More Research, Ramsey Stuff

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For a set X and a cardinal n, we define

$$[X]^n = \{A \subseteq X : |A| = n\}$$

(The brackets are to indicate that these may also be thought of as increasing sequences yes?) In particular then the set $[\omega]^\omega$ is the set of all sequences of natural numbers. We will use lower case letters $a, b, c, ...$ to denote finite sets of natural numbers, and upper case letters $A, B, C, ...$ to denote infinite sets. Define $a < A$ to mean that $max(a) < min(A)$. Then for $a < A$, define

$$[a, A] = \{S \subseteq [\omega]^\omega : a \subseteq S \subseteq a \cup A\}$$

Note that $[\varnothing, A] = [A]^\omega$. We call the set a the **standard** and the set A the **body**. Sets of this form form a basis for what is known as the **Ellentuck topology**. Note that there are continuum many such sets.

Lemma 0.1. *$[a, A] \subseteq [b, B]$ iff $b \subseteq a, a - b \subseteq B$, and $A \subseteq B$. That is to say:*

- *The standard of the containing set is a subset of the standard of the contained set*
- *The 'thinned out' elements from the standard of the contained set must appear in the body of the containing set*
- *The body of the containing set must contain the body of the contained set*

Proof. First note that if $b \subseteq a$, $A \subseteq B$, and $a - b \subseteq B$, then $a \cup A \subseteq b \cup B$, and so if $S \in [a, A]$ we have

$$b \subseteq a \subseteq S \subseteq a \cup A \subseteq b \subseteq B$$

And thus $S \in [b, B]$, so $[a, A] \subseteq [b, B]$. For the reverse direction, suppose $[a, B]$ and $[b, B]$ had the property that b was not a subset of a. If the sets are incomparable, then $[a, A]$ and $[b, B]$ have to in fact be disjoint, since no set S from one could possibly start with the same standard as the other. Else, assume that $a \subset b$ is proper.

Let $a = \{a_1, ..., a_n\}$, and $b = \{a_1, ..., a_n, b_1, ..., b_m\}$. If $A \cap B$ were finite or empty, then it couldn't possibly be the case that these sets were comparable, since any sequence from $[a, A]$ would contain infinitely many elements not present in B and vice versa. Thus assume $A \cap B = \{i_1, i_2, ...\}$, and note that since $b < B$, $b < A \cap B$ as well, so that i_1 is greater than all the elements of b. We can then see that the set $S = \{a_1, ..., a_n, i_1, i_2, ...\}$ is a member of $[a, A]$ but cannot possibly be in $[b, B]$, since b is not a subset of S. Thus it cannot be the case that $[a, A] \subseteq [b, B]$. Thus, $b \subseteq a$ is a precondition for containment. For the rest of the proof we will assume $a = \{a_1, ..., a_n\}$, and that $b = \{a_1, ..., a_m\}$, where $m \leq n$. Next, suppose that A is not a subset of B. Then there is something in A which is not in B, and since $a \cap A = \varnothing$ it must be something also outside of both standards. Clearly then if we simply consider the set S beginning with the standard of a, followed by this one element, then regardless of what follows we've created something which is in $[a, A]$ but not $[b, B]$, so $[a, A]$ is not contained in $[b, B]$. Thus another precondition for containment is that $A \subseteq B$. Finally, suppose that $a - b$ is not contained in B. Then, once more, there must be something in a which is not in b and not in B. Then there is no possible way for a set from $[b, B]$ to begin with the string a, and thus in fact nothing in $[a, A]$ is in $[b, B]$. Thus all three preconditions are confirmed. $\square$

Theorem 0.1 (Galvin-Prikrey). *Let $[\omega]^\omega = P_0 \cup P_1 \cup \ldots \cup P_{k-1}$ be a partition of $[\omega]^\omega$, with each P_i Borel. Then there exists an infinite $H \subseteq \omega, i < k$, such that $[H]^\omega \subseteq P_i$. We call the set a* ***homogeneous*** *set with respect to the partition.*

To prove this we will define some new terminology and shift our perspective somewhat.

Definition 0.1. Say a set X is **Ramsey** if there exists an infinite set H such that $[\varnothing, H] \subseteq X$ or $[\varnothing, H] \subseteq X^c$. Say the set is **completely Ramsey** if there exists an A such that for all $a < A$, there exists an $H \subseteq A$ such that $[a, H] \subseteq X$ or $[a, H] \subseteq \neg X$. Note that every set which is completely Ramsey is also Ramsey - just fix $a = \varnothing$.

