Homework 1

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1

Thoughtfully read

2

Nonlinear Feedback Design for Fixed-Time Stabilization of Linear Control Systems

The title of the paper is quite colorful in the description of the paper as it paints the full picture. It introduces finite time controllers for global-finite time stability of closed loop systems and a guaranteed convergence time of system trajectories into a neighborhood. These controllers are of two different "types". Having taken nonlinear control and optimal control that math does not look foreign, but there is a whole lot in this paper. With time I could probably get through it.

Cooperative Output Feedback Tracking Control of Stochastic Linear Heterogeneous Multiagent Systems

This paper is an odd one. It has a set of heterogeneous leaders and followers. Each agent has incomplete measurable dynamics, so they propose a set of observation strategies to be able to predict the states of the collective. Again, this is a long paper with a lot of dynamics. I have taken a multiagent systems course, optimal estimation, and the previously mentioned courses in control. With time, I could probably get though the paper.

Time-Variation in Online Nonconvex Optimization Enables Escaping From Spurious Local Minima

This paper alludes me. I understand what meant when they throw out nonconvex optimization, stability analysis, and time-varying optimization, but that is about as far as I go. I believe the goal is to use something similar to that of a gradient search for time dependent systems. They are allowing the gradient to vary with time and include inertial terms as to not allow the line the algorithm is tracking to be immediately swayed by any change to the system.

3

Problem Statement

Show that the following are convex:

1. The set of $n \times n$ Toeplitz matrices
2. The set of monic polynomials of the same degree
3. The set of symmetric matrices

Solution

Begin by defining what a complex set is:

> A set S is convex if for any two points $p, q \in S$, then all points of the form
>
> $$\lambda p + (1 - \lambda)q$$
>
> for $0 \leq \lambda \leq 1$, are also in S.

3.1: Toeplitz

Begin by defining what a Toeplitz matrix is

> A Toeplitz matrix is a diagonal-constant matrix, which means all elements along a diagonal have the same value. For a Toeplitz matrix A we have $A_{ij} = a_{i-j}$ which results in the form
>
> $$\begin{bmatrix} a & b & c & \cdots \\ e & a & b & \cdots \\ f & e & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Consider Toeplitz matrices P and Q of dimension $n \times n$ as let S be the set of $n \times n$ Toeplitz matrices. Lets now apply the definition of the convex set:

$$\lambda P + (1 - \lambda)Q$$

There are two operations being applied to the matrices: addition

> If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same size, then their sum $A + B$ is the matrix obtained by adding the corresponding elements of the matrix A and B.

and multiplication of a matrix by a number

> If $A = [a_ij]$ is a matrix and c is a number, then cA is the matrix obtained by multiplying each element of A by c.

Therefore, we can make the statements

- Any Toeplitz matrix multiplied by a scalar is also Toeplitz
- Any two $n \times n$ Toeplitz matrices being added together is also Toeplitz

The following statement can then be made: The set of $n \times n$ Toeplitz matrices is convex.

$$\lambda P + (1 - \lambda)Q \in S$$

3.2: Monic Polynomials of Degree n

Begin by defining what a monic polynomial is

> A polynomial is monic if the coefficient of the highest order term is 1.

Suppose p and q are monic polynomials of degree n and S is the set of all monic polynomials of degree n. It can be written

$$\lambda p(x) + (1 - \lambda)q(x) \in S$$

Expanding the above gives

$$\begin{aligned}&\lambda(x^n - ax^{n-1} + \cdots + bx + c) + (1 - \lambda)(x^n + dx^{n-1} + \cdots + ex + f)\\&= 2\lambda x^n + (a + dx^{n-1}) + \cdots\end{aligned}$$

Dividing by 2λ produces another monic polynomial of degree n. Therefore, the set is convex, i.e $\lambda p(x) + (1 - \lambda)q(x) \in S$.

3.3: Symmetric Matrices

Define what a symmetric matrix is

> A matrix A is symmetric $\iff A = A^T$.

