1)
Given:

$$y = Ax + e$$

- A is known
- e is measurement noise with Gaussian distribution, mean of 0 and covariance R.

$$e \sim N(0, R)$$

Find:

a) $f_{y|x}(y|x)$

b) $\hat{x}_{ML} = \arg\max_x f_{y|x}(y|x)$. Find this in 2 ways

- Computing gradients
- Completing the square

c) Using Baye's rule, write down $f_{x|y}(x|y)$

d) Show that

$$\hat{x}_{MAP} = \arg\max_x f_{x|y}(x|y) = \arg\max_x f_{y|x}(y|x) f_x(x)$$

e) Perform maximization of $\hat{x}_{map}$ using gradients or completing the square

f) Show

$$\hat{x}_{map} = x_0 + (\text{some matrix})^{-1}(\text{some vector})$$

Solution:

a)

$$f_{y}(\underline{y}|\underline{x}) = \frac{1}{(2\pi)^{k/2}|R|^{1/2}} \exp\left[-\frac{1}{2}(y - A\underline{x})^T R^{-1}(y - Ax)\right]$$

b) Gradient

$$\arg\max_x \underbrace{\frac{1}{(2\pi)^{k/2}|R|^{1/2}}}_{\text{Just a constant}} \exp\left[-\frac{1}{2}(y - Ax)^T R^{-1}(y - Ax)\right]$$

$$\arg\min_x \ (y-Ax)^T R^{-1} (y-Ax)$$

$$\arg\min_x \ y^T R^{-1} y - y^T R^{-1} Ax - x^T A^T y + x^T A^T R^{-1} Ax$$

$$\arg\min_x \ y^T R^{-1} y - 2x^T A^T R^{-1} y + x^T A^T R^{-1} Ax = \tilde{A}$$

$$\frac{\partial \tilde{A}}{\partial x} = 0 - 2A^T R^{-1} y + 2A^T R^{-1} Ax$$

Find where gradient derivative is 0

$$A^T R^{-1} y = A^T R^{-1} Ax$$

$$x = (A^T R^{-1} A)^{-1} (A^T R^{-1}) y$$

Complete square

Starting with

$$(x - \tfrac{1}{2}A^{-1}b)^T A (x - \tfrac{1}{2}A^{-1}b) + c - \tfrac{1}{4} b^T A^{-1} b$$

we have most of the square values, except b. To find that expand

$$(y-Ax)^T R^{-1} (y-Ax)$$

to get

$$y^T R^{-1} y - \underbrace{2y^T R^{-1} A}_{b^T} x + w^T \underbrace{A^T Q^{-1} A}_{A} w$$

$$x = \frac{-1}{2} (A^T Q^{-1} A)^{-1} (2A^T Q^{-1} y)$$

c)

$$f_{x|y}(x|y) = \frac{f(y|x)\,f(x)}{f(y)}$$

$$= \frac{\exp\left[-\frac{1}{2}(y-Ax)^T Q^{-1}(y-Ax)\right]\exp\left[-\frac{1}{2}(x-x_0)^T Q^{-1}(x-x_0)\right]}{\exp\left[-1/2\,(y-\mu_y)^T R_y^{-1}(y-\mu_y)\right]}$$

d) When performing an optimization, we want to maximize the numerator to obtain the largest value.

Therefore it is safe to state

$$\arg\max_x f(x|y) = \arg\max_x f(y|x)\,f_x(x)$$

e)

$$\arg\max_x \left[-(y-Ax)^T Q^{-1}(y-Ax)\right] + \left[(x-x_0)^T Q^{-1}(x-x_0)\right] = \tilde{A}$$

$$(y^T Q^{-1} y - 2x^T A^T Q^{-1} y + x^T A^T Q^{-1} Ax) +$$

$$(x^T Q^{-1} x - 2x_0^T Q^{-1} x + x_0^T Q^{-1} x_0) = \tilde{A}$$

$$\frac{\partial \tilde{A}}{\partial x} = (0 - 2A^T Q^{-1} y + 2A^T Q^{-1} Ax) + (2Q^{-1}x - 2x_0^T Q^{-1} + 0)$$

$$-2A^T Q^{-1} y + 2A^T Q^{-1} Ax + 2Q^{-1}x - 2x_0^T Q^{-1}$$

$$x = (2A^{T}Q^{-1}A + 2Q^{-1})^{-1}(2x_0^{T}Q^{-1} - 2A^{T}Q^{-1}y)$$

d) $(A^{T}Q^{-1}A + Q^{-1})^{-1}(x_0^{T}Q^{-1} - A^{T}Q^{-1}y)$

$$(A^{T}Q^{-1}A)^{-1}(A^{T}Qy^{-1} + x_0^{-T}Q + y^{-1}) + x_0$$

2)

Given:

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_{n-1} & r_n \\ \bar{r}_1 & r_0 & \cdots & r_{n-2} & r_{n-1} \\ \vdots & \cdots & \ddots & & \\ \bar{r}_n & \bar{r}_{n-1} & \cdots & r_1 & r_0 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_{n+1} \end{bmatrix}$$

- Where the overbar defines the complex conjugate
- The matrix is complex, Hermitian symmetric Toeplitz

Find:

a) Derive the "Fast" algorithm for solving the system of equations

b) Code algorithm in MATLAB and verify it works.

