1)

As $\rho \to 0$ the circle "collapses" in a dot at 0.

As $\rho \to 0$ then be create squares with rounded edges.

A.5-17)

Given:

Let (X, d_2) be a metric space of functions defined on $\mathbb{R}$ with the Euclidean metric

$$d_2^2(f,g) = \int_{-\infty}^{\infty} (f(t)-g(t))^2\,dt$$

Define the mapping $\Phi_\phi : X \to \mathbb{R}$ by

$$\Phi_\phi(x) = \int_{-\infty}^{\infty} x(t)\,\phi(t)\,dt$$

Find:

Show that if $\phi(t)$ is square integrable, that is,

$$\int_{-\infty}^{\infty} \phi(t)^2\,dt < \infty$$

then Φ_ϕ is a continuous mapping

Solution:

Def: A function $g: \mathbb{R} \to \mathbb{R}$ is continuous if $\forall \epsilon > 0 \; \exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x-y| < \delta$$

$$\Phi_\phi(x) = \int_{-\infty}^{\infty} x(t)\phi(t)\,dt \le \left(\int_{-\infty}^{\infty} x(t)^2\,dt\right)^{1/2}\left(\int_{-\infty}^{\infty} \phi(t)^2\,dt\right)^{1/2}$$

$$\int_{-\infty}^{\infty} |\Phi_\phi(x)|^2\,dt \le \left(\int_{-\infty}^{\infty} x(t)^2\,dt\right)^{1/2}\left(\int_{-\infty}^{\infty} \phi(t)^2\,dt\right)^{1/2}$$

$$\|\Phi_\phi(x)\|^2 = \int_{-\infty}^{\infty} |\Phi_\phi(x)|^2\,dt$$

$$\le \left(\int_{-\infty}^{\infty} x(t)^2\,dt\right)^{1/2}\left(\int_{-\infty}^{\infty} \phi(t)^2\,dt\right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty} x(t)^2\,dt\right)^{1/2} \|\phi(t)\|^2$$

Hence $\Phi_\phi(x)$ is a bounded linear operator on the Hilbert space. and is continuous.

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2.1-1)

Given:

$$x = [1\ 2\ 3\ 4\ 5\ 6]^T$$

Find:

Compute l_p metric $d_p(x,0)$ for $p = 1, 2, 4, 10, 100, \infty$.

Comment on why $d_p(x,0) \to \max(x_i)$ as $p \to \infty$

Solution:

Generally

$$d_p(x,y) = \left[\int_a^b |x(t) - y(t)|^p \, dt \right]^{1/p}$$

$p = 1$

$$d_1(x,0) = \left[1 + 2 + 3 + 4 + 5 + 6 \right] = 21$$

$p = 2$

$$d_2(x,0) = \left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 \right]^{1/2} = 9.5394$$

$p = 4$

$$d_4(x,0) = \left[1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 \right]^{1/4} = 6.9063$$

$p = 10$: $d_{10}(x,0) = 6.1001$

$p = 100$: $d_{100}(x,0) = 6.000$

$p_\infty = 6$

As $p \to \infty$, the value is converging to the largest value in the set

//

2.1-4)

Given

a) $x, y \in \mathbb{R}$

$$|x+y| \le |x| + |y|$$

b) $x, y \in \mathbb{R}$

$$\|x+y\| \le \|x\| + \|y\|$$

Find:

Prove (a) & (b)

a) What is the condition for equality?

b) Hint $\sum_{i=1}^{n} x_i y_i \le \|x\| \, \|y\|$

Solution

a) $a^2 + b^2 + 2|a||b| \ge a^2 + b^2 + 2ab$

$$(|a| + |b|)^2 \ge |a+b|^2$$

$$|a| + |b| \ge |a+b| \quad //$$

Equality occurs when $x = \alpha y$. //

b) $\|x+y\|^2 = \|x\|^2 + 2(xy) + \|y\|^2$

Cauchy-Schwartz Inequality $\underline{|xy| \le \|x\| \, \|y\|}$ $\Rightarrow \|x\|\|y\| > xy$

$$\|x\|^2 + 2(xy) + \|y\|^2 \le \|x\|^2 + 2(\|x\|\|y\|) + \|y\|^2$$

$$\|x\|^2 + 2(xy) + \|y\|^2 \le (\|x\| + \|y\|)^2$$

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2 \Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

//

2.1-5)

Given:

Let (X, d) be a metric space

Find:

Show $d_0(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ is a metric on X.

~~What significant feature does this metric possess?~~

Solution:

Def 2.2: A metric space (X, d) is a set X together with a metric d.