Note another way to describe Ramsey sets is to say that a set X is Ramsey if the partition $[\omega]^\omega = X \cup X^c$ has a homogeneous set. A simple induction confirms that showing Galvin-Prikrey for the special case of a two element partition proves the case for any finite partition. Thus the Galvin-Prikrey theorem can in fact be restated as: All Borel sets are Ramsey. In fact, all Borel sets are completely Ramsey, as we will show. Recall first the following definition:

Definition 0.2. A set X has the **Baire property** with respect to a particular topology τ if it differs meagerly by an open set in that topology. That is to say, there exists an open set U and a meager set Y such that $X = U\Delta Y$, where Δ denotes the **symmetric difference**: $U\Delta Y = U \cup Y - U \cap Y$.

The following theorem is the key component to linking everything together:

Theorem 0.2 (Ellentuck). *Let $X \subseteq [\omega]^\omega$. Then X is completely Ramsey iff X has the Baire property in the Ellentuck topology.*

Since all Borel sets have the Baire property(?), it would follow from Ellentuck that all Borel sets are completely Ramsey and thus Ramsey, from which Galvin-Prikrey follows. Thus it remains for us to prove Ellentuck. We begin with the following lemma:

Lemma 0.2. *All open sets in the Ellentuck topology are completely Ramsay*

Proof. Fix an open set, U. We will define $[a, A]$ to be *good* if there exists a $B \subseteq A$ such that $[a, B] \subseteq U$. Otherwise, we'll call it *bad*. Additionally, we will call it *very bad* if it is not only bad, but for all $n \in A$, $[a \cup \{n\}, A/n]$ is bad, where $A/n = \{x \in A : x > n\}$. I.e. a set is very bad if no amount of 'thinning out' of the body will make it good. Note that if $[a, A]$ is good, then we are done. Thus we will assume that it's bad, and show that on this assumption we can find a $B \subseteq A$ such that $[a, B] \subseteq U^c$.

Before beginning to do this, we claim that if $[a, A]$ is bad, then there exists a $B \subseteq A$ such that $[a, B]$ is very bad, i.e. we can always 'thin out' the body of a bad set to produce a very bad set. Suppose this is not the case, and fix an arbitrary $B \subseteq A$. By hypothesis then this is not very bad, meaning there exists a $n_0 \in B$ such that $[a \cup \{n_0\}, B/n_0]$ is good, that is to say there is a $B_0 \subseteq B/n_0$ such that $[a \cup \{n_0\}, B_0] \subseteq U$. Likewise we know that $[a, B_0]$ is not very bad, so there exists an $n_1 \in B_0$ such that $[a \cup \{n_1\}, B_0/n_1]$ is good, and again this means there exists a $B_1 \subseteq B_0/n_1$ such that $[a \cup \{n_1\}, B_1] \subseteq U$. Note that $n_1 > n_0$, since $n_1 \in B_0/n_0$. Thus continuing in this manner generates an increasing sequence $\{n_0, n_1, ...\} := C \subseteq A$. We claim that $[a, C] \subseteq U$. To see this, first note that each of the B_i sets contains a tail of C, beginning with n_{i+1}. Consider an arbitrary element $S \in [a, C]$. Let $n_i = min(S - a)$. Then the body of S is a subset of $\{n_i, n_{i+1}, ...\}$ and so $S \in [a \cup \{n_i\}, B_i] \subseteq U$, i.e. $S \in U$. Thus $[a, C] \subseteq U$, but since $C \subseteq A$, we've shown that $[a, A]$ is good, contradicting ourselves. Thus, the claim is proven.

Now we make repeated use of the claim. First, since $[a, A]$ is bad, there exists a $B_0 \subseteq A$ such that $[a, B_0]$ is very bad. Likewise let $n_0 = min(B_0)$. Then by definition $[a \cup \{n_0\}, B/n_0]$ is bad, so using the claim again we can find a $B_1 \subseteq B_0/n_0$ such that $[a \cup \{n_0\}, B_1]$ is very bad. Let $n_1 = min(B_1)$, and note that $[a \cup \{n_0, n_1\}, B_1/n_1]$ is bad. Continue in this manner, to produce a decreasing sequence $A \supseteq B_0 \supseteq B_1 \supseteq \ldots$, and more importantly a collection $B := \{n_i\}_{i\in\omega}$. The crucial property which the set B has is that for all $i \in omega$ and for all $b \subseteq \{n_0, n_1, ..., n_i\}$, $[a \cup b, B_i/b]$ is bad. (Need a little induction for EVERY subset)

we claim that $[a, B] \subseteq U^c$. Suppose by way of contradiction that it isn't. Then $[a, B] \cap U \neq \varnothing$, but this is the intersection of two open sets, and thus there must exist a a', B' such that $[a', B'] \subseteq [a, B]$ and $[a', B'] \subseteq U$. Now, making use of our earlier observations about containment of these basic Ellentuck sets, note that $a \subseteq a'$, and $a' - a \subseteq B = \{n_i\}_{i \in \omega}$. Thus for some $i \in \omega$, there must exist a $b \subseteq \{n_0, ..., n_i\}$, $a' = a \cup b$, so we have

$$[a \cup b, B'/b] = [a', B'] \subseteq U$$

A contradiction of the crucial property of B which we noted. Thus we must conclude that $[a, B] \subseteq U^c$. $\square$