Solution:

a)

To begin, show that given the Hermitian matrix, show that it is persymmetric

Def: A matrix is persymmetric if, for an $n \times n$ matrix

$$a_{ij} = a_{n-j+1,\, n-i+1} \quad \forall\ i,j$$

To show a matrix is persymmetric

Def: Let $J = \begin{bmatrix} [0] & & 1 \\ & \iddots & \\ 1 & & [0] \end{bmatrix}_{n \times n}$

A matrix, T, is persymmetric if

$$T = J T^T J$$

$$T^T = \begin{bmatrix} r_0 & \bar{r}_1 & \bar{r}_2 & \cdots & \bar{r}_{n-1} & \bar{r}_{n-2} \\ r_1 & r_0 & \bar{r}_1 & \cdots & \bar{r}_{n-2} & \bar{r}_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_n & r_{n-1} & r_{n-2} & \cdots & r_1 & r_0 \end{bmatrix}$$

$$JT^T = \begin{bmatrix} r_n & r_{n-1} & r_{n-2} & \cdots & r_1 & r_0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ r_1 & r_0 & \bar{r}_1 & \ddots & \bar{r}_{n-2} & \bar{r}_{n-1} \\ r_0 & \bar{r}_1 & \bar{r}_2 & \cdots & \bar{r}_{n-1} & \bar{r}_n \end{bmatrix} = \tilde{T}$$

$$\tilde{T} J = \begin{bmatrix} r_0 & r_1 & r_2 & \cdots & r_{n-1} & r_n \\ r_1 & r_0 & r_1 & \cdots & r_{n-2} & r_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{r}_n & \bar{r}_{n-1} & \bar{r}_{n-2} & \cdots & \bar{r}_1 & r_0 \end{bmatrix} = T$$

From the notes we have the fact about persymmetric matrizes.

Fact: if T is persymmetric and J is the counter-identity, then

$$JT^{-1} = T^{-T} J$$

Now, the moment we have all been waiting for!

Suppose we know the solution to

$$\underline{r}_n = T_n \underline{w}_n$$

and we want to find the solution to

$$T_{n+1}\underline{w}_{n+1} = \underline{r}_{n+1}$$

Let $w_{n+1} = \begin{bmatrix} \underline{z}_n \\ \alpha_n \end{bmatrix}$ where $\underline{z}_n$ is a vector and α_n is a scalar

$$T_{n+1} = \left[\begin{array}{c|c} T_n & \begin{matrix} r_{n+1} \\ r_n \\ \vdots \\ r_1 \end{matrix} \\ \hline \bar{r}_{n+1}\ \bar{r}_n \cdots \bar{r}_1 & r_0 \end{array}\right] = \begin{bmatrix} [T_n] & [J_n \underline{r}_n] \\ [\bar{\underline{r}}_n^T J] & r_0 \end{bmatrix}$$

$$\begin{bmatrix} [T_n] & [J_n \underline{r}_n] \\ [\bar{\underline{r}}_n^T J] & r_0 \end{bmatrix} \begin{bmatrix} \underline{z}_n \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \bar{\underline{r}}_n \\ \bar{r}_{n+2} \end{bmatrix}$$

$$T_n z_n + J_n \underline{r}_n \alpha_n = \bar{\underline{r}}_n \qquad (1)$$

$$\bar{\underline{r}}_n^T J \underline{z}_n + r_0 \alpha_n = \bar{r}_{n+2} \qquad (2)$$

(1) Solve for z_n : $z_n = T_n^{-1}(\underline{r}_n - J_n \bar{r}_n \alpha_n)$

$$= T_n^{-1}\underline{r}_n - T_n^{-1} J_n \bar{r}_n \alpha_n$$

$$= w_n - \alpha_n J_n \bar{w}_n$$

Aside

Because T is persymmetric

$T^{-1} J = J T^{-1}$

Substitute back into (2) and solve for α_n

$$\bar{r}_n^T J_n (w_n - \alpha_n J_n \bar{w}_n) + r_0 \alpha_n = \bar{r}_{n+2}$$

$$\Rightarrow \alpha_n = \frac{\bar{r}_{n+2} - \bar{r}_n^T J_n \bar{w}_n}{r_0 - \bar{r}_n^T w_n}$$

1.4-13)

Given:

$$H(z) = \frac{1-5z^{-1}-6z^{-2}}{1-1.5z^{-1}+0.56z^{-2}}$$

Find:

Write in controller canonical form and draw a realization

Solution:

