Homework 8 Signals Brown, Alexander

4.1-2) Given:

$$\dddot{x}(t) - 2\ddot{x}(t) - \dot{x}(t) = b(t)$$

Find:

Write the DEQ in operator form

Solution

$$A = \frac{d^3}{dt^3} - 2\frac{d^2}{dt^2} - \frac{d}{dt}$$

$$Ax(t) = b(t)$$

4.1-3)

Given:

$$f(x) = \int_0^1 a(t) \int_0^t b(s)\, x(s)\, ds\, dt$$

Let the functional f be defined on $L_2[0,1]$ where $a, b \in L_2[0,1]$

Find:

Show that f is a bounded functional on $L_2[0,1]$ and find an element $y \in L[0,1]$ such that

$$f(x) = \langle x, y \rangle$$

Solution:

$$f(x) = \int_0^1 a(t) \int_0^t b(s)\, x(s)\, ds\, dt$$

$$= \int_0^1 a(t)\, y(t)\, dt$$

$$\Rightarrow \|f(x)\|^2 = \left| \int_0^1 a(t) y(t)\, dt \right|^2 = |\langle a, y \rangle|^2$$

$$\leq |\langle a, a \rangle| \, |\langle y, y \rangle$$

$$= \int_0^1 a(t)^2\, dt \int_0^1 y(t)\, dt < \infty \;//$$

4.1-4)

Given:

$$A_1 = \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Find:

Determine the l_1, l_2, frobenius, and l_∞ norms

Solution

a) $A_1 = \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}$

$$l_1 = \max_j \sum_i |a_{ij}| = 9$$

$$l_2 = \sqrt{\rho(A^H A)} \; ; \; \rho(A) = \max_i |\lambda_i|$$

$$\text{eig}(A^H A) = 3.3772, \underline{66.62}$$

$$\Rightarrow l_2 = \sqrt{66.62} = 8.1623$$

$$l_{frob} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} = 8.366$$

$$l_\infty = \max_i \sum_j |a_{ij}| = 9$$

b)

$l_1 = 4$

$l_2 = 3.2566$

$l_F = 3.7417$

$l_\infty = 3$

c)

$l_1 = 3$

$l_2 = 2.4142$

$l_F = 2.4495$

$l_\infty = 3$

4.2-5) Given:

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}| \qquad (1)$$

Find:

Show (1) is true

Solution:

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty = \max_{\|x\|_\infty = 1} \max_i \left| \sum_j a_{ij} x_j \right|$$

$\|x\|_\infty = 1$, thus

$$\|A\|_\infty = \max_i \max_{\|x\|_\infty = 1} \left| \sum_j a_{ij} x_j \right| = \max_i \sum_j |a_{ij}|$$

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4.2-6) Given

$$\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 = \max_j \sum_i |a_{ij}| \qquad (1)$$

Find:

Show (1) is true

Solution:

$$\|A_1\| = \max_{\|x\|_1=1} \|Ax\|_1 = \max_j \max_{\|x\|_1=1} \left|\sum_i a_{ij} x_j\right|$$

$$= \max_j \sum_i |a_{ij}| \quad //$$

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4.2-7) Given:

$$\|AB\|_p \leq \|A\|_p \|B\|_p \qquad (1)$$

Find:

Provide an example of a norm that does not satisfy (1)

Solution:

Let $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$

Let $\|M\| = \max_{ij}(|m_{ij}|)$

$\|A \cdot A\| > \|A\| \|A\| \iff 8 > 4$

4.2-10) Given:

- Square matrix F
- $\|F\| < 1$

$$\|(I-F)^{-1}\| \leq \frac{1}{1-\|F\|} \qquad (1)$$

Find:

Show that (1) is true

Solution:

Theorem: Suppose $\|\cdot\|$ is a norm satisfying the submultiplicative property and $A: x \to x$ is an operator with $\|A\| < 1$. Then $(I-A)^{-1}$ exists and

$$(I-A)^{-1} = \sum_{i=0}^{\infty} A^i$$

Fact: If $x \in \mathbb{R}$ and $\|x\| < 1$

$$1 + x + x^2 + \dots + x^n = (1-x)^{-1}$$

$\|F\| \in \mathbb{R}$, using our fact and Theorem we get

$$\sum_{i=1}^{\infty} \|F^i\| = \|(I-F)^{-1}\| \leq (\|I\| - \|F\|)^{-1} = (1 - \|F\|)^{-1} //$$

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4.2-12) Given:

- F is a square matrix
- $\|F\| < 1$ and satisfies submultiplicitive property

Find:

$$\| I - (I-F)^{-1} \| \leq \frac{\|F\|}{1 - \|F\|}$$

Hint: Show $I - (I-F)^{-1} = -F(I-F)^{-1}$

Solution:

$$I - (I-F)^{-1} = I - \frac{I}{(I-F)} = \frac{I(I-F)}{(I-F)} - \frac{I}{I-F}$$

$$= \frac{-F}{(I-F)} = -F(I-F)^{-1} /$$

$$\| F(I-F)^{-1} \| = \left\| \frac{F}{I-F} \right\| \leq \frac{\|F\|}{I - \|F\|} //$$

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4.2-15) Given:

$$\|A\|_F^2 = tr(A^H A) \qquad (1)$$

Find:

Show (1) is true

Solution:

$$\|A\|_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}}_{A^H} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} = \begin{bmatrix} a_{11}a_{11} + a_{21}a_{21} & \cdot \\ \cdot & a_{12}a_{12} + a_{22}a_{22} \end{bmatrix}$$

$$\|A_F\|^2 = \sum_{i=1}^{m} a_i^H a_i = tr(A^H A)$$

4.2-16) Given:

- m×m matrix A
- nonzero m×1 vector x

Find:

$$\left\| A\left(I - \frac{xx^H}{x^Hx}\right)\right\|_F^2 = \|A\|_F^2 - \frac{\|Ax\|_2^2}{x^Hx}$$

Solution:

$$\left\| A - \frac{Axx^H}{x^Hx}\right\|_F^2 = \|A\|_F^2 - \left\|\frac{Axx^H}{x^Hx}\right\|_F^2$$

$$= \|A\|_F^2 - \frac{\|Axx^H\|_F^2}{x^Hx} \longleftarrow \text{As shown in (4.2-15)}$$

$\|A\|_2 \leq \|A\|_F$ by (4.10)

$$\Rightarrow \left\| A\left(I - \frac{xx^H}{x^Hx}\right)\right\|_F^2 = \|A\|_F^2 - \frac{\|Ax\|_2^2}{x^Hx} \quad //$$

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4.2-17) Given:

B is a submatrix of A

Find:

Show $\|B\|_2 \le \|A\|_2$

Solution:

If B is a submatrix of A then necessarily the elements of B are a subset of A. Therefore for the l_1, l_∞, and l_F norms, the inequalities must hold

$$\max_j \sum_i |a_{ij}| \ge \max_l \sum_k |b_{kl}| \quad \text{for } l \le j \text{ and } k \le i$$

$$\max_i \sum_j |a_{ij}| \ge \max_k \sum_l |b_{kl}| \quad \text{for } l \le j \text{ and } k \le i$$

$$\left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \ge \left(\sum_i^{k} \sum_j^{l} |b_{ij}|^2 \right)^{1/2}$$

Furthermore by the Cauchy Interlace theorem:

Theorem: Let A be a Hermitian matrix of order n, and let B be a principal submatrix of A order n−1. If $\lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_1$ lists the eigenvalues of A and $\mu_n \le \mu_{n-1} \le \cdots \le \mu_2$ lists the eigenvalues of B then

$$\lambda_n \le \mu_n \le \lambda_{n-1} \le \mu_{n-1} \cdots \le \lambda_2 \le \mu_2 \le \lambda_1$$

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4.2-19) Given: mxm matrix D

$$\frac{1}{\sqrt{m}} |tr(D)| \leq \|D_F\| \qquad (1)$$

Find:

Show (1) is true

Solution

Let $\langle x, y \rangle = \frac{tr(x^T y)}{\sqrt{m}}$ $\quad x, y \in \mathbb{R}^{3\times 3}$

$$\langle I, D \rangle = \frac{tr(D)}{\sqrt{m}} \leq \langle D, D \rangle \langle I, I \rangle$$

$$= \frac{\|D\|_F}{\sqrt{m}} \cdot \frac{1}{\sqrt{m}}$$

$$\leq \|D\|_F \quad //$$

4.2-22) Given:

Matrix A in $\|Ax\| = \|x\|$ is called isometric or norm-preserving.

Find:

Show that a square matrix is isometric iff it is orthogonal (or unitary if A is complex).

Note: An orthogonal matrix A satisfies $A^T A = I$. A unitary matrix satisfies $A^H A = I$.

