

1)

As $\rho \rightarrow 0$ the circle "collapses" in a dot at O.

As $\rho \rightarrow 0$ then we create squares with rounded edges.



J

A.5-17)

Given:

Let (X, d_2) be a metric space of functions defined on \mathbb{R} with the Euclidean metric

$$d_2^2(f, g) = \int_{-\infty}^{\infty} (f(t) - g(t))^2 dt$$

Define the mapping $\Phi_\phi: X \rightarrow \mathbb{R}$ by

$$\Phi_\phi(x) = \int_{-\infty}^{\infty} x(t) \phi(t) dt$$

Find:

Show that if $\phi(t)$ is square integrable, that is,

$$\int_{-\infty}^{\infty} \phi(t)^2 dt < \infty$$

then Φ_ϕ is a continuous mapping

Solution:

Def: A function $g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous if $\forall \epsilon > 0 \exists \delta > 0$
s.t.

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x-y| < \delta$$

$$\Phi_\phi(x) = \int_{-\infty}^{\infty} x(t) \phi(t) dt \leq \left(\int_{-\infty}^{\infty} x(t)^2 dt \right)^{1/2} \left(\int_{-\infty}^{\infty} \phi(t)^2 dt \right)^{1/2}$$

$$\int_{-\infty}^{\infty} |\Phi_\phi(x)|^2 dt \leq \left(\int_{-\infty}^{\infty} x(t)^2 dt \right)^{1/2} \left(\int_{-\infty}^{\infty} \phi(t)^2 dt \right)^{1/2}$$

$$\|\Phi_\phi(x)\|^2 = \int_{-\infty}^{\infty} |\Phi_\phi(x)|^2 dt$$

$$\leq \left(\int_{-\infty}^{\infty} x(t)^2 dt \right)^{1/2} \left(\int_{-\infty}^{\infty} \phi(t)^2 dt \right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty} x(t)^2 dt \right)^{1/2} \|\phi(t)\|^2$$

Hence $\mathbb{B}_f(x)$ is a bounded linear operator on the Hilbert space. and is continuous.



Q.1-1)

Given:

$$x = [1 \ 2 \ 3 \ 4 \ 5 \ 6]^T$$

Find:

Compute ℓ_p metric $d_p(x, 0)$ for $p = 1, 2, 4, 10, 100, \infty$.

Comment on why $d_p(x, 0) \rightarrow \max(x_i)$ as $p \rightarrow \infty$

Solution:

Generally

$$d_p(x, y) = \left[\int_a^b |x(t) - y(t)|^p dt \right]^{1/p}$$

$p=1$

$$d_1(x, 0) = [1+2+3+4+5+6] = 21$$

$p=2$

$$d_2(x, 0) = [1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2]^{1/2} = 9.5394$$

$p=4$

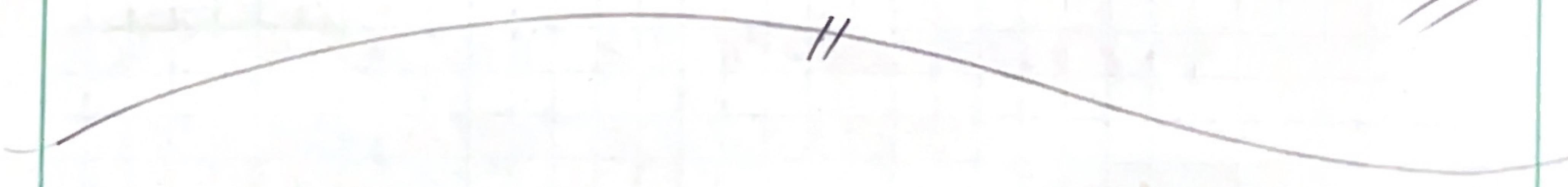
$$d_4(x, 0) = [1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4]^{1/4} = 6.9063$$

$$p=10 : d_{10}(x, 0) = 6.1001$$

$$P_\infty = 6$$

$$p=100 : d_{100}(x, 0) = 6.000$$

As $p \rightarrow \infty$, the value is converging to the largest value in the set