Begin with Partial Fraction expansion

$$H(z) = \frac{128.71}{x-0.7} - \frac{117.0}{x-0.8}$$

To write $H(z)$ in CCF we write

$$u(k) = 0.56g(k+2) - 1.5g(k+1) + g(k)$$

$$g(k+2) = (1.5g(k+1) - g(k) + u(k))/0.56$$

$$= 2.68g(k+1) - 1.79g(k) + 1.79u(k)$$

$$y(k) = -6g(k-2) - 5g(k-1) + g(k)$$

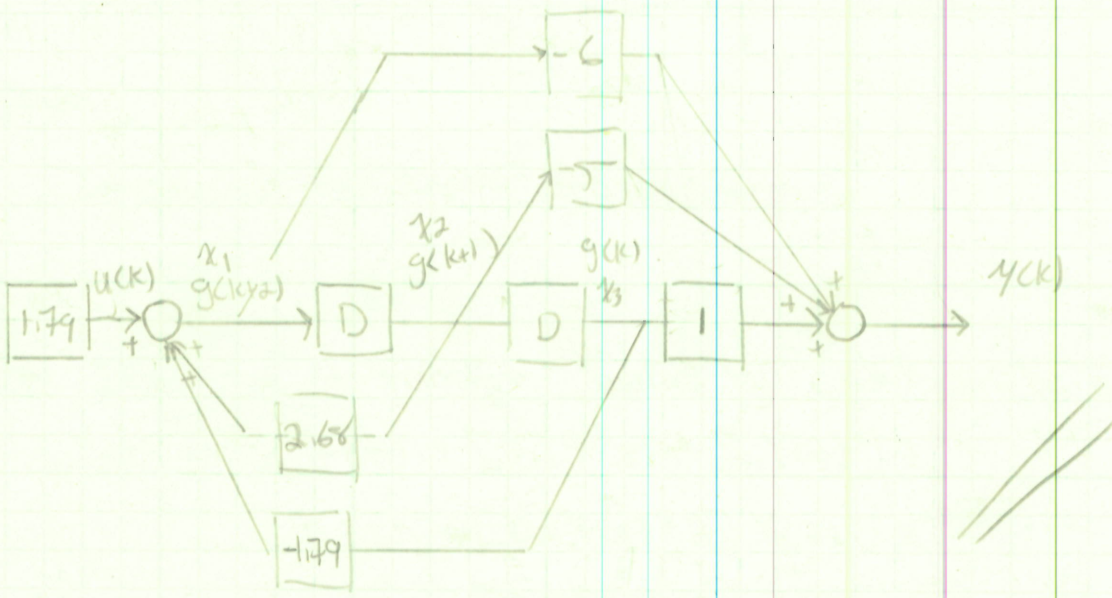
Now to define

$$x(k+1) = Ax(k) + bu(k) \; ; \; y(k) = cx(k)$$

Let $x_1[k] = g(k+1)$ $\quad x_2(k+1) = x_1(k)$ $\quad x_3[k+1] = x_2(k)$

$$x(k+1) = \begin{bmatrix} x_3(k+1) \\ x_2(k+1) \\ x_1(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1.79 & 2.68 \end{bmatrix} \begin{bmatrix} x_3(k) \\ x_2(k) \\ x_1(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1.79 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & -5 & -6 \end{bmatrix} x(k)$$


1.79
u(k)
x1
g(k+2)
D
x2
g(k+1)
D
g(k)
x3
1
-6
-5
2.68
-1.79
y(k)
+

1.4-14)

Given:

Introduction to observer canonical form

Find:

a) Show the z-transform implied by (1.2) can be written as

$$Y(z) = \bar{b}_0 F(z) + [\bar{b}_1 F(z) - \bar{a}_1 Y(z)] z^{-1} + [\bar{b}_2 F(z) - \bar{a}_2 Y(z)] z^{-2} + \cdots [\bar{b}_p F(z) - \bar{a}_p Y(z)] z^{-p} \quad (1.81)$$

b) Draw a block diagram representing (1.81)

c) Label the outputs of the delay elements from right to left as $x_1, x_2, \ldots, x_p$. Show that the system can be written as

$$A = \begin{bmatrix} -\bar{a}_1 & 1 & 0 & \cdots & 0 \\ -\bar{a}_2 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -\bar{a}_{p-1} & 0 & 0 & \cdots & 1 \\ -\bar{a}_p & 0 & 0 & \cdots & 0 \end{bmatrix} \quad b = \begin{bmatrix} \bar{b}_1 - \bar{a}_1 \bar{b}_0 \\ \bar{b}_2 - \bar{a}_2 \bar{b}_0 \\ \vdots \\ \bar{b}_p - \bar{a}_p \bar{b}_0 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$d = \bar{b}_0$$

d) Draw the block diagram in observer canonical form for

$$H(z) = \frac{2 + 3z^{-1} + 4z^{-2}}{1 + z^{-1} - 6z^{-2} - 7z^{-3}}$$

Solution:

a) $$\sum_{k=0}^{p} \bar{a}_k y[t-k] = \sum_{k=0}^{p} \bar{b}_k f[t-k] \quad \text{and } a_0 = 1$$

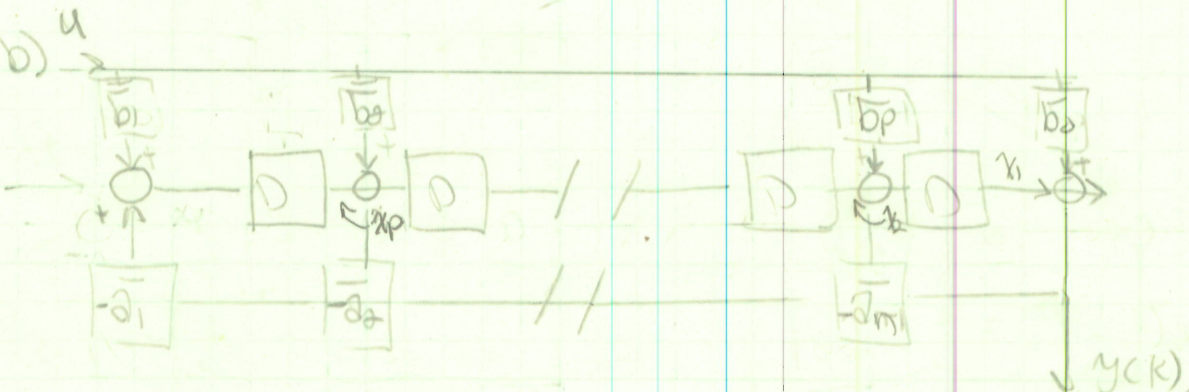
$$y[t-k] + \sum_{k=1}^{p} \bar{a}_k y[t-k] = \bar{b}_p f[t] z^{-p} + \sum_{k=1}^{p} \bar{b}_k f[t-k]$$