It obviously follows symmetrically that all closed sets in the Ellentuck topology are also completely Ramsey. A simple induction which we will now do will confirm from this that all Borel sets are completely Ramsey:

Lemma 0.3. *If X is nowhere dense, then for any $[a, A]$, there is a $B \subseteq A$ such that $[a, B] \subseteq X^c$*

Proof. Let $[a, A]$ be arbitrary. By above our observation, the closure of X $\bar{X}$ is closed and thus completely Ramsay. Thus there exists a $B \subseteq A$ such that $[a, B] \subseteq \bar{X}$ or $[a, B] \subseteq \bar{X}^c$. But $[a, B]$ is open and thus since X is nowhere dense, we must conclude that $[a, B] \subseteq \bar{X}^c$. But since $X \subseteq \bar{X}$, $\bar{X}^c \subseteq X^c$, and thus $[a, B] \subseteq X^c$. $\square$

Lemma 0.4. *If X is meager, then for any $[a, A]$. Then there is a $B \subseteq A$ such that $[a, B] \subseteq \bar{X}$.*

Proof. Let $[a, A]$ be arbitrary, and $X = \bigcup_{n \in \omega} X_n$, with each X_n nowhere dense. Let $A_1 \subseteq A$ be such that $[a, A_1] \subseteq X_1$. Let $n_1 = \min(A_1)$, and consider $[a \cup \{n_1\}, A_1/n_1]$. $\square$

Lemma 0.5. *All Borel sets in the Ellentuck topology are completely Ramsey.*

Proof. Suppose that $V \in \mathbf{\Sigma}^0_{i+1}$, i.e. $V = \bigcup_{j \in \omega} F_j$, where $F_j \in \mathbf{\Pi}^0_i$ for each $j \in \omega$. Let $a, A \subseteq \omega$ be arbitrary. By the inductive hypothesis, for each F_j, there exists a $B_j \subseteq F_j$ such that $[a, B_j] \subseteq F_j$ or $[a, B_j] \subseteq F_j^c$. Without loss of generality we can assume the former, since if it is the latter, then $\square$

Definition 0.3. The **Mathias forcing** is the partial order $\mathbb{P} = (P, \leq)$ where a condition $p \in P$ is of the form $p = (s, A)$ where $s \in \omega^{<\omega}$, $A \subseteq \omega$ is infinite, and $\min(A) > \max(s)$. We define $(s', A') \leq (s, A)$ if s' extends s, $A' \subseteq A$, and $s' - s \subseteq A$. (That is to say, if $[s', A'] \subseteq [s, A]$.)

Lemma 0.6 (Prikry Property)**.** *Suppose that ϕ is a sentence in the forcing language, and $p = (s, A)$ a condition in the Mathias partial order. Then there is an $A' \subseteq A$ such that $p' = (s, A')$ decides phi. That is to say, either $p' \Vdash \phi$ or $p' \Vdash \neg\phi$.*

Proof. If $A' \subseteq A$ is such that $(s, A) \Vdash \phi$, then we're done. Thus suppose that no such A' exists. We will from this show that there is an $A' \subseteq A$ such that $(s, A') \Vdash \neg\phi$.

To begin, we first claim then that in such a situation, there must always be an $a_0 \in A$ and an $A_0 \subseteq A/\{a_0+1\}$ such that for all $A' \subseteq A_0$, we have $\neg(s \cup \{a_0\}, A_0) \Vdash \phi$. To show this, suppose this is not the case. Let $b_0 = min(A)$. Then there exists a $B_0 \subseteq A/\{a_0+1\}$ such that $(s \cup \{b_0\}, B_0) \Vdash \phi$. But then of course we can do the same thing for (s, B_0), letting $b_1 = min(B_0) > b_1$ and having $B_1 \subseteq B_0/\{b_1+1\}$ be such that $(s \cup \{b_1\}, B_1) \Vdash \phi$. And so forth, to generate an increasing sequence $b_0 < b_1 < \ldots$ and a decreasing chain $B_0, B_1, ...$ such that $(s \cup \{b_n\}, B_n) \Vdash \phi$ for each n. Let $B = \{b_1, b_2, ...\}$. We claim that $(s, B) \Vdash \phi$. To see this, we will show that $\{(t, B') : (t, B') \Vdash \phi\}$ is dense below (s, B). Given any $(t, B') \leq (s, B)$, without loss of generality assume that t strictly extends s (If it doesn't, just pick something below which does.) Then the first character past the initial segment s must be from B, i.e. is b_n for some $n \in \omega$. Of course, B' must only include numbers above the largest element of t, and so we can assume that $B' \subseteq B_n$, and thus we have $(t, B') \leq (s \cup \{b_n\}, B_n) \Vdash \phi$, meaning that $(t, B') \Vdash \phi$. Thus clearly the set is dense and we have $(s, B) \Vdash \phi$, but this contradicts our very initial assumption that no such subset B of A exists.

