1)

Given:

$$P(T,t) = a_1(T) + a_2(T)t + a_3(T)t^2 + \cdots + a_n(t)t^{n-1}$$

where $a_i(T)$ is dependent on the observation time (T).

Find:

Show that the set $\{a_i(T)\}$ need not be completely recomputed for each T, but rather can be continuously updated with

$$\frac{da_i(T)}{dT} = \frac{b_i \varepsilon(T)}{T^i} \quad \text{where } \varepsilon(T) = x(T) - P(T,T)$$

$$b_i \in \mathbb{R}$$

Solution:

The coefficients are selected to minimize

$$\int_0^T (x(t) - P(T,t))^2 \, dt$$

Let the inner product be

$$\int_0^T f(t)g(t)\,dt \Rightarrow \int_0^T e(T,t)\, t^i = \langle e(T,t), t^i \rangle = 0$$

where $e(T,t) = x(t) - P(T,t)$ and t^i are our basis functions.

Theorem: When differentiating an integral with its limits of integration depend on the variable derivative is

$$\frac{d}{da}\int_{p(a)}^{q(a)} f(x,a)\,dx = f(q,a)\frac{dq}{da} - f(p,a)\frac{dp}{da} + \int_{p(a)}^{q(a)} \frac{\partial f(x,a)}{da}\,dx$$

Lebinie integral rule

Lets write out our inner product again

$$\int_0^T \underbrace{[x(t) - P(T,t)]\, t^i}_{f(t,T)}\, dt = \int_0^T f(t,T)\,dt$$

Now lets apply Lebniz Integral rule

$$\frac{d}{dT}\int_{p(T)}^{q(T)} f(t,T)\,dt = f(q,T)\frac{dq}{dT} - f(p,T)\frac{dp}{dT} + \int_{p(T)}^{q(T)} \frac{\partial f(t,T)}{\partial T}\,dt$$

where $q(T) = T$; $p(T) = 0$. So we can simplify this expression

$$= \underbrace{f(q,T)\,1}_{\frac{\partial T}{\partial T}=1} - \underbrace{0}_{\frac{\partial 0}{\partial T}=0} + \underbrace{\int_0^T \frac{\partial f(t,T)}{\partial T}\,dt}_{\text{Lets focus on this for a moment}} \qquad (1)$$

$$\int_0^T \frac{d}{dT}\left[(x(t) - P(T,t))\,t^i\right] dt$$

$$= \int_0^T -\frac{dP(T,t)}{dT}\,t^i\,dt = \int_0^T \frac{-\partial\, a_i(T)\, t^{i!}\, t^i}{\partial T}\,dt$$

$$= \int_0^T -\frac{\partial a_i(T)}{\partial T}\,t^i = -\frac{1}{b_i}\frac{da_i(T)}{dT}\,t^i\Big|_0^T = -\frac{1}{b_i}\frac{da_i(T)}{dT}\,T^{2i}$$

$\hookrightarrow b_i = \frac{1}{2i}$

Plugging this back in to (1) we get

$$f(T,T) - \frac{da_i(T)}{dT}\,T^i = \underbrace{[x(T) - P(T,T)]}_{\varepsilon(T)}\,T^i - \frac{1}{b_i}\frac{da_i(T)}{dT}\,T^{2i}$$

Since we want $\langle e(\cdot), t^i\rangle = 0$ we can equate the two

$$\varepsilon(T)\,T^i = \frac{1}{b_i}\frac{da_i(T)}{dT}\,T^{2i} = \frac{b_i\,\varepsilon(T)\,T^i}{T^{2i}} = \frac{da_i(T)}{dT}$$

$$\Rightarrow \frac{b_i\,\varepsilon(T)}{T^i} = \frac{da_i(T)}{dT}$$

1/1

2)

Given:

$$\lim_{n\to\infty} x_n = x \qquad \lim_{n\to\infty} y_n \to y \qquad \|x_n\| \leq M$$

$\{x_n\}$ and $\{y_n\}$ are bounded sequences

Find:

Show $\lim_{n\to\infty} \langle x_n, y_n \rangle = \langle x, y \rangle$

Solution:

Definition: A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy Sequence if for any $\epsilon > 0$ $\exists$ $N > 0$ such that $d(x_n, x_m) < \epsilon$ $\forall$ $m, n > N$

Which we can leverage to show that the limit converges.

