Homework 4 Signals Alexander Brown

2.2-28)

Given:

S is a finite dimensional vector space with $\dim(S) = m$.

Find:

Show that every set containing $m+1$ points is linearly dependent. Use induction.

Solution:

Theorem: A linear system of equations has either

- A unique solution
- No solution
- Infinitely many solutions

Theorem: Every homogenous linear system with more variables than equations has infinitely many solutions.

Theorem: An ordered set of non-zero vectors, $v_1, \ldots, v_n$ is linearly dependent iff on vector v_k is expressable as a linear combination of the preceding vectors.

Let $c_i = 0 \;\forall\; i = 1, 2, \ldots, m$ for the linearly independent set

$$c_1v_1 + c_2v_2 + \cdots + c_{m-1}v_{m-1} + c_mv_m = 0$$

Now increase $\dim(S) = m+1$. We can then write

$$c_1v_1 + c_2v_2 + \cdots + c_{m-1}v_{m-1} + c_mv_m + c_{m+1} = 0$$

We know $v_1 - v_m$ are linearly independent

$$c_1v_1 + c_2v_2 + \cdots + c_{m-1}v_{m-1} + c_mv_m = -c_{m+1}v_{m+1} = 0$$

$$v_{m+1} = \frac{0}{c_{m+1}} \quad \therefore \quad c_{m+1} \in \mathbb{R} \quad \text{(infinitely many solutions)}$$

2.2 - 31)

Given:

$S = L_2[0, 2\pi]$ and $T = \{x(t) = e^{jnt} ; n = 0, 1, \dots\}$

Find:

- Show T is linearly independent.
- Conclude that $L_2[0, 2\pi]$ is an infinite-dimensional space.

Hint: ~~Assume that~~ $c_1 e^{jn_1 t} + c_2 e^{jn_2 t} + \dots + c_m e^{jn_m t} = 0$ for $n_i \neq n_j$ when $i \neq j$. Differentiate $(m-1)$ times and use the properties of Vandermonde matrixes (section 8.4).

Solution:

Let

$$\frac{d}{dt}(e^{jn_1 t}) = \underbrace{jn_1}_{z_1} e^{jn_1 t}$$

$$\frac{d^2}{dt^2}(e^{jn_2 t}) = \underbrace{(jn_2)^2}_{z_1^2} e^{jn_2 t}$$

$$\vdots$$

$$\frac{d^{m-1}}{dt^{m-1}}(e^{jnt}) = \underbrace{(jn)^{m-1}}_{z_1^{m-1}} e^{jnt}$$

$$V = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_{m-1} \\ z_0^2 & z_1^2 & \cdots & z_{m-1}^2 \\ \vdots & \vdots & & \vdots \\ z_0^{m-1} & z_1^{m-1} & \cdots & z_{m-1}^{m-1} \end{bmatrix}$$

The detormanent of V, which is of the form of a Vandermonde matrix, is

$$\det(V) = \prod_{\substack{i,j=1 \\ i>j}}^{n} (z_i - z_j)$$

$\therefore \det(V) \neq 0 \iff$ linearly independent //

1/1

2.2-32)

Given:

The set $\{1, t, t^2, \ldots, t^m\}$

Find:

Show the set is linearly independent.

Hint: The fundamental theorem of algebra states that a polynomial f(x) of degree m has exactly m roots, counting multiplicity.

Solution

Let

$f(t) = a_0 t^0 + a_1 t^1 + a_2 t^2 + \cdots + a_m t^m = 0$

To show linear independence, show that there are finite solutions. Rewrite the function as

$a_0 + a_1 t^1 + a_2 t^2 + \cdots + a_m t^m = 0$

$a_0 + t(a_1 + a_2 t) + t^3(a_3 + a_4 t) + \cdots + t^{m-1}(a_{m-1} + a_m t) = 0$

which as infinitaly many roots, therefore the only set $a_i \ \forall i = 1-m$ s.t. $f(t) = 0$ is $a_i = 0 \ \forall i = 1-n$.