Similarly to 3.1,

- Let P and Q are $n \times n$ symmetric matrices
- S is the set of $n \times n$ symmetric matrices
- Any symmetric matrix multiplied by a scalar is also symmetric
- Any two $n \times n$ symmetric matrices being added together is also symmetric

Therefore, S is convex because $\lambda P + (1 - \lambda)Q \in S$.

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Problem Statement

The set of even integers can be represented as $2\mathbb{Z}$. Show that $|2\mathbb{Z}| = |\mathbb{Z}|$. Similarly show that there are as many odd integers as there are integers.

Solution

Let S and T be two different sets. T and S have the same cardinality if there is a bijection f from S to T. Therefore, we need to show $f : \mathbb{Z} \to 2\mathbb{Z}$. Let the mapping $f(n)$ be defined as

$$f(n) = 2n$$

It now needs to be shown that $f(n)$ is both one-to-one and onto. To show that $f(n)$ is one-to-one begin by defining how to show a mapping is one-to-one

> A function f from A onto B is one-to-one if each element of B has at most one element of A mapped into it. That is, $f(x) = f(y)$, then $x = y$.

From this if we suppose $f(a) = f(b)$, then $2a = 2b$ so $a = b$. Thus, f is one-to-one. Now we need to show f is onto. Begin by defining onto

> A function is onto if each element of B has at least one element of A that is mapped into it. That is, $\forall b \in B$ there is an $a \in A$ such that $f(a) = b$.

Take $b = 2n$ for some a, then $f(n) = 2n = b$ which shows that f is onto. Therefore, $f(n)$ is a bijection and $|\mathbb{Z}| = |2\mathbb{Z}|$.

Similarly, for the odd we need to show $f : \mathbb{Z} \to 2\mathbb{Z} + 1$ is a bijection. To show $f(n)$ is one-to-one let $f(a) = f(b)$, then $2a + 1 = 2b + 1$, so $a = b$. To show f is onto let $b = 2n + 1$, then $f(n) = 2n + 1 = b$. Therefore, $f(n)$ is a bijection and $|\mathbb{Z}| = |2\mathbb{Z} + 1|$.

5

Problem Statement

Show that $|(0, 1]| = |\mathbb{R}|$.

Solution

A simple way to go about this is to first show that $\big|[0, 1)\big| = \big|[-\pi/2, \pi/2)\big|$. Suppose $f(x) = \pi x - \pi/2$. To show that $f(x)$ is one-to-one

$$\begin{aligned}
&f(x) = f(y) \\
&\pi x - \pi/2 = \pi y - \pi/2 \\
&\pi x = \pi y \\
&x = y
\end{aligned}$$

Therefore, $f(x)$ is one-to-one. Now to show that $f(x)$ is also onto.

$$
\begin{aligned}
&f(x) = y \\
&\pi x - \pi/2 = y \\
&x = y/\pi + 1/2
\end{aligned}
$$

And because we know that $0 < x \leq 1$ we can show that x written above is in that range by saying

$$
\begin{aligned}
&-\pi/2 < y \leq \pi/2 \\
&-1/2 < y/\pi \leq 1/2 \\
&0 < y/\pi + 1/2 \leq 1
\end{aligned}
$$

Therefore, the function is also onto. Now to show that $\big|[-\pi/2, \pi/2)\big| = \big|\mathbb{R}\big|$. Let $g(x) = tan(x)$ it can be shown that $tan(x)$ is always increasing.

> **Fact**: If $g(x)$ is always increasing, then $g(x)$ is one-to-one.

By taking the derivative of $g'(x) = sec^2(x) > 0$, therefore $g(x)$ is one-to-one. To show that $g(x)$ is onto, we will use the intermediate value theorem

> If $g(x)$ is continuous on an interval $[a, b]$, then $g(x)$ contains all the values between $g(a)$ and $g(b)$.