Let $\epsilon > 0$ and $n > N$ such that we obtain

$$\|x_n - x\| < \epsilon \quad \text{and} \quad \|y_n - y\| < \epsilon$$

Now we write

$$\|\langle x_n, y_n \rangle - \langle x, y \rangle\| = \|\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle\|$$

By Cauchy-Schwarz

$$\leq \|\langle x_n, y_n \rangle - \langle x_n, y \rangle\| + \|\langle x_n, y \rangle - \langle x, y \rangle\|$$

$$\leq \underbrace{\|x_n\|}_{\leq M} \underbrace{\|y_n - y\|}_{\leq \epsilon} + \underbrace{\|x_n - x\|}_{\leq \epsilon} \|y\|$$

$$\leq M\epsilon + \|y\|\epsilon$$

Which tells us the sequence is coming together. Thus, we can say

$$\lim_{n\to\infty} \langle x_n, y_n \rangle = \langle x, y \rangle$$

3)

Given

$$x_1 + x_2 + x_3 = 0 \qquad (1)$$

Find;

Find a orthogonal projection matrix P that projects $\mathbb{R}^3$ to the space orthogonal to (1).

Solution:

The formula for the projection matrix is

$$P = A(A^T A)^{-1} A^T$$

Theorem: Let P be a projection matrix defined on a linear space S. Then the range and nullspace of P are disjoint linear subspaces of S, and $S = R(P) + N(P)$. That is $R(P)$ and $N(P)$ are algebraic compliments.

Furthermore

$$x = \underbrace{Px}_{\in V} + \underbrace{(I-P)x}_{\in W}$$

Definition: Let P be a projection operator on an inner product space S. P is said to be an orthogonal projection if its range and nullspace are orthogonal, $R(P) \perp N(P)$.

Thus

$$x = \underbrace{Px}_{\in R(P)} + \underbrace{(I-P)x}_{\in N(P)}$$

So a projection matrix onto $x_1 + x_2 + x_3 = 0$ can be written as follows. We find a set of linearly-independent and in the plane.

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

from which we construct $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$

So $P = \begin{bmatrix} 0.667 & -0.333 & -0.333 \\ -0.333 & 0.667 & -0.333 \\ -0.333 & -0.333 & 0.667 \end{bmatrix}$

from which we can show that this is a projection onto the plane $x_1 + x_2 + x_3 = 0$ by

$$P\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

As per the theorem, if we want to get the projection orthogonal to the space we write

$$\hat{P} = I - P = \begin{bmatrix} 0.333 & 0.333 & 0.333 \\ 0.333 & 0.333 & 0.333 \\ 0.333 & 0.333 & 0.333 \end{bmatrix}$$

Which should should now result in a 0 vector if we say

$$\hat{P}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we are trying to project a vector a_1 to a vector space orthogonal to it.

See P3.m

1/1

4)

Given:

- x_k and y_k
- y_k is quadratic

Find:

- Determine a best fit quadratic
- Express measure of fitness $\|e\|_2^2$

Solution

$$y_i \approx a_0 + a_1 x_i + a_2 x_i^2$$

$$y_i = \sum_{j=0}^{2} a_j x_i^j + e$$

$$y = [y_1 \ y_2 \cdots y_n]^T$$

$$e = [e_1, \ldots, e_n]^T$$

$$c = [a_0 \ a_1 \ a_2]^T$$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix}$$

By least squares we find

$$c = (A^H A)^{-1} A^H y$$

By plugging in the values and solving for c we find

$$c = [0.8538 \ \ 4.0499 \ \ 3.6242]^T$$

See P4.m

5)

Given:

$\mathcal{A}$ is a vector space of real $p \times q$ matrices.