//

//

2.3-33)

Show that a normed linear space

$$\big|\,\|x\| - \|y\|\,\big| \leq \|x-y\|$$

Solution:

$$\|x-y\|^2 = \|x-y\|\,\|x-y\| = x^2 - 2xy + y^2$$

$$x^2 - 2xy + y^2 \geq x^2 - 2|x||y| + y^2$$

$$\|x-y\|^2 \geq (\|x\| - \|y\|)^2$$

$$\|x-y\| \geq \big|\,\|x\| - \|y\|\,\big|$$

//

1/1

2.3-34)

Show that a norm is a convex function

Solution:

Def A.5: A function f is convex over an open set D if $\forall\ s,t \in D$

$$f(\lambda s + (1-\lambda)t) \le \lambda f(s) + (1-\lambda) f(t)$$

Let V be a vector space $\|\cdot\| : V \to \mathbb{R}$ and let $v, w \in V$

$$\|\lambda v + (1-\lambda)w\| \le \|\lambda v\| + \|(1-\lambda)w\| = \lambda\|v\| + (1-\lambda)\|w\|$$

$\therefore$ By definition the norms are convex. //

2.3-35)

Show that every Cauchy sequence $\{x_n\}$ in a normed linear space is bounded.

Solution:

Normed Linear Space

1) $||x|| > 0$

2) $||x|| = 0 \Rightarrow x = 0$

3) $||\alpha x|| = |\alpha| \, ||x||$

4) $||x+y|| \leq ||x|| + ||y||$

If a sequence is cauchy then given a metric space (X, d), the sequence

$$x_1, x_2, \ldots$$

is Cauchy $\forall \epsilon > 0$ there is a positive integer N s.t. $\forall m, n > N$, the distance

$$d(x_m, x_n) < \epsilon$$

For normed linear spaces we have the induced norm $d(x,y) = ||y - x||$. Thus, if a sequence is cauchy, then the induced norm

$$\lim_{m,n \to \infty} d(x_m, x_n) = 0$$

2.3 - 38)

Given:

S is a normed linear space

Find:

Show that the norm function

$$\|\cdot\| : S \to \mathbb{R}$$

is continuous.

Hint: See exercise 2.3-33

Solution

Let $(X, \|\cdot\|)$ be a normed space. Let $\epsilon > 0$ and $\{x_n\}$ be an arbitrary sequence in X that converges to $x \in X$.

$$\exists N \in \mathbb{N} : n > N \Rightarrow \|x_n - x\| < \epsilon$$

$$\|x_n - x\| \geq \big|\, \|x_n\| - \|x\| \,\big|$$

$$\exists N \in \mathbb{N} : n > N \Rightarrow \big|\, \|x_n\| - \|x\| \,\big| < \epsilon$$

$\therefore$ It is continuous //

//

1/1

2.5-43)

Given;

Induced norm $\|\cdot\|$ over a real vector space

Find:

show that it is true for the following

a) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

b) $\langle x,y\rangle = \dfrac{\|x+y\|^2 - \|x-y\|^2}{4}$

Solution:

a) $\|x+y\|^2 + \|x-y\|^2$

$= \|x\|^2 + \|2xy\| + \|y\|^2 + \|x\|^2 - \|2xy\| + \|y\|^2$

$= 2\|x\|^2 + 2\|y\|^2$ //

b) $\dfrac{\|x+y\|^2 - \|x-y\|^2}{4} = \dfrac{\|x\| + 2\|xy\| + \|y\|^2 - \|x\| + 2\|xy\| - \|y\|}{4}$

$= \dfrac{4\|xy\|}{4} = \|xy\| = \langle x,y\rangle$ //

2.8-51)

Show by integration that

$$\int_{-1}^{1} \frac{1}{\sqrt{1+t^2}} T_n(t) T_m(t)\, dt = \begin{cases} \pi & n=m=0 \\ \pi/2 & n=m \; n\neq 0 \\ 0 & n\neq m \end{cases}$$