Let the range of interest be $[-\pi/2 + \epsilon, \pi/2 - \epsilon]$. $g(x)$ is continuous within the range, therefore it obtains all values $g(-\pi/2 + \epsilon)$ to $g(\pi/2 - \epsilon)$. If we let $\epsilon \to 0$ then $g(x) \to \mathbb{R}$. Therefore, $\big|(0, 1]\big| = \big|\mathbb{R}\big|$.

6

Problem Statement

Show that the intersection of a convex set is convex.

Solution

Let A and B be two convex sets, and let $C = A \cup B$. Now let $p, q \in C$.

- If $p, q \in C$ then $p, q \in A$ and A is convex
- If $p, q \in C$ then $p, q \in B$ and B is convex
- Therefore C must be complex

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Problem Statement

If S and T are convex sets both in $\mathbb{R}^n$, show that the set sum is convex.

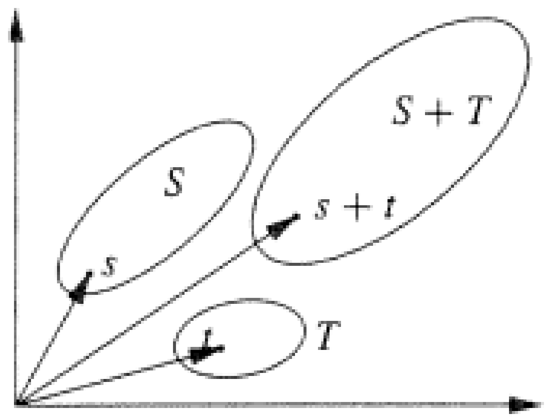

Figure A.7: The set sum.

Solution

The set sum is defined as

$$S + T = \{x : x = s + t, s \in S, t \in T\}$$

Let S and T be convex sets and $S+T \in C$, let $s_1, s_2 \in S$ and $t_1, t_2 \in T$, and let $s = s_1 + t_1$ and $t = s_2 + t_2$, then

$$\lambda s + (1-\lambda)t\lambda s_1 + \lambda t_1 + s_2(1-\lambda) + t_2(1-\lambda)\lambda s_1 + (1-\lambda)t_1 + \lambda s_2 + (1-\lambda)t_2 \in C$$

Therefore, the set sum is convex.

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Problem Statement

Show that the polytope in n dimensions is defined by

$$P_n = \{x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$$

Solution

Let is take the case of $n = 1$ to start. Let $p = x1$ and $q = y_1$ then using the definition used before we get

$$\lambda p + (1-\lambda)q$$

Which must be convex because it is a single point. Now let $n = 3$

$$\lambda p + (1-\lambda)q$$
$$\lambda(x_1, x_2, x_3) + (1-\lambda)(y_1, y_2, y_2) = (z_1, z_2, z_3)$$

Because z must add up to 1, the set must be convex.

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Problem Statement

For the polytope P_n of the previous problem, let $(a_1, a_2, \cdots, a_n) \in P_n$. Show by induction that

$$n^2 \leq \sum_{i=1}^{n} \frac{1}{a_i}$$

Solution

Begin with the base case, $n = 1$.

$$1^2 \leq \sum_{i=1}^{1} \frac{1}{1}$$
$$1 \leq 1$$

which is true. Now let

$$n^2 \leq \sum_{i=1}^{n} \frac{1}{a_i}$$

be true. We now need to show that the following is true

$$(n+1)^2 \leq \sum_{i=1}^{n+1} \frac{1}{a_i}$$

Begin by defining an element from P_N: $p = (a_1, a_2, \cdots, a_n)$. To make p an element in the P_{n+1} space let $p = (a_1, a_2, \cdots, a_n, 0)$. Let's define another point $q = (0, 0, \cdots, 0, 1)$. Now let's define the line between the points p and q

$$\lambda p + (1-\lambda)q$$
$$\lambda(a_1, a_2, \cdots, a_n, 0) + (1-\lambda)(0, 0, \cdots, 0, 1) = (b_1, b_2, \cdots, b_{n+1})$$