$\mathcal{B}$ is the vector space of polynomials of degree r or less.

A point for $M \in \mathcal{A}$ and $p(t) \in \mathcal{B}$ are

$$M = \begin{bmatrix} m_{11} & \cdots & m_{1q} \\ \vdots & & \\ m_{p1} & & m_{pq} \end{bmatrix} \qquad p(t) = p_0 + p_1 t + \dots + p_r t^r \quad p_i \in \mathbb{R}$$

The space is defined by the direct sum $S = \mathcal{A} \oplus \mathcal{B}$.

Let $\sigma = (M, p(t))$ and $\rho = (N, q(t))$ be points in S.

Let the inner product be defined as

$$\langle \sigma, \rho \rangle = \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} N_{ij} + \sum_{k=1}^{r} p_k q_k \qquad (1)$$

Let $s_1, \dots, s_m$ be a basis for a subspace $V < S$ where

$$s_i = (M_i, P_i(t)) \qquad i = 1, 2, \dots, m$$

Find:

a) Show that (1) is actually an inner product.

b) Given $x \in S$ write down normal equations for finding $\hat{x} \in V$ that is closest to x where $\hat{x} = c_1 s_1 + \dots + c_m s_m$

c). What is the $(i,j)^{th}$ element of the Grammian Matrix? What is the i^{th} element of the cross correlation vector p?

d) Is the Grammian positive definite? Why or why not?

Solution:

a) To show that (1) is an inner product lets define what an inner product is.

Definition: Let S be a vector space defined over a scalar field $\mathbb{R}$. An inner product is a function $\langle \cdot, \cdot \rangle : S \times S \to \mathbb{R}$ with the following properties

1) For real vectors $\langle x, y \rangle = \langle y, x \rangle$

2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \forall \alpha \in \mathbb{R}$

3) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

4) $\langle x, x \rangle > 0$ if $x \neq 0$ and

$\langle x, x \rangle = 0$ iff $x = 0$

Show (1) satisfies all the properties

1)
$$\langle \rho, \sigma \rangle = \sum_{i=1}^{p} \sum_{j=1}^{q} \underbrace{N_{ij} M_{ij}}_{= M_{ij} N_{ij}} + \sum_{k=1}^{r} \underbrace{q_k p_k}_{= p_k q_k} \quad \text{for } M_{ij} = m_{ij}$$

$$\Rightarrow \langle \sigma, \rho \rangle = \langle \rho, \sigma \rangle \quad //$$

2)
$$\langle \alpha\sigma, \rho \rangle = \sum_{i=1}^{p} \sum_{j=1}^{q} \alpha M_{ij} N_{ij} + \sum_{k=1}^{r} \alpha p_k q_k$$

$$= \alpha \left[\sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} N_{ij} + \sum_{k=1}^{r} p_k q_k \right]$$

$$= \alpha \langle \sigma, \rho \rangle \quad //$$

3) $\langle \sigma + \rho, \phi \rangle$ for $\phi = (A, b)$ and

$\sigma + \rho = (M + N, p + q) = (L, l)$

This can be done because

Definition: The direct sum of linear spaces V and W ($V \oplus W$) is defined on the cartesian product $V \times W$, so a point in $V \oplus W$ is an ordered pair (v, w) with $v \in V$ and $w \in W$. Addition is component wise:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$

Applies to (2)

and scalar multiplication is $\alpha(v, w) = (\alpha v, \alpha w)$

So we write

$$\langle \sigma + \rho, \phi \rangle = \sum_{i=1}^{p} \sum_{j=1}^{q} L_{ij} A_{ij} + \sum_{k=1}^{r} l_k b_k$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} (M_{ij} + N_{ij}) A_{ij} + \sum_{k=1}^{r} (p_k + q_k) b_k$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} A_{ij} + N_{ij} A_{ij} + \sum_{k=1}^{r} p_k b_k + q_k b_k$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} A_{ij} + \sum_{k=1}^{r} p_k b_k + \sum_{i=1}^{p} \sum_{j=1}^{q} N_{ij} A_{ij} + \sum_{k=1}^{r} q_k b_k$$

