7.1-1) Given:

- Theorem 7.1
- $\langle u_i, u_j \rangle = \delta_{i-j}$ (1)

Find:

Show (1) is true

Solution:

$u_i = \frac{Av_i}{\sqrt{\lambda_i}}$ where λ_i is an eigenvalue of A
v_i is an eigenvector of A

$\langle u_i, u_j \rangle = \delta_{i-j}$ where δ is the kronecker delta.

Let $\langle a, b \rangle = b^T a$

$\Rightarrow \langle u_i, u_j \rangle = \frac{v_j^H A^H A v_i}{\sqrt{\lambda_i}\sqrt{\lambda_j}} = \frac{v_j^H A^H \lambda_i v_i}{\sqrt{\lambda_i}\sqrt{\lambda_j}}$

We know $Av = \lambda v \Rightarrow v^H A^H = \lambda v^H$

$\Rightarrow \frac{\lambda_i \lambda_j v_j^H v_i}{\sqrt{\lambda_i}\sqrt{\lambda_j}}$ and we know $v_i \perp v_j$ if $i \neq j$

$\Rightarrow \langle u_i, u_j \rangle = \delta_{i-j}$ //

7.1-2) Given:

- Matrix 2 norm and Frobenius norm

Find:

$$\|A\|_F^2 = \sum_{i=1}^{p} \sigma_i^2$$

$$\|A\|_2 = \sigma_1$$

are true.

Solution:

$A = U\Sigma V^H$ $\quad$ U is unitary

V is unitary

$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} \; ; \; \|A\|_F^2 = tr(AA^H)$$

$\|A\|_2 = \sqrt{\rho(A^H A)}$ and if A is Hermitian $\|A\|_2 = \rho(A)$

Because U and V are unitary, they drop out of the norm leaving

$$\|A\|_F^2 = tr(AA^H) = \sum_{i=1}^{p} \sigma_i^2 \;//$$

$\Sigma_1 = \Sigma_1^4$ because it is diagonal thus

$\|A\|_2 = \rho(A)$ where $\rho(A) = \max_i |\lambda_i|$

Since Σ_1 has σ_i in descending order

$\|A\|_2 = \rho(\Sigma_1) = \sigma_1$ //

7.2-3) Given:

$$R(A) = \text{span}(U_1)$$
$$N(A) = \text{span}(V_2) \qquad (1)$$
$$R(A^H) = \text{span}(V_1)$$
$$N(A^H) = \text{span}(U_2)$$

Find:

Show that the SVD can be used to determine the fundamental subspaces in (1)

Solution: Note the end for definition of U_1, U_2, V_1, V_2.

Let $A = U\Sigma V^H$ be the SVD of A.

$$R(A) = b : b = U\Sigma V^H x$$
$$= U\Sigma y$$
$$= U_1 y = \text{span}\{U_1\}$$

$$R(A^H) = V\Sigma U^H x = V\Sigma y = V_1 y = \text{span}(V_1)$$

$$N(A) = U\Sigma V^H x = 0$$
$$= V_2 x = \text{span}(V_2)$$

$$N(A^H) = V\Sigma U^H = U_2 x = \text{span}(U_2)$$

Note

$$U\Sigma_1 V^H = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_{12} \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

$$\Sigma_{12} = 0$$

7.3-5) Given:

The LS solution to $\|Ax-b\|_2^2$ is

$$E^2_{min} = \|U_2^H b\|_2^2 \qquad (1)$$

Find:

- Show (1)
- Interpret (1) in terms of the four fundamental subspaces.

Solution:

$$\|Ax-b\|_2^2 = \|U\Sigma V^H - b\|_2^2$$

$$= \|U(\Sigma V^H - U^H b\|_2^2$$

$$= \|\Sigma v - \hat{b}\|_2^2$$

Where $\underline{v} = V^H x$ and $\hat{b} = U^H b$

$$\Rightarrow \min \|\Sigma_1 \underline{v} - \hat{b}\|_2^2 = \left\| \begin{matrix} \Sigma_1 v_1 = \hat{b}_1 \\ \Sigma_2 \underline{v_2} = \hat{b}_2 \end{matrix} \right\|_2^2$$

$$\Rightarrow \|v = \Sigma_1^+ \hat{b}\|_2^2 = \|\Sigma_1^+ U^H \underline{b}\|_2^2$$

$$= \Sigma_1^+ \|U^H \underline{b}\|_2^2 = \Sigma_1^+ \left\| \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\|_2^2$$

Σ_2 is 0, but to get the minimum norm we want

V_2 to be 0. This means that, in terms of the fundamental subspaces, no "data" is lost because V_2 is 0.

7.5-7) Given:

$$\|A-B\|_2^2 \overset{(1)}{\geq} \|Az\|_2^2 = \sum_{i=1}^{k+1} \sigma_i^2 (v_i^H z)^2 \overset{(2)}{\geq} \sigma_{k+1} \qquad (1)$$

Find:

a) Show the inequalities in (1) are correct

b) Show the conditions for achieving the lower bound are correct.