Solution:

$$\|Ax\|_2^2 = x^T A^T A x$$

if $A^T A = I$ then — If A is orthogonal or unitary

$$\|Ax\|_2^2 = x^T x = \|x\|$$

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4.3-27) Given:

- $\dot{x} = Ax(t) + bf(t)$
- $x(0) = 0 \quad x(T) = x_T$
- $x(T) = \int_0^T e^{A(T-t)} bf(t)\,dt$
- $L: L_2[0,T] \to \mathbb{R}^n \quad Lf = \int_0^T e^{A(T-t)} bf(t)\,dt$

Find:

a) Show $L^* = b^T e^{A^T(T-t)}$

b) Show $LL^* = \int_0^T e^{A(T-t)} bb^T e^{A^T(T-t)}\,dt$

c) We wish to minimize

$$\int_0^T f^2(t)\,dt$$

Determine an expression for the minimum energy $f(t)$. Hint: Theorem 4.4 $f = L^* z$

Solution:

a) $$Lf = \int_0^T \underbrace{e^{A(T-t)} b}_{L} f(t)\,dt$$

$$= \int_0^T f(t) \underbrace{b^T e^{A^T(T-t)}}_{L^*}\,dt = L^* f \quad //$$

b) By substitution $f(t) = b^T e^{A^T(T-t)}$

$$LL^* = \int_0^T e^{A(T-t)} bb^T e^{A^T(T-t)}\,dt \quad //$$

c) $\dot{x} = Ax(t) + bf(t)$

$$x(t) = 0 + \int_0^t e^{A(t-\tau)} bf(\tau)\, d\tau$$

$$\Rightarrow x(T) = \int_0^T e^{A(T-\tau)} bf(t)\, dt = Lf = x_T$$

$$Lf = z \Rightarrow f = L^* z$$

$$(LL^*) z = x_T \Rightarrow z = (LL^*)^{-1} x_T$$

$$\Rightarrow f = L(LL^*)^{-1} x_T$$

$$LL^* = \int_0^T e^{A(T-t)} bb^T e^{A^T(T-t)}\, dt$$

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4.5-29) Given:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \\ 3 & 12 \end{bmatrix}$$

Find:

The four fundamental subspaces

Solution

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$R(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} \right)$$

$$N(A) = \{0\}$$

row space is span $\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} \right)$

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1.5 \end{bmatrix}$$

$$N(A^T) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1.5 \end{bmatrix} \right)$$

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4.5-31) Given:

The solution $Ax = b$ (if it exists) is unique iff the only solution of $Ax = 0$ is $x = 0$, that is, if $N(A) = \{0\}$.

Find:

Prove the above theorem

Solution:

Definition: The vectors $v_1, v_2, \dots, v_k$ in a vector space V are said to be linearly independent provided

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$. That is, the only linear combination that represents the 0 vector is the trivial combination

Thus the theory is implying that A has a rank of m ($A \in \mathbb{R}^{m \times m}$ for example), meaning the vectors are linerly independent. If $\text{rank}(A) < m$, that would imply infinite solutions.

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4.6-34) Given:

$$(AB)^{-1} = B^{-1}A^{-1} \quad (1)$$

Find:

Prove (1)

Solution:

We know $AA^{-1} = I$, thus if

$$(AB)^{-1} = B^{-1}A^{-1}$$

then the following must be true

1) $(AB)^{-1}AB = I$

2) $B^{-1}A^{-1}AB = I$

1) This is trivial, let $C = AB$ //

$\Rightarrow C^{-1}C = I$

2) $B^{-1}\underbrace{A^{-1}A}_{I}B = B^{-1}B = I$ //

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4.6-36) Given:

$AB = 0$ for matrices A and B

Find:

Show $R(B) \subset N(A)$

Solution:

Let c_i be the columns of B and v_i be in the correct vector space. We can write

$$Bv = \sum_i v_i c_i$$

Then we can say

$$ABv = \sum_i v_i A c_i = 0 \qquad (1)$$

(1) is $0 \ \forall \ v_i$ if $Ac_i = 0$ $\therefore$ $R(B) \subset N(A)$ //

4.9-38) Given:

$$A^{+}A \text{ \& } AA^{+} \qquad (1)$$

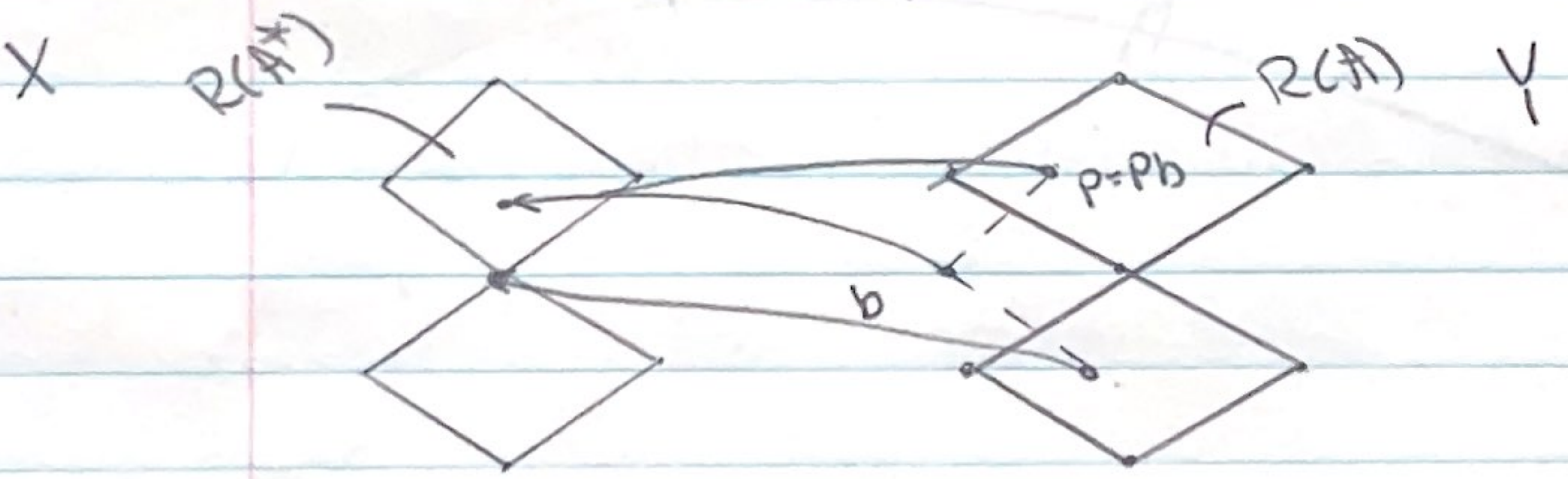
Find:

Explain why (1) are projection matrices. What fundamental subspace to they project to?

Solution:

Let $A: X \to Y$ s.t. $p = Ab \in Y$

Let A^{+} be a pseudoinvers that maps $b \in Y$ to $\hat{x} \in R(A^{*})$



Definition: Let $A: X \to Y$ be a bounded Linear operator, where X and Y are Hilbert spaces, and let $R(A)$ be closed. For some $b \in Y$ let $\hat{x}$ be the vector of minimum norm $\|\hat{x}\|_2$ that minimizes $\|Ax - b\|$. The pseudoinverse A^{+} is the operator mapping b to $\hat{x}$ for each $b \in Y$.

Because A^+ is of minimum norm, A^+A and AA^+ projection matrices onto $R(A)$ and $R(A^*)$. That is

$$A : X \to Y$$

$$A^+ : Y \to R(A^*)$$

The components that are in the null space of Y are lost

Thus A^TA and AA^T are elements of $R(A^*)$. //

4.10-39) Given:

$$\kappa(AB) \le \kappa(A)\,\kappa(B) \qquad (1)$$

$$\kappa(\alpha A) = \alpha\,\kappa(A) \qquad \alpha \in \mathbb{R}^+ \qquad (2)$$

Find:

Show (1) and (2) are true

Solution:

$$\kappa(A) = \|A\|\,\|A^{-1}\|$$

(1)

Submultiplicative property

$$\|AB\|_p \le \|A\|_p\,\|B\|_p$$

If A & B are square then

$$\|AB\|\,\|(AB)^{-1}\| \le \|A\|\,\|B\|\,\|B^{-1}\|\,\|A^{-1}\|$$

$$= \kappa(A)\,\kappa(B) \quad //$$

(2)

$$\kappa(\alpha A) = \alpha\|A\|\,\|A^{-1}\| = \alpha\,\kappa(A) \quad //$$

4.10-40) Given:

The determinant of a matrix cannot be used to determine the ill conditioning of a matrix.

Find:

a) The determinant and $K_\infty(B_n)$ for

$$B_n = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & -1 & & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

K_∞ denotes the condition number computed using the L^∞ norm

b) Find the determinant and condition number of

$$D_n = \text{diag}(10^{-1}, \ldots, 10^{-1}) \in \mathbb{R}^{n \times n}$$

Solution:

a)

Theorem: Let T_n be an upper triangular matrix of order n. Let $\det(T_n)$ be the determinant of T_n. Then $\det(T_n)$ is equal to the product of all the diagonal elements of T_n. That is

$$\det(T_n) = \prod_{k=1}^{n} a_{kk}$$