12.2.2.1)

Given:

$$\dot{x}_1 = x_1 + x_2$$

$$\dot{x}_2 = 3x_1^2 x_2 + x_1 + u$$

$$y = -x_1^3 + x_2$$

Find:
Design a state feedback controller to stabilize the origin.

Solution:

Since we are doing state feedback, ignore output $y$.

$$\dot{x} = \underbrace{\begin{bmatrix} x_1 + x_2 \\ 3x_1^2 x_2 + x_1 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ u \end{bmatrix}}_{g(x)}$$

Note that: $n = 2$ for this problem

$$ad_f^0 g(x) = g(x)$$

$$ad_f g(x) = [f, g](x) = \nabla g(x) f(x) - \nabla f(x) g(x)$$

$$ad_f^2 g(x) = [f, ad_f g](x)$$

$$\vdots$$

$$\Rightarrow \quad ad_f^0 g(x) = \begin{bmatrix} 0 \\ u \end{bmatrix} \qquad ad_f g(x) = [0] - \begin{bmatrix} 1 & 1 \\ 6x_1 x_2 + 1 & 3x_1^2 \end{bmatrix}$$

$$G(x) = \begin{bmatrix} 0 & -1 & -1 \\ u & -6x_1x_2 - 1 & -3x_1^2 \end{bmatrix}$$ has $\underline{\text{rank}(G(x)) = 2}$

and

$$\Delta := \text{span}\{g(x), ad_f g\} \in D$$

↳ Involutive

To find a satisfactory $h(x)$ we must satisfy

$L_g L_f^0 h = 0 \Rightarrow \nabla h(x) g(x) = 0$

$L_g L_f h \neq 0 \Rightarrow \nabla(\nabla h(x) f(x))\, g(x) \neq 0$

$h(0) = 0$

$$\nabla h(x) g(x) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{\frac{\partial h}{\partial x_2} = 0}$$

$h(x)$ is independent of $x_2$

$$\Rightarrow \nabla h(x) f(x) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} x_1 + x_2 \\ 3x_1^2 x_2 + x_1 \end{bmatrix} = \frac{\partial h}{\partial x_1}(x_1 + x_2)$$

$$\Rightarrow \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{\frac{\partial h}{\partial x_1} \neq 0}$$

Choose $\underline{h(x) = x_1}$ $\Rightarrow$ $\underline{h(0) = 0}$

$\underline{z_1} = h(x) = x_1$ ; $\underline{z_2} = \dot{z}_1 = \dot{x}_1 = x_1 + x_2$

$\Rightarrow \dot{z}_1 = z_2 = \underline{x_1 + x_2}$ ; $\dot{z}_2 = \dot{x}_1 + \dot{x}_2 = x_1 + x_2 + 3x_1^2 x_2 + x_1 + u$

$\bar{u}$ $= \underline{3x_1^2 x_2 + 2x_1 + x_2 + u}$

$u = -(3x_1^2 x_2 + 2x_1 + x_2) + v$

We are left w/

$$\dot{z} = Az + Bv = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

Choose a $v$ such that $A-Bk$ is Hurwitz.

$v = -kz$ and let $k = [10 \ \ 10]$

$\Rightarrow \dot{z} = (A - Bk) z \Rightarrow eig(A - Bk) = -1.12, -8.87$

$\Rightarrow u = \bar{u} - v = -3x_1^2 x_2 + 2x_1 + x_2 - kz$

(12.2.2.2)

Given:

$$\dot{x}_1 = x_1 + x_2$$

$$\dot{x}_2 = x_1 x_2^2 - x_1 + x_3$$

$$\dot{x}_3 = u$$

Find:

See (12.2.2.1)

Solution:

This time lets try linearization

$$J = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u$$

$$\frac{\partial f}{\partial x} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ x_2^2 - 1 & 2x_1x_2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\Bigg|_{\bar{x}=[0]}}_{A} \delta x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} v$$

$$\Rightarrow \dot{\bar{x}} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \bar{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} v$$

