Homework 9 Signals Brown, Alexander

6.23) Given:

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

Find:

Show $\lambda(T) = \lambda(T_{11}) \cup \lambda(T_{22})$

Solution:

$$\det(T - \lambda I_n) = \det\left(\begin{bmatrix} T_{11} - \lambda I_n & T_{12} \\ 0 & T_{22} - \lambda I_n \end{bmatrix}\right)$$

$$= \det(T_{11} - \lambda I_n) \cdot \det(T_{22} - \lambda I_n) + 0$$

Thus $\lambda(T) = \lambda(T_{11}) \cup \lambda(T_{22})$ //

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6.2-4) Given:

The determinant of an nxn matrix is the product of the eigenvalues.

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

Solution

If A and B are similar matrices, then they have the same eigenvalues.

That is if we can row reduce (or Σ in the SVD) a matrix A to have values only on the diagonals, call it B, then the following is true

A and B are similar matrices $\therefore$

$$\det(A) = \det(B) = \prod_{i=1}^{n} \lambda_i$$

Where λ_i are elements on the diagonal of B.

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6.2-7) Given:

A is a rank 1 matrix formed by $A = ab^T$

Find:

- Determine the eigenvalues and eigenvectors of A.
- Also show that if A is rank 1 then

$$\det(I + A) = 1 + tr(A)$$

Solution:

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & & & \vdots \\ a_nb_1 & a_nb_2 & & a_nb_n \end{bmatrix}$$

Because A is rank 1 we know $\lambda = 0$ with $n-1$ multiplicity. We can also say

$$Aa = (ab^T)a = \underbrace{a}_{x}\underbrace{(b^Ta)}_{\lambda}$$

the eigenvector $x = a$ and eigenvalue $\lambda = b^Ta$ //

Thus $\det(I + A) = 1 + \det(A) = 1 + \begin{bmatrix} b^Ta & & 0 \\ 0 & \ddots & \\ & & 0 \end{bmatrix}$

$= 1 + tr(A)$ //

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6.2-8) Given:

λ^* is an eigenvalue of matrix A.

Find:

Show $\lambda^* + r$ is an eigenvalue of $A + rI$ and $A + rI$ and A have the same eigenvectors

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Solutions:

Begin with $Av = \lambda v = \lambda^* v$

If we add rv to both sides

$(A + rI)v = (\lambda^* + r)v$ //

$\lambda = \lambda^*$ $\qquad\qquad$ $\lambda = \lambda^* + r$

$Av = \lambda^* I v$ // $\qquad$ $(A + \cancel{rI} - \lambda I - \cancel{rI})v = 0$

$Av = \lambda^* I v$

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6.2-9) Given:

If λ is an eigenvalue of A then λ^n is an eigenvalue of A^n, and A^n has the same eigenvecters as A.

Find:

Show the statement above is true

Solution:

$\det(A - \lambda I) = 0 \Rightarrow \lambda$

Let a be diagonal, that is $a_{ii} = \lambda_i$

Thus A^n has eigenvalues λ_i^n //

$A^n v = A^{n-1}(Av) = A^{n-1} \lambda v \cdots \lambda^n v$ //

6.2-10) Given:

If λ is a nonzero eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1} (1)

The eigenvectors of A corresponding to nonzero eigenvalues are eigenvectors of A^{-1} (2)

Find:

Show (1) and (2) are true

Solution:

$$Av = \lambda v = \underbrace{A^{-1}A}_{I} v = \lambda A^{-1} v$$

$$\Rightarrow A^{-1} v = \frac{1}{\lambda} v$$

This shows (1) and (2) by

a) relating $A^{-1}v = 1/\lambda v$ and

b) by pre multiplying by A^{-1} we have shown the equation is equivalent, thus v is the same for

$Av = \lambda v$ and $A^{-1} v = \frac{1}{\lambda} v$

6.2.11) Given

- $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalues of A
- $g(x)$ is a scalar polynomial

