4.22)

Given:

$$PA + A^T P = -C^T C$$ where (A, C) is observable

Find:

- Show A is Hurwitz iff $P = P^T > 0$ that satisfies the equation
- Show that if A is Hurwitz, the Lyapunov function will have a unique solution

Solution:

Hint: Apply LaSalle's Theorem and recall that for an observable pair (A, C), the vector $C\exp(At)x \equiv 0 \ \forall t$ iff $x = 0$.

LaSalle's Theorem: Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (1)

$$\dot{x} = f(x)$$

Let $V: D \to R$ be a continuously differentiable function s.t. $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Where: Positively Invariant: $x(0) \in M \Rightarrow x(t) \in M \ \forall t \geq 0$

Invariant Set: $x(0) \in M \Rightarrow x(t) \in M \ \forall t \in R$

Theorem: A matrix is Hurwitz (2)

i) $(\forall \lambda_i, \ Re(\lambda_i) < 0)$

iff

ii) $\forall Q = Q^T > 0 \ \exists P = P^T > 0$ s.t.

$$PA + A^T P = -Q$$

(3)

Corrollary 4.2: Let $x = 0$ be an equilibrium point for
(Invariance Principle)

$$\dot{x} = f(x)$$

Let $V: \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable, radially unbounded, positive definite function s.t. $\dot{V}(x) \leq 0 \ \forall \ x \in \mathbb{R}^n$. Let $S = \{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically on S, other than the trivial solution $x(t) \equiv 0$. Then the origin is A.S.

Lets use a similar approach used to prove Hurwitz in class (pg 80)

i) Suppose $V(x) = x^T P x$

$$\Rightarrow \quad \dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x$$

$$\dot{x} = Ax$$

Now substitute $PA + A^T P = -C^T C$

$$\Rightarrow \quad \dot{V}(x) = x^T(-C^T C)x = -x^T C^T C x$$

By invariance principle: if $C^T x \equiv 0 \Rightarrow \dot{V}(x) = 0$

ii) Now suppose A is Hurwitz (all eigs < 0)

Define

$$P = \int_0^\infty e^{A^T t} Q e^{At} \, dt$$

Now to show P is positive definite, suppose that it is not

$$\exists x \in \mathbb{R}^n \setminus \{0\} \text{ s.t. } x^T P x = 0$$

$$\Rightarrow \quad \int_0^\infty e^{A^T t} Q e^{At} x \, dt = 0$$

$$\Rightarrow \quad e^{At} x = 0 \quad \Rightarrow \quad x = 0$$

Which contradicts what was stated above. ∴ P is positive definite.

This can be said because (A, C) is observable.

Now to show that it is a solution of the Lyapanov Equation we use direct substitution

$$PA + A^T P = \int_0^\infty e^{A^T t} C^T C e^{At} A dt + \int_0^\infty A e^{A^T t} C^T C e^{At} dt$$

$$= \int_0^\infty \frac{d}{dt}\left(e^{A^T t} C^T C e^{At}\right) dt$$

$$= e^{A^T t} C^T C e^{At} \Big|_0^\infty = -C^T C$$

Which is a unique solution

4.27)

Given:

$$\dot{x}_1 = -x_2 x_3 + 1$$

$$\dot{x}_2 = x_1 x_3 - x_2$$

$$\dot{x}_3 = x_3^2 (1 - x_3)$$

Find:

a) Show that the system has a unique equilibrium point

b) Using linearization, show that that the equilibrium point is asymptotically stable. Is it G.A.S.?

Solution:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_2 x_3 + 1 \\ x_1 x_3 - x_2 \\ x_3^2 (1 - x_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3^2 (1 - x_3) \Rightarrow x = 0, 1$$

$\underline{x_3 = 0:}$

$$-x_2(0) + 1 = 0 \qquad\qquad x_1(0) - x_2 = 0$$

$$\underline{1 \neq 0}$$

$\underline{x_3 = 1:}$

$$-x_2(1) + 1 = 0 \qquad\qquad x_1(1) - x_2 = 0$$

$$-x_2 = -1 \qquad\qquad \underline{x_1 = x_2 = 1}$$

$$\underline{x_2 = 1}$$

The only equilibrium point is $x^{eq} = [1 \;\; 1 \;\; 1]^T$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -x_3 & -x_2 \\ x_3 & -1 & x_1 \\ 0 & 0 & 2x_3 - 3x_3^3 \end{bmatrix} \Bigg|_{x = x^*}$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} := A$$

$$\text{eig}(A) = \frac{-1 \pm i\sqrt{3}}{2} \text{ or } -1$$

Because $\text{Re}(\lambda_i) < 0$ then we can say it is A.S.
However for $x_3 = 0$, x_1 grows unbounded. ∴ the system is not G.A.S.

