

Given:

$$\dot{x}_1 = -x_1 - x_2 ; \dot{x}_2 = 2x_1 - x_2^3$$

Find:

See (4.3.1)

Solution:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$$

$$\dot{V}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1(-x_1 - x_2) + x_2(2x_1 - x_2^3)$$

$$= -x_1^2 - \underbrace{x_1 x_2}_{\cancel{+ 2x_1 x_2}} - x_2^4$$

$$= -x_2^4 - x_1^2 + x_1 x_2 \quad \text{If it were } 2x_1 x_2, \text{ we could cancel out the terms}$$

$$\text{Suppose } V(x) = x_1^2 + \frac{1}{2}x_2^2$$

$$\begin{aligned}\dot{V}(x) &= 2x_1 \dot{x}_1 + x_2 \dot{x}_2 = 2x_1(-x_1 - x_2) + x_2(2x_1 - x_2^3) \\ &= -2x_1^2 - \cancel{2x_1 x_2} + \cancel{2x_1 x_2} - x_2^4 \\ &= -x_2^4 - 2x_1^2\end{aligned}$$

$V(x)$  is Negative Definite  $\therefore$  the origin is G.A.S. (See (2))

4.4)

Given:

$$J_1 \dot{\omega}_1 = (J_2 - J_3) \omega_2 \omega_3 + u_1$$

$$J_2 \dot{\omega}_2 = (J_3 - J_1) \omega_3 \omega_1 + u_2$$

$$J_3 \dot{\omega}_3 = (J_1 - J_2) \omega_1 \omega_2 + u_3$$

Find:

a) With  $u_1 = u_2 = u_3 = 0$  the origin  $\omega = 0$  is stable.  
Is it A.S.?

b) Suppose  $u_i = -k_i \omega_i$  where  $k_i > 0$ . Show that the closed-loop system is G.S.

Solution:

$$\text{Suppose } V(\omega) = \frac{1}{2} (J_1 \omega_1^2 + J_2 \omega_2^2 + J_3 \omega_3^2)$$

$$\dot{V}(\omega) = J_1 \omega_1 \dot{\omega}_1 + J_2 \omega_2 \dot{\omega}_2 + J_3 \omega_3 \dot{\omega}_3$$

$$\dot{V}(\omega) = J_1 \omega_1 \left( \frac{J_2 - J_3}{J_1} \omega_2 \omega_3 + u_1 \right)$$

$$+ J_2 \omega_2 \left( \frac{J_3 - J_1}{J_2} \omega_3 \omega_1 + u_2 \right)$$

$$+ J_3 \omega_3 \left( \frac{J_1 - J_2}{J_3} \omega_1 \omega_2 + u_3 \right)$$

$$\text{a) } u_1 = u_2 = u_3 = 0$$

$$\dot{V}(\omega) = (\underline{J_2} - \underline{J_3}) \underline{\omega_1} \underline{\omega_2} \underline{\omega_3} + (\underline{J_3} - \underline{J_1}) \underline{\omega_1} \underline{\omega_2} \underline{\omega_3} + (\underline{J_1} - \underline{J_2}) \underline{\omega_1} \underline{\omega_2} \underline{\omega_3} =$$

$$\underline{\dot{V}(\omega)} = 0$$

$\therefore$  The origin is stable but is not A.S. because  $\dot{V}(\omega) \not\equiv 0$

$\hookrightarrow$  See page 71 in class notes

b)  $U_i = -k_i w_i$

$$\dot{V}(\omega) = (J_2 - J_3) w_1 w_2 w_3 - k_1 w_1^2 + (J_2 - J_3) w_1 w_2 w_3 - k_2 w_2^2 \\ + (J_1 - J_2) w_1 w_2 w_3 - k_3 w_3^2$$

$$= w_1 ((J_2 - J_3) w_2 w_3 - k_1 w_1)$$

$$+ w_2 ((J_2 - J_3) w_1 w_3 - k_2 w_2)$$

$$+ w_3 ((J_1 - J_2) w_1 w_2 - k_3 w_3)$$

$$\Rightarrow \dot{V}(\omega) = -k_1 w_1^2 - k_2 w_2^2 - k_3 w_3^2 < 0$$

$\therefore$  The origin is G.H.S. as per (2)

4.9)

Given:

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

Find:

a)  $V(x) \rightarrow \infty$  as  $\|x\| \rightarrow \infty$  along the lines  $x_1 = 0$   
or  $x_2 = 0$ .

b) Show that  $V(x)$  is not radially unbounded

Solution:

a)

$$\underline{x_1 = 0}$$

$$V(x) = \frac{x_2^2}{1 + x_2^2} + x_1^2$$

$$\lim_{\|x\| \rightarrow \infty} V(x) \Big|_{x_1=0} = \infty$$

$$\underline{x_2 = 0}$$

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$

$$\lim_{\|x\| \rightarrow \infty} V(x) \Big|_{x_2=0} = \infty$$

b)

Definition:

(4)

A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  for which

$$\|x\| \rightarrow \infty \Rightarrow f(x) \rightarrow \infty$$

is said to be radially unbounded.

Suppose  $x_1 = x_2$

$$V(x) = \frac{(x_1 + x_2)^2}{1 + (x_1 + x_2)^2} + (x_1 - x_2)^2$$

$$= \frac{2x_1^2}{1 + 2x_1^2} + 0 \quad \rightarrow \text{Try plugging big numbers in}$$

$\therefore$  as  $|x| \rightarrow \infty$   $V(x) \rightarrow 1$ , which is not radially unbounded

4.16)

Given:

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = -x_1^3 - x_2^3$$

Find:

The origin is G.A.S.

Solution:

Suppose  $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\begin{aligned} \dot{V}(x) &= x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1 x_2 + x_2 (-x_1^3 - x_2^3) \\ &= x_1 x_2 - x_1^3 x_2 - x_2^4 \end{aligned}$$

Want to remove

To the  $\dot{V}(x)$  used in class

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^3 \Rightarrow \dot{V}(x) = x_1^3 \dot{x}_1 + x_2 \dot{x}_2$$

$$\begin{aligned} \Rightarrow \dot{V}(x) &= \cancel{x_1^3 x_2} - \cancel{x_1^3 x_2} - x_2^4 \\ &= -x_2^4 \end{aligned}$$

$$\text{If } x_2 \equiv 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow x_1 \equiv 0$$

$\therefore$  By (3) the origin is f.s. Furthermore because  $V(x)$  is radially unbounded the origin is G.A.S. as per (2)