Going back to the $(n+1)^2 \leq \sum_{i=1}^{n+1} \frac{1}{a_i}$, let's plug this in for b for a: $(n+1)^2 = \sum_{i=1}^{n+1} \frac{1}{b_i}$. Note that the $(1-\lambda)$ is non-zero at $n+1$, so we can rewrite this as $(n+1)^2 = \frac{1}{1-\lambda} + \sum_{i=1}^{n+1} \frac{1}{\lambda a_i}$. Now to remove the λ:

$$\frac{1}{1-\lambda} + \sum_{i=1}^{n+1} \frac{1}{\lambda a_i} \leq \sum_{i=1}^{n+1} \frac{1}{a_i}$$

Therefore, $(n+1)^2 \leq \sum_{i=1}^{n+1} \frac{1}{a_i}$.

10

Problem Statement

Show that $(AB)^T = B^T A^T$ is true.

Solution

Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. And let $A = (a_{ij})$ and $A^T = (a_{ji})$, the same can be said for B. If we look at the multiplication of $(AB)^T$

$$(AB)^T = \sum_{k=1}^{n} (a_{ik} b_{ki})^T$$

Which denotes the row/column multiplication/addition of matrix multiplication for transposed matrices. Now if we transpose the summed values

$$(AB)^T = \sum_{k=1}^{n} (a_{ik} b_{ki})^T = \sum_{k=1}^{n} (a_{kj} b_{ki})$$

Reversing the multiplication order we get

$$(AB)^T = \sum_{k=1}^{n} (b_{ki} a_{kj})^T = B^T A^T$$

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Problem Statement

Show that the following are true

Solution

$A_{i:} = \sum_j a_{ij} e_j$

Begin with definition of unit vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ n \end{bmatrix}$$

Now outline the form of $A_{i:} = [a_{i1}, a_{i2}, a_{i3}, \cdots, a_{in}]$ which denotes the all the elements of row i. To show that is equivalent to the sum, begin by expanding the sum. Let k be the column of interest.

$$\sum_j a_{ij}e_j = a_{i1}e_1 + a_{i2}e_2 + \cdots + a_{ik}e_k + a_{in}e_n$$

Referring back to the definition of e, we see that only e_k is nonzero therefore the only value returned is a_{ik}. Extrapolating this for all columns n in the matrix we get the vector $[a_{i1}, a_{i2}, a_{i3}, \cdots, a_{in}]$.

$A_{:j} = \sum_i a_{ij}e_i$

This is very similarly to the previous problem; however, now we are summing over the columns. $A_{:j} = [a_{1j}, a_{2j} \cdots, a_{mj}]^T$. Now taking the sum version, we find

$$\sum_i a_{ij}e_i = a_{1j}e_1 + a_{2j}e_2 + \cdots + a_{kj}e_k + a_{nj}e_m$$

Where the only nonzero value in e is e_k, therefore we are returned akj when $i = k$. Doing this for all m elements returns the vector $[a_{1j}, a_{2j}, \cdots, a_{mj}]^T$

$A_{i:}^T = \sum_j a_{ij}e_j^T$

This is nearly the same as $A_{i:} = \sum_j a_{ij}e_j$, but now because A is transposed, the unit vectors must also be transposed to keep the dimensions connect (column vector to row). Therefore, in a similar vein we can state $(A_{i:}^T) = (a_{:i}) = [a_{1i}, a_{2i}, \cdots, a_{ni}]^T$. Taking the summed version we find

$$\sum_j a_{ij}e_j^T = a_{1i}e_1 + a_{2i}e_2 + \cdots + a_{ki}e_k + a_{ni}e_n$$

Again, because k is the index of interest the only value that is returned is a_{ki}. Extrapolating out, as we have done before, we find that the vector that is returned is the column vector of $[a_{1i}, a_{2i}, \cdots, a_{ni}]^T$.

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Problem Statement

Show that $(A^{-1})^T = (A^T)^{-1}$.