Take z transformation

$$Y(z) + \sum_{k=1}^{p} \bar{a}_k Y(z) z^{-p} = b_p F(z) z^{-p} + \sum_{k=1}^{p} \bar{b}_k F(z) z^{-p}$$

$$\Rightarrow Y(z) = b_p F(z) z^{-p} + \sum_{k=1}^{p-1} (\bar{b}_{k+1} F(z) - \bar{a}_k Y(z)) z^{-p}$$

b)



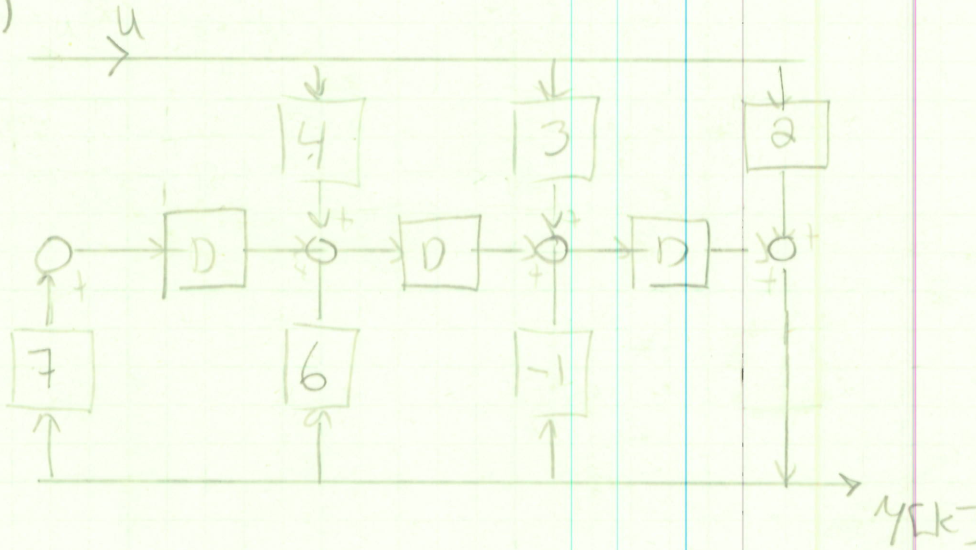
c) Follow diagram to get

$$x_p[k+1] = -a_p x_p[k] + \bar{b}_1 \underline{u}[k] - \bar{a}_p \bar{b}_0 [k]$$

So and so forth

$$\Rightarrow X[k] = [x_1 \;\; x_2 \;\; x_3 \cdots x_p]$$

d)



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1.4-16)

Given:

$$x[t+1] = Ax[t] + bf[t]$$
$$y[t] = c^T x[t] + df[t]$$ (1.21)

$$z[t+1] = \bar{A}z[t] + \bar{b}f[t]$$
$$y[t] = \bar{c}^T z[t] + \bar{d}f[t]$$ (1.22)

$\bar{A} = T^{-1}AT \quad \bar{b} = T^{-1}b \quad \bar{c} = T^T c \quad \bar{d} = d \quad z = T^{-1}x$

Find: $\quad x = Tz$

Show (1.21) and (1.22) have the same transfer function

Solution

$$\hat{G}(z) = C(zI - A)^{-1}B + D$$ where $\hat{G}(z)$ is the transfer function

$$\hat{G}(x) = C(xI - A)^{-1}B + d$$

Now

$$\bar{\hat{G}}(x) = \bar{C}(xI - \bar{A})^{-1}\bar{B} + \bar{d}$$

$$= cT(xI - T^{-1}AT)^{-1}T^{-1}B + d$$
$$= C(T^T xIT^{-1} - T^T T^{-1}ATT^{-1})^{-1}B + d$$
$$= C(TxIT^{-1} - IAI)^{-1}B + d$$
$$= C(xI - A)^{-1}B + d$$

//

1.4-18)

Given: 2 systems (A_1, b_1, c_1^T) and (A_2, b_2, c_2^T)

Find:

(A, b, c^T) by connecting the systems in:

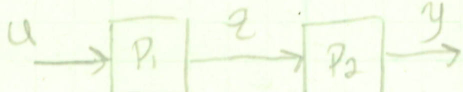
a) Series

b) Parallel

c) A feed forward configuration with (A_1, b_1, c_1^T) and feedback configuration with (A_2, b_2, c_2^T)

Solution:

a)

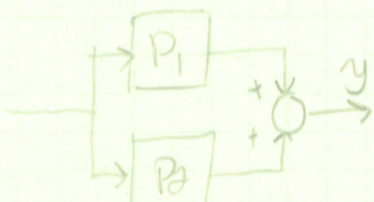


$\dot{x}_1 = A_1 x_1 + b_1 u \qquad \dot{x}_2 = A_2 x_2 + b_2 z$

$z = c_1^T x \qquad y = c_2^T x_2$

$$\dot{x} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad y = [C_1 \ C_2] x$$