Find:

$g(A)$ has eigenvalues of $g(\lambda_1), g(\lambda_2), \ldots, g(\lambda_m)$

Solution

Theorem: $A \in M_n(F)$ where F is algebraically closed. If the spectrum of A is $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $P \in F[x]$ is a polynomial with coefficients in F, then

$$\det(P(A)) = \prod_{i=1}^{n} P(\lambda_i)$$

By the theorem

$$\det(P(A) - yI) = \det(f(A)) = \prod_{i=1}^{n} f(\lambda_i)$$

$$= \prod_{i=1}^{n} (y - p(\lambda_i))$$ //

Thus, $\{P(\lambda_1), P(\lambda_2), \ldots, P(\lambda_n)\}$ is in the spectrum of $P(A)$.

(6.2-12) Given: Projection matrix P

Find:

Show that the eigenvalues of a projection matrix is either 0 or 1.

Solution:

For projection matrices $P^2 = P$

$$\lambda^2 v = P^2 v = Pv = \lambda v$$

$$\Rightarrow \lambda^2 = \lambda \Rightarrow \lambda = 0, 1 \,//$$

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6.2-13) Given:

A and B are square matrices

Find:

a) $eig(BA) = eig(AB)$

b) A & B have n linearly independent vectors. Then $AB = BA$

Solution:

a) We know similar matrices have the same eigenvalues, then we can say

$$AB = ABAA^{-1} = A(BA)A^{-1}$$

Which says AB and BA are similar $\therefore$

$$eig(AB) = eig(BA) \quad //$$

b)

$$Av = \lambda I v = Bv = ABx = BAx \qquad s.t. \; v = Bx$$

Let $\{x_1, \ldots, x_n\}$ be an eigenbasis $E\lambda$ and let $v \in E\lambda$

$$B\{x_1, \ldots, x_n\} = \{Bx_1, \ldots, Bx_n\} = \underline{x}\begin{bmatrix} C & D \\ 0 & F \end{bmatrix}$$

Let $P = \{x_1, \dots, x_n\} \Rightarrow P^{-1}BP = \begin{bmatrix} C & D \\ 0 & F \end{bmatrix}$

$$\det(B - \lambda I) = \det(P^{-1}BP - \lambda I)$$

$$= \begin{bmatrix} C - \lambda I & D \\ 0 & F - \lambda I \end{bmatrix} = \det(C - \lambda I)\det(F - \lambda I)$$

if μ is an eigenvalue and a is an eigenvecter, Thus

$$\det(C - \mu I) = 0 \Rightarrow \det(B - \mu I) = 0$$

such that μ produces the eigenvecter $\underline{a}$. Now define

$$y = a_1 x_1 + \dots + a_k x_k \in E_\lambda$$

$$BP = P\begin{bmatrix} C & D \\ 0 & F \end{bmatrix} \Rightarrow \underbrace{BP\underline{a}}_{By} = \underbrace{P\begin{bmatrix} C & D \\ 0 & F \end{bmatrix}\underline{a}}_{\mu y}$$

$$\Rightarrow By = \mu y$$

Thus y is an eigenvecter of A and B. //

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6.2-15) Given:

a) $A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0 & 0.7 \\ 0.3 & 0.7 & 0.1 \end{bmatrix}$ Probability transition matrix

b) $\phi(t) = \sum_{k=0}^{N-1} c_k \, \phi(2t-k)$

$\phi(t) = 0$ if $t \leq 0$ and $t \geq N-1$

Find:

a) Determine steady-state probability p such that $Ap = p$

b) Write an equation of the form

$$\begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(N-2) \end{bmatrix} = A \begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(N-2) \end{bmatrix}$$

A is the matrix of wavelet coefficients c_k. Specify A and describe how to solve the equation. Describe how to find $\phi(t)$ and all dyadic rational numbers

