Homework 1

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Problem Statement

Show that the following are convex:

1. The set of $n \times n$ Toeplitz matrices
2. The set of monic polynomials of the same degree
3. The set of symmetric matrices

Solution

Begin by defining what a complex set is:

> A set S is convex if for any two points $p, q \in S$, then all points of the form
>
> $$\lambda p + (1 - \lambda) q$$
>
> for $0 \leq \lambda \leq 1$, are also in S.

3.1: Toeplitz

Begin by defining what a Toeplitz matrix is

> A Toeplitz matrix is a diagonal-constant matrix, which means all elements along a diagonal have the same value. For a Toeplitz matrix A we have $A_{ij} = a_{i-j}$ which results in the form
>
> $$\begin{bmatrix} a & b & c & \cdots \\ e & a & b & \cdots \\ f & e & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Consider Toeplitz matrices P and Q of dimension $n \times n$ as let S be the set of $n \times n$ Toeplitz matrices. Lets now apply the definition of the convex set:

$$\lambda P + (1 - \lambda) Q$$

There are two operations being applied to the matrices: addition

> If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same size, then their sum $A + B$ is the matrix obtained by adding the corresponding elements of the matrix A and B.

and multiplication of a matrix by a number

> If $A = [a_ij]$ is a matrix and c is a number, then cA is the matrix obtained by multiplying each element of A by c.

Therefore, we can make the statements

- Any Toeplitz matrix multiplied by a scalar is also Toeplitz
- Any two $n \times n$ Toeplitz matrices being added together is also Toeplitz

The following statement can then be made: The set of $n \times n$ Toeplitz matrices is convex.

$$\lambda P + (1 - \lambda)Q \in S$$

3.2: Monic Polynomials of Degree n

Begin by defining what a monic polynomial is

> A polynomial is monic if the coefficient of the highest order term is 1.

Suppose p and q are monic polynomials of degree n and S is the set of all monic polynomials of degree n. It can be written

$$\lambda p(x) + (1 - \lambda)q(x) \in S$$

Expanding the above gives

$$\begin{aligned}&\lambda(x^n - ax^{n-1} + \cdots + bx + c) + (1 - \lambda)(x^n + dx^{n-1} + \cdots + ex + f)\\&= 2\lambda x^n + (a + dx^{n-1}) + \cdots\end{aligned}$$

Dividing by 2λ produces another monic polynomial of degree n. Therefore, the set is convex, i.e $\lambda p(x) + (1 - \lambda)q(x) \in S$.

3.3: Symmetric Matrices

Define what a symmetric matrix is

> A matrix A is symmetric $\iff A = A^T$.

Similarly to 3.1,

- Let P and Q are $n \times n$ symmetric matrices
- S is the set of $n \times n$ symmetric matrices
- Any symmetric matrix multiplied by a scalar is also symmetric
- Any two $n \times n$ symmetric matrices being added together is also symmetric

Therefore, S is convex because $\lambda P + (1 - \lambda)Q \in S$.

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Problem Statement

The set of even integers can be represented as $2\mathbb{Z}$. Show that $|2\mathbb{Z}| = |\mathbb{Z}|$. Similarly show that there are as many odd integers as there are integers.

Solution

Let S and T be two different sets. T and S have the same cardinality if there is a bijection f from S to T. Therefore, we need to show $f : \mathbb{Z} \to 2\mathbb{Z}$. Let the mapping $f(n)$ be defined as

$$f(n) = 2n$$

It now needs to be shown that $f(n)$ is both one-to-one and onto. To show that $f(n)$ is one-to-one begin by defining how to show a mapping is one-to-one

> A function f from A onto B is one-to-one if each element of B has at most one element of A mapped into it. That is, $f(x) = f(y)$, then $x = y$.

From this if we suppose $f(a) = f(b)$, then $2a = 2b$ so $a = b$. Thus, f is one-to-one. Now we need to show f is onto. Begin by defining onto

> A function is onto if each element of B has at least one element of A that is mapped into it. That is, $\forall b \in B$ there is an $a \in A$ such that $f(a) = b$.

Take $b = 2n$ for some a, then $f(n) = 2n = b$ which shows that f is onto. Therefore, $f(n)$ is a bijection and $|\mathbb{Z}| = |2\mathbb{Z}|$.

Similarly, for the odd we need to show $f : \mathbb{Z} \to 2\mathbb{Z} + 1$ is a bijection. To show $f(n)$ is one-to-one let $f(a) = f(b)$, then $2a + 1 = 2b + 1$, so $a = b$. To show f is onto let $b = 2n + 1$, then $f(n) = 2n + 1 = b$. Therefore, $f(n)$ is a bijection and $|\mathbb{Z}| = |2\mathbb{Z} + 1|$.