Q.1-4)

Given

a) $x, y \in \mathbb{R}$

$$|x+y| \leq |x| + |y|$$

b) $x, y \in \mathbb{R}$

$$\|x+y\| \leq \|x\| + \|y\|$$

Find:

Prove (a) \Rightarrow (b)

a) What is the condition for equality?

b)
Hint $\sum_{i=1}^n x_i y_i \leq \|x\| \|y\|$

Solution

a) $a^2 + b^2 + 2|ab| \geq a^2 + b^2 + 2ab$

$$(|a| + |b|)^2 \geq |a+b|^2$$

$$|a| + |b| \geq |a+b| \quad //$$

Equality occurs when $x = \alpha y$. //

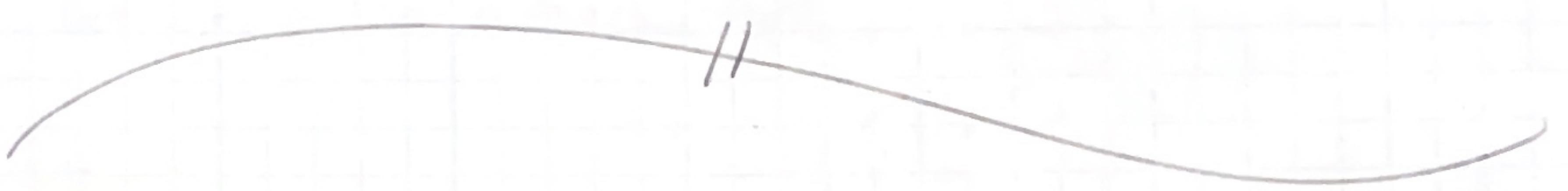
b) $\|x+y\|^2 = \|x\|^2 + 2(xy) + \|y\|^2$

Cauchy-Schwarz Inequality $|xy| \leq \|x\| \|y\| \Rightarrow \|x\| \|y\| > xy$

$$\|x\|^2 + 2(xy) + \|y\|^2 \leq \|x\|^2 + 2(\|x\| \|y\|) + \|y\|^2$$

$$\|x\|^2 + 2(xy) + \|y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2 \Rightarrow \|x+y\| \leq \|x\| + \|y\|$$



2.1-5)

Given:

Let (X, d) be a metric space

Find:

Show $d_0(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ is a metric on X .

What significant feature does this metric possess?

Solution:

Def 2.2: A metric space (X, d) is a set X together with a metric d . //

We then need to show $d_0(x, y)$ is a metric

Def 2.1: A metric $d: X \times X \rightarrow \mathbb{R}$ is a function that is used to measure distance between elements in X . In order to be a metric, it must satisfy the following properties $\forall x, y \in X$

$$1) d(x, y) = d(y, x)$$

$$2) d(x, y) \geq 0$$

$$3) d(x, y) = 0 \text{ iff } x = y$$

$$4) \forall x, y, z \in X$$

$$d(x, z) \leq d(x, y) + d(y, z)$$

1)

$$d_0(x, y) = d_0(y, x) \Rightarrow \frac{d(x, y)}{1 + d(x, y)} = \frac{d(y, x)}{1 + d(y, x)}$$