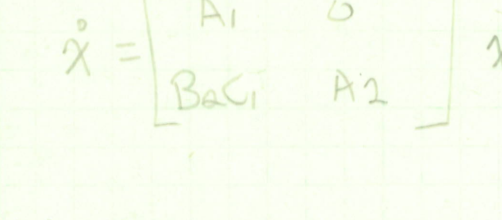
b)



$$\dot{x} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = [C_1 \ C_2] \ x$$

c)



$u' = u + y_2 \Rightarrow u = u' - y_2$

$\dot{x}_1 = A_1 x + B_1 u = A_1 x_1 + B_1 (u' - y_2)$

$y_1 = c_1^T x_1$

$\dot{x}_2 = A_2 x_2 + B_2 y_1 = A_2 x_2 + B_2 (c_1^T x_1) \qquad y_2 = c_2^T x_2$

$$\dot{x} = \begin{bmatrix} A_1 & -C_2^T \\ C_1^T & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u' \qquad y_1 = C_1^T x_1$$

1/1

1.4-19)

Given:

(1) $\begin{bmatrix} A & A_1 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} [c^T \; q^T]$

Both have same transfer function

(2) $\begin{bmatrix} A & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} b \\ q \end{bmatrix} [c^T \; 0]$

Find:

Show that realizations can have different state numbers

Solution

Transfer function from a SS realization is

$$G(z) = C(zI - A)^{-1}B + D$$

Plug values for (1) in

$$G_1(z) = \frac{bc}{-A + x}$$

Similarly for (2)

$$G_2(z) = \frac{bc}{-A + x}$$

1.4-21)

Given:

(A, b, c^T, d) and $d \neq 0$. describes a system $H(s)$ state space form.

Find:

Show that

$$(A - bc^T/d,\ b/d,\ -c^T/d,\ 1/d)$$

describes $1/H(s)$

Solution:

$G(s)\hat{G}(s)$ has to be 1 if $\hat{G}(s) = G(s)^{-1}$

$$\left[c^T(sI - A)^{-1}B + d\right]\left[-\frac{c^T}{d}\left(sI - A + \frac{bc^T}{d}\right)^{-1}\frac{b}{d} + \frac{I}{d}\right]$$

$$\left(\frac{BC}{-A+s}D\right)\left(-\frac{BC}{D^2\left(-A + \frac{BC}{D} + s\right)} + \frac{1}{D}\right)$$

Multiplying out and simplifying Gives $G(s)\hat{G}(s) = 1$ //

1.4-26)

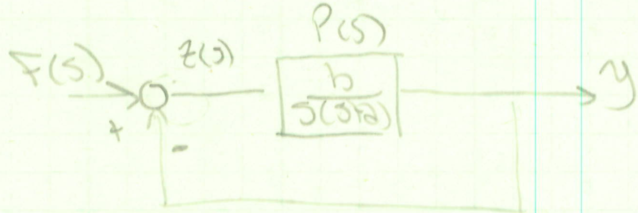
Given

Assume a system has an open-loop transfer function

$$H_o(s) = \frac{b}{s(s+a)}$$

Find:

a) Show that a system with the configuration



has the transfer function

$$H_c(s) = \frac{Y(s)}{F(s)} = \frac{1}{1 + \left(\frac{a}{b}\right)s + \left(\frac{1}{b}\right)s^2}$$

b) Show

$$\frac{1}{H_c(j\omega)} = A(j\omega) \angle \phi(j\omega)$$

where

$$A(j\omega) = \frac{1}{b}\sqrt{(b-\omega^2)^2 + (a\omega)^2} \quad \text{and} \quad \tan\phi = \frac{a\omega}{b-\omega^2}$$

$A(j\omega)$ and $\phi(j\omega)$ correspond to the reciprocal amplitude and the phase difference between input and output

c)

Show that if amplitude/phase measurements are taken at n frequencies a and b can be solved by

$$\begin{bmatrix} A(j\omega_1) & -\omega_1\sqrt{1+1/\tan^2\phi(j\omega_1)} \\ \tan\phi(j\omega_1) & -\omega_1 \\ \vdots & \vdots \\ A(j\omega_n) & -\omega_n\sqrt{1+1/\tan^2\phi(j\omega_n)} \\ \tan\phi(j\omega_n) & -\omega_n \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_1^2\tan\phi(j\omega_1) \\ \vdots \\ 0 \\ \omega_n^2\tan\phi(j\omega_n) \end{bmatrix}$$

Solution

a)

$$Y(s) = Z(s)P(s)$$

$$Z(s) = F(s) - Y(s)$$

$$F(s) = Z(s) + Y(s)$$

$$= Z(s) + Z(s)P(s)$$

$$\frac{Y(s)}{F(s)} = \frac{Z(s)P(s)}{Z(s) + P(s)Z(s)}$$

$$= \frac{P(s)}{1+P(s)} = \frac{\frac{b}{s(s+a)}}{1+\frac{b}{s(s+a)}}\left(\frac{\frac{s(s+a)}{b}}{\frac{s(s+a)}{b}}\right)$$

$$= \frac{1}{\frac{s(s+a)}{b}+1}$$

b)

$A(j\omega)$ is amplitude, so rewrite

$$\frac{1}{b}(b + aj\omega - \omega^2)$$

take magnitude of real and imaginary to get

$$\frac{1}{b}\sqrt{(b-\omega^2)^2 + (a\omega)^2}$$

Similarly phrase is $\tan^{-1}\left(\frac{\mathbb{I}}{\mathbb{R}}\right)$ $\left(\frac{\text{imaginary}}{\text{real}}\right)$

$$\phi = \tan^{-1}\left(\frac{a\omega}{b-\omega^2}\right)$$

c)

