Homework 1

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1.1

Given

A mathematical model that describes a wide variety of physical nonlinear systems is the nth-order differential equation

$$y^n = g(t, y, \dot{y}, ..., y^{n-1}, u)$$

where u and y are scalar variables. With u as input and y as output, find a sate model.

Solution

Suppose that $y = x_1$, $\dot{y} = x_2$, etc. Continuing to differentiate gives:

$$\begin{aligned}
&\dot{x_1} = x_2 = \dot{y} \\
&\dot{x_2} = x_3 \\
&... \\
&\dot{x}_{n-1} = x_n \\
&\dot{x}_n = g(t, y, \dot{y}, ..., y^{n-1}, u)
\end{aligned}$$

1.2

Given

Consider a single-input-single-output system described by the nth-order differential equation

$$y^n = g_1(t, y, \dot{y}, ..., y^{n-1}, u) + g_2(t, y, \dot{y}, ..., y^{n-1})\dot{u}$$

where g_2 is a differentiable function of its arguments. With u as input and y as output, find a state model.

Hint: Take $x_n = y^{n-1} - g_2(t, y, \dot{y}, ..., \dot{y}^{n-1})u$

Solution

Lets begin by defining $x_1 = y$, $x_2 = y^{(1)}$, ..., $x_{n-1} = y^{(n-2)}$, $x_n = y^{(n-1)} - g_2(t, y, \dot{y}, ..., y^{(n-2)})u$. Therefore

$$
\begin{aligned}
&\dot{x}_1 = y^{(1)} = x_2 \\
&\dot{x}_2 = x_3 \\
&... \\
&\dot{x}_{n-1} = y^{(n-1)} = x_n + g_2(t, x_1, x_2, ..., x_{n-1})u \\
&\dot{x}_n = y^{(n)} - \dot{g}_2(t, x_1, x_2, ..., x_{n-q})u - g_2(t, x_1, x_2, ..., x_{n-q})\dot{u}
\end{aligned}
$$

Where $\dot{x}_{n-1}$ is found by solving $x_n = y^{n-1} - g_2(t, y, \dot{y}, ..., \dot{y}^{n-1})u$ for $y^{(n-1)}$ and $\dot{x}_n = y^{(n)} - \dot{g}_2(t, x_1, x_2, ..., x_{n-q})u - g_2(t, x_1, x_2, ..., x_{n-q})\dot{u}$ is found by using the chain rule and product rule.

Chain Rule: $h'(x) = f'(g(x))g'(x)$

Product Rule: $(u \cdot v)' = u' \cdot v + u' \cdot v$

$$
\begin{aligned}
&\dot{x}_n = y^{(n)} - \dot{g}_2(t, x_1, x_2, ..., x_{n-q})u - g_2(t, x_1, x_2, ..., x_{n-q})\dot{u} = \\
&g_1(t, x_1, x_2, ..., x_n + g_2(t, x_2, x_3, ..., x_{n-2})u, u) - \\
&(\tfrac{\partial g_2}{\partial t} + \tfrac{\partial g_2}{\partial x_1}\dot{x}_1 + \tfrac{\partial g_2}{\partial x_2}\dot{x}_2 + ... + \tfrac{\partial g_2}{\partial x_{n-1}}\dot{x}_{n-1})u - g_2(t, x_1, x_2, ..., x_{n-q})\dot{u}
\end{aligned}
$$

1.5

Given

The nonlinear dynamic equations for a single-link manipulator with flexible joints, damping ignored, is given by

$$
\begin{aligned}
&I\ddot{q}_1 + MgLsin(q_1) + k(q_1 - q_2) = 0 \\
&J\ddot{q}_2 - k(q_1 - q_2) = u
\end{aligned}
$$

Solution

Sources

Chain Rule Product Rule