Solution

Let $A^{-1} = B$. Then we can write

$$B^T = (A^T)^{-1}$$

Inverting both sides and stating the fact that $(A^{-1})^{-1} = A$ we get

$$A^T = (B^T)^{-1}$$

Substituting the result from above back into the original equation we get

$$((B^T)^{-1})^{-1} = B^T$$

Using the definition that the inverse of an inverse is the original matrix for an inverterable matrix we get

$$B^T = B^T$$

Therefore, $(A^{-1})^T = (A^T)^{-1}$.

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Problem Statement

Show that $\text{tr}(AB) = \text{tr}(BA)$

Solution

Define what the trace of a matrix is

> The trace of a matrix $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$. In other words, the trace is the sum of the elements along the main of the diagonal

The trace can be written as

$$\text{tr}(AB) = (AB)_{ii} = \sum_{k=1}^{m}(AB)_{ii} = \sum_{i=1}^{m}\sum_{k=1}^{n} A_{ik}B_{ki}$$

Reversing the summations we get

$$\sum_{k=1}^{n}\sum_{i=1}^{m} B_{ki}A_{ik} = \sum_{k=1}^{n}(BA)_{kk} = \text{tr}(BA)$$

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Problem Statement

Define the offset trace as a generalization of the usual trace

$$\text{tr}(C, l) = \sum_{i} C_{i,i+l}$$

where the usual trace is obtained when $l = 0$, and for $l > 0$, the sum is taken on the lth superdiagonal. Show that for $l \neq 0$

$$\text{tr}(AB, l) = \text{tr}(B^T A^T, l)$$

Solution

To begin we state the fact that was proven before.

$$(AB)^T = B^T A^T$$

Now we need to show that $(A)_{i,i+1} = ((A)_{i+1,i})^T$. The obvious case is when $j = 0$, when $l > 0$. Let $j = i+l$, we know that

$$(a_{i,j}) = (a_{j,i})^T$$

substituting $j = i + 1$ is then obvious. Putting these facts together, let $C = AB$

$$\text{tr}(C,l) = \sum_i C^T_{i+l,i} = \sum_i (B^T A^T)_{i+l,i}$$

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Problem Statement

Let two complex numbers be defined as $z_1 = a + jb$ and $z_2 = c + jd$. Let $z_3 = z_1 z_2 = e + jf$. Show

1. The product can be written as

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

1. The complex product can also be written as

$$e = (a-b)d + a(c-d) \quad f = (a-b)d + b(c+d)$$

1. Show that this modified scheme can be expressed in matrix notation as

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (c-d) & 0 & 0 \\ 0 & (c+d) & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Solution

1.4-1.1

Complex matrix multiplication can be written as

$$z_1 z_2 = (a + jb)(c + jd)$$

Expanding and combining real and imaginary terms

$$\begin{gathered} z_1 z_2 = ac + ajd + cjb + bdj^2 \\ (ac - bd) + (ajd + cjb) \end{gathered}$$

Now lets expand the matrix form shown in the problem statement

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca - bd \\ da + cb \end{bmatrix}$$

Note that the grouped pairs match for real and imaginary parts.

1.4-1.2

This can be found by simply expanding and simplifying. Lets begin with e

$$\begin{aligned} e &= (a - b)d + a(c - d) \\ e &= ad - bd + ac - ad \\ e &= ac - bd \end{aligned}$$

Which matches the two solutions found before. Similarly for f

$$\begin{aligned} f &= (a - b)d + b(c + d) \\ f &= ad - bd + bc + bd \\ f &= ad + bc \end{aligned}$$

Which, again, matches what was found before.

1.4-1.3

Once again, we can show that they are equivalent by expansion and simplification. We will work from left to right performing matrix multiplication

$$\begin{gathered} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (c - d) & 0 & 0 \\ 0 & (c + d) & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} (c - d) & 0 & d \\ 0 & (c + d) & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} (c - d) & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca - bd \\ da + cb \end{bmatrix} \end{gathered}$$

Which is equivalent to what was found in the previous problems.

