3.14 - 28)

Given:

System has impulse response $h(t) = 3e^{-2t} + 4e^{-5t}$
with initial conditions of

$$y(0) \; ; \; \dot{y}(0) = 0.$$

Find:

Determine an input s.t.

$$y(2) = 2 \; ; \; \int_0^2 y(t)\,dt = 3$$

Solution:

$$\int_0^2 (\overbrace{3e^{-2(2-\tau)} + 4e^{-5(2-\tau)}}^{y_1})\, x(\tau)\, d\tau$$

Where we define the inner product

$$\langle f, g \rangle = \int_0^2 f(\tau) g(\tau)\, d\tau$$

So our first inner product is $\langle x, y_1 \rangle = 2$

$$y_2 = \int_0^t h(\tau)\, d\tau = -\frac{3}{2} e^{-2t} - \frac{4}{5} e^{-5t} + 2.3$$

So we get $\langle x, y_2 \rangle$ and $y_2 = -\frac{3}{2} e^{-2(2-\tau)} - \frac{4}{5} e^{-5(2-\tau)} + 2.3$

We write the grammian

$$\begin{bmatrix} \langle y_1, y_1 \rangle & \langle y_1, y_2 \rangle \\ \langle y_1, y_2 \rangle & \langle y_2, y_2 \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow [c_1 \;\; c_2]^T = [0.15 \;\; 0.35]$$

see P3-28.m

3.14-29)

Given:

- $h[t] = (0.2)^t + 3(0.4)^t \quad t \geq 0$
- Initial conditions are 0

Find:

Determine a sequence $x[t]$ such that the output

$y[t] = h[t] * x[t]$ satisfies

1) $y[10] = 5$ 2) $\sum_{j=0}^{10} y[t] = 2$

and such that the energy $\sum_{j=0}^{10} |x[t]|^2$ is minimized.

Formulate this as a dual approximation problem and find the minimizing sequence $x[t]$.

Solution:

$$\sum_{j=0}^{10} h[j]\,x[j] = 5 \iff h^T x = 5$$

$$y_1 = \sum_{m=0}^{N-1} (0.2)^{10-m}\; 3(0.4)^{10-m}$$

define the inner product $\langle x, y_1\rangle = x^T y$ so we get

$$\langle x, y_1\rangle = 5 \checkmark$$

Let $y_2 = \sum_{j=0}^{10} h[t] = h[t]$

$$\langle x, y_2\rangle = 2 \checkmark$$

Write out the Grammian

$$\begin{bmatrix} \langle y_1, y_1 \rangle & \langle y_1, y_2 \rangle \\ \langle y_1, y_2 \rangle & \langle y_2, y_2 \rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$[c_1 \quad c_2]^T = [2.19 \quad 0.86]$$

See p3-29.m

3.15-30)

Given:

Minimize $x^H Q x$

s.t. $Ax = b$

Find:

Using the projection theorem, solve the finite dimensional problem.

Solution:

$x^H Q x$ is the weighted norm $\langle x, x \rangle_Q$. If we write A in terms of its rows we see

$$A = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Which we see will lead to

$$\langle y_1^H, x \rangle_Q = b_1 \qquad \langle y_2^H, x \rangle_Q = b_2 \quad \cdots \quad \langle y_m^H, x \rangle_Q = b_m$$

$$x = A^H Q c \implies (A^H Q A)^{-1} c = b$$

$$\implies x = A^H W (A^H Q A)^{-1} b$$

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3.15-31

$x \in S$; S is a Hilbert space

$\{x_1, x_2, \ldots x_m\}$ and $\{y_1, y_2, \ldots, y_m\}$ are sets of linearly independent vectors in S.

We desire to minimize $\|x - \hat{x}\|$ while satisfying

$$\hat{x} \in M = \text{span}(\{x_1, x_2, \ldots, x_m\})$$

and $\langle \hat{x}_i, y \rangle = c_i$, $i = 1, 2, \cdots, n$.

Find:

Find equations for the solution which are similar to the normal equations.

