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Homework 4 Signals  
2.2-28)

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Given:

$S$  is a finite dimensional vector space with  $\dim(S) = m$ .

Find:

Show that every set containing  $m+1$  points is linearly dependent. (Use induction.)

Solution:

Theorem: A linear system of equations has either

- A unique solution
- No solution
- Infinitely many solutions



Theorem: Every homogeneous linear system with more variables than equations has infinitely many solutions.



Theorem: An ordered set of non-zero vectors,  $v_1, \dots, v_n$  is linearly dependent iff one vector  $v_k$  is expressible as a linear combination of the preceding vectors.



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Let  $c_i = 0 \forall i = 1, 2, \dots, m$  for the linearly independent set

$$c_1v_1 + c_2v_2 + \dots + c_{m+1}v_{m+1} + c_mv_m = 0$$

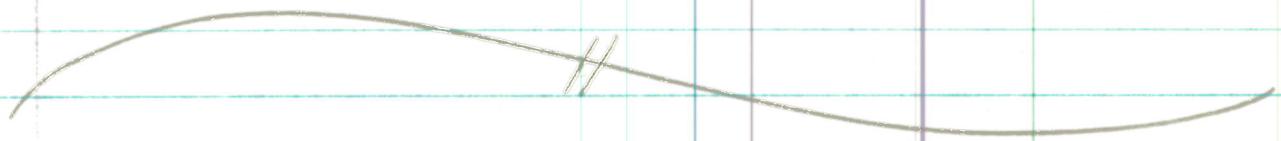
Now increase  $\dim(S) = m+1$ . We can then write

$$c_1v_1 + c_2v_2 + \dots + c_{m+1}v_{m+1} + c_mv_m + c_{m+1}v_{m+1} = 0$$

We know  $v_1 - v_{m+1}$  are linearly independent

$$c_1v_1 + c_2v_2 + \dots + c_{m+1}v_{m+1} + c_mv_m + -c_{m+1}v_{m+1} = 0$$

$$v_{m+1} = \frac{0}{c_{m+1}} \therefore c_{m+1} \in \mathbb{R} \quad (\text{infinitely many solutions})$$



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2.2 - 31)

Given:

$$S = L_2[0, 2\pi] \text{ and } T = \{x(t) = e^{jnt} : n=0, 1, \dots\}$$

Find:

- Show  $T$  is linearly independent.
- Conclude that  $L_2[0, 2\pi]$  is an infinite-dimensional space.

Hint: Assume that  $c_0e^{jn_0t} + c_1e^{jn_1t} + \dots + c_me^{jn_mt} = 0$  for  $n_i \neq n_j$  when  $i \neq j$ . Differentiate  $(m+1)$  times and use the properties of Vandermonde matrices (Section 8.4).

Solution:

Let

$$\frac{d}{dt}(e^{jnt}) = jn, e^{jnt} \quad \begin{matrix} \curvearrowleft \\ \curvearrowright \\ 2i \end{matrix}$$

$$\frac{d^2}{dt^2}(e^{jnt}) = (jn)^2 e^{jnt} \quad \begin{matrix} \curvearrowleft \\ \curvearrowright \\ 2i \\ \vdots \\ 0 \end{matrix}$$

$$\frac{d^{m+1}}{dt^{m+1}}(e^{jnt}) = (jn)^{m+1} e^{jnt} \quad \begin{matrix} \curvearrowleft \\ \curvearrowright \\ 2i \\ \vdots \\ m+1 \end{matrix}$$

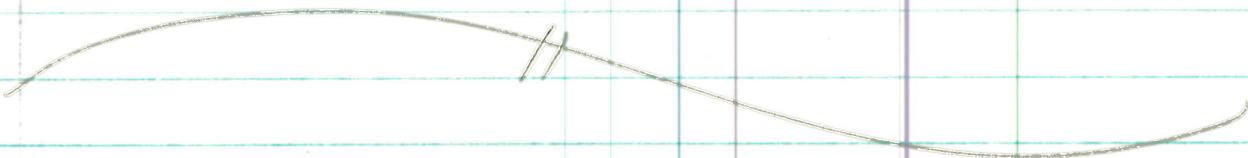
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$$V = \begin{bmatrix} 1 & z_1 & \dots & z_{m-1} \\ z_0 & z_1 & \dots & z_{m-1} \\ z_0^2 & z_1^2 & \dots & z_{m-1}^2 \\ \vdots & \vdots & & \vdots \\ z_0^{m-1} & z_1^{m-1} & \dots & z_{m-1}^{m-1} \end{bmatrix}$$

The determinant of  $V$ , which is of the form of a Vandermonde matrix, is

$$\det(V) = \prod_{\substack{i,j=1 \\ i>j}}^n (z_i - z_j)$$

$\therefore \det(V) \neq 0 \Leftrightarrow$  linearly independent



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2.2 - 32)

Given:

The set  $\{1, t, t^2, \dots, t^m\}$ 

Find:

Show the set is linearly independent.