Now, using this result, let $a_0 \in A$ and $A_0 \subseteq A/\{a_0+1\}$ be as above, i.e. such that for all $A' \subseteq A_0$, we have that it is not the case that $(s \cup \{a_0\}, A_0) \Vdash \phi$. Note in fact we also has the property that for all $A' \subseteq A_0$, it cannot be the case that $(s, A') \Vdash \phi$, since this would contradict our initial assumption. Thus we actually have an increasing 'sequence' $a_0, ..., a_n$ and a decreasing 'chain' $A_0, ..., A_n$ (namely, for $n = 0$) such that for all $t \subseteq \{a_0, ..., a_n\}$, it is not the case that there exists an $A' \subseteq A_n$ such that $(s \cup t, A') \Vdash \phi$. This will serve as the base case for an induction. We claim we can continue this for $n+1$. That is to say, we can find an a_{n+1} and a set $A_{n+1} \subseteq A_n/\{a_n+1\}$ such that for all $t \subseteq \{a_0, ..., a_{n+1}\}$, there doesn't exist an $A' \subseteq A_{n+1}$ with $(s \cup t, A') \Vdash \phi$. Suppose not. Let $b_0 = min(A_n)$ Then there exists a $t_0 \subseteq \{a_0, ..., a_n, b_0\}$ and a $B_0 \subseteq A_n/\{b_0+1\}$ such that $(s \cup t_0, B_0) \Vdash \phi$. Similarly, letting $b_1 = min(B_0)$, again there must be a $t_1 \subseteq \{a_0, ..., a_n, b_1\} \subseteq \{a_0, ..., a_n, b_0, b_1\}$ and a $B_1 \subseteq B_n/\{b_1 + 1\}$ such that $(s \cup t_1, B_1) \Vdash \phi$. And so on, to generate the increasing sequence $a_n < b_0 < b_1 < \dots$ and the decreasing chain $B_0, B_1, ...$ such that for any m, there always exists a $t_m \subseteq \{a_0, ..., a_n, b_0, ..., b_m\}$ such that $(s \cup t_m, B_m) \Vdash \phi$. (Note that t_m always contains b_m, and $b_m \in B_{m-1}$ for each m.) Now, there are only finitely many choices for $t_m \cap \{a_0, ..., a_n\}$, but infinitely many t_m. Thus, at least one of these finite a_i sequences must end up being the initial a-stuff segment for infinitely many of the t_m. Let $b'_0 < b'_1 < \dots$ be the subsequence of b_m's such that the associated t_m uses this fixed a-stuff initial segment, and call this segment t, and finally let B' be the entire subcollection of b's. Then note that for all m, we clearly have $(s \cup t \cup \{b_m\}, B'/\{b_m + 1\}) \Vdash \phi$ (since $s \cup t \cup \{b_m\} = s \cup t_m$, and $B'/\{b_m + 1\} \subseteq B_m$, i.e. $(s \cup t \cup \{b_m\}, B'/\{b_m + 1\}) \leq (s \cup t_m, B_m) \Vdash \phi$.) But this can only mean that $(s \cup t, B') \Vdash \phi$, contradicting the inductive hypothesis.

Thus, we've proven, arduously, that we can construct an increasing sequence $a_0 < a_1 < \dots$ and a decreasing chain $A_0, A_1, ...$ such that for any n and for any $t \subseteq \{a_0, ..., a_n\}$, there does not exist an $A'' \subseteq A_n$ such that $(s \cup t, A'') \Vdash \phi$. We use two primes because we are going to next define $A' = \{a_n\}_{n \in \omega}$. Note that it immediately follows that for any n and any $t \subseteq \{a_0, ..., a_n\}$, there cannot exist an $A'' \subseteq A'/\{a_n + 1\}$ such that $(s \cup t, A'') \Vdash \phi$. But this is precisely what it means for $(s, A') \Vdash \neg\phi$, and so we are done. $\square$

Let M be a model of ZF, $\mathbb{P}^M$ the Mathias forcing as defined in M, and G a generic for $\mathbb{P}$ over M. As with the Cohen forcing, this generic defines effectively a real number, x_G, in the following way: Note that for each n, the set $D_n = \{(s, A) : |s| \geq n\}$ is clearly dense in $\mathbb{P}$. (Given any $(r, B) \in \mathbb{P}$, if $|r| \geq n