Homework 5 Brown, Alexander

2.11-54)

Given:

$$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

Find:

Range and Nullspace

Solution

$$RREF(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow R(A) = span\left(\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}\right)$$

$$N(A) = span([0, 0, 1]^T) \quad //$$

$$RREF(B) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow R(B) = span\left(\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = -x_3 \\ x_2 = x_1 = -x_3 \end{matrix}$$

$$N(B) = span([0 \quad 0 \quad 1]^T) \quad //$$

2.12-57)

Show that if V and W are subspaces of a vector space S, then their intersection $V \cap W$ is a subspace

Solution:

Theorem: The nonempty subset W of vector space V is a subspace of V iff it satisfies

1) $u, v \in W$ then $u+v \in W$
2) $u \in W$, $c \in \mathbb{R}$ then $cu \in W$

Let $u, v \in V \cap W \Rightarrow u, v \in V$ and $u, v \in W$

V is a subspace $\therefore$ $u+v \in V$
W is a subspace $\therefore$ $u+v \in W$

Therefore $u+v$ is in both V and W, or

$$u+v \in V \cap W$$

A similar argument can be made for scalar multiplication
Let $\alpha \in \mathbb{R}$

$$\alpha u \in V \; ; \; \alpha u \in W \Rightarrow \alpha u \in V \cap W$$

Therefore $V \cap W$ is a subspace of S //

2.12-60)

Show that

a) If V and W are orthogonal subspaces, then they are disjoint

b) If V and W are disjoint, then they are not necessarily orthogonal

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Solution

a) Def: Let S be a vector space S and let V and W be subspaces of S. V and W are orthogonal if every vector $v \in V$ and $w \in W$ are orthogonal; $\langle v, w \rangle = 0$.

Def: Two linear subspaces are disjoint if $V \cap W = \{0\}$. That is, the only vector they have in common is the zero vecter.

By the two definitions, an orthogonal set must be disjoint

Furthermore suppose V is a subspace and $W = V^{\perp}$, by theorem 2.5

If $x \in V \cap W$ then $x = 0$

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b) By Lemma 2.3:

Let V and W be subspaces of vector space S. Then for each $x \in V+W$ there is a unique $v \in V$ and $w \in W$ such that $v+w = x$ iff V and W are disjoint.

Proof

$$F = \{ N \subset X : N \text{ is a linear subspace of } X \text{ and } M \text{ and } M \cap N = \{0\} \}$$

Let $(N_\alpha)_{\alpha \in \Gamma}$ be a chain of linear subspaces in F and let

$$N = \text{span}\{ N_\alpha : \alpha \in \Gamma \}$$

We want to show $M \cap N = \{0\}$. Let $x \in M \cap N$, then

$$x = \sum_{k=1}^{n} a_k m_k$$

where $a_k \in \mathbb{C}$ and $m_k \in N_{\alpha_k}$. $(N_\alpha)_{\alpha \in \Gamma}$ is a chain of sets in F, there must be a $\beta \in \Gamma$ that contains all the elements $m_1, \ldots, m_n$. Thus, x is a linear combination of elements in N_β. $x \in M$ as well so $x \in M \in N_\beta = \{0\}$. and $x = 0$. Therefore N is an upper bound for the chain $(N_\alpha)_{\alpha \in \Gamma}$. By Zorn's Lemma, there is a maximal element $M' \in F$. Therefore

$$M \cap M' = \{0\}$$

Clearly $\mathcal{X} \supseteq \mathcal{M} \cap \mathcal{M}'$. Suppose $\mathcal{X} \not\subseteq \mathcal{M} + \mathcal{M}'$ $\exists$ $x \in \mathcal{X} \setminus (\mathcal{M} + \mathcal{M}')$ and furthermore $\mathcal{X} \neq 0$.