Hint: Use $t=\cos(x)$ in the integral

Solution:

$T_n(t) = \cos(n\cos^{-1}(t))$

$n=m=0$: $T_0(t)=1$

$$\int_{-1}^{1} \frac{1}{\sqrt{1+t^2}}\, dt = \sin^{-1}(t)\Big|_{-1}^{1} = \sin^{-1}(1) - \sin^{-1}(-1)$$

$$\approx \pi //$$

$n=m\neq 0$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} T_n^2(t)\, dt$$

Let $t=\cos(x)$

$T_n(\cos(x)) = \cos(nx)$ /

$$\int_{-1}^{1} \frac{1}{\sqrt{1-\cos(x)^2}} \cos(nx)^2 \, dx$$

is the same as saying

$$\int_{-1}^{1} \frac{t^2}{\sqrt{1-t^2}} \, dt = \frac{1}{2}\left(\sin^{-1}(t) - t\sqrt{1-t^2}\right)\Big|_{-1}^{1}$$

$$\approx \pi/2 \; //$$

$n \neq m$:

Let $m=0$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} T_n(t)\, T_m(x) = \int_{-1}^{1} \frac{1}{\sqrt{1-\cos(x)^2}} \cos(nx) \cdot 1 \, dx$$

$$\int_{-1}^{1} \frac{t}{\sqrt{1-t^2}} \, dt = -\sqrt{1-x^2}\Big|_{-1}^{1} = 0 \; //$$

2.10 - 52)

Show that the orthogonal compliment to a subspace is a subspace

Solution:

Let V be a subspace of S

Show

1) $V^{\perp}$ is nonempty

2) closed under scalar multiplication

3) closed under vector addition

1) $V^{\perp}$ must at least contain $\{0\}$.

2) Let $[P_1 - P_n] = V^{\perp}$

$$\alpha(P_i + P_j) = \alpha P_i + \alpha P_j = 0 + 0 = 0 \in V^{\perp}$$

3)

$$P_i + P_j = 0 + 0 = 0$$

//

2.3-39)

Given:

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2 \qquad ①$$

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty \qquad ②$$

$$\|x\|_\infty \le \|x\|_1 \le n\,\|x\|_\infty \qquad ③$$

(a) (b)

Find:

For each inequality determine a nonzero vector $x^n \in \mathbb{R}^n$ for which each inequality is achieved (seperately) with equality.

Solution

1)

1.a) $x = [1\ 0\ 0\ \dots]$

$$\sqrt{1^2 + 0^2 + \dots} = 1 + 0 + \dots$$

$$1 = 1$$

1.b) $x = [1\ 1\ 1\ 1]$

$$|1+1+1+1| = \sqrt{4}\,\sqrt{1^2+1^2+1^2+1^2}$$

$$4 = 2 \cdot 2 = 4$$

2/2

2)

2.a) $[2\ 0 \cdots]$

$$\max(x) = \sqrt{2^2 + 0 + \cdots}$$
$$2 = 2$$

2.b) $x = [1\ 1\ 1\ 1]$

$$\sqrt{1^2+1^2+1^2+1^2} = \sqrt{4}\ \max(x)$$
$$\sqrt{4} = \sqrt{4}\,(1)$$
$$2 = 2$$

3)

3.a) $x = [1\ 0 \cdots]$

$$\max(x) = |1 + 0 + 0 + \cdots|$$
$$1 = 1$$

3.b) $x = [1\ 1\ 1\ 1]$

$$|1+1+1+1| = 4\max(x)$$
$$4 = 4(1)$$
$$4 = 4$$

A.7-27)

a) For $x > 0$ and an integer $n > 1$, Show that

$$(1+x)^n > 1+nx$$

Hint: Use binomial theorem

b) The geometric mean of a set of numbers $\{z_i : i = 1-n\}$ is defined by

$$G_n = \left(\prod_{i=1}^{n} z_i \right)^{1/n}$$

The arithmetic mean is the set of numbers

$$A_n = \frac{1}{n} \sum_{i=1}^{n} z_i$$

The following holds:

The geometric mean of n positive real numbers is less than or equal to its arithmetic mean, with equality iff all the numbers are equal.