Hint: The fundamental theorem of algebra states that a polynomial  $f(x)$  of degree  $m$  has exactly  $m$  roots, counting multiplicity.

Solution

Let

$$f(t) = a_0t^0 + a_1t^1 + a_2t^2 + \dots + a_mt^m = 0$$

To show linear independence, show that there are finite solutions. Rewrite the function as

$$a_0 + a_1t^1 + a_2t^2 + \dots + a_mt^m = 0$$

$$a_0 + t(a_1 + a_2t) + t^2(a_3 + a_4t) + \dots + t^{m-1}(a_{m-1} + a_mt) = 0$$

which has infinitely many roots, therefore the only set  $+a_i \neq 0 \forall i = 1 - m$  s.t.  $f(t) = 0$  is  $a_i = 0 \forall i = 1 - n$ .



$y_1$

Q.3 - 33)

Show that a normed linear space

$$\boxed{\|x\| - \|y\| \leq \|x-y\|}$$

Solution:

$$\|x-y\|^2 = \|x-y\| \|x-y\| = x^2 - 2xy + y^2$$

$$x^2 - 2xy + y^2 \geq (x - y)^2$$

$$\|x-y\|^2 \geq (\|x\| - \|y\|)^2$$

$$\|x-y\| \geq \sqrt{(\|x\| - \|y\|)^2}$$



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Q.3-34)

Show that a norm is a convex function

Solution:

Def A.5: A function  $f$  is convex over an open set

$D$  if  $\forall s, t \in D$

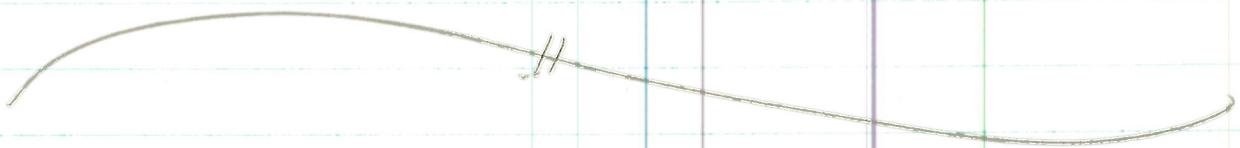
$$f(\lambda s + (1-\lambda)t) \leq \lambda f(s) + (1-\lambda)f(t)$$

Let  $V$  be a vector space  $\|\cdot\| : V \rightarrow \mathbb{R}$   
and let  $v, w \in V$

$$\|\lambda v + (1-\lambda)w\| \leq \|\lambda v\| + \|(1-\lambda)w\| = \lambda\|v\| + (1-\lambda)\|w\|$$

∴ By definition the norms are convex.

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Q3-35)

Show that every Cauchy sequence  $\{x_n\}$  in a normed linear space is bounded.

Solution:

Named Linear Space

- 1)  $\|x\| > 0$
- 2)  $\|x\| = 0 \Rightarrow x = 0$
- 3)  $\|\alpha x\| = |\alpha| \|x\|$
- 4)  $\|x+y\| \leq \|x\| + \|y\|$

If a sequence is Cauchy then given a metric space  $(X, d)$ , the sequence

$x_1, x_2, \dots$

is Cauchy  $\forall \epsilon > 0$  there is a positive integer  $N$  s.t.  $\forall m, n > N$ , the distance

$$d(x_m, x_n) < \epsilon$$

For normed linear spaces we have the induced norm  $d(x, y) = \|y - x\|$ . Thus, if a sequence is Cauchy, then the induced norm

$$\lim_{m, n \rightarrow \infty} d(x_m, x_n) = 0$$

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2.3 - 38)

Given:

 $S$  is a normed linear space

Find:

Show that the norm function

$$\|\cdot\|: S \rightarrow \mathbb{R}$$

is continuous.

Hint: See exercise 2.3-33

Solution

Let  $(X, \|\cdot\|)$  be a normed space. Let  $\epsilon > 0$  and  $\{x_n\}$  be an arbitrary sequence in  $X$  that converges to  $x \in X$ .

$$\exists N \in \mathbb{N}: n > N \Rightarrow \|x_n - x\| < \epsilon$$

$$\|x_n - x\| \geq |\|x_n\| - \|x\||$$

$$\exists N \in \mathbb{N}: n > N \Rightarrow |\|x_n\| - \|x\|| < \epsilon$$

$\therefore$  It is continuous //

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2.5 - 43)

Given:

Induced norm  $\|\cdot\|$  over a real vector space

Find:

Show that it is true for the following

$$\text{a)} \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$\text{b)} \quad \langle x, y \rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

Solution:

$$\text{a)} \quad \|x+y\|^2 + \|x-y\|^2$$

$$= \|x\|^2 + \|2xy\| + \|y\|^2 + \|x\|^2 - \|2xy\| + \|y\|^2$$

$$= 2\|x\|^2 + 2\|y\|^2 //$$

$$\text{b)} \quad \frac{\|x+y\|^2 - \|x-y\|^2}{4} = \frac{\|x\|^2 + 2\|xy\| + \|y\|^2 - \|x\|^2 - 2\|xy\| + \|y\|^2}{4}$$

$$= \frac{4\|xy\|}{4} = \|xy\| = \langle x, y \rangle //$$

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2.8-51)

Show by integration that

$$\int_{-1}^1 \frac{1}{\sqrt{1+t^2}} T_n(t) T_m(t) dt$$

$$\begin{cases} \int_0^\pi \\ 0 & n=m \\ 0 & n \neq m \end{cases}$$

$$n=m=0$$

$$n=m \neq 0$$

$$n \neq m$$

Hint: Use  $t = \cos(x)$  in the integral

Solution:

$$T_n(t) = \cos(n \cos^{-1}(t))$$

$$n=m=0 : T_0(t) = 1$$

$$\int_{-1}^1 \frac{1}{\sqrt{1+t^2}} dt = \sin^{-1}(t) \Big|_{-1}^1 = \sin^{-1}(1) - \sin^{-1}(-1) \approx \pi //$$

$$n=m \neq 0$$

$$\int_{-1}^1 \frac{1}{\sqrt{1-t^2}} T_n^2(t) dt$$

$$\text{let } t = \cos(x)$$

$$T_n(\cos(x)) = \cos(nx)$$

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$$\int_{-1}^1 \frac{1}{\sqrt{1 - \cos(x)^2}} \cos(nx)^2 dx$$

is the same as saying

$$\int_{-1}^1 \frac{t^2}{\sqrt{1 - t^2}} dt = \frac{1}{2} (\sin^{-1}(t) - t\sqrt{1-t^2}) \Big|_{-1}^1$$

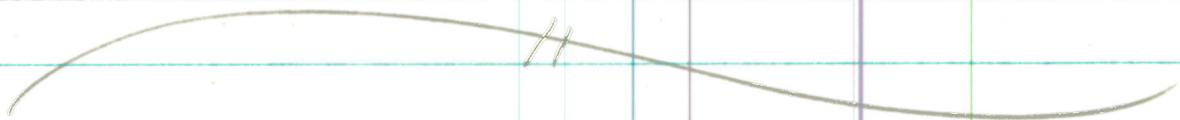
$$\approx \pi/2 //$$

$n \neq m$ :

Let  $m=0$

$$\int_{-1}^1 \frac{1}{\sqrt{1-t^2}} T_n(t) T_m(x) = \int_{-1}^1 \frac{1}{\sqrt{1-\cos(x)^2}} \cos(nx) \cdot 1 dx$$

$$\int_{-1}^1 \frac{t}{\sqrt{1-t^2}} dt = -\sqrt{1-x^2} \Big|_{-1}^1 = 0 //$$



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2.10 - 52)

Show that the orthogonal complement to a subspace is a subspace

Solution:

Let  $V$  be a subspace of  $S$

Show

- 1)  $V^\perp$  is nonempty
- 2) closed under scalar multiplication
- 3) closed under vector addition

1)  $V^\perp$  must at least contain  $\{0\}$ .

2) Let  $[P_1 - P_n] = V^\perp$

$$\alpha(P_i + P_j) \in \alpha P_i + \alpha P_j = 0 + 0 = 0 \in V^\perp$$

3)

$$P_i + P_j = 0 + 0 = 0$$



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2.3 - 39)

Given:

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2 \quad \textcircled{1}$$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty \quad \textcircled{2}$$

$$\|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty \quad \textcircled{3}$$

(a) : (b)

Find:

For each inequality determine a nonzero vector  $x^n \in \mathbb{R}^n$  for which each inequality is achieved (separately) with equality.