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Problems Statement

Show that

$$k_j = \frac{1}{p^j j!}(-1)^j \frac{d^j}{d(z^{-1})^j}(1 - pz^{-}1)^r H(z)\Big|_{z=p}$$

for the partial fraction expansion of a Z-transform with repeated roots is correct.

Solution

For repeated roots, PFE is of the form

$$k_j = \frac{1}{p^j j!}(-1)^j \frac{d^j}{d(z^{-1})^j}(1 - pz^{-}1)^r H(z)$$

with descending degrees in the denominator. Now to show the form of k_i

$$\begin{aligned} k_0 &= (1 - pz^{-1})^n H(z)|_{z=p} \\ k_1 &= (1 - pz^{-1})^n H(z)|_{z=p} \\ k_2 &= \tfrac{1}{2p^2} \tfrac{d^2}{(dz^1)^2} (1 - pz^{-1})^n H(z)|_{z=p} \\ &\vdots \\ k_j &= \tfrac{1}{2p^j} (-1)^j \tfrac{d^j}{(dz^1)^j} (1 - pz^{-1})^n H(z)|_{z=p} \end{aligned}$$

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Problem Statement

Determine the PFE for

1. $H(z) = \frac{1-5z^{-1}-6z^{-2}}{1-1.5z^{-1}+0.56^{-2}}$
2. $H(z) = \frac{5-6z^{-1}}{(1-0.3z^{-1})^2(1-0.4z^{-1})}$

Solution

1.4-3.1

The degree of the numerator is the same as the denominator, so we perform long division to find

$$H(z) = 10.714 + \frac{-21.07z^{-1} - 11.714}{1 - 1.5z^{-1} + 0.56^{-2}}$$

Finding of the roots of the denominator and finding a common denominator we get

$$-21.07z^{-1} - 11.714 = A(1 - 0.7z^{-1}) + B(1 - z^{-1} - 0.8)$$

Let $z^{-1} = 1.43$ and solve for $A = 128.81$. Similarly, let $z^{-1} = 1.25$ and solve for $B = -116.998$

Octave check:

```
pkg load signal;
residuez([1,-5,-6],[1,-1.5,0.56])
```

128.7142857142874
-117.0000000000017

1.4-3.2

The PFE form is of the form

$$\frac{5-6z^{-1}}{(1-0.3z^{-1})^2(1-0.4z^{-1})} = \frac{A}{(1-0.3z^{-1})^2} + \frac{B}{(1-0.3z^{-1})} + \frac{C}{(1-0.4z^{-1})}$$
$$\frac{5-6z^{-1}}{(1-0.3z^{-1})^2(1-0.4z^{-1})} = A(1-0.4z^{-1}) + (1-0.3z^{-1})(1-0.4z^{-1})B + (1-0.3z^{-1})C$$

Let $z^{-1} = 2.5$ and solve for $C = -160$, similarly set $z^{-1} = 3.\bar{3}3$ and solve for $A = 45.17$. To find B begin be grouping liked variables and then set the coefficients equal to one another

$$5 = A + B + C$$
$$B = 120$$

Octave check:

```
residuez([5,-6],[1,-1,0.33,-0.036])
```

120.0000000000038
45.00000000000062
-160.0000000000044

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Problem Statement

Show that the autocorrelation function

$$r_{yy}[k] = E[y[t]\overline{y[t-k]}]$$

has the property

$$r_{yy}[k] = \overline{r}_{yy}[-k]$$

Solution

Fact about even functions: $f(x)\overline{f(x)}$

$$\begin{aligned}\overline{r_{yy}} &= \overline{E[y[t]\overline{y[t+k]}]} \\ \overline{r_{yy}} &= E[\overline{y[t]}y[t+k]] = R_{yy}[k]\end{aligned}$$

Doing that again yields $\overline{R_{yy}[-k]} = r_{yy}k$

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Problem Statement

Show that for the MA process

$$y[t] = f[t] + 2f[t-1] + 3f[t-2]$$

where $f[t]$ is a zero-mean white random process with $\sigma_f^2 = 0.1$ determine the 3×3 autocorrelation matrix R