If $(m' + \lambda x) \in \mathcal{M}$ for some $m' \in \mathcal{M}'$ then $\lambda x \in \mathcal{M} + \mathcal{M}'$. But $\lambda = 0$ since $x \neq 0$ so $m' \in \mathcal{M}$ but $m' \in \mathcal{M} \cap \mathcal{M}' = \{0\}$, so $m' = 0$. Therefore $(\mathcal{M}' + \operatorname{span}(x)) \in \mathcal{F}$, but that contradicts the maximality of $\mathcal{M}$. Thus

$$\mathcal{X} = \mathcal{M} + \mathcal{M}'$$

Hence, $\mathcal{M}$ has an algebraic compliment //

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2.13-64)

If v is a vector, show that the matrix which projects onto span(v) is

$$P_v = \frac{vv^H}{v^H v}$$

Solution:

Let v be a vector in $v \in V$. Suppose

$x = vc$ s.t. c is an operator $P: S \to V$

We wish to minimize

$\min \|\hat{x} - x\|$ s.t. $\hat{x} \in V$

This is minimized when

$\langle \hat{x} - vc, v \rangle = 0$

$\langle \hat{x}, v \rangle - \langle vc, v \rangle = 0$

$\langle \hat{x}, v \rangle = \langle vc, v \rangle$

$\hat{x}^H v = c v^H v$

$c = \dfrac{v^H \hat{x}}{v^H v} \longrightarrow$ Plug back in $\quad x = vc = v \dfrac{v^H \hat{x}}{v^H v}$

$= \dfrac{vv^H}{v^H v} \hat{x}$ //

2.13-65)

Show the matrix

$$P_A = A(A^H A)^{-1} A^H$$

is a projection matrix

Solution

Def: A linear transformation P of a linear space into itself is a projection if $P^2 = P$.

$$P_A^2 = (A(A^H A)^{-1} A^H)^2 = (A(A^H A)^{-1} A^H)(A(A^H A)^{-1} A^H)$$

$$= A(A^H A)^{-1} \overbrace{\cancel{A^H A}}^{I}(A^H A)^{-1} A^H$$

$$= A\overbrace{\cancel{(A^H A)^{-1}(A^H A)}}^{I}(A^H A)^{-1} A^H$$

$$= A(A^H A)^{-1} A^H$$

//

2.13-66)

Given:

$$P_{A,W} = A(A^H W A)^{-1} A^H W$$

Find:

a) Show $P_{A,W}^2 = P_{A,W}$

b) Show $P_{A,W}^{\perp} = I - P_{A,W}$ is orthogonal to $P_{A,W}$ using the weighted inner product.

Solution:

a)

$$P_{A,W}^2 = A(A^H W A)^{-1} \underbrace{A^H W A(A^H W A)^{-1}}_{I} A^H W$$

$$= A(A^H W A)^{-1} A^H W$$

b) $\langle P_{A,W}, I - P_{A,W} \rangle = \langle P_{A,W}, I \rangle - \langle P_{A,W}, P_{A,W} \rangle$

$= P_{A,W} - P_{A,W}^2 = P_{A,W} - P_{A,W} = 0$

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2.13-67)

Given

$P_1 = [1\ 2\ 3\ 4]^T \quad P_2 = [4\ -2\ -6\ -7]^T$

$P_3 = [3\ 4\ -2\ 1]^T$

$x = [1\ -2\ \ 3\ 7]^T$

Find:

The nearest vector $\hat{x}$ in span$[P_1, P_2, P_3]$. Also determine the orthogonal compliment of x in span$[P_1, P_2, P_3]$

Solution:

Let $A = [P_1\ \ P_2\ \ P_3]$

We know $\quad P_A = A(A^H A)^{-1} A^H$

$\hat{x} = P_A x = A(A^H A)^{-1} A^H x$

$\hat{x} = [0.8946\ \ 2.9665\ \ 4.1247\ \ 5.6946]^T$

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow N(A) = \mathrm{span}([0\ 0\ 0\ 1]^T)$$

2.13-69)

Given:

Two orthogonal projections are said to be orthogonal if $P_A P_B = 0$. ($P_A \perp P_B$).

Find:

a) P_A and P_B are orthogonal iff their ranges are orthogonal.

b) $(P_A + P_B)$ is a projection operator iff P_A and P_b are orthogonal.