Solution

a) $eig(A) = 1$ and $Av = \lambda v = v$

$\lambda = 1 \quad (A - I)p = 0$

$$\Rightarrow \begin{bmatrix} -0.5 & 0.3 & 0.2 \\ 0.2 & -1 & 0.7 \\ 0.3 & 0.7 & -0.9 \end{bmatrix} p = 0$$

see P15.m

$p = [0.57199 \quad 0.54469 \quad 0.61384]^T$ //

b) $\phi(t)$ follows the shift orthogonality property. That is

$$\phi(t) \perp \phi(t-n) \quad \text{s.t. } n \in \mathbb{Z}$$

Thus

$$A = \begin{bmatrix} c_1 & & \\ & c_2 & 0 \\ 0 & \ddots & \\ & & c_{N-2} \end{bmatrix}$$

Thus we have the eigenvectors of A. Similarly to before we can say

$(A - I\lambda)p = 0$ where $p = \begin{bmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \end{bmatrix}$

and we can substitute c_k for λ. //

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6.3-18) Given:

A and B are similar $B = T^{-1}AT$

Find:

a) A and B have the same eigenvalues and the same characteristic equations

b) If x an eigenvector of A then $z = T^{-1}x$ is an eigenvector of B.

c) If C and D are similar with $D = T^{-1}CT$ then $A+C$ is similar to $B+D$

Solution:

a,b)

$$B = T^{-1}AT \iff TBT^{-1} = A$$

$$TBT^{-1}v = \lambda v \implies BT^{-1}v = \lambda T^{-1}v$$

Thus if v is an eigenvector of A with eigenvalue λ $T^{-1}v$ is an eigenvector B with the same eigenvalue.

c) $B = T^{-1}AT \implies B+D = T^{-1}AT + T^{-1}CT$

$$A+C = TBT^{-1} + TDT^{-1}$$

$$(A+C)v = \lambda v = (TBT^{-1} + TDT^{-1})v = \lambda v$$

$$= (BT^{-1} + DT^{-1})v = \lambda T^{-1}v$$

Which raises the same argument as before //

6.3-21) Given:

A self-adjoint matrix is positive definite iff all of its eigenvalues are ≥ 0 (1)

Find:

- Show (1) is true
- Also show that if all the eigenvalues are positive, then the matrix is positive definite

Solution:

A self adjoint matrix has the property

$$A^* = A^H$$

A real matrix that is said to be self-adjoint is said to be symmetric.

A symmetric matrix is always diagonalizable, thus its eigenvectors form a basis. Let the set of eigenvectors be $\{e_1, \dots, e_n\}$ be an orthonormal basis corresponding to the eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

$$x = c_1 e_1 + c_2 e_2 + \dots + c_n e_n$$

So $x^T A x = \{c_1 e_1^T + \dots + c_n e_n^T\}\{c_1 A e_1 + \dots + c_n A e_n\}$
$= c_1^2 \lambda_1 + \dots + c_n^2 \lambda_n \geq 0$ //

Similarly, if the eigenvalues are strictly positive, then we can say

$$c_1^2\lambda_1 + \cdots + c_n^2\lambda_n > 0 \quad //$$

6.3-24) Given:

For a self-adjoint matrix, the eigenvectors corresponding to distinct eigenvalues are orthogonal.

Find:

Prove the theorem above by

a) Show if A is self-adjoint and U is unitary, then $T = U^H A U$ is also self-adjoint

b) Show that if a self-adjoint matrix is triangular, then it must be diagonal.

Solution

a) A matrix is unitary if

$$U^H U = I = UU^{-1} = UU^H$$

A matrix is self-adjoint if $A = A^H$

$T^H = U^H A U$ and $T^H T = TT^H = T^2$

$$TT^H = U^H A U U^H A^H U = U^H A^2 U = T^2 \quad //$$

b)

If $A = A^H$ and is triangular, suppose

$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ which is triangular.