At this point there are only 2 unknowns let

$A_n = A(j\omega_n)$; $\tan\phi(j\omega_n) = \phi_n$; $\xi_n = b\,\omega_n\sqrt{1 + 1/\tan^2\phi(j\omega_1)}$

and $a_n = A_n a$

Write augmented matrix

$$\left[\begin{array}{cc|c} a_1 & \xi_1 & 0 \\ \phi_1 & -\omega_1 & \omega_1^2\phi_1 \\ a_2 & \xi_2 & 0 \\ \phi_2 & -\omega_2 & \omega_2^2\phi_2 \\ \vdots & \vdots & \vdots \\ a_n & \xi_n & 0 \\ \phi_n & \phi_n & \omega_n^2\phi_n \end{array}\right]$$

For the case of 2 you get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Which can be solved because its full rank

1.4-30)

Given:

The following data is measured from a third order system

$$y = [0.32, 0.25, 0.1, -0.022, 0.0006, -0.0012, 0.0005, -0.0001]$$

Assume first time index is 0, so $y[0] = 0.32$

Find

a) Determine the modes in the system and then plot them in the complex plane

b) The data can be written as

$$y[t] = c_1 (p_1)^t + c_2 (p_2)^t + c_3 (p_3)^t \quad t \geq 0$$

Find c_1, c_2, c_3

c) Add Gaussian random noise to the system with $\sigma^2 = 0.01$ then find modes of noisy data.

Do this several times and comment.

Solution:

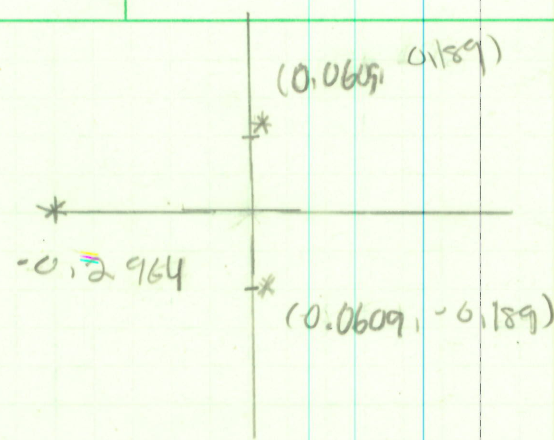
a)

$$\begin{bmatrix} -y_2 & -y_1 & -y_0 \\ -y_3 & -y_2 & -y_1 \\ -y_4 & -y_3 & -y_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

$a_1 = 1.75 \cdot 10^{-1}$, $a_2 = 3.24 \cdot 10^{-3}$ $1.166 \cdot 10^{-2}$

$$z^3 + 1.75 \cdot 10^{-1} z^2 + 3.24 \cdot 10^{-3} z + 1.166 \cdot 10^{-2}$$

$z = -0.2964$ $z = 0.0609 \pm 0.189j$



b)

$$z^2 + a_1 z_1 + a_2 = (z - p_1)(z - p_2)$$

$y[t] = c_1(p_1)^t + c_2(p_2)^t$ p_1 and p_2 are based on initial conditions

$$y[t] = c_1 e^{0.0609t}\cos(0.189t) + c_2 e^{0.0609t}\sin(0.189t) + c_3 e^{-0.2964t}$$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \end{bmatrix} = \begin{bmatrix} p_1|_0 & p_2|_0 & p_3|_0 \\ p_1|_1 & p_2|_1 & p_3|_1 \\ p_1|_2 & p_2|_2 & p_3|_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.0439 & 0.1997 & 0.7448 \\ 1.0498 & 0.4169 & 0.555 \\ 1.0126 & 0.6448 & 0.413 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$c_1 = 0.9263$ $c_2 = -0.8972$ $c_3 = -0.6281$

c)

$\sigma^2 = 0.01$

modes = [-0.4211 , 0.0345 ± 0.37j]

$\sigma^2 = 0.05$

modes = [-0.3466, 0.1927 ± 0.162j]

$\sigma^2 = 0.1$

modes = -0.374 , 0.063 ± 0.181j]

$\sigma^2 = 0.0001$

modes = [- 0.609 , 0.0537 ± 0.49j] .

The smaller the variance, the larger the modes seem to be in magnitude.

1/1

1.4-31

Given:

$$y[t] = A\cos(\omega_1 t + \theta_1) + B\cos(\omega_2 t + \theta_2)$$

And the frequencies are known

Find:

Determine the means of computing the amplitudes and phases at $t_1, \ldots, t_n$.

