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Homework 9 Signals

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6.23)

Given:

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

Find:

Show $\lambda(T) = \lambda(T_{11}) \cup \lambda(T_{22})$

Solution:

$$\det(T - \lambda I_n) = \det \begin{pmatrix} T_{11} - \lambda I_n & T_{12} \\ 0 & T_{22} - \lambda I_n \end{pmatrix}$$

$$= \det(T_{11} - \lambda I_n) \cdot \det(T_{22} - \lambda I_n) + 0$$

Thus $\lambda(T) = \lambda(T_{11}) \cup \lambda(T_{22}) //$

6.2-4) Given:

The determinant of an $n \times n$ matrix is the product of the eigenvalues.

$$\det(A) = \prod_{i=1}^n \lambda_i$$

Solution

If A and B are similar matrices, then they have the same eigenvalues.

or Σ in the SVD

That is if we can row reduce^V a matrix A to have values only on the diagonals, call it B , then the following is true

A and B are similar matrices \therefore

$$\det(A) = \det(B) = \prod_{i=1}^n \lambda_i$$

Where λ_i are elements on the diagonal of B .

6a-7) Given:

A is a rank 1 matrix formed by $A = ab^T$

Find:

- Determine the eigenvalues and eigenvectors of A.
- Also show that if A is rank 1 then

$$\det(I + A) = 1 + \text{tr}(A)$$

Solution:

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} [b_1 \ b_2 \ \dots \ b_n] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & a_n b_2 & \dots & a_n b_n \end{bmatrix}$$

Because A is rank 1 we know $\lambda = 0$ with $n-1$ multiplicity. We can also say

$$Aa = (ab^T)a = \underbrace{a}_{x} \underbrace{(b^T a)}_{\lambda}$$

the eigenvector $x = a$ and eigenvalue $\lambda = b^T a //$

$$\begin{aligned} \text{Thus } \det(I + A) &= 1 + \det(A) = 1 + \begin{bmatrix} b^T a & 0 \\ 0 & \ddots & 0 \end{bmatrix} \\ &= 1 + \text{tr}(A) // \end{aligned}$$

11

62-8) Given:

λ^* is an eigenvalue of matrix A.

Find:

Show $\lambda^* + r$ is an eigenvalue of $A + rI$ and
 $A + rI$ and A have the same eigenvectors

Solutions:

$$\text{Begin with } Av = \lambda v = \lambda^* v$$

If we add rV to both sides

$$(A + rI)v = (\lambda^* + r)v //$$

$$\lambda = \lambda^*$$

$$\lambda = \lambda^* + r$$

$$Av = \lambda^* Iv //$$

$$(A - \lambda^* I - rI)v = 0$$

$$Av = \lambda^* Iv$$

62-9) Given:

If λ is an eigenvalue of A then λ^n is an eigenvalue of A^n , and A^n has the same eigenvectors as A .

Find:

Show the statement above is true.

Solution:

$$\det(A - \lambda I) = 0 \Rightarrow \lambda$$

Let A be diagonal, that is $a_{ii} = \lambda_i$

Thus A^n has eigenvalues $\lambda_i^n //$

$$A^n v = \overset{n-1}{\underset{A}{\overbrace{A(Av)}}} = \overset{n-1}{\underset{\lambda}{\overbrace{A\lambda v}}} = \lambda^n v //$$

1/1

6.2-10) Given:

If λ is a nonzero eigenvalue of A then $1/\lambda$ is an (1)
eigenvalue of A^{-1}

The eigenvectors of A corresponding to nonzero eigenvalues (2)
are eigenvectors of A^{-1}

Find:

Show (1) and (2) are true

Solution:

$$\overbrace{Av}^{\text{I}}$$

$$Av = \lambda v = A^{-1}A v = \lambda A^{-1}v$$

$$\Rightarrow A^{-1}v = \frac{1}{\lambda}v \quad //$$

This shows (1) and (2) by

a) relating $A^{-1}v = 1/\lambda v$ and

b) by pre multiplying by A^{-1} we have shown the
equation is equivalent, thus v is the same for

$$Av = \lambda v \quad \text{and} \quad A^{-1}v = \frac{1}{\lambda}v$$

//

6(II) Given

- $\lambda_1, \lambda_2, \dots, \lambda_m$ are eigenvalues of A .
- $g(x)$ is a scalar polynomial.

Find:

$g(A)$ has eigenvalues of $g(\lambda_1), g(\lambda_2), \dots, g(\lambda_m)$

Solution

Theorem: $A \in M_n(F)$ where F is algebraically closed.