$$= \langle \sigma, \phi \rangle + \langle \rho, \phi \rangle \quad //$$

4) $$\langle \sigma, \sigma \rangle = \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij}^2 + \sum_{k=1}^{r} p_k^2$$

if $M_{ij} \neq 0$ or $p_k \neq 0 \Rightarrow \langle \sigma, \sigma \rangle \neq 0$

if $M_{ij} = 0$ and $p_k = 0 \Rightarrow \langle \sigma, \sigma \rangle = 0$ //

Therefore, (1) is an inner product! //

b)

$\hat{x} = c_1 s_1 + c_2 s_2 + \cdots + c_m s_m \in V$ and $x \in S$

So $x \approx \hat{x}$ and $x = \hat{x} + e \Rightarrow e = x - \hat{x}$. We want to minimize the normed error

$$\|e\| = \|x - \hat{x}\| = \left\| x - \sum_{i=1}^{m} c_i s_i \right\|$$

That is to say $e \perp s_j$ which leads us to write

$$\left\langle x - \sum_{i=1}^{m} c_i s_i , s_j \right\rangle = 0$$

$$= \left\langle \sum_{i=1}^{m} c_i s_i , s_j \right\rangle = \langle x, s_j \rangle$$

From which we can produce the Grammian

$$\underbrace{\begin{bmatrix} \langle s_1, s_1\rangle & \langle s_2, s_1\rangle & \cdots & \langle s_m, s_1\rangle \\ \langle s_1, s_2\rangle & \langle s_2, s_2\rangle & \cdots & \langle s_m, s_2\rangle \\ \vdots & & & \vdots \\ \langle s_1, s_m\rangle & \langle s_2, s_m\rangle & \cdots & \langle s_m, s_m\rangle \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} \langle x, s_1\rangle \\ \langle x, s_2\rangle \\ \vdots \\ \langle x, s_m\rangle \end{bmatrix}}_{p} \qquad (2)$$

Using (1) as our induced norm in (2) we found our set of normal equations.

c)

i^{th} matrix in $\{s\}$

j^{th} matrix in $\{s\}$

$$\langle s_i, s_j\rangle = \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} N_{ij} + \sum_{k=1}^{r} p_k q_k$$

for $s_i = (M_i, P_i(t))$ $\quad s_j = (M_j, P_j(t))$

$$\langle x, s_i\rangle = \sum_{i=1}^{p} \sum_{j=1}^{q} A_{ij} M_{ij} + \sum_{k=1}^{r} b_k p_k$$

for $s_i = (M_i, P_i(t))$ and $x = (A, b)$

d)

Theorem: A Grammian matrix is always PSD ($x^H R x \geq 0 \;\forall x \in \mathbb{C}^M$)
It is PD iff $p_1, \dots, p_m$ are linearly independent.

We have no guarantees other than the values are in $\mathbb{R}$.

It is not PD. If $m_{ij} \in \mathbb{R}$ and $p_i \in \mathbb{R}$ then we can have cases such that

$$x^H R x = 0 \text{ for } x \neq 0$$

6)

Given

$$\min_x \left[(x - x_0)^T \Pi^{-1} (x - x_0) + \| b - Ax \|_W^2 \right] \quad (1)$$

Find:

a) Determine a set of linear equations that satisfy (1)

b) Show the optimum solution of the form

$$\hat{x} = x_0 + \text{"stuff"}$$

Solution:

$\langle x, y \rangle_W = y^H W x$; using this we can rewrite (1) as

$$\min_x \left[\| x - x_0 \|_{\Pi^{-1}}^2 + \| b - Ax \|_W^2 \right]$$

Which we can write as a least squares problem. Let

$$\tilde{A} = \begin{bmatrix} A \\ I \end{bmatrix} \qquad \tilde{W} = \begin{bmatrix} W \\ \Pi^{-1} \end{bmatrix} \qquad \tilde{b} = \begin{bmatrix} b \\ x_0 \end{bmatrix}$$