If $d(y, x)$ is an λe type metric. //

2) If $d(x,y) \leq \lambda_x(x,y)$ the smallest it can be is

$$\frac{0}{1+0} = 0 //$$

3) If $x=y$ and we assume $\lambda_x(x,y)$ we get

$$d(x,y) = \frac{0}{1+0} = 0 //$$

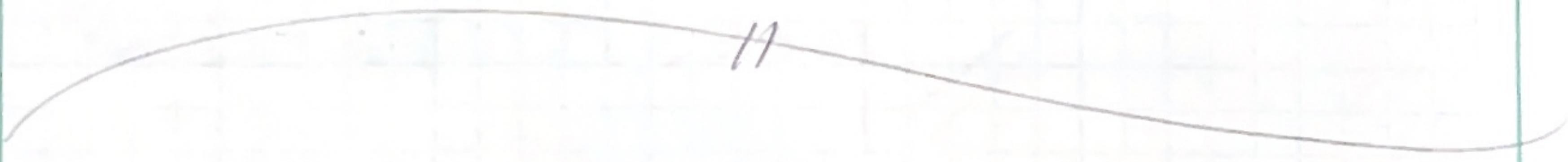
4) Let $d(x,y) \leq d(z,w)$

$$d(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$$

$$= \frac{d(x,z)}{1+d(x,z)+d(y,z)} + \frac{d(z,y)}{1+d(x,z)+d(y,z)}$$

$$\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(y,z)}{1+d(y,z)} = d(x,z) + d(z,y) //$$

This metric space is always less than 1.



2.1-6)

Given:

$$\text{Let } d_m(x,y) = \min(1, d(x,y))$$

Find:

Show that this is a metric space (X, d_m)

What is the significance?

Solution

We must show $d_m(x,y)$ is a metric

1) We know $d(x,y) = d(y,x)$ then

$$\min(1, d(y,x)) = \min(1, d(x,y)) //$$

2) By virtue the minimum is be used, $d(x,y)$ can be at the least 0.

3)

$$d(x,x) = 0 \Rightarrow \min(1, d(x,x)) = 0 //$$

4)

let $d(x,y) \geq 1$ then

$$d(x,y) + d(y,z) \geq 1 \geq d(x,z) //$$

The metric is another way to have a range between 0 and 1



2.1-9)

Given:

If requirement M3 in the definition of a metric is relaxed to

$$d(x,y) = 0 \text{ if } x = y$$

allowing $d(x,y) = 0$ even if $x \neq y$ then a pseudometric is obtained.

Find:

Let $f: X \rightarrow \mathbb{R}$ be an arbitrary function defined on the set X . Show that $d(x,y) = |f(x) - f(y)|$ is a pseudo metric.

Solution

$$\| \cdot \| = \sqrt{a^2 + b^2 + \dots}$$

Then

$$1) \quad \sqrt{f(x)^2 - f(y)^2} = \sqrt{f(y)^2 - f(x)^2} //$$

$$2) \quad \text{Let } f(x) \leq f(y)$$

$$\sqrt{f(x)^2 - f(y)^2} \geq 0 //$$

$$3) \quad \text{Let } \{z \in X : f(z) = 0\}$$

$$\forall z \in X \quad d(x,z) = 0 //$$

$$4) \quad d(x,z) \leq d(x,y) + d(z,y) \text{ identically} //$$

2.1-14)

Given:

$$A = (0, 4), B = (0, \infty), C = (-\infty, 5]$$

Find

inf and sup for the given sets

Solution

a)

$$\inf(A) = 0$$

$$\sup(A) = 4 //$$

b)

$$\inf(B) = 0$$

$$\sup(B) = \infty //$$

c)

$$\inf(C) = -\infty$$

$$\sup(C) = 5 //$$

2.1-23)

Given:

$$x_n = \int_1^n \frac{\cos(t)}{t^2} dt \quad d(x,y) = |x - y|$$

Find:

Show the sequence x_n is convergent using $d(x,y)$.

Hint:

Show that x_n is a Cauchy sequence. Use the fact that

$$\int \left| \frac{\cos t}{t^2} \right| dt \leq \int \frac{1}{t^2} dt$$

Solution

$$x_n = \int_1^n \frac{\cos(t)}{t^2} dt \leq \int_1^n \frac{1}{t^2} dt \leq \int_1^n \frac{1}{a^2} dt$$

$$x_n \leq \int_1^n \frac{1}{a^2} dt$$

$$\text{if } d(x,y) = |x - y| = \sqrt{\left(\sum_{a=1}^n \frac{1}{a^2} \right)^2 + \left(\sum_{b=1}^m \frac{1}{b^2} \right)^2}$$

$$d(x,y) = \sqrt{\sum_{a=1}^n \frac{1}{a^2} + \sum_{b=1}^m \frac{1}{b^2}} \leq \sqrt{\sum_{a=1}^n \frac{1}{a^2}} < \epsilon$$

Therefore it must converge //

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