Solution:

Let $L : X \to Y$ be an operator. The range space $R(L)$ is

$$R(L) = \{ y = Lx : x \in X \}$$

The null space is

$$N(L) = \{ Lx = 0 ; x \in X \}$$

Let $R(P_A)$ and $R(P_B)$ be the ranges of P_A and P_B, respectively. Let $R(P_A) \perp R(P_B)$. Now suppose

$$P_A P_B x = \hat{x} \in R(P_A) \cap R(P_B)$$

If $R(P_A) \perp R(P_B)$ then $\hat{x} \in \{0\}$
$\therefore P_A P_B = 0$.

2/2

b) $(P_A + P_B)^2 = P_A + P_B$

$(P_A + P_B)(P_A + P_B) = P_A^2 + 2P_AP_B + P_B^2$

If $P_AP_B = 0$ then

$P_A^2 + P_B^2 = P_A + P_B$ //

2.13-71)

Given:

$P_1, P_2, \ldots, P_m$ are a set of orthogonal projections.

$$P_i P_j = 0 \quad \forall\ i,j = 1,2,\ldots,m \text{ and } i \neq j$$

Find:

Show $Q = P_1 + P_2 + \ldots + P_m$ is an orthogonal projection

Solution:

Need to show $Q^2 = Q$

$$Q^2 = (P_1 + P_2 + \ldots + P_m)(P_1 + P_2 + \ldots + P_m)$$

We know that $P_{ij} = 0$ for $i \neq j$. Therefore, all cross terms go to zero and we are left with

$$P_1^2 + P_2^2 + \ldots + P_m^2 = P_1 + P_2 + \ldots + P_m$$

2.13 - 73)

Given:

S is a vector space and let $V_1, V_2, \ldots, V_n$ be linear subspaces such that V_i is orthogonal to $\sum_{j \neq i} V_j$ for each i and where

$$S = V_1 + V_2 + \ldots + V_n$$

Let P_j be the projection on S for which $R(P_j) = V_j$ and $N(P_j) = \sum_{j \neq k} V_k$. Define an operator

$$P = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n$$

Find:

a) Show that if $x \in V_j$ then $Px = \lambda_j x$

b) Show that P is a projection iff λ_j is 0 or 1.

Solution:

$$V_i \perp \sum_{j \neq i} V_j \qquad ; N(P_j) = \sum_{j \neq k} V_k$$

$$R(P_j) = V_j$$

a) If $x \in V_j$ and we know $N(P_j) = \sum_{j \neq k} V_k$ then

$$Px = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_j P_j + \ldots + \lambda_n P_n)x$$
$$= \lambda_j P_j x$$

We know $R(P_j) = V_j$ and $x \in V_j$

so P_j is redundant

$$Px = \lambda_j x \quad //$$

b) $P = P^2 = (\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_n P_n)(\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_n P_n)$

We know that $V_i \perp \sum_{i \neq k} V_i$, so the cross terms go to 0

$$\lambda_1^2 P_1^2 + \lambda_2^2 P_2^2 + \lambda_3^2 P_3^2 + \dots + \lambda_n^2 P_n^2 = P^2$$

if $x \in V_j$ then

$$P^2 = \lambda_j^2 P_j^2$$

if $\lambda_j = 0$ then $x \notin V_j$ and $P = P^2 = \{0\}$

if $\lambda_j = 1$ then $x \in V_j$ and $P = P^2 = P_j$ //

2.15 - 80)

Given:

Let

$$A = [p_1 \ p_2 \ \cdots \ p_n] \in \mathbb{R}^{m \times n}$$

at the k^{th} step of the Gram-Schmidt algorithm

$$e_k = p_k - \sum_{i=1}^{k-1} \langle p_{k_i}, q_i \rangle q_i$$

with $q_k = e_k / \|e_k\|$. We can stack the results of the computations as

$$A = [q_1 \ldots q_n] \begin{bmatrix} \|p_1\| & \langle p_2, q_1 \rangle & \langle p_3, q_1 \rangle & \langle p_4, q_1 \rangle & \cdots & \langle p_n, q_1 \rangle \\ 0 & \|e_2\| & \langle p_3, q_2 \rangle & \langle p_4, q_2 \rangle & \cdots & \langle p_n, q_2 \rangle \\ \vdots & 0 & \|e_3\| & \langle p_4, q_3 \rangle & \cdots & \langle p_n, q_3 \rangle \\ \vdots & & & & & \\ \vdots & \cdots & \cdots & \cdots & \cdots & \|e_n\| \end{bmatrix}$$

$$= QR$$

where $Q^TQ = I$ (orthogonal), and R is the upper triangular matrix. The Gram-schmidt process works but is poor numerically. We will reformulate the problem to have better numerical properties. In this new formulation at each step a new column and row are obtained.