Solution:

$$y[0] = A\cos(\omega_1 t + \theta_1) + B\cos(\omega_2 t + \theta_2)$$
$$y[1] = A\cos(\omega_1 t + \theta_1) + B\cos(\omega_2 + \theta_2)$$
$$y[3] = A\cos(\omega_1 t + \theta_1) + B\cos(\omega_2 + \theta_2)$$
$$y[4] = A\cos(\omega_1 t + \theta_1) + B\cos(\omega_2 + \theta_2)$$

Now we have 4 eq's and 4 unknowns

//

//

B.1-1)

Given:

The characteristic function of a random variable X is the (conjugate of) Fourier transform of its density

$$\Phi_x(\omega) = \int f_x(x) e^{j\omega x} dx$$

Find:

a) Show that a Gaussian density with

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right]$$

has the characteristic function

$$\Phi_x(\omega) = \exp\left[j\mu\omega - \frac{1}{2}\omega^2\sigma^2 \right]$$

b) Show that the n^{th} moment of X can be obtained from its characteristic function by

$$E[X^n] = \frac{1}{j^n} \left. \frac{d^n \Phi_x(\omega)}{d\omega^n} \right|_{\omega=0}$$

Solution:

Look at exponent

$$E = \frac{1}{2\sigma^2}\left(X^2 + \mu^2 - 2\mu X + 2\sigma^2 \omega X\right)$$

$$= \frac{1}{2\sigma^2}\left(X^2 - 2(\mu - \sigma^2\omega)X + \mu^2 + (\mu - \sigma^2\omega)^2 - (\mu - \sigma^2\omega)^2\right)$$

$$= \frac{1}{2\sigma^2}\left(X - (\mu - \sigma^2\omega)\right)^2 - (\mu - \sigma^2\omega)^2 + \mu^2$$

$$\mathcal{F}(f_x(x)) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(\mu - \sigma^2\omega j)^2 + \mu^2\right] + \cancel{\int \exp\left[\frac{x-(\mu-\sigma^2\omega)}{2\sigma^2}\right]}$$

$$= \exp\left[\frac{1}{2\sigma^2}((\mu - j\sigma^2\omega)^2 + \mu^2)\right] =$$

$$= \exp\left[-\frac{\sigma^2\omega^2}{2} + j\mu\omega\right]$$

//

b)

$\phi_x(\omega) = E[e^{j\omega x}]$ Expand using Taylor series

$$= E\left[1 + \frac{j\omega x}{1!} + \frac{(j\omega x)^2}{2!} + \frac{(j\omega x)^3}{3!}\right]$$

$$= E[1] + E\left[\frac{j\omega x}{1!}\right] + E\left[\frac{(j\omega x)^2}{2!}\right] + \cdots$$

if $n=0$ then no derivatives are taken. Then the expected value is

$$E[1]$$

taking one derivative leaves

$$0 + E\left[\frac{jx}{1!}\right] + E\left[\frac{2(j\omega x)}{2!}\right] + \cdots \Bigg|_{\omega=0}$$

$$= E\left[\frac{jx}{1!}\right]$$

//

Taking subsequent derivatives then removes the previous value, The next term loses all ω's.

//

B.1-2)

Given:

$$\frac{1}{\sqrt{2\pi(1-\rho^2)}\,\sigma_x} \exp\left[-\frac{1}{2\sigma_x^2(1-\rho^2)}\left(x-\left(\mu_x+\frac{\sigma_x}{\sigma_y}\rho(y-\mu_y)\right)\right)^2\right]$$

Show:

That the above eq is correct

Solution

$$f(x|y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho}{\sigma_x\sigma_y}(x-\mu_x)(y-\mu_y)\right)\right]$$

Lets look @ this

$$-\frac{1}{2(1-\rho^2)\sigma^2}\left((x-\mu_x)^2+\frac{(y-\mu_y)^2}{\sigma_y^2}\sigma_x^2-\frac{2\rho\sigma_x}{\sigma_y}(x-\mu_x)(y-\mu_y)\right)$$

forgot this term

$$-\frac{1}{2(1-\rho^2)\sigma_x^2}\left(x^2-2\mu_x x+\mu_x^2-\frac{2\rho\sigma_x}{\sigma_y}(xy-x\mu_y-y\mu_x+\mu_x\mu_y)\right)$$

$$-\frac{1}{2(1-\rho^2)\sigma_x^2}\left(x^2-2\left[\mu_x+\frac{\rho\sigma_x}{\sigma_y}(y-\mu_y)\right]x+\mu_x^2-\frac{2\rho\sigma_x}{\sigma_y}(-y\mu_x+\mu_x\mu_y\right.$$

$$-\frac{1}{2(1-\rho^2)\sigma_x^2}\left(\left(x-\left[\mu_x+\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right]\right)^2+\frac{\sigma_x^2(y-\mu_y)^2}{\sigma_y^2}\right)$$

$$f(x|y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{1}{2(1-\rho^2)\sigma_x^2}\left(x-\left(\mu_x+\frac{\sigma_x}{\sigma_y}\rho(y-\mu_y)\right)\right)^2\right)+$$

$$\exp\left(\frac{(y-\mu_y)^2\sigma_x^2}{\sigma_y^2}\right)$$

$$\frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left[-\frac{1}{2\sigma_y^2}(y-\mu_y^2)\right]$$

$$f(x|y) = \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_x}} \exp\left[-\frac{1}{2\sigma_x^2(1-\rho^2)}\left(x - \left(\mu_x + \frac{\sigma_x}{\sigma_y}\rho(y-\mu_x)\right)\right)^2\right.$$

B.2-3)

Given

$(x+z)^T A (x+z) + d$; $z = \frac{1}{2} A^{-1} y$; $d = c - \frac{1}{2} y^T z$

Find:

$f(x|y)$ when

$$Y = x + N$$

and x and N are indepenent Gaussian random vectors w/

$x \sim \mathcal{N}(\mu, R_x)$ $N \sim \mathcal{N}(0, R_n)$

Identify the mean of $X|Y$ and variance $X|Y$

Solution

$$E = \frac{1}{2}\left[(y-x)^T R_n^{-1} (y-x) + (x-\mu_x)^T R_x^{-1} (x-\mu_x) \right]$$

$$= -\frac{1}{2}\Big[x^T \big(\underbrace{(R_n^{-1} + R_x^{-1})}_{A} x - 2 \underbrace{(R_n^{-1} y + R_x^{-1} \mu_x)}_{y} \big) \Big] + c_1$$

$(x+z)^T A (x+z) + d$

$z = \frac{1}{2} (R_n + R_x)(R_n^{-1} y + R_x^{-1}) \mu_x$ $A = (R_n^{-1} + R_x^{-1})$

$d = c_1 - \frac{1}{2} y^T z$

$\mu_{x|y} = -z$ // $R_{x|y} = R_n + R_x$ //

1.8-41)

Show that there are an infinite number of primes

Hint: Use proof by contradiction.