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Problem Statement

Show that $|(0, 1]| = |\mathbb{R}|$.

Solution

A simple way to go about this is to first show that $\big|[0,1)\big| = \big|[-\pi/2, \pi/2)\big|$. Suppose $f(x) = \pi x - \pi/2$. To show that $f(x)$ is one-to-one

$$\begin{aligned}&f(x) = f(y)\\&\pi x - \pi/2 = \pi y - \pi/2\\&\pi x = \pi y\\&x = y\end{aligned}$$

Therefore, $f(x)$ is one-to-one. Now to show that $f(x)$ is also onto.

$$\begin{aligned}&f(x) = y\\&\pi x - \pi/2 = y\\&x = y/\pi + 1/2\end{aligned}$$

And because we know that $0 < x \leq 1$ we can show that x written above is in that range by saying

$$\begin{aligned}&-\pi/2 < y \leq \pi/2\\&-1/2 < y/\pi \leq 1/2\\&0 < y/\pi + 1/2 \leq 1\end{aligned}$$

Therefore, the function is also onto. Now to show that $\big|[-\pi/2, \pi/2)\big| = \big|\mathbb{R}\big|$. Let $g(x) = tan(x)$ it can be shown that $tan(x)$ is always increasing.

> **Fact**: If $g(x)$ is always increasing, then $g(x)$ is one-to-one.

By taking the derivative of $g'(x) = sec^2(x) > 0$, therefore $g(x)$ is one-to-one. To show that $g(x)$ is onto, we will use the intermediate value theorem

> If $g(x)$ is continuous on an interval $[a, b]$, then $g(x)$ contains all the values between $g(a)$ and $g(b)$.

Let the range of interest be $[-\pi/2 + \epsilon, \pi/2 - \epsilon]$. $g(x)$ is continuous within the range, therefore it obtains all values $g(-\pi/2 + \epsilon)$ to $g(\pi/2 - \epsilon)$. If we let $\epsilon \to 0$ then $g(x) \to \mathbb{R}$. Therefore, $\big|(0,1]\big| = \big|\mathbb{R}\big|$.

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Problem Statement

Show that the intersection of a convex set is convex.

Solution

Let A and B be two convex sets, and let $C = A \cup B$. Now let $p, q \in C$.

- If $p, q \in C$ then $p, q \in A$ and A is convex

- If $p, q \in C$ then $p, q \in B$ and B is convex
- Therefore C must be complex

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Problem Statement

If S and T are convex sets both in $\mathbb{R}^n$, show that the set sum is convex.

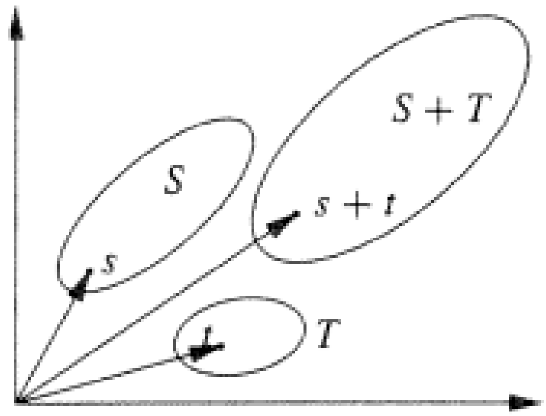

Figure A.7: The set sum.

Solution

The set sum is defined as

$$S + T = \{x : x = s + t, s \in S, t \in T\}$$

Let S and T be convex sets and $S + T \in C$, let $s_1, s_2 \in S$ and $t_1, t_2 \in T$, and let $s = s_1 + t_1$ and $t = s_2 + t_2$, then

$$\lambda s + (1-\lambda)t\lambda s_1 + \lambda t_1 + s_2(1-\lambda) + t_2(1-\lambda)\lambda s_1 + (1-\lambda)t_1 + \lambda s_2 + (1-\lambda)t_2 \in C$$

Therefore, the set sum is convex.

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Problem Statement

Show that the polytope in n dimensions is defined by

$$P_n = \{x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$$

Solution

Let is take the case of $n = 1$ to start. Let $p = x1$ and $q = y_1$ then using the definition used before we get

$$\lambda p + (1 - \lambda)q$$

Which must be convex because it is a single point. Now let $n = 3$

$$\lambda p + (1 - \lambda)q$$
$$\lambda(x_1, x_2, x_3) + (1 - \lambda)(y_1, y_2, y_2) = (z_1, z_2, z_3)$$

Because z must add up to 1, the set must be convex.