Let $v = -k\bar{x}$

and $k = [10 \quad 10 \quad 5]$

$$\Rightarrow \dot{\bar{x}}_1 = (A - Bk)\bar{x} \quad \Rightarrow \quad \text{eig}(A - Bk) = -0.82 \pm 1.202i, \; -2.353$$

14.31)

Given:

$$\dot{x}_1 = x_2 + a + (x_1 - a^{1/3})^3$$

$$\dot{x}_2 = x_1 + u$$

a is a known constant

Find:
Using backstepping, find a state feedback controller to globally stabilize the system

Solution:

$$\eta = x_1 \; ; \; \xi = x_2$$

$$f_a(\eta) = a + (x_1 - a^{1/3})^3 \; ; \; g_a(\eta) = 1$$

$$f_b(\eta) = x_1 \; ; \; g_b(\eta, \xi) = 1$$

Suppose that there is a smooth function $\xi = \phi(\eta)$; $0 = \phi(0)$ s.t.

$$\dot{\eta} = f_a(\eta) + g_a(\eta)\phi(\eta)$$

is asymptotically stable.

To make life easy lets start by saying

$$\phi(\eta) = \underbrace{-a - (x_1 - a^{1/3})^3}_{} + \psi(\eta)$$

$g_a(\eta) = 1$ so this cancels out $f_a(\eta)$

Now let $\underline{\psi(\eta) = -x_1}$ to make $\underline{\dot{\eta} = f_a + g_a \phi \text{ A.S.}}$

$$\Rightarrow \underline{\phi(\eta) = -a - (x_1 - a^{1/3})^3} - x_1$$

Now suppose there is a Lyapunov function $V_a$ and positive definite function $W$ s.t.

$$\forall \eta \in D, \; \nabla V_a(\eta)[f_a(\eta) + g_a(\eta)\phi(\eta)] \leq -W(\eta)$$

Let $V(x) = \frac{1}{2}x_1^2 \Rightarrow \dot{V}(x) = x_1\dot{x}_1$

Where $\dot{x}_1 = -x_1 \rightarrow \dot{V}(x) = -x_1^2 \leq 0$

To backstep define

$$z := \xi - \phi(\eta) = x_2 + a + (x_1 - a^{1/3})^3 + x_1$$

$$= x_1 + x_2 + 3a^{2/3}x_1 + x_1^3 - 3a^{1/3}x_1^2$$

$$\Rightarrow \dot{z} = \dot{x}_1 + \dot{x}_2 + 3a^{2/3}\dot{x}_1 + 3x_1^2\dot{x}_1 - 6a^{1/3}x_1\dot{x}_1$$

If $a = 0$

$$\dot{z} = \dot{x}_1 + \dot{x}_2 + 3x_1^2\dot{x}_1 = x_2 + x_1 + u + 3x_1^2x_2$$

Let $V = \frac{1}{2}x_1^2 + \frac{1}{2}z^2$

$$\Rightarrow \dot{V} = x_1\dot{x}_1 + z\dot{z} = x_1x_2 + z(x_2 + x_1 + u + 3x_1^2x_2)$$

Let $u = -x_1 - x_2 - 3x_1^2x_2 - z$

Then $\dot{V} = -x_1^2 - z^2 \leq 0$

14.34)

Given:

$$\dot{x}_1 = -x_1 + x_2$$

$$\dot{x}_2 = x_1 - x_2 - x_1 x_3 + u$$

$$\dot{x}_3 = x_1 + x_1 x_2 - 2x_3$$

Find:

a) Starting w/ $\dot{x}_1$ and stepping back into $\dot{x}_2$, design a state feedback controller $u = \psi(x)$ s.t. $x = [0]$ is G.A.S.

b) Show that the controller found causes the system to be G.A.S.