If the spectrum of A is $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ and $p \in F[x]$ is a polynomial with coefficients in F , then

$$\det(p(A)) = \prod_{i=1}^n p(\lambda_i)$$

By the theorem

$$\begin{aligned}\det(g(A) - yI) &= \det(f(A)) = \prod_{i=1}^n f(\lambda_i) \\ &= \prod_{i=1}^n (y - p(\lambda_i)) //\end{aligned}$$

Thus, $\{p(\lambda_1), p(\lambda_2), \dots, p(\lambda_n)\}$ is in the spectrum of $p(A)$.

6.2-12) Given: Projection matrix P

Find:

Show that the eigenvalues of a projection matrix
is either 0 or 1.

Solution:

For projection matrices $P^2 = P$

$$\lambda^2 v = P^2 v = Pv = \lambda v$$

$$\Rightarrow \lambda^2 = \lambda \Rightarrow \lambda = 0, 1 //$$

1/2

6.2-13)

Given:

A and B are square matrices

Find:

a) $\text{eig}(BA) = \text{eig}(AB)$

b) A & B have n linearly independent vectors. Then

$AB = BA$

Solution:

- a) We know similar matrices have the same eigenvalues, then we can say

$$AB = ABAA^{-1} = A(BA)A^{-1}$$

which says AB and BA are similar. \therefore

$$\text{eig}(AB) = \text{eig}(BA) //$$

b)

$$Av = \lambda Iv = Bv = ABx = BAx \quad \text{s.t. } v = Bx$$

Let $\{x_1, \dots, x_n\}$ be an eigenbasis of A
and let $v \in \mathbb{C}^n$

$$B\{x_1, \dots, x_n\} = \{Bx_1, \dots, Bx_n\} = x \begin{bmatrix} C & D \\ 0 & F \end{bmatrix}$$

2/2

$$\text{Let } P = \{x_1, \dots, x_n\} \Rightarrow P^{-1}BP = \begin{bmatrix} C & D \\ 0 & F \end{bmatrix}$$

$$\det(B - \lambda I) = \det(P^{-1}BP - \lambda I)$$

$$= \begin{bmatrix} C - \lambda I & D \\ 0 & F - \lambda I \end{bmatrix} = \det(C - \lambda I) \det(F - \lambda I)$$

If μ is an eigenvalue and a is an eigenvector, Thus

$$\det(L - \mu I) = 0 \Rightarrow \det(B - \mu I) = 0$$

such that μ produces the eigenvector \underline{a} . Now define

$$y = a_1x_1 + \dots + a_kx_k \in E_A$$

$$BP = P \begin{bmatrix} C & D \\ 0 & F \end{bmatrix} \Rightarrow \underbrace{BP \underline{a}}_{By} = P \begin{bmatrix} C & D \\ 0 & F \end{bmatrix} \underline{a} \underbrace{\underline{a}}_{\mu y}$$

$$\Rightarrow By = \mu y$$

Thus y is an eigenvector of A and B , //

6.2-15) Given:

a) $A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0 & 0.7 \\ 0.3 & 0.7 & 0.1 \end{bmatrix}$ Probability transition matrix

b) $\phi(t) = \sum_{k=0}^{N-1} c_k \phi(2t-k)$

$\phi(t) = 0$ if $t \leq 0$ and $t \geq N-1$

Find:

a) Determine steady-state probability p such that
 $Ap = p$

b) Write an equation of the form

$$\begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(N-2) \end{bmatrix} = A \begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(N-2) \end{bmatrix}$$

A is the matrix of wavelet coefficients c_k .

Specify A and describe how to solve the equation.

Describe how to find $\phi(t)$ and all dyadic rational numbers

Solution

a) $\text{eig}(A)=1$ and $Av = \lambda v = v$

$$\lambda = 1 \quad (A - I)v = 0$$

$$\Rightarrow \begin{bmatrix} -0.5 & 0.3 & 0.2 \\ 0.2 & -1 & 0.7 \\ 0.3 & 0.7 & -0.9 \end{bmatrix} v = 0$$

see P15,m

$$v = [0.57199 \quad 0.54469 \quad 0.61384]^T //$$

b) $\phi(t)$ follows the shift orthogonality property. That is

$$\phi(t) \perp \phi(t-n) \text{ s.t. } n \in \mathbb{Z}$$

Thus

$$A = \begin{bmatrix} c_1 & & & \\ & c_2 & 0 & \\ 0 & \ddots & & \\ & & c_{n-2} & \end{bmatrix}$$

Thus we have the eigenvectors of A . Similarly to before we can say

$$(A - I\lambda)v = 0 \text{ where } v = \begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \end{bmatrix}$$