We then need to show $d_0(x,y)$ is a metric

Def 2.1: A metrix $d: X \times X \to \mathbb{R}$ is a function that is used to measure distance between elements in X. In order to be a metric, it must satisfy the following properties $\forall\, x,y \in X$

1) $d(x,y) = d(y,x)$
2) $d(x,y) \geq 0$
3) $d(x,y) = 0$ iff $x=y$
4) $\forall\, x,y,z \in X$

$$d(x,z) \leq d(x,y) + d(y,z)$$

1)

$$d_0(x,y) = d_0(y,x) \Rightarrow \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)}$$

If $d(y,x)$ is an l_p type metric.

2) If $d(x,y) \triangleq \lambda_p(x,y)$ the smallest it can be is

$$\frac{0}{1+0} = 0 \quad //$$

3) If $x=y$ and we assume $\lambda_p(x,y)$ we get

$$d_b(x,y) = \frac{0}{1+0} = 0 \quad //$$

4) Let $d(x,y) \leq d(z,w)$

$$d(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$$

$$= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)}$$

$$\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(y,z)}{1+d(y,z)} = d(x,z)+d(z,y) \quad //$$

This metric space is always less than 1.

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2.1-6)

Given:

Let $dm(x,y) = \min(1, d(x,y))$

Find:

Show that this is a metric space (X, dm)

What is the significance?

Solution

We must show $dm(x,y)$ is a metric

1) We know $d(x,y) = d(y,x)$ then

$\min(1, d(y,x)) = \min(1, d(x,y))$ //

2) By virtue the minimum is be used, $d(x,y)$ can be at the least 0.

//

3)

$d(x,x) = 0 \Rightarrow \min(1, d(x,x)) = 0$ //

4) Let $d(x,y) \geq 1$ then

$d(x,y) + d(y,z) \geq 1 \geq d(x,z)$ //

The metric is another way to have a range between 0 and 1

2.1-9)

Given:

If requirement M3 in the definition of a metric is relaxed to

$$d(x,y) = 0 \quad \text{if } x = y$$

allowing $d(x,y) = 0$ even if $x \neq y$ then a pseudometric is obtained.

Find:

Let $f : X \to \mathbb{R}$ be an arbitrary function defined on the set X. Show that $d(x,y) = |f(x) - f(y)|$ is a pseudo metric.

Solution

$$|\cdot| = \sqrt{a^2 + b^2 + \dots}$$

Then

1) $\sqrt{f(x)^2 - f(y)^2} = \sqrt{f(y)^2 - f(x)^2}$ //

2) Let $f(x) \leq f(y)$

$$\sqrt{f(x)^2 - f(y)^2} \geq 0 \quad //$$

3) Let $\{s \in X : f(s) = 0\}$

$\forall s \in X \quad d(x,y) = 0$ //

4) $d(x,z) \leq d(x,y) + d(z,y)$ identically //

//

2.1-14)

Given:

$A = (0,4)$, $B = (0, \infty)$, $C = (-\infty, 5]$

Find

inf and sup for the given sets

Solution

a)

$\inf(A) = 0$

$\sup(A) = 4$ //

b)

$\inf(B) = 0$

$\sup(B) = \infty$ //

c)

$\inf(C) = -\infty$

$\sup(C) = 5$ //

2.1-23)

Given:

$$x_n = \int_1^n \frac{\cos(t)}{t^2}\,dt \qquad d(x,y) = |x-y|$$

Find:

Show the sequence x_n is convergent using $d(x,y)$.

Hint:

Show that x_n is a Cauchy sequence. Use the fact that

$$\int \frac{|\cos t|}{t^2}\,dt \le \int \frac{1}{t^2}\,dt$$

Solution

$$x_n = \int_1^n \frac{\cos(t)}{t^2}\,dt \le \int_1^n \frac{|\cos(t)|}{t^2}\,dt \le \int_1^n \frac{1}{t^2}\,dt$$

$$x_n \le \int_1^n \frac{1}{t^2}\,dt$$

$$\text{if } d(x,y) = |x-y| = \sqrt{\left(\sum_{a=1}^{n} \frac{1}{a^2}\right)^2 + \left(\sum_{b=1}^{m} \frac{1}{b^2}\right)^2}$$

$$d(x,y) = \sum_{a=1}^{n} \frac{1}{a^2} + \sum_{b=1}^{m} \frac{1}{b^2} \le \sum_{a=1}^{n} \frac{1}{a^2} < \epsilon$$

Therefore it must converge //