Hint: Use input-to-state properties of the third equation

Solution:

a)

$\eta = x_1 \quad \xi = x_2$

$f_a(\eta) = -x_1 \; ; \; g_a(\xi) = 1$

$f_b(\eta, \xi) = x_1 - x_2 - x_1 x_3 \; ; \; g_b(\eta, \xi) = 1$

$\dot{\eta} = f_a(\eta) + g_a(\eta)\,\phi(\eta) \; ; \;$ Let $\phi(\eta) = 0$

$\Rightarrow \dot{\eta} = -x_1$

Let $V(x) = \frac{1}{2}x_1^2 \Rightarrow \dot{V}(x) = x_1 \dot{x}_1 = \underline{-x_1^2 \leq 0}$

$z := \xi - \phi(\eta) = x_2 - 0 = x_2$

$\Rightarrow \dot{z} = \dot{x}_2 = x_1 - x_2 - x_1 x_3 + u$

Let $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}z^2$

$\Rightarrow \dot{V}(x) = x_1\dot{x}_1 + z\dot{z} = -x_1^2 + z(x_1 - x_2 - x_1x_3 + u)$

$\Rightarrow u = -x_1 + x_2 + x_1x_3 - z$

$\Rightarrow \dot{V}(x) = -x_1^2 + z(-z) = \underline{-x_1^2 - x_2^2}$

Therefore the system

$\dot{x}_1 = -x_1 + x_2$

$\dot{x}_2 = -x_2$

is G.A.S

b) To show the system is G.A.S. begin by observing

$\dot{x}_3 = x_1 + x_1x_2 - 2x_3$

Where we have guaranteed $x_1 \to 0$ and $x_2 \to 0$. The uncontrolled variable $x_3$ is what remains.

$V(x) = \frac{1}{2}x_3^2 \Rightarrow x_3\dot{x}_3 = x_3(x_1 + x_1x_2 - 2x_3) = \dot{V}(x)$

$\Rightarrow \dot{V}(x) = \cancelto{0}{x_3x_1} + \cancelto{0}{x_1x_2x_3} - 2x_3^2 \leq 0$

Therefore the system is G.A.S.

14.42)

Given:

$$\dot{x} = Ax + Bu$$

$$PA + A^TP \le 0 \text{ and } P > 0$$

Find:
A globally stabilizing feedback law $u = -\psi(x)$ such that $\|\psi(x)\| < k \;\forall\, x$ where $k$ is given as a positive constant

Solution:

$$\dot{x} = Ax + Bu \;;\; y = Cx$$

Let $V(x) = \frac{x^T P x}{2}$

$$\Rightarrow \dot{V}(x) = x(P + P^T)\dot{x} = x^T(P + P^T)(Ax + Bu)$$

$$= x^T(PAx + PBu + P^TAx + P^TBu)$$

$$= x^T(PA + P^TA)x + x^T(PB + P^TB)u$$

Note $A^TB = B^TA$ if symmetric

$$= x^T(PA + A^TP)x + x^TPBu = -B^TPu$$

$$\Rightarrow \quad x^T(PA + A^TP)x + x^TPBu \le B^TPxu = y^Tu$$

Let $B^TPx = C$ then

$x^T(PA + A^TP)x + x^TPBu \le yu$ $\therefore$ The system is passive

Also note that if $u = 0$ then when $x = 0 \;; y = 0$

$\therefore$ The system is zero state observable.

Choose $u = -ky$

14.43)

Given:

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1^3 + \psi(u)$

$\psi$ is:

- Locally Lipshitz
- $\psi(0) = 0 \quad u\psi(u) > 0 \ \forall \ u \neq 0$

Find:

Design a globally stabilizing state feedback control

Solution:

Let $V(x) = \frac{1}{2} x x^T \Rightarrow \dot{V}(x) = x^T \dot{x} = x_1 \dot{x}_1 + x_2 \dot{x}_2$

$= x_1 x_2 + x_2(-x_1^3 + \psi(u))$

$\Rightarrow \dot{V}(x) = x_1 x_2 - x_1^3 x_2 + x_2 \psi(u)$

$\Rightarrow -x_1 x_2 + x_1^3 x_2 = x_2 \psi(u)$ Let $x_2 = y$

Therefore the system is passive

If $u = 0$ and $y = 0 \Rightarrow x_2 = 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow x_1 = 0$

$\therefore$ The system is zero state observable

Let $u = -y^2$