Solution

1)

1.a)  $x = [1 \ 0 \ 0 \ \dots]$

$$\sqrt{1^2 + 0^2 + \dots} = 1 + 0 + \dots$$

$$1 = 1$$

1.b)  $x = [1 \ 1 \ 1 \ 1]$

$$\|1+1+1+1\| = \sqrt{4} \cdot \sqrt{1^2 + 1^2 + 1^2 + 1^2}$$

$$4 = 2 \cdot 2 = 4$$

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2)

2.a)  $x = [2, 0 \dots]$

$$\max(x) = \sqrt{2^2 + 0 + \dots}$$

$$2 = 2$$

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2.b)  $x = [1, 1, 1, 1]$

$$\sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} \max(x)$$

$$\sqrt{4} = \sqrt{4}(1)$$

$$2 = 2$$

//

3)

3.a)  $x = [1, 0 \dots]$

$$\max(x) = |1 + 0 + 0 + \dots|$$

$$1 = 1$$

3.b)  $x = [1, 1, 1, 1]$

$$|1+1+1+1| = 4 \max(x)$$

$$4 = 4(1)$$

$$4 = 4$$

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A.7-27)

a) For  $x > 0$  and an integer  $n \geq 1$ , show that

$$(1+x)^n > 1+nx$$

Hint: Use binomial theorem

b) The geometric mean of a set of numbers  $\{z_i : i=1-n\}$  is defined by

$$G_n = \left( \prod_{i=1}^n z_i \right)^{1/n}$$

The arithmetic mean is the set of numbers

$$A_n = \frac{1}{n} \sum_{i=1}^n z_i$$

The following holds:

The geometric mean of  $n$  positive real numbers is less than or equal to its arithmetic mean with equality iff all the numbers are equal.

We show now that for  $x \geq -1$  and  $0 < \alpha < 1$

$$(1+x)^\alpha \leq 1 + \alpha x$$

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Assume  $\alpha = m/n$  we can write

$$(1+x)^\alpha = \underbrace{(1+x)(1+x)\dots}_{m \text{ factors}} \underbrace{1 \cdot 1 \cdot 1 \dots}_{n-m \text{ factors}}$$

Using the inequality relating the geometric and arithmetic means, show that

$$(1+x)^{mn} \leq \frac{m(1+x) + n - m}{n} = 1 + \frac{m}{n}x$$

c) Employ a continuity argument to extend the result +  $\alpha$  s.t.  $0 < \alpha < 1$

d) Finally, to establish (A), let  $x = y-1$   
Solution:

d)

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Let  $n=2, x=1$

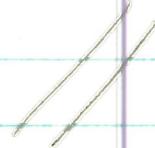
$$\begin{aligned} (1+1)^2 &> 1+2 \\ 4 &> 2 \end{aligned}$$

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Suppose  $(1+x)^n > 1+nx$

$$(1+x)^{n+1} \geq (1+nx)(1+x)$$

$$\begin{aligned}(1+nx)(1+x) &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x\end{aligned}$$



b)  $(1+x)^{m/n} = \underbrace{[(1+x)(1+x) \cdots (1 \cdot 1)]^{1/n}}$

Geometric mean with different values : //

$$(1+x)^{m/n} \leq \frac{m(1+x) + n-m}{n} = 1 + \frac{mx}{n}$$



c) Let  $\alpha = \frac{m}{n}$  and  $0 < \alpha < 1$  as was described in the problem statement

$$(1+x)^\alpha \leq 1 + \alpha x \quad \text{s.t. } 0 < \alpha < 1$$



d) Let  $x = y-1$

$$y^\alpha \leq 1 + \alpha(y-1) //$$



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A.7-29)

Given:

$$\left| \int_a^b f(t)g(t)dt \right| \leq \left( \int_a^b |f(t)|^p dt \right)^{\frac{1}{p}} \left( \int_a^b |g(t)|^q dt \right)^{\frac{1}{q}}$$

s.t.  $p, q > 0$  and  $\frac{1}{p} + \frac{1}{q} = 1$

Find:

Prove the above inequality and integrate from a to b.

Hint:

$$A = \frac{|f(t)|^p}{\int_a^b |f(t)|^p dt} \quad B = \frac{|g(t)|^q}{\int_a^b |g(t)|^q dt}$$

Solution:

Begin with Young's Inequality for Products

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q} = \frac{|f|^p}{p \|f\|_p^p} + \frac{|g|^q}{q \|g\|_q^q}$$

$$\int_a^b AB dx = \frac{\|f\|^p}{p \|f\|_p^p} + \frac{\|g\|^q}{q \|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \frac{|f(t)|^p}{p \|f(t)\|_p^p} + \frac{|g(t)|^q}{q \|g(t)\|_q^q} \geq \frac{f(t)g(t)}{\|f(t)\|_p \|g(t)\|_q}$$

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A.7-32)

Given:

$$\ln x \leq x - 1$$

Find:

Prove the "information theory inequality"

Solution:

$$\text{Let } f(x) = \ln(x) - x + 1$$

$$f'(x) = \frac{1}{x} - 1 = 0 \Rightarrow x = 1$$

$$f''(x) = -\frac{1}{x^2} < 0 \quad \forall x > 0 \therefore \text{Max point}$$

at  $x = 1$

$$\ln(x) = x - 1$$

$\ln(x) < x - 1$  everywhere else