Solution:

(1)

$$\|A-B\| \geq \|(A-B)z\|_2^2 = \|Az\|_2^2$$

where $z \in \text{span}\{v_1, v_2, \ldots, v_{k+1}\}$

(2)

$$\|Az\|_2^2 = \sum_{i=1}^{k+1} \sigma_i^2 (v_i^H z)^2 \geq \sigma_{k+1}^2 \sum_{i=1}^{k+1} (v_i^H z)^2$$

$$= \sigma_{k+1}^2$$

b) Let $B = \sum_{i=1}^{k} \sigma_i u_i v_i^H$ and $z = v_{k+1}$

$$A = U\Sigma V = \sum_{i=1}^{r} \sigma_i u_i v_i^H$$

$$\|A-B\| = \left\| \sum_{i=1}^{r} \sigma_i u_i v_i^H - \sum_{i=k}^{k} \sigma_i u_i v^H \right\|$$

$$= \left\| \sum_{i=k}^{r} \sigma_i u_i v_i^H \right\|$$

$$\left\| \sum_{i=k}^{r} (\sigma_i u_i v_i) z_i \right\| \quad \text{where } z = v_{k+1}$$

$$\Rightarrow \left\| \sum_{i=k}^{r} (\sigma_i u_i v_i) v_{k+1} \right\| = 0 \quad //$$

7.7 - 12) Given:

P is on the plane $\perp$ normal vector $n \in \mathbb{R}^{n+1}$

$$P = \{ r \in \mathbb{R}^{n+1} : r^T n = 0 \}$$

Let $n = \begin{bmatrix} x \\ -1 \end{bmatrix}$ and $p = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{n+1}$

Find:

Show the shortes squared distance from p to P is

$$\frac{(x^T a - b)^2}{x^T x + 1}$$

Solution:

$$[A + E \mid b + r] \begin{bmatrix} x \\ -1 \end{bmatrix} = 0$$

$$(\underbrace{[A \mid b]}_{C} + \underbrace{[E \mid r]}_{\Delta}) \begin{bmatrix} x \\ -1 \end{bmatrix} = 0$$

$$S_c = R(\tilde{V}) = \operatorname{span}(V_{k+1}, V_{k+2}, \dots, V_{n+1})$$

$$\min_x \frac{\| [A|b] \begin{bmatrix} x \\ -1 \end{bmatrix} \|^2}{\| \begin{matrix} x \\ -1 \end{matrix} \|^2} = \min_x \sum \frac{|a_i^H x - b_i|^2}{x^H x + 1}$$

$$= \min_x \sum \frac{(a_i^H x - b_i)^2}{x^H x + 1} = \min_x \sum \frac{(x^H a_i - b)^2}{x^T x + 1}$$

Because $n \in \mathbb{R}^{n+1}$

$$= \min_x \sum \frac{(x^T a_i - b)^2}{x^T x + 1}$$ since $p = [a \; b]$ is a single point

$$= \min_x \frac{(x^T a - b)^2}{x^T x + 1}$$ noting that $x^T a = b$ we want an x that minimizes the square distance with b.

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7.7-13) Given:

$\bar{V} = [v_k, v_{k+1}, \dots, v_{n+1}] \in \mathbb{R}^{(m+1)(n-k+2)}$. In TLS we sought an element $y \in R(\bar{V})$ such that $x = \bar{I} y$ of minimum norm where

$$\bar{I} = \begin{bmatrix} I_m & \\ & 0 \end{bmatrix}$$

picks out the first m elements of y, and $y_{m+1} = -1$.

Find:

Formulate the problem of minimum-norm x as a constrained optimization problem, and determine the solution.

Solution:

$$\max \|\bar{I} y\|^2 \quad \text{s.t.} \quad y_{m+1} = -1$$

$$\bar{I} = \sum_{i=1}^{m} \sigma_i u_i v_i^H \qquad m = \text{rank}(I_m)$$

$$v_{m+1} \perp v_i \qquad i = 1, 2, \dots, m$$

$\Rightarrow$ $y = -v_{m+1}$ recall $y_{m+1} = -1$ $\therefore$

$$y = -\frac{\bar{v}_{m+1}}{v_{m+1,m+1}}$$

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7.9-15) Given:

$$A = \begin{bmatrix} 4.85 & -2.47 & 0.55 & -4.32 \\ 0.81 & -3.46 & -0.25 & 2.97 \end{bmatrix}$$

$$B = \begin{bmatrix} 4.51 & -2.19 & 0.59 & -4.72 \\ 1.95 & -3.64 & -0.131 & 2.29 \end{bmatrix}$$

A and B are rotations of each other

Find:

The amount of rotation between the data sets.

Solution:

maximize $\mathrm{tr}(Q^{H}B^{H}A)$ Q is unitary

The maximizing Q can be found by means of SVD of $B^H A$.

Theorem: If

$$B^H A = U \Sigma_1 V^H$$

then the maximizing Q for the orthogonal Procrustes problem is

See P715.m

$$Q = UV^T$$

$$A - BQ = \begin{bmatrix} -0.029 & 0.293 & 0.0281 & -0.313 \\ -0.453 & 0.266 & -0.047 & 0.360 \end{bmatrix}$$

//

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7.10-17) Given:

$A = B + jC$ B and C are real

Find:

Determine a means of finding the SVD of A in terms of the SVD of

$$\begin{bmatrix} B & -C \\ C & B \end{bmatrix}$$

Solution:

$$U^H A V = \begin{bmatrix} B \\ 0 \end{bmatrix} + j \begin{bmatrix} C \\ 0 \end{bmatrix}$$

Where we attempt to get the diagonals to be eigenvalues of A. Much like in the book.

$$\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

But because C is complex it will have positive and negative eigenvalue pairs resulting in

$$\begin{bmatrix} 0 & -C \\ C & 0 \end{bmatrix}$$

Thus by adding them we get

$$\begin{bmatrix} B & -C \\ C & B \end{bmatrix}$$

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