Solution

Because the matrix R is Toeplitz and symmetric, only 3 points need to be found

$$\begin{aligned}E[y[t]y[t]] &= 0.1 + 0.4 + 0.9 \\ E[y[t]y[t-1]] &= 0.2 + 0.6 \\ E[y[t]y[t-2]] &= 0.3\end{aligned}$$

Therefore the matrix is

$$R = \begin{bmatrix} 1.4 & 0.8 & 0.3 \\ 0.8 & 1.4 & 0.8 \\ 0.3 & 0.8 & 1.4 \end{bmatrix}$$

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Problem Statement

For the first-order real AR process

$$y[t+1] + a_1 y[t] = f[t+1]$$

with $|a_1| < 1$ and $E[f[t]] = 0$, show that

$$\sigma_y^2 = E[y^2[t]] = \frac{\sigma_f^2}{1 - a^2}$$

Solution

$$
\begin{aligned}
&E[y[t+1]^2] = E[(-a_1 y[t] + f[t+1])^2] \\
&E[y[t+1]^2] = E[a_1^2 y^2[t] - 2y[t]f[t+1] + f[t+1]] \\
&a_1^2\sigma_y^2 + \sigma_f^2 = \sigma_f^2 \\
&\sigma_f^2 = \sigma_y^2(1 - a_1^2) \\
&\sigma_y^2 = \tfrac{\sigma_f^2}{1-a_1^2}
\end{aligned}
$$

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Problem Statement

Consider the second-order real AR process

$$y[t+2] + a_1 y[t+1] + a_2 y[t] = f[t+2]$$

where $f[t]$ is a zero-mean white-noise sequence. The difference equation in (1.14) has a characteristic equation with the roots

$$p_1, p_2 = \frac{1}{2}(-a_1 \pm \sqrt{a_1^2 - 4a2})$$

a

Using the Yule-Walker equations, show that if the autocorrelation values

$$r_{yy}[l-k] = E[y[t-k]\overline{y}[t-l]]$$

are known, then the model parameters may be determined

$$
\begin{aligned}
a_1 &= -\tfrac{r_1(r_0 - r_2)}{r_0^2 - r_1^2} \\
a_2 &= -\tfrac{r_0 r_2 - r_1^2}{r_0^2 - r_1^2}
\end{aligned}
$$

where $r_k = r_{kk}[k]$.

b

One the other hand, if $\sigma_f^2 = r_0$ and a_1 and a_2 are known, show that the autocorrelation value are expressed as

$$
\begin{aligned}
r_1 &= -\tfrac{a_1}{1+a_2}\sigma_y^2 \\
r_2 &= \sigma_y^2(\tfrac{a_1^2}{1+a_2} a_2)
\end{aligned}
$$

c

Using

$$\sigma_f^2 = \sum_{i=0}^{p} a_i r_i$$

and the results of this problem, show that

$$r_0 = \sigma_y^2 = \frac{1+a_2}{1-a_2}\frac{\sigma_f^2}{[(1+a_2)^2 - a_1^2]}$$

d

Using $r_0 = \sigma_y^2$ and $r_1 = -a_1\frac{\sigma_y^2}{1+a_2}$ as initial conditions, find an explicit solution to the Yule-Walker difference equation

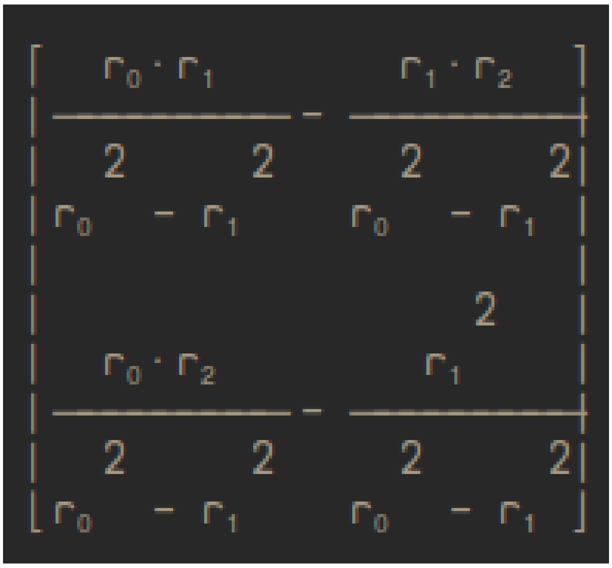
$$r_k + a_1 r_{k-1} + a_2 r_{k-2} = 0$$

in terms of p_1, p_2, and σ_y^2.