Solution:

$\langle \hat{x}_i, y \rangle = c_i$ implies that we are working with a linear variety

$$\Rightarrow \hat{x} = \sum_{i=1}^{m} c_i y_i \quad \Rightarrow \quad \begin{matrix} \langle \hat{x}, y_1 \rangle = c_1 \\ \langle \hat{x}, y_2 \rangle = c_2 \\ \vdots \end{matrix}$$

and saying $\langle x, y \rangle = 0$

We write

$$\begin{bmatrix} \langle y_1, y_1 \rangle & \langle y_2, y_1 \rangle & \cdots & \langle y_m, y_1 \rangle \\ \langle y_1, y_2 \rangle & \langle y_2, y_2 \rangle & \cdots & \vdots \\ \vdots & & & \vdots \\ \langle y_1, y_m \rangle & \cdots & & \langle y_m, y_m \rangle \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

3.17.32)

Given:

$$P_k[t] = \frac{1}{\sqrt{N}} e^{\frac{j2\pi kt}{N}}$$

is orthonormal in respect to the inner product

$$\langle x[t]\ y[t] \rangle = \sum_{t=0}^{N-1} x[t]\,\bar{y}[t]$$

Find:

Show that $P_k[t]$ is an orthonormal set

Solution:

$$\langle P_k, P_l \rangle = \sum_{t=0}^{N-1} \frac{1}{\sqrt{N}} e^{\frac{j2\pi kt}{N}} \frac{1}{\sqrt{N}} e^{\frac{-j2\pi lt}{N}}$$

$$= \sum_{t=0}^{N-1} \frac{1}{N} e^{\frac{j2\pi t}{N}(k-l)}$$

If $k = l \rightarrow \langle P_k, P_l \rangle = 1$ //

If $k \neq l \rightarrow \langle P_k, P_l \rangle = 0$ //

//

3.17-33)

Given

$g(t) = e^{-t/2}$ for $0 \le t \le \pi$ and

$$f(t) = \sum_k^{\infty} g(t - k\pi)$$

Find:

a) Fourier series coefficients of $f(t)$

b) The sum of the series

$$\sum_n \left(\frac{2^n}{\pi^2} \cdot \frac{1}{1+16n^2} \right)$$

Hint: Use Parseval's Theorem

Solution:

a) $f(t) = \sum_k g(t-k\pi) = \sum_k^{\infty} e^{-(t-km)/2}$

$b_n = \int_0^{\pi} e^{-\left(\frac{t-km}{2}\right)} e^{-jn\omega_0 t}\, dt \qquad \omega_0 = 2$

A function with period T_0 has the general formula

b) If we let the inner product be

$$\sum_t x[t]\,\bar{y}[t]$$

then by Parseval's theorem

$$\sum_{n=-\infty}^{\infty} a_n \bar{b}_n = \int_{-b}^{\pi} A(x)\,\overline{B(x)}\,dx$$

$$A(x) = \int_{-\infty}^{\infty} \frac{a^2}{\pi^2} e^{j2\pi t} \quad , \quad B(x) = \int_{-\infty}^{\infty} \frac{1}{1+16\pi^2} e^{j2\pi t}$$

$$= \frac{a^2}{\pi^2} \delta(f) \quad , \quad = \frac{\pi}{a} e^{-16f}$$

$$\int_0^{\pi} \frac{a^2}{\pi^2} \delta(f) \frac{\pi}{a} e^{-16f} \, df = 0 \quad //$$

2)

a) Show for $P_k(t) = \text{sinc}(2B(t - k/2B))$

Show $\langle P_k, P_l \rangle = \frac{1}{2B} \delta_{k,l}$. Also show

$$\int_{-\infty}^{\infty} \frac{\sin t}{t} \frac{\sin(t-z)}{t-z} dt = \frac{\pi \sin z}{z}$$

b) Show that the following is correct for bandlimited functions

$$\frac{\langle f, P_k \rangle}{\langle P_k P_k \rangle} = f\left(\frac{k}{2B}\right)$$

c) Show that if $f(t)$ is bandlimited to B Hz

$$f(t) = 2B \int_{-\infty}^{\infty} f(t) P_0(t-z) dt$$

Solution

a) $\int_{-\infty}^{\infty} \mathcal{F}[P_k] \overline{\mathcal{F}[P_l]} df$

$$= \int_{-\infty}^{\infty} \frac{1}{2B} e^{-j\frac{2\pi f k}{2B}} \text{rect}\left(\frac{f}{2B}\right) e^{j\frac{2\pi f l}{2B}} \frac{1}{2B} \text{rect}\left(\frac{f}{2B}\right) df$$