3/3

and we can substitute c_k for λ . //



6.3-18) Given:

$$A \text{ and } B \text{ are similar } B = T^{-1}AT$$

Find:

- A and B have the same eigenvalues and the same characteristic equations
- If x an eigenvector of A then $z = T^{-1}x$ is an eigenvector of B.
- If C and D are similar with $D = T^{-1}CT$ then $A+C$ is similar to $B+D$

Solution:

a,b)

$$B = T^{-1}AT \Leftrightarrow TBT^{-1} = A$$

$$TBT^{-1}v = \lambda v \Rightarrow BT^{-1}v = AT^{-1}v$$

Thus if v is an eigenvector of A with eigenvalue λ $T^{-1}v$ is an eigenvector B with the same eigenvalue.

$$\begin{aligned} c) B &= T^{-1}AT \Rightarrow B+D = T^{-1}AT + T^{-1}CT \\ &\quad ATC = TBT^{-1} + TDT^{-1} \end{aligned}$$

$$(A+C)v = \lambda v = (TBT^{-1} + TDT^{-1})v = \lambda v$$

$$= (BT^{-1} + DT^{-1})v = \lambda T^{-1}v$$

Which raises the same argument as before //

1/8

6.3-21) Given:

A self-adjoint matrix is positive definite iff all of its (1)
eigenvalues are ≥ 0

Find:

- Show (1) is true
- Also show that if all the eigenvalues are positive,
then the matrix is positive definite.

Solution:

A self adjoint matrix has the property

$$A^* = A^\dagger$$

A real matrix that is said to be self-adjoint is said to
be symmetric.

A symmetric matrix is always diagonalizable, thus its
eigenvectors form a basis. Let the set of eigenvectors
be $\{e_1, \dots, e_n\}$ be an orthonormal basis corresponding
to the eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

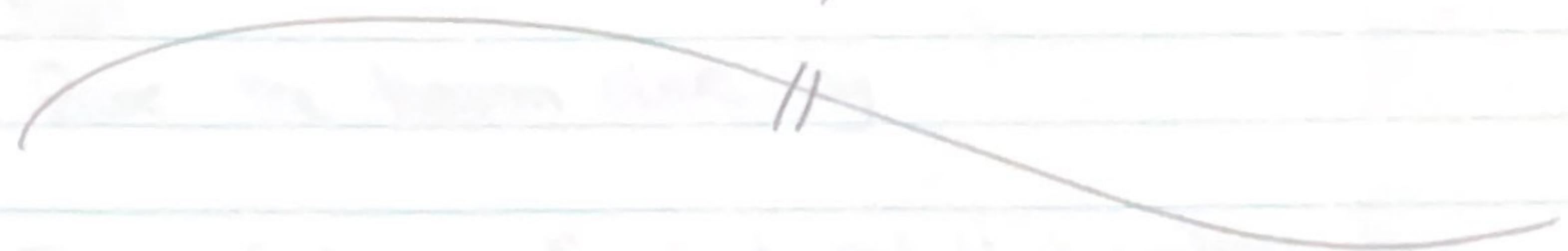
$$x = c_1 e_1 + c_2 e_2 + \dots + c_n e_n$$

$$\begin{aligned} \text{So } x^T A x &= [c_1 e_1^T + \dots + c_n e_n^T] \underbrace{\frac{1}{2} (A + A^\dagger)}_{= I} [c_1 e_1 + \dots + c_n e_n] \\ &= c_1^2 \lambda_1 + \dots + c_n^2 \lambda_n \geq 0 // \end{aligned}$$

2/2

Similarly, if the eigenvalues are strictly positive, then we can say

$$c_1^2\lambda_1 + \dots + c_n^2\lambda_n > 0 //$$



1/2

6.3-24) Given:

For a self-adjoint matrix, the eigenvectors corresponding to distinct eigenvalues are orthogonal.

Find:

Prove the theorem above by

- a) Show if A is self-adjoint and U is unitary, then $T = U^H A U$ is also self-adjoint
- b) Show that if a self-adjoint matrix is triangular, then it must be diagonal.

Solution

- a) A matrix is unitary if

$$U^H U = I = U U^{-1} = U U^H$$

A matrix is self-adjoint if $A = A^H$

$$T^H = U^H A U \quad \text{and} \quad T^H = T T^H = T^2$$

$$T T^H = U^H A U U^H A^H U = U^H A^2 U = T^2 //$$

- b) If $A = A^H$ and is triangular, suppose

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{which is triangular.}$$

2/2

$A \neq A^H$ unless $a = 0$, thus A must be diagonal.
Extrapolating, any $n \times n$ matrix

$$A = \begin{bmatrix} 1 & \cdots & a \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

For $A^H = A$, a must be 0, similarly for any other off-diagonal terms.