Let the ith row of the R matrix be denoted by r_i^T

find:

a) Show $A = \sum_{i=1}^{n} q_i r_i^T$

b) Let $A^{(k)}$ be an $m \times (n-k+1)$ matrix defined by

$$A - \sum_{i=1}^{k-1} q_i r_i^T = [\, 0_{m \times k-1} \quad A^{(k)} \,]$$

Show that

$$[\, 0_{m \times k-1} \quad A^{(k)} \,] = \sum_{i=k}^{n} q_i r_i^T$$

c) Write

$$A^{(k)} = [\, e_k \quad B^{(k)} \,]$$

Where $B^{(k)}$ is an $m \times n-k$ matrix. Then $q_k = e_k / \|e_k\|$, recall that when $k=1$, $e_k = p_1$ and observe that $A^{(1)} = A$).

Show that

$$q_k^T B^{(k)} = [\, r_{k,k+1} \quad r_{k,k+2} \quad r_{k,k+3} \; \cdots \; r_{k,n} \,]$$

d). Explain why we can move to the next step by computing

$$A^{(k+1)} = B^{(k)} - q_k [\, r_{k,k+1} \quad r_{k,k+2} \; \ldots \; r_{k,n} \,]$$

(Then we set $k \leftarrow k+1$ and return to (c))

e) Code the Graham-Schmidt Algorithm

Solution:

a)

$A = QR$ where

$$Q = [\,q_1 \; q_2 \; \cdots \; q_n\,]_{m\times n} \qquad R = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_n^T \end{bmatrix}_{n\times n}$$

$$A = [\,q_1 \; q_2 \; \cdots \; q_n\,] \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_n^T \end{bmatrix} = \sum_{i=1}^{n} q_i r_i^T \;//$$

b) We know R is an upper triangular matrix, so the first $k-1$ elements are 0. If we want to produce a row/column matrix starting at the k^{th} element then the matrix must be padded with $(m \times k-1)$ zeros. The $A^{(k)}$ matrix is defined as (a), but starting at $i=k$. Thus,

$$[\,0_{m\times k-1} \quad A^{(k)}\,] = \sum_{i=k}^{n} q_i r_i^T$$

Furthermore let us write out

$$\sum_{i=1}^{n} q_i r_i^T - \sum_{i=1}^{k-1} q_i r_i^T$$

Then we know the first $k-1$ elements are 0 leaving us with the summation $\sum_{i=k}^{n} q_i r_i^T$. The first $(m \times k-1)$ values are 0

so we (again) find that

$$[\,0_{m\times k-1} \quad A^{(k)}\,] = \sum_{i=k}^{n} q_i r_i^T$$

c) $q_k = \frac{e_k}{\|e_k\|}$ which means that it is normalized.

$B^{(k)} = [\,p_{k+1} \quad p_{k+2} \quad \cdots \quad p_n\,]$

$$q_k^T [\,p_{k+1} \quad p_{k+2} \quad \cdots \quad p_n\,] = [\,\langle p_{k+1}, q_k\rangle \quad \langle p_{k+2}, q_n\rangle \cdots \langle p_n, q_k\rangle\,]$$

$$= [\,r_{k,k+1} \quad r_{k,k+2} \quad \cdots \quad r_{k,n}\,]$$

d) Going to $[\,0_{m\times k-1} \quad A^{(k)}\,] = \sum_{i=k}^{n} q_i r_i^T$

If we increment k by 1 it is equivalent to

$$A^{(k)} = \sum_{i=k+1}^{n} q_i r_i^T + q_k r_k^T = B^{(k)}$$

$$\sum_{i=k+1}^{n} q_i r_i^T = B^{(k)} - q_k r_k^T = A^{(k+1)}$$

e) See gramschmidt.m