A normal LS solution is of the form

$\tilde{A} x = \tilde{b}$; but we have a weighted inner product so multiply both sides by $\tilde{A}^T \tilde{W}$

$$\tilde{A}^T \tilde{W} \tilde{A} x = \tilde{A}^T \tilde{W} \tilde{b} \Rightarrow x = (\tilde{A} \tilde{W} \tilde{A})^{-1} \tilde{A}^T \tilde{W} \tilde{b}$$

From which we expand to find

$$x = \left([A^T \ I^T] \begin{bmatrix} W \\ \Pi^{-1} \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} \right)^{-1} [A^T \ I^T] \begin{bmatrix} W \\ \Pi^{-1} \end{bmatrix} \begin{bmatrix} b \\ x_0 \end{bmatrix}$$

$$= (A^T W A + \Pi^{-1})^{-1} (A^T W b + \Pi^{-1} x_0)$$

$$= x_0 + (A^T W A)^{-1} A^T W b + (A^T W A)^{-1} \Pi^{-1} x_0 + \Pi A^T W b$$

7)

Given:

- $\hat{x}_n = \arg\min_x \|y - H_n x\|_2^2$ $\quad x_n$ is an $n \times 1$ vector
- H_n is full rank (columns are linearly independent)
  ↳ $m \times n$
- y is $m \times 1$
- $H_{n+1} = [H_n \ h_{n+1}]$ (still full rank)
- $\hat{x}_{n+1} = \arg\min_{\begin{bmatrix} x \\ x_{n+1} \end{bmatrix}} \left\| y - [H_n \ h_{n+1}] \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \right\|_2^2$
- $\hat{h}_{n+1} = P_H h_{n+1} = H_n (H_n^T H_n)^{-1} H_n^T h_{n+1} = H_n a$.
- $\tilde{h}_{n+1} = h_{n+1} - \hat{h}_{n+1} = h_{n+1} - H_n a$

Find:

a) Prove $R([H_n \ \tilde{h}_{n+1}]) = R(H_n \ h_{n+1})]$

b) Prove $\tilde{h}_{n+1}$ is orthogonal to $R(H_n)$

c) Explain how the original LS problem can be replaced by

$$\hat{x}_{n+1} = \arg\min_{\begin{bmatrix} x \\ x_{n+1} \end{bmatrix}} \left\| y - [H_n \ \tilde{h}_{n+1}] \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \right\|_2^2 \qquad (1)$$

d) Set up and solve normal equations in (1). Find a relationship between $\hat{x}_{n+1}$ and x_n.

Solution:

a) We know $h_{n+1} = \tilde{h}_{n+1} + \hat{h}_{n+1}$

Because h_{n+1} is the orthogonal projection of h_{n+1} onto H_n then

$\hat{h}_{n+1} \perp h_{n+1}$

Definition: Let P be a projection operator on an inner product space S. P is said to be an orthogonal projection matrix if is range and nullspace are orthogonal $R(P) \perp N(P)$

And by Theorem 2.7: $S = R(P) + N(P)$

So this tells us $\hat{h}_{n+1} \in \mathcal{N}(h_{n+1})$. That is we can say

$$h_{n+1} = h_{n+1_W} + h_{n+1_{W^\perp}} \Rightarrow h_{n+1} - h_{n+1_{W^\perp}} = h_{n+1_W}$$

Let $h_{n+1_W} = \tilde{h}_{n+1}$ and $h_{n+1_{W^\perp}} = \hat{h}_{n+1}$, So we get

$\tilde{h}_{n+1} = R(h_{n+1})$ $\therefore$ $R([H_n\ h_{n+1}])$ and $R(H_n\ \tilde{h}_{n+1}])$ are the same

b)

very much like "update" $e \perp$

Continuing on from the previous statement

$\hat{h}_{n+1} \in R(H_n)$ (as per the problem statement)

because we projected h_{n+1} onto $R(H_n)$. As it was a orthogonal projection. We know