$$= \int_{-\infty}^{\infty} e^{j\frac{2\pi f}{2B}\overbrace{(l-k)}^{z}} \frac{1}{2B} \text{rect}\left(\frac{f}{2B}\right) \frac{1}{2B} \text{rect}\left(\frac{f}{2B}\right) df$$

$$= \int_{-\infty}^{\infty} \frac{\sin(\pi 2B(t - \frac{z}{2B}))}{\pi 2B(t - z/2B)} \frac{\sin(\pi 2Bt)}{\pi 2Bt} dt = \frac{1}{2B} \delta_{k,l}$$

convolution leaves us with the e^x value and we are left with the

I'm not quite sure how to prove

$$\int_{-\infty}^{\infty} \frac{\sin(t)}{t} \frac{\sin(t-z)}{t-z} dt = \frac{\pi \sin z}{z}$$

But it makes sense that our functions look like the sinc functions scaled by π. Now we are left with a convolution, so what is left over is

$$\frac{\pi \sin z}{z}$$

2/

b)

$$\int_{-\infty}^{\infty} P_k f \, dt = \int_{-\infty}^{\infty} \mathcal{F}[P] \mathcal{F}[f] \, df$$

$$= \frac{1}{2B} \int_{-B}^{B} e^{\frac{-j2\pi f k}{2B}} \operatorname{rect}\left(\frac{f}{2B}\right) F(f) e^{\frac{j2\pi f k}{2B}}$$

$$= \frac{1}{2B} \int_{-B}^{B} \operatorname{rect}\left(\frac{f}{2B}\right) F(f) e^{j2\pi f t} = f\left(\frac{k}{2B}\right)$$

c) From the previous result the inverse fourier gives

$$f(\tau) = \int_{-\infty}^{\infty} f\left(\frac{k}{2B}\right) \operatorname{sinc}\left(2B\left(\tau - \frac{k}{2B}\right)\right) dt$$

$$= 2B \int f(t) \, P_0(t-\tau) \, dt \quad //$$

3)

a) Determine $\frac{\partial}{\partial X} tr(XAX^T)$

b) Determine $\frac{\partial}{\partial X} tr(AX^TB)$

c) Use these derivatives to find

$$\frac{\partial J(w_0, w_1)}{\partial w_0} \quad \text{and} \quad \frac{\partial J(w_0, w_1)}{\partial w_1}$$

Where $J(w_0, w_1) = \| w_1 z_1[n] + w_0 z_0[n-1] \|^2$

Solution:

a) Using (9) from pg 907 (transpose and remove B)

$$\frac{\partial}{\partial X} tr(XAX^T) = XA^T + XA$$

b) Using (8) (transpose)

$$\frac{\partial}{\partial X}(B^T X A^T) = \frac{\partial}{\partial X}(AX^TB) = (BA)^T$$

c)

$$\|w_1 z_1 + w_0 z_0\|^2 = \langle w_1 z_1 + w_0 z_0, w_1 z_1 + w_0 z_0 \rangle$$

Let $\langle A, B \rangle = trace(AB^H)$

$$tr(w_1 z_1 z_1^T w_1^T + 2 w_1 z_1 z_0^T w_0^T + w_0 z_0 z_0^T w_0^T)$$

$$\underbrace{\underbrace{tr(w_1 z_1 z_1^T w_1)}_{AB_1} + \underbrace{tr(2 w_1 z_1 z_0^T w_0^T)}_{AB_2} + \underbrace{tr(w_0 z_0 w_0^T z_0^T)}_{AB_3}}_{AB}$$

$$\frac{\partial AB}{\partial w_1} = \underbrace{\frac{\partial trAB_1}{\partial w_1}}_{XAX^T} + \underbrace{\frac{\partial trAB_2}{\partial w_1}}_{\substack{A^TX \\ XA^T}} + \frac{\partial trAB_3}{w_1}$$

$$\frac{\partial J}{\partial W_0} = 2 W_1 z_1 z_0^T + W_0 (z_0^T z_0^T)^T + W_0 z_0 z_0^T$$

$$\frac{\partial J}{\partial W_1} = W_1 (z_1 z_1^T)^T + W_1 z_0 z_1^T + 2 z_1 z_1^T W_0^T$$