4.32.1)

Given:

$\dot{x}_1 = -x_1 + x_1^2 \qquad \dot{x}_2 = -x_2 + x_3^2 \qquad \dot{x}_3 = x_3 - x_1^2$

Find:

Investigate whether the origin is stable, asymptotically stable, or unstable

Solution:

To determine stability of the origin we can try to linearize about it to see if any information can be extracted.

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \\ \vdots & \frac{\partial f_2}{\partial x_2} & \\ \vdots & & \ddots \end{bmatrix} = \begin{bmatrix} 2x_1 - 1 & 0 & 0 \\ 0 & -1 & 2x_3 \\ -2x_1 & 0 & 1 \end{bmatrix}\Bigg|_{x=[0]}$$

$$\Rightarrow J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{eig}(J) = -1, -1, \underline{1}$$

The origin is unstable because:

Theorem: Let $x=0$ be an equilibrium point for (4)

$$\dot{x} = f(x)$$

where $f: D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(0)$$

Let λ_i denote an eigenvalue of A

i) If $\forall \lambda_i$ $Re(\lambda_i) < 0$, then the origin is A.S.

ii) If $\exists \lambda_i$ s.t. $Re(\lambda_i) > 0$ then the origin is not stable.

4.32.2)

Given:

$\dot{x}_1 = x_2 \quad \dot{x}_2 = -\sin x_3 + x_1[-2x_3 - sat(y)]^2 \quad \dot{x}_3 = -2x_3 - sat(y)$

where $y = -2x_1 = 5x_2 + 2x_3$

Find:

See (4.32.1)

Solution:

If we assume $y(0) = 0$ we get

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ -2x_3^2 & 0 & -\cos x_3 - 4x_3 x_1 \\ 0 & 0 & -2 \end{bmatrix} \Big|_{x=0}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & +2 \end{bmatrix} \Rightarrow eig(J) = \text{No good}$$

Lets try assuming that y is not saturated near the origin.

$sat(y) = y = -2x_1 - 5x_2 + 2x_3$

$\Rightarrow \quad \dot{x}_1 = x_2$

$\dot{x}_2 = -\sin x_3 + x_1[-2x_3 - (-2x_1 - 5x_2 + 2x_3)]^2$

$\dot{x}_3 = -2x_3 - (-2x_1 - 5x_2 + 2x_3)$

$$J = \begin{bmatrix} 0 & 1 & 0 \\ (2x_1 + 5x_2 - 4x_3)^2 + x_1(8x_1 + 20x_2 - 16x_3) & x_1(20x_1 + 50x_2 - 40x_3) & -\cos x_3 - x_1(16x_1 - 40x_2 - 32x_3) \\ 2 & 5 & -4 \end{bmatrix}$$

$$J|_{x=[0]} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 2 & 5 & -4 \end{bmatrix} \Rightarrow eig(J) = \begin{matrix} -0.3146 \pm 0i \\ 2.1573 \pm 1.205i \end{matrix}$$

Which implies origin is unstable

4.32.3)

Given:

$\dot{x}_1 = -2x_1 + x_1^3 \qquad \dot{x}_2 = -x_2 + x_1^2 \qquad \dot{x}_3 = -x_3$

Find:

See (4.3.2.1)

Solution:

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 3x_1^2 - 2 & 0 & 0 \\ 2x_1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\Bigg|_{x=[0]}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \text{eig}(J) = -2, \underline{-1, -1}$$

By (4), the system is A.S.

4.32.4)

Given:

$\dot{x}_1 = -x_1 \qquad \dot{x}_2 = -x_1 - x_2 - x_3 - x_1 x_3 \qquad \dot{x}_3 = (x_1 + 1) x_2$

Find:

See (4.32.1)

Solution:

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 0 & 0 \\ -x_3 - 1 & -1 & -x_1 - 1 \\ x_2 & x_1 + 1 & 0 \end{bmatrix} \Bigg|_{x = [0]}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \text{eig}(J) = -0.5 \pm 0.866i, \; -1$$

By (4) the system is A.S.