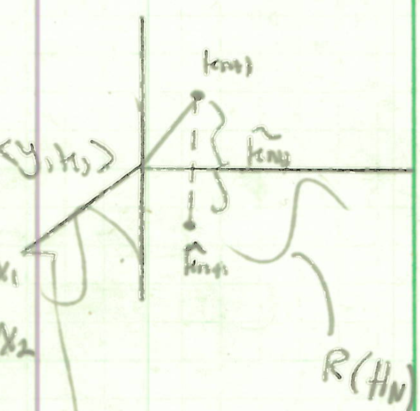
$\hat{h}_{n+1} \perp \tilde{h}_{n+1}$ thus $\tilde{h}_{n+1} \perp R(H_n)$

c)

We know $[H_n\ h_{n+1}]$ is full rank. By removing $\hat{h}_{n+1}$ and keeping $\tilde{h}_{n+1}$ which is $R(h_{n+1})$ we know that $[H_n\ \tilde{h}_{n+1}]$ is still full rank and has the same space as $R(h_{n+1})$.

d)

$\langle y - H_{n+1}x_{n+1}, \tilde{h}_i \rangle = 0 \Rightarrow \langle H_{n+1}x_{n+1}, h_i \rangle = \langle y, h_i \rangle$



$$\left[\begin{array}{cccc|c} \langle \tilde{h}_1 \tilde{h}_1 \rangle & \langle \tilde{h}_2 \tilde{h}_1 \rangle & \cdots & \langle \tilde{h}_n \tilde{h}_1 \rangle & \langle \tilde{h}_{n+1} \tilde{h}_1 \rangle \\ \langle \tilde{h}_1 \tilde{h}_2 \rangle & \langle \tilde{h}_2, \tilde{h}_2 \rangle & & \langle \tilde{h}_n \tilde{h}_2 \rangle & \langle \tilde{h}_{n+1} \tilde{h}_2 \rangle \\ \vdots & & & & \\ \langle \tilde{h}_1 \tilde{h}_n \rangle & \langle \tilde{h}_2 \tilde{h}_n \rangle & & \langle \tilde{h}_n \tilde{h}_n \rangle & \langle \tilde{h}_{n+1} \tilde{h}_n \rangle \\ \hline \langle \tilde{h}_1 \tilde{h}_{n+1} \rangle & \langle \tilde{h}_2 \tilde{h}_{n+1} \rangle & \cdots & \langle \tilde{h}_n \tilde{h}_{n+1} \rangle & \langle \tilde{h}_{n+1} \tilde{h}_{n+1} \rangle \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \langle \tilde{h}_1 y \rangle \\ \langle \tilde{h}_2 y \rangle \\ \vdots \\ \langle \tilde{h}_n y \rangle \\ \langle \tilde{h}_{n+1} y \rangle \end{bmatrix}$$

$\tilde{h}_{n+1} \perp R(H_n) \Rightarrow \langle \tilde{h}_{n+1}, h_i \rangle = 0$

$i \neq n+1$

$$\begin{bmatrix} H & — & O \\ | & & | \\ O & — & \langle f(n+1), f(n+1) \rangle \end{bmatrix} \begin{bmatrix} \underline{x_n} \\ \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \underline{f_n} \; \underline{y_n} \\ \\ f(n), y(n+1) \end{bmatrix}$$

8)

Given:

a) - S is the set of 2x2 real matrizes

- $M \subset S$ of 2x2 symmetric matrices

- $A, B \in S$ let $\langle A, B \rangle = tr(AB^T)$

- $X = \begin{bmatrix} 2 & 5 \\ 6 & 7 \end{bmatrix}$

b)

- U vector space of 5x4 real matrizes

- $T \subset U$ is the subset of 5x4 Toeplitz matrizes

- $\langle A, B \rangle = tr(AB^T)$

-
$$X = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 2 & 2 & 2 \\ 4 & 7 & 5 & 3 \\ 5 & 4 & 3 & 2 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

Find:

a) What is the orthogonal projection of X onto M?

b) What is the projection of X onto T?