Solution

a

```
pkg load symbolic;
syms r0 r1 r2 a1 a2;
R = [r0, r1; r1, r0];
r = [r1; r2];
inv(R)*r
```



$$\begin{bmatrix} \dfrac{r_0 \cdot r_1}{r_0^2 - r_1^2} - \dfrac{r_1 \cdot r_2}{r_0^2 - r_1^2} \\ \dfrac{r_0 \cdot r_2}{r_0^2 - r_1^2} - \dfrac{r_1^2}{r_0^2 - r_1^2} \end{bmatrix}$$

b

```
p = [-a1; -a2];
R*r
```

$$\begin{bmatrix} -a_1 \cdot r_0 - a_2 \cdot r_1 \\ \\ -a_1 \cdot r_1 - a_2 \cdot r_0 \end{bmatrix}$$

Now solving for r_1

$$r_1 + a_2 r_1 = -a_1 r_0$$
$$r_1 = \frac{-a_1}{1+a_1}\sigma_y^2$$

Similarly for r_2

$$r_2 = -a_1 r_1 - a_2 r_0$$
$$r_2 = -a_1(\frac{-a_1}{1+a_2}\sigma_y^2)a_2\sigma_y^2$$
$$r_2 = (\frac{a_1^2}{1+a_2} - a_2)\sigma_y^2$$

c

$$\sigma_y^2 - a_1\frac{a1}{1+a_2}\sigma_y^2 + a_2(\frac{a_1^2}{1+a_2} - a_2)\sigma_y^2$$
$$\sigma_y^2 - \frac{a1^2}{1+a_2}\sigma_y^2 + (\frac{a_1^2 a_2}{1+a_2} - a_2^2)\sigma_y^2$$
$$\sigma_y^2 = \frac{\sigma_f^2}{1-\frac{a_1^2}{1+a_2}+(\frac{a_1^2 a_2}{1+a_2}-a_2^2)}$$
$$\sigma_y^2 = (1+a_2)\frac{\sigma_f^2}{1+a_2-a_1^2+a_1^2 a_2-a_2^2(1+a_2)}$$
$$\sigma_y^2 = (1+a_2)\frac{\sigma_f^2}{(1-a_2)(a_2+1)^2-a_1^2(1-a_2)}$$
$$\sigma_y^2 = \frac{1+a_2}{1-a}\frac{\sigma_f^2}{[(1+a_2)^2-a_1^2]}$$

d

Similarly to differential equations, we can write the solution of the difference equation in the form

$$r_k = c_1 p_1^k + c_2 p_2^k$$

Given the initial conditions $r_0 = \sigma_y^2$ and $r_1 = -\frac{a_1}{1+a_2}\sigma_y^2$. We can solve for c_1 and c_2. Plugging r_0 gives

$$\sigma_y^2 = c_1 + c_2$$

Similarly for r_1 we can start solving for c_1 and c_2

$c_1 = \frac{a_1}{1+a_2}\sigma_y^2 - c_2 p_2$
plug in to find
$c_2 = \frac{\sigma_y^2 p_1 - \frac{a_1}{1+a_1}\sigma_y^2}{p_1 - p_2}$
plug back into c_1 to get
$c_1 = \frac{a_1}{1+a_2}\sigma_y^2 - \frac{\sigma_y^2 p_1 - \frac{a_1}{1+a_1}\sigma_y^2}{p_1 - p_2} p_2$