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Problem Statement

For the polytope P_n of the previous problem, let $(a_1, a_2, \cdots, a_n) \in P_n$. Show by induction that

$$n^2 \leq \sum_{i=1}^{n} \frac{1}{a_i}$$

Solution

Begin with the base case, $n = 1$.

$$1^2 \leq \textstyle\sum_{i=1}^{1} \frac{1}{1}$$
$$1 \leq 1$$

which is true. Now let

$$n^2 \leq \sum_{i=1}^{n} \frac{1}{a_i}$$

be true. We now need to show that the following is true

$$(n+1)^2 \leq \sum_{i=1}^{n+1} \frac{1}{a_i}$$

Begin by defining an element from P_N: $p = (a_1, a_2, \cdots, a_n)$. To make p an element in the P_{n+1} space let $p = (a_1, a_2, \cdots, a_n, 0)$. Let's define another point $q = (0, 0, \cdots, 0, 1)$. Now let's define the line between the points p and q

$$\lambda p + (1 - \lambda)q$$
$$\lambda(a_1, a_2, \cdots, a_n, 0) + (1 - \lambda)(0, 0, \cdots, 0, 1) = (b_1, b_2, \cdots, b_{n+1})$$

Going back to the $(n+1)^2 \leq \sum_{i=1}^{n+1} \frac{1}{a_i}$, let's plug this in for b for a: $(n+1)^2 = \sum_{i=1}^{n+1} \frac{1}{b_i}$. Note that the $(1-\lambda)$ is non-zero at $n+1$, so we can rewrite this as $(n+1)^2 = \frac{1}{1-\lambda} + \sum_{i=1}^{n+1} \frac{1}{\lambda a_i}$. Now to remove the λ:

$$\frac{1}{1-\lambda} + \sum_{i=1}^{n+1} \frac{1}{\lambda a_i} \leq \sum_{i=1}^{n+1} \frac{1}{a_i}$$

Therefore, $(n+1)^2 \leq \sum_{i=1}^{n+1} \frac{1}{a_i}$.

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Problem Statement

Show that $(AB)^T = B^T A^T$ is true.

Solution

Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. And let $A = (a_{ij})$ and $A^T = (a_{ji})$, the same can be said for B. If we look at the multiplication of $(AB)^T$

$$(AB)^T = \sum_{k=1}^{n} (a_{ik} b_{ki})^T$$

Which denotes the row/column multiplication/addition of matrix multiplication for transposed matrices. Now if we transpose the summed values

$$(AB)^T = \sum_{k=1}^{n} (a_{ik} b_{ki})^T = \sum_{k=1}^{n} (a_{kj} b_{ki})$$

Reversing the multiplication order we get

$$(AB)^T = \sum_{k=1}^{n} (b_{ki} a_{kj})^T = B^T A^T$$

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Problem Statement

Show that the following are true

Solution

$A_{i:} = \sum_j a_{ij} e_j$

Begin with definition of unit vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ n \end{bmatrix}$$

Now outline the form of $A_{i:} = [a_{i1}, a_{i2}, a_{i3}, \cdots, a_{in}]$ which denotes the all the elements of row i. To show that is equivalent to the sum, begin by expanding the sum. Let k be the column of interest.

$$\sum_j a_{ij}e_j = a_{i1}e_1 + a_{i2}e_2 + \cdots + a_{ik}e_k + a_{in}e_n$$

Referring back to the definition of e, we see that only e_k is nonzero therefore the only value returned is a_{ik}. Extrapolating this for all columns n in the matrix we get the vector $[a_{i1}, a_{i2}, a_{i3}, \cdots, a_{in}]$.

$A_{:j} = \sum_i a_{ij}e_i$

This is very similarly to the previous problem; however, now we are summing over the columns. $A_{:j} = [a_{1j}, a_{2j} \cdots, a_{mj}]^T$. Now taking the sum version, we find

$$\sum_i a_{ij}e_i = a_{1j}e_1 + a_{2j}e_2 + \cdots + a_{kj}e_k + a_{nj}e_m$$

Where the only nonzero value in e is e_k, therefore we are returned akj when $i = k$. Doing this for all m elements returns the vector $[a_{1j}, a_{2j}, \cdots, a_{mj}]^T$

$A_{i:}^T = \sum_j a_{ij}e_j^T$

This is nearly the same as $A_{i:} = \sum_j a_{ij}e_j$, but now because A is transposed, the unit vectors must also be transposed to keep the dimensions connect (column vector to row). Therefore, in a similar vein we can state $(A_{i:}^T) = (a_{:i}) = [a_{1i}, a_{2i}, \cdots, a_{ni}]^T$. Taking the summed version we find

$$\sum_j a_{ij}e_j^T = a_{1i}e_1 + a_{2i}e_2 + \cdots + a_{ki}e_k + a_{ni}e_n$$

Again, because k is the index of interest the only value that is returned is a_{ki}. Extrapolating out, as we have done before, we find that the vector that is returned is the column vector of $[a_{1i}, a_{2i}, \cdots, a_{ni}]^T$.