$A \neq A^H$ unless $a = 0$, thus A must be diagonal.
Extrapolating, any $n \times n$ matrix

$$A = \begin{bmatrix} 1 & \cdots & a & \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

For $A^H = A$, a must be 0, similarly for any other off-diagonal terms. //

6.3-28) Given:

$$F = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & \cdots & e^{-2\pi(N-1)/N} \\ 1 & e^{-j4\pi/N} & \cdots & e^{-4\pi(N-1)/N} \\ \vdots & & & \vdots \\ 1 & e^{-2\pi(N-1)/N} & \cdots & e^{-j2\pi(N-1)^2/N} \end{bmatrix}$$

that if $f_{ij} = e^{-j2\pi ij/N}$, for a vector $\underline{x}$

$$X = F\underline{x}$$

find:

a) Prove $F/\sqrt{N}$ is unitary. Hint: Show

$$\sum_{n=0}^{N-1} e^{j2\pi nk/N} = \begin{cases} N & k \equiv 0 \mod N \\ 0 & k \not\equiv 0 \mod N \end{cases}$$

b) A matrix

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ \vdots & & & & \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix}$$ circulant matrix

Show C is diagonizable by F, $CF = F\Lambda$ where Λ is diagonal. Comment on the eigenvalues and eigenvectors of a circulant matrix.

Solution:

a) $\sum_{n=0}^{N-1} e^{j2\pi nk/N}$ if $k \equiv 0 \bmod N$ $= 0$

$\Rightarrow \sum_{n=0}^{N-1} 1 = N$

If $k \not\equiv 0 \bmod N$ then

$\sum_{n=0}^{N-1} e^{j2\pi nk/N}$ by Example 3.17.2 forms an

orthonormal set thus is 0. for $n=m$.
Therefore if we let $k \equiv 0 \bmod N$ then

$$F = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \Rightarrow \frac{F}{\sqrt{N}} = \begin{bmatrix} 1/\sqrt{N} & 1/\sqrt{N} & \cdots & 1/\sqrt{N} \\ 1/\sqrt{N} & 1/\sqrt{N} & \cdots & 1/\sqrt{N} \\ & & & \vdots \\ 1/\sqrt{N} & 1/\sqrt{N} & \cdots & 1/\sqrt{N} \end{bmatrix}$$

$\Rightarrow$ Let $\frac{F}{\sqrt{N}} = B$ and $BB^* = BB^H = I$ //

b) $CF = F\Lambda \Rightarrow C = F\Lambda F^{-1}$

$\Rightarrow \Lambda = F^{-1}CF$ which is said to be a diagonalization of C by F. Thus the eigenvectors are not linearly dependent and the eigenvalues are the diagonal terms. //

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6.4-30) Given:

Construct 3x3 matrices given the following conditions

Find:

a) $\lambda_1 = \lambda_2 = 1$ $\lambda_3 = 2$

$R_1 = \text{span}([1,2,1]^T, [-2,1,0]^T)$
$R_2 = \text{span}([-1,-2,5]^T)$

Find eigenvalues and vectors of constructed matrix.

b) $\lambda_1 = 1$ $\lambda_2 = 4$, $\lambda_3 = 9$ with corresponding eigenvectors

$$x_1 = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} \quad x_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\\0\end{bmatrix} \quad x_3 = \frac{1}{\sqrt{70}}\begin{bmatrix}-3\\-6\\5\end{bmatrix}$$

Solution:

a) See P30.m.

b) See P30.m

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6.7-39) Given:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 7 \end{bmatrix} \qquad (1)$$

Find:

Determine regions in the complex plane where the eigenvalues of (1)

Solution:

$$R_i(A) = \left\{ x \in \mathbb{C} : |x - a_{ii}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \right\}$$

$$|x-3| \le 2 \; ; \quad |x-4| \le 3 \; ; \quad |x-7| \le 3$$

6.7-41) Given:

Real $m \times m$ matrix with disjoint Gershgorin circles

Find:

Show the eigenvalues of A are real

Solution:

Let $a \in A$ and λ be an eigenvalue of A. Let disk q be created from any row in A and let q be disjoint from all other disks. If

$$\lambda = x + jy$$

and $x, y \neq 0$ then another eigenvalue of A is

$$\lambda = x - jy$$

since it is in q. However that means that there are 2 eigenvalues in q which contradicts the theorem that states if the discs are disjoint, then there is only 1 eigenvalue.