1/2

6.3-28) Given:

$$F = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e^{-j2\pi/N} & \dots & e^{-j2\pi(N-1)/N} \\ 1 & e^{-j4\pi/N} & \dots & e^{-j4\pi(N-1)/N} \\ \vdots & & & \vdots \\ 1 & e^{-j2\pi(N-1)/N} & \dots & e^{-j2\pi(N-1)^2/N} \end{bmatrix}$$

that if $\tilde{x}_{ii} = e^{-j2\pi i^2/N}$, for a vector \underline{x}

$$\underline{x} = F \underline{\tilde{x}}$$

find:

a) Prove F/\sqrt{N} is unitary. Hint: Show

$$\sum_{n=0}^{N-1} e^{j2\pi n k / N} = \begin{cases} N & k = 0 \pmod{N} \\ 0 & k \neq 0 \pmod{N} \end{cases}$$

b) A matrix

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \dots & c_{N-2} \\ \vdots & & & & \\ c_1 & c_2 & c_3 & \dots & c_0 \end{bmatrix}$$

↙ circulant matrix

Show C is diagonalizable by F , $CF = F\Lambda$ where Λ is diagonal. Comment on the eigenvalues and eigenvectors of a circulant matrix.

2/2

Solution:

a)

$$\sum_{n=0}^{N-1} e^{j\theta \sin k n} \quad \text{if } k \equiv 0 \pmod{N} = 0$$

$$\Rightarrow \sum_{n=0}^{N-1} 1 = N$$

If $k \not\equiv 0 \pmod{N}$ then

$$\sum_{n=0}^{N-1} e^{j\theta \sin k n} \quad \text{by Example 3.17.2 forms an}$$

orthonormal set thus is 0. for $n=m$.Therefore if we let $k \equiv 0 \pmod{N}$ then

$$F = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & & & \\ 1 & 1 & \dots & 1 \end{bmatrix} \Rightarrow \frac{F}{\sqrt{N}} = \begin{bmatrix} \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \dots & \frac{1}{\sqrt{N}} \\ \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \dots & \frac{1}{\sqrt{N}} \\ \vdots & & & \\ \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \dots & \frac{1}{\sqrt{N}} \end{bmatrix}$$

$$\Rightarrow \text{Let } \frac{F}{\sqrt{N}} = B \text{ and } BB^* = BB^H = I //$$

$$b) CF = FA \Rightarrow C = F \lambda F^{-1}$$

$\Rightarrow \lambda = F^{-1}CF$ which is said to be a diagonalization of C by F . Thus the eigenvectors are not linearly dependent and the eigenvalues are the diagonal terms. //

11

6.4-30) Given:

Construct 3×3 matrices given the following conditions

Find:

a) $\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$

$$R_1 = \text{span}([1, 2, 1]^T, [-2, 1, 0]^T)$$

$$R_2 = \text{span}([-1, -2, 5]^T)$$

Find eigenvalues and vectors of constructed matrix.

b) $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 9$ with corresponding eigenvectors

$$x_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \frac{1}{\sqrt{40}} \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}$$

Solution:

a) See Q30.m.

b) See P30.m



6.7-39) Given:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 7 \end{bmatrix} \quad (1)$$

Find:

Determine regions in the complex plane where the eigenvalues of (1)

Solution:

$$R_i(A) = \left\{ x \in \mathbb{C} : |x - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \right\}$$

$$|x - 3| \leq 2 ; |x - 4| \leq 3 ; |x - 7| \leq 3$$



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6.7-41) Given:

Real mxm matrix with disjoint Gershgorin circles

Find:

Show the eigenvalues of A are real

Solution:

Let $a \in A$ and λ be an eigenvalue of A . Let disk g be created from any row in A and let g be disjoint from all other disks. If

$$\lambda = x + iy$$

and $x, y \neq 0$ then another eigenvalue of A is

$$\lambda = x - iy$$

since it is in g . However that means that there are 2 eigenvalues in g which contradicts the theorem that states if the discs are disjoint, then there is only 1 eigenvalue.

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6.8-45) Given:

$$x_1 = [1, 5]^T \quad x_2 = [-5, 1]^T$$

$$\lambda_1 = 10, \lambda_2 = 2$$

find

- a) Construct a symmetric matrix with the properties above
- b) Generate and plot 200 points of zero-mean Gaussian data that have covariance R
- c) Form an estimate of R
- d) Plot the principal components.

Solution:

See P45.m