We show now that for $x \geq -1$ and $0 < \alpha < 1$

$$(1+x)^\alpha \leq 1 + \alpha x$$

Assume $\alpha = m/n$ we can write

$$(1+x)^{\alpha} = \sqrt[n]{\underbrace{(1+x)(1+x)\cdots}_{m \text{ factors}} \underbrace{1 \cdot 1 \cdot 1 \cdots}_{n-m \text{ factors}}}$$

Using the inequality relating the geometric and arithmetic means, show that

$$(1+x)^{m/n} \leq \frac{m(1+x)+n-m}{n} = 1+\frac{m}{n}x$$

c) Employ a continuity argument to extend the result $\forall \alpha$ s.t $0<\alpha<1$

d) Finally, to establish (A.14), let $x=y-1$

Solution:

a)

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Let $n=2$, $x=1$

$$(1+1)^2 > 1+2$$
$$4 > 2$$

Suppose $(1+x)^n \geq 1+nx$

$(1+x)^{n+1} \geq (1+nx)(1+x)$

$(1+nx)(1+x) = 1+(k+1)x+kx^3$
$\geq 1+(k+1)x$ //

b) $(1+x)^{m/n} = \underbrace{[(1+x)(1+x)\cdots 1\cdot 1\cdot 1]^{1/n}}$

Geometric mean with different values $\therefore$

$(1+x)^{m/n} \geq \dfrac{m(1+x)+n-m}{n} = 1+\dfrac{m}{n}x$ //

c) Let $\alpha = m/n$ and $0<\alpha<1$ as was described in the problem statement

$(1+x)^\alpha \leq 1+\alpha x$ s.t. $0<\alpha<1$ //

d) Let $x = y-1$

$y^\alpha \leq 1+\alpha(y-1)$ //

A.7-29)

Given:

$$\left| \int_a^b f(t)\,g(t)\,dt \right| \le \left(\int_a^b |f(t)|^p\,dt \right)^{1/p} \left(\int_a^b |g(t)|^p\,dt \right)^{1/q}$$

S.t. $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$

Find:

Prove the above inequality and integrate from a to b.

Hint:

$$A = \frac{|f(t)|^p}{\int_a^b |f(t)|^p\,dt} \qquad B = \frac{|g(t)|^q}{\int_a^b |g(t)|^q\,dt}$$

Solution:

Begin with Young's Inequality for Products

$$AB \le \frac{A^p}{p} + \frac{B^q}{q} = \frac{|f|^p}{p\,\|f\|_p^p} + \frac{|g|^q}{q\,\|g\|_q^q}$$

$$\int_a^b AB\,dx \le \frac{\|f\|^p}{p\,\|f\|_p^p} + \frac{\|g\|^q}{q\,\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1$$

//

$$\Rightarrow \frac{|f(t)|^p}{p\,\|f(t)\|_p^p} + \frac{|g(t)|^p}{p\,\|g(t)\|_p^p} \ge \frac{f(t)\,g(t)}{\|f(t)\|_p\,\|g(t)\|_p}$$

//

A.7-32)

Given:

$\ln x \le x-1$

Find:

Prove the "information theory inequality"

Solution:

Let $f(x) = \ln(x) - x + 1$

$f'(x) = \frac{1}{x} - 1 = 0 \Rightarrow x = 1$

$f''(x) = -\frac{1}{x^2} < 0 \;\forall\, x > 0 \;\therefore$ Max point

at $x=1$

$\ln(x) = x-1$

$\ln(x) < x-1$ everywhere else //