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6.8-45) Given:

$x_1 = [1, 5]^T$ $\quad x_2 = [-5, 1]^T$

$\lambda_1 = 10$, $\lambda_2 = 2$

Find

a) Construct A symmetric matrix with the properties above

b) Generate and Plot 200 points of zero-mean Gaussian data that have covariance R

c) Form an estimate of R

d) Plot the principal components.

Solution:

See P45.m

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7.1-1) Given:

- Theorem 7.1
- $\langle u_i, u_j \rangle = \delta_{i-j}$ (1)

Find -

Show (1) is true

Solution:

$u_i = \frac{A v_i}{\sqrt{\lambda_i}}$ where λ_i is an eigenvalue of A
v_i is an eigenvector of A

$\langle u_i, u_j \rangle = \delta_{i-j}$ where δ is the Kronecker delta.

Let $\langle a, b \rangle = b^T a$

$\Rightarrow \langle u_i, u_j \rangle = \frac{v_j^H A^H A v_i}{\sqrt{\lambda_i}\sqrt{\lambda_j}} = \frac{v_j^H A^H \lambda_i v_i}{\sqrt{\lambda_i}\sqrt{\lambda_j}}$

We know $Av = \lambda v \Rightarrow v^H A^H = \lambda v^H$

$\Rightarrow \frac{\lambda_i \lambda_j v_j^H v_i}{\sqrt{\lambda_i}\sqrt{\lambda_j}}$ and we know $v_i \perp v_j$ if $i \neq j$

$\Rightarrow \langle u_i, u_j \rangle = \delta_{i-j}$ //

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7.1-2) Given:

- Matrix 2 norm and Frobenius norm

Find:

$$\|A\|_F^2 = \sum_{i=1}^{p} \sigma_i^2$$

$$\|A\|_2 = \sigma_1$$

are true.

Solution:

$A = U \Sigma V^H$

U is unitary
V is unitary
$\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p)$

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} ; \|A\|_F^2 = tr(AA^H)$$

$\|A\|_2 = \sqrt{\rho(A^H A)}$ and if A is Hermitian $\|A\|_2 = \rho(A)$

Because U and V are unitary, they drop out of the norm leaving

$$\|A\|_F^2 = tr(AA^H) = \sum_{i=1}^{p} \sigma_i^2 \quad //$$

$\Sigma_1 = \Sigma_1^4$ because it is diagonal thus

$\|A\|_2 = \rho(A)$ where $\rho(A) = \max_i |\lambda_i|$

Since Σ_1 has σ_i in descending order

$\|A\|_2 = \rho(\Sigma_1) = \sigma_1$ //

7.2-3) Given:

$$R(A) = \text{span}(U_1)$$
$$N(A) = \text{span}(V_2) \quad (1)$$
$$R(A^H) = \text{span}(V_1)$$
$$N(A^H) = \text{span}(U_2)$$

Find:

Show that the SVD can be used to determine the fundamental subspaces in (1)

Solution: Note the end for definition of U_1, U_2, V_1, V_2.

Let $A = U\Sigma V^H$ be the SVD of A.

$$R(A) = b : b = U\Sigma V^H x$$
$$= U\Sigma y$$
$$= U_1 y = \text{span}\{U_1\}$$

$$R(A^H) = V\Sigma U^H x = V\Sigma y = V_1 y = \text{span}(V_1)$$

$$N(A) = U\Sigma V^H x = 0$$
$$= V_2 x = \text{span}(V_2)$$

$$N(A^H) = V\Sigma U^H = U_2 x = \text{span}(U_2)$$

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Note

$$U\Sigma_1 V^H = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$

$$\Sigma_{12} = 0$$

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7.3-5) Given:

The LS solution to $\|Ax - b\|_2^2$ is

$$E^2_{min} = \|U_2^H b\|_2^2 \qquad (1)$$

Find:

- Show (1)
- Interpret (1) in terms of the four fundamental subspaces.