Explain why this is a reasonable projection.

Solution:

a)

Lets begin with finding a basis.

A symmetric matrix is of the form $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$

We can expand this to

$$A = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

If we let $b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $b_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

such that $B = \{ b_1 \; b_2 \; b_3 \}$.

Furthermore because A is written as a linear combination of matrices in B, we can say any symmetric matrix in M can be expressed as linear combinations of B. The set is also linearly independent. So, B is a basis for M.

We define the Grammian in the usual way

$$\begin{bmatrix} \langle b_1 \; b_1 \rangle & \langle b_2 \; b_1 \rangle & \langle b_3 \; b_1 \rangle \\ \langle b_1 \; b_2 \rangle & \langle b_2 \; b_2 \rangle & \langle b_3 \; b_2 \rangle \\ \langle b_1 \; b_3 \rangle & \langle b_2 \; b_3 \rangle & \langle b_3 \; b_3 \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \langle x \; p_1 \rangle \\ \langle x \; p_2 \rangle \\ \langle x \; p_3 \rangle \end{bmatrix}$$

$c = [2 \;\; 5.5 \;\; 7]^T$ where the matrix is built via

$$x' = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 2 & 5.5 \\ 5.5 & 7 \end{bmatrix} \;//$$

b)

Letting the basis sets be

$$b_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad b_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \ldots \quad b_8 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Then creating the 8x8 Grammian similar to before

$$\begin{bmatrix} \langle b_1 \; b_1 \rangle & \cdots & \langle b_8 \; b_1 \rangle \\ \vdots & & \vdots \\ \langle b_1, b_8 \rangle & \cdots & \langle b_8, b_8 \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_8 \end{bmatrix} = \begin{bmatrix} \langle x \; p_1 \rangle \\ \vdots \\ \langle x \; p_8 \rangle \end{bmatrix}$$

The solution is creating a Toeplitz matrix where the diagonals are the the average of the diagonal values in X. We got the result

$$c = [4, 2, 2.67, 2.5, 4.25, 4, 4, 2]$$

Where c_1 is the top right value and you work your way down.

See P6.m

9)

Given:

$h(t) = (e^{-4t} + e^{-3t})u(t)$ where $u(t)$ is the unit step function

$y(0) = 0 \quad \dot{y}(0) = 0$

Find:

Find a signal $x(t)$ s.t. $y(3) = 2$ and $\int_0^3 y(\tau) = 4$

energy of x is minimized

Solution:

$y(t) = h(t) * x(t)$

$$\int_0^3 h(3-\tau)x(\tau)\,d\tau = \int_0^3 e^{-4(3-\tau)} + e^{-3(3-\tau)}\,x(\tau)\,d\tau$$

drop $u(t)$ because

$u(0) = 0$

$u(t) > 0 = 1$

$y_1 = e^{-4(3-\tau)} + e^{-3(3-\tau)}$

$\Rightarrow \langle x, y_1 \rangle = 2$

where $\langle \cdot, \cdot \rangle = \int_0^3 f(\tau)g(\tau)\,d\tau$

$$k(t) = \int_0^t h(\tau)\,d\tau = -\frac{1}{3}e^{-3t} - \frac{1}{4}e^{-4t} + C$$

$y(0) = 0 = C - \frac{1}{3} - \frac{1}{4} \Rightarrow C = \frac{7}{12}$

$y_2 = \frac{7}{12} - \frac{1}{3}e^{-3t} - \frac{1}{4}e^{-4t}$ $\Rightarrow \langle x, y_2 \rangle = 4$

$$\begin{bmatrix} \langle y_1\, y_1 \rangle & \langle y_1\, y_2 \rangle \\ \langle y_1\, y_2 \rangle & \langle y_2\, y_2 \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow c = [2.236 \quad 4.168]^T$$

See P9.m.