Solution

Suppose that there are a finite set of prime numbers

$$P = \{2, 3, 5, 7, \ldots, p_n\}$$

where p_n is the last prime in the set.

Let $q = 1 + \prod_{p_i \in P} p_i$

Stating the fact that every natural number, $n > 1$, is divisible by some prime number.

Now going back to q. This new number we created has to either be prime or not prime. Using our set P and the fact stated above suppose we divide q by every $p_i \in P$

$$\forall p_i \in P \quad \frac{q}{p_i}$$

They would all have a remainder which violates the fact we stated. Therefore, there must be another prime that our number q is divisible by.

1/1

1.8-43)

Given:

$$\sum_{i=0}^{n} 2^i$$

Find:

a) By trial and error, find a formula for what is above

b) Prove by induction that your formula is correct.

Solution:

a)

Suppose the formula of the form

$f(n) = 2^{n+1} - 1$

Basic checks

1 : $2^0 + 2^1 = 3$ $\quad f(1) = 2^2 - 1 = 3$

2 : $2^0 + 2^1 + 2^2 = 7$ $\quad f(2) = 2^3 - 1 = 7$

b)

Base case

0 : $2^0 = 1$ $\quad 2^1 - 1 = 1$

Suppose $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Then add 2^{n+1} to each side

$$\sum_{i=0}^{n+1} 2^i = 2^{n+2} - 1$$

Which concludes the proof //

(1.8-45)

Given:

$$\sum_{i=1}^{n} \frac{1}{i^2+n}$$

Find:

a) Formula for the above written summation

b) Prove the formula by induction

Solution:

a)

Suppose the formula of the form

$$f(n) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Basic checks

$1: \frac{1}{1^2+1} = \frac{1}{2}$ $\quad f(1) = 1 - \frac{1}{2} = \frac{1}{2}$

$2: \frac{1}{2^2+2} + \frac{1}{2}$ $\quad f(2) = 1 - \frac{1}{3} = \frac{2}{3}$

$= \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$

b)

Suppose $\displaystyle\sum_{i=1}^{n} \frac{1}{i^2+1} = 1 - \frac{1}{n+1}$

then

$$\sum_{i=1}^{n+1} \frac{1}{i^2+i} = \sum_{i=1}^{n} \frac{1}{i^2+1} + \frac{1}{\quad}$$

$$\sum_{i=1}^{n+1} \frac{1}{i^2+i} = \sum_{i=1}^{k} \frac{1}{i^2+i} + \frac{1}{(n+1)^2(n+1)} = \frac{n}{n+1} + \frac{1}{(n+1)^2(n+1)}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2} \quad //$$

//

1.8-46)

Given: $n^3 - n$

Show: $\forall \mathbb{N} \mid n > 0,\ n \in \mathbb{N}$ that $n^3 - n$ is divisible by 3.

Solution

Base case

0, 1: $1^3 - 1 = 0$; $\frac{0}{3} = 0$

2: $2^3 - 2 = 6$; $\frac{6}{3} = 2$

Suppose $3k = n^3 - n$

Then $3j = (n+1)^3 - (n+1)$

$= (n^3 - n) + 3n^2 + 3n$

$= 3(k + n^2 + n)$

Which must be divisible by 3. //

1.8-47)

Given:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{where} \quad n \geq k$$

Find:

Show by induction for $1 \leq k \leq n$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Solution

If $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ then show base case

$k=1, n=1$

$$\binom{2}{1} = \binom{1}{1} + \binom{1}{0}$$

$$\frac{2!}{1!(2-1)!} = \frac{1!}{1!(0)!} + \frac{1!}{0!\,1!}$$

$$\frac{2}{1} = 1 + 1 = 2 \checkmark$$

Let $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ be true

Suppose then we start with $\begin{pmatrix} n \\ k-1 \end{pmatrix} = \begin{pmatrix} n & n \\ k-1 & k-2 \end{pmatrix}$

then by induction

$$\begin{pmatrix} n+1 \\ k \end{pmatrix} = \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-(k-1))!\,(k-1)!}$$

$$= \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-k+1)!\,(k-1)!}$$

$$= \frac{(n-k+1)\,n!}{(n-k+1)(n-k)!\,k!} + \frac{n!\,k}{(n-k+1)!\,(k-1)!}$$

$$= \frac{n \cdot n! - k n! + n! + n!\,k}{(n-k+1)!\,k!}$$

$$= \frac{n \cdot n! + n!}{(n-k+1)!\,k!} = \frac{n!\,(n+1)}{(n-k+1)!\,k!} = \frac{(n+1)!}{((n+1)-k)!\,k!} = \begin{pmatrix} n+1 \\ k \end{pmatrix}$$