Solution:

$$\|Ax - b\|_2^2 = \|U\Sigma V^H \cdot b\|_2^2$$

$$= \|U(\Sigma V^H - U^H b\|_2^2$$

$$= \|\Sigma \underline{v} - \hat{b}\|_2^2$$

Where $\underline{v} = V^H x$ and $\hat{b} = U^H b$

$$\Rightarrow \min \|\Sigma \underline{v} - \hat{b}\|_2^2 = \left\| \begin{matrix} \Sigma_1 \underline{v_1} = \hat{b}_1 \\ \Sigma_2 \underline{v_2} = \hat{b}_2 \end{matrix} \right\|_2^2$$

$$\Rightarrow \|v = \Sigma_1^+ \hat{b}\|_2^2 = \|\Sigma_1^+ U^H \underline{b}\|_2^2$$

$$= \Sigma_1^+ \|U^H \underline{b}\|_2^2 = \Sigma_1^+ \left\| \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\|_2^2$$

Σ_2 is 0, but to get the minimum norm we want

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V_2 to be 0. This means that, in terms of the fundamental subspaces, no "data" is lost because V_2 is 0.

7.5-7) Given:

$$\|A-B\|_2^2 \overset{(1)}{\geq} \|Az\|_2^2 = \sum_{i=1}^{k+1} \sigma_i^2 (v_i^H z)^2 \overset{(2)}{\geq} \sigma_{k+1} \quad (1)$$

Find:

a) Show the inequalities in (1) are correct

b) Show the conditions for achieving the lower bound are correct.

Solution:

(1)

$$\|A-B\| \geq \|(A-B)z\|_2^2 = \|Az\|_2^2$$

where $z \in \text{span}\{v_1, v_2, \dots, v_{k+1}\}$

(2)

$$\|Az\|_2^2 = \sum_{i=1}^{k+1} \sigma_i^2 (v_i^H z)^2 \geq \sigma_{k+1}^2 \sum_{i=1}^{k+1} (v_i^H z)^2$$

$$= \sigma_{k+1}^2$$

b) Let $B = \sum_{i=1}^{k} \sigma_i u_i v_i^H$ and $z = v_{k+1}$

$$A = U \Sigma V = \sum_{i=1}^{r} \sigma_i u_i v_i^H$$

$$\|A-B\| = \left\| \sum_{i=1}^{r} \sigma_i u_i v_i^H - \sum_{i=k}^{k} \sigma_i u_i v^H \right\|$$

$$= \left\| \sum_{i=k}^{r} \sigma_i u_i v_i^H \right\|$$

$$\left\| \sum_{i=k}^{r} (\sigma_i u_i v_i) z_i \right\| \quad \text{where } z = v_{k+1}$$

$$\Rightarrow \left\| \sum_{i=k}^{r} (\sigma_i u_i v_i) v_{k+1} \right\| = 0 \quad //$$

7.7-12) Given:

P is on the plane $\perp$ normal vector $n \in \mathbb{R}^{n+1}$

$$P = \{ r \in \mathbb{R}^{n+1} : r^T n = 0 \}$$

Let $n = \begin{bmatrix} x \\ -1 \end{bmatrix}$ and $p = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^{n+1}$

Find:

Show the shortes squared distance from p to P is

$$\frac{(x^T a - b)^2}{x^T x + 1}$$

Solution:

$$[A + E \mid b + r] \begin{bmatrix} x \\ -1 \end{bmatrix} = 0$$

$$(\underbrace{[A \mid b]}_{C} + \underbrace{[E \mid r]}_{\Delta}) \begin{bmatrix} x \\ -1 \end{bmatrix} = 0$$

$$S_c = R(\bar{V}) = \text{span}(V_{k+1}, V_{k+2}, \ldots, V_{n+1})$$

$$\min_x \frac{\left\| [A|b] \begin{bmatrix} x \\ -1 \end{bmatrix} \right\|^2}{\left\| \begin{matrix} x \\ -1 \end{matrix} \right\|^2} = \min_x \sum \frac{|a_i^H x - b_i|^2}{x^H x + 1}$$

$$= \min_x \sum \frac{(a_i^H x - b_i)^2}{x^H x + 1} = \min_x \sum \frac{(x^H a_i - b)^2}{x^T x + 1}$$

Because $n \in \mathbb{R}^{n+1}$

$$= \min_x \sum \frac{(x^T a_i - b)^2}{x^T x + 1}$$ since $p = [a\ b]$ is a single point

$$= \min_x \frac{(x^T a - b)^2}{x^T x + 1}$$ noting that $x^T a = b$ we want an x that minimizes the square distance with b.

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7.7-13) Given:

$\bar{V} = [V_k, V_{k+1}, \ldots, V_{n+1}] \in \mathbb{R}^{(m+1)(n-k+2)}$. In TLS we sought an element $y \in R(\bar{V})$ such that $x = \bar{I}y$ of minimum norm where

$$\bar{I} = \begin{bmatrix} I_m & \\ & 0 \end{bmatrix}$$

picks out the first m elements of y, and $y_{m+1} = -1$.

Find:
Formulate the problem of minimum-norm x as a constrained optimization problem, and determine the solution.

Solution:

$\max \|\bar{I}y\|^2$ s.t. $y_{m+1} = -1$

$\bar{I} = \sum_{i=1}^{m} \sigma_i u_i v_i^H \qquad m = \text{rank}(I_m)$

$v_{m+1} \perp v_i \qquad i = 1, 2, \ldots, m$

$\Rightarrow y = -v_{m+1}$ recall $y_{m+1} = -1 \therefore$

$$y = \frac{-\bar{V}_{m+1}}{V_{m+1,m+1}}$$

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7.9-15) Given:

$$A = \begin{bmatrix} 4.85 & -2.47 & 0.55 & -4.32 \\ 0.81 & -3.46 & -0.25 & 2.97 \end{bmatrix}$$

$$B = \begin{bmatrix} 4.51 & -2.19 & 0.59 & -4.72 \\ 1.95 & -3.64 & -0.131 & 2.29 \end{bmatrix}$$

A and B are rotations of each other

Find:

The amount of rotation between the data sets.

Solution:

maximize $tr(Q^{H} B^{H} A)$ Q is unitary

The maximizing Q can be found by means of SVD of $B^{H}A$.

Theorem: If

$$B^{H}A = U \Sigma V^{H}$$

then the maximizing Q for the orthogonal Procrustes problem is

See P715.m

$$Q = UV^{T}$$

$$A - BQ = \begin{bmatrix} -0.029 & 0.293 & 0.0281 & -0.313 \\ -0.453 & 0.266 & -0.047 & 0.360 \end{bmatrix}$$

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7.10-17) Given:

$A = B + jC$ B and C are real

Find:

Determine a means of finding the SVD of A in terms of the SVD of

$$\begin{bmatrix} B & -C \\ C & B \end{bmatrix}$$

Solution:

$$U^H A V = \begin{bmatrix} B \\ 0 \end{bmatrix} + j \begin{bmatrix} C \\ 0 \end{bmatrix}$$

Where we attempt to get the diagonals to be eigenvalues of A. Much like in the book.

$$\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

But because C is complex it will have positive and negative eigenvalue pairs resulting in

$$\begin{bmatrix} 0 & -C \\ C & 0 \end{bmatrix}$$

Thus by adding them we get

$$\begin{bmatrix} B & -C \\ C & B \end{bmatrix}$$

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