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Problem Statement

Show that $(A^{-1})^T = (A^T)^{-1}$.

Solution

Let $A^{-1} = B$. Then we can write

$$B^T = (A^T)^{-1}$$

Inverting both sides and stating the fact that $(A^{-1})^{-1} = A$ we get

$$A^T = (B^T)^{-1}$$

Substituting the result from above back into the original equation we get

$$((B^T)^{-1})^{-1} = B^T$$

Using the definition that the inverse of an inverse is the original matrix for an inverterable matrix we get

$$B^T = B^T$$

Therefore, $(A^{-1})^T = (A^T)^{-1}$.

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Problem Statement

Show that $\text{tr}(AB) = \text{tr}(BA)$

Solution

Define what the trace of a matrix is

> The trace of a matrix $\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$. In other words, the trace is the sum of the elements along the main of the diagonal

The trace can be written as

$$\text{tr}(AB) = (AB)_{ii} = \sum_{k=1}^{m} (AB)_{ii} = \sum_{i=1}^{m} \sum_{k=1}^{n} A_{ik} B_{ki}$$

Reversing the summations we get

$$\sum_{k=1}^{n} \sum_{i=1}^{m} B_{ki} A_{ik} = \sum_{k=1}^{n} (BA)_{kk} = \text{tr}(BA)$$

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Problem Statement

Define the offset trace as a generalization of the usual trace

$$\text{tr}(C, l) = \sum_i C_{i,i+l}$$

where the usual trace is obtained when $l = 0$, and for $l > 0$, the sum is taken on the lth superdiagonal. Show that for $l \neq 0$

$$\text{tr}(AB, l) = \text{tr}(B^T A^T, l)$$

Solution

To begin we state the fact that was proven before.

$$(AB)^T = B^T A^T$$

Now we need to show that $(A)_{i,i+1} = ((A)_{i+1,i})^T$. The obvious case is when $j = 0$, when $l > 0$. Let $j = i+l$, we know that

$$(a_{i,j}) = (a_{j,i})^T$$

substituting $j = i + 1$ is then obvious. Putting these facts together, let $C = AB$

$$\text{tr}(C, l) = \sum_i C^T_{i+l,i} = \sum_i (B^T A^T)_{i+l,i}$$

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Problem Statement

Let two complex numbers be defined as $z_1 = a + jb$ and $z_2 = c + jd$. Let $z_3 = z_1 z_2 = e + jf$. Show

1. The product can be written as

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

1. The complex product can also be written as

$$e = (a - b)d + a(c - d) \quad f = (a - b)d + b(c + d)$$

1. Show that this modified scheme can be expressed in matrix notation as

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (c-d) & 0 & 0 \\ 0 & (c+d) & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Solution

1.4-1.1

Complex matrix multiplication can be written as

$$z_1 z_2 = (a + jb)(c + jd)$$

Expanding and combining real and imaginary terms

$$\begin{gathered} z_1 z_2 = ac + ajd + cjb + bdj^2 \\ (ac - bd) + (ajd + cjb) \end{gathered}$$

Now lets expand the matrix form shown in the problem statement

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca - bd \\ da + cb \end{bmatrix}$$

Note that the grouped pairs match for real and imaginary parts.

1.4-1.2

This can be found by simply expanding and simplifying. Lets begin with e

$$\begin{aligned} e &= (a - b)d + a(c - d) \\ e &= ad - bd + ac - ad \\ e &= ac - bd \end{aligned}$$

Which matches the two solutions found before. Similarly for f

$$\begin{aligned} f &= (a - b)d + b(c + d) \\ f &= ad - bd + bc + bd \\ f &= ad + bc \end{aligned}$$

Which, again, matches what was found before.

1.4-1.3

Once again, we can show that they are equivalent by expansion and simplification. We will work from left to right performing matrix multiplication

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} (c-d) & 0 & 0 \\ 0 & (c+d) & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} (c-d) & 0 & d \\ 0 & (c+d) & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} (c-d) & -d \\ d & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca - bd \\ da + cb \end{bmatrix}$$

Which is equivalent to what was found in the previous problems.

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Problems Statement

Show that

$$k_j = \frac{1}{p^j j!}(-1)^j \frac{d^j}{d(z^{-1})^j}(1 - pz^{-}1)^r H(z)\Big|_{z=p}$$

for the partial fraction expansion of a Z-transform with repeated roots is correct.

Solution