2.1.1)

Given:

$\dot{x}_1 = -x_1 + 2x_1^3 + x_2 \qquad \dot{x}_2 = -x_1 - x_2$

Find:

- Determine type of isolated equilibrium
- Plot vector field

Solution:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_1^3 + x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\underline{x_1 = -x_2}$

$$-x_1 + 2x_1^3 + x_2 = 0$$

$$-(-x_2) + 2(-x_2)^3 + x_2 = 0$$

$$x_2 - 2x_2^3 + x_2 = 0$$

$$\cancel{2}x_2 - \cancel{2}x_2^3 = 0$$

$$x_2(1 - x_2^2) = 0$$

$$\underline{x_2 = 0, \pm 1}$$

$x^{eq} = [0,0] ; [1,-1] ; [-1,1]$

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 6x_1^2 - 1 & 1 \\ -1 & -1 \end{bmatrix}$$

i) [0,0]

$$J\Big|_{[0,0]} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow eig(J) = \underline{-1 \pm 1i}$$

Stable Focus

ii) [1,-1]

$$J\Big|_{[1,-1]} = \begin{bmatrix} 6-1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow eig(J) = \begin{matrix} 4.82 \\ -0.82 \end{matrix}$$

Saddle ($\pm$ eigen value)

Same as before

iii) [-1,1]

$$J\Big|_{[-1,1]} = \begin{bmatrix} 6-1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow eig(J) = \begin{matrix} 4.82 \\ -0.82 \end{matrix}$$

Saddle

2.1.2)

Given:

$\dot{x}_1 = x_1 + x_1 x_2$ $\quad\quad$ $\dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$

Find:

See (2.1.1)

Solution:

$$\dot{x} = \begin{bmatrix} x_1 + x_1 x_2 \\ -x_2 + x_2^2 + x_1 x_2 - x_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(x_1 + x_1 x_2) = x_1(1 + x_2) = 0 \Rightarrow \underline{x_1 = 0 \text{ or } x_2 = -1}$$

$x_1 = 0$

$-x_2 + x_2^2 + (0)x_2 - 0^3 = 0$

$-x_2 + x_2^2 = 0$

$x_2(-1 + x_2) = 0$

$\underline{x_2 = 0, 1}$

$x_2 = -1$

$-(-1) + (-1)^2 - x_1 - x_1^3 = 0$

$2 - x_1 - x_1^3 = 0$

$\underline{x_1 = 1}$

$$\Rightarrow x_{eq} = [0, 0]; [0, 1]; [1, -1]$$

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} 1 + x_2 & x_1 \\ x_2 - 3x_1^2 & -1 + 2x_2 + x_1 \end{bmatrix}$$

i) [0,0]

$$J|_{[0,0]} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow eig(J) = \pm 1$$

Saddle

ii) [0,1]

$$J|_{[0,1]} = \begin{bmatrix} 1+1 & 0 \\ 1-0 & -1+2+0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow eig(J) = 1, 2$ Unstable node

iii) [1,-1]

$$J|_{[1,-1]} = \begin{bmatrix} 1+(-1) & 1 \\ -1-3(1) & -1+2(-1)+1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}$$

$\Rightarrow eig(A) = -1 \pm 1.7321i$ Stable focus

Given:

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -0.8x_1 - 10x_1^2 x_2 + u$$

Find:

Develop an approximate, higher order linearization of the following system.

Solution:

Lets attempt a simple redefinition of

$$\tilde{x}_1 = x_2 \; ; \quad \tilde{x}_2 = -0.8x_1 \qquad \tilde{x}_3 = -10x_1^2 x_2$$

Differentiating gives

$$\dot{\tilde{x}}_1 = \dot{x}_2 = -0.8x_1 - 10x_1^2 x_2 + u = \tilde{x}_2 + \tilde{x}_3 + u$$

$$\dot{\tilde{x}}_2 = \underline{-0.8\dot{x}_1} = -0.8x_2 = \underline{-0.8\tilde{x}_1}$$

$$\dot{\tilde{x}}_3 = -10\,2x_1\dot{x}_1 x_2 - 10x_1^2 \dot{x}_2$$

$$= \frac{20\tilde{x}_2 x_1^2}{0.8} - \frac{10x_2^2 x_1}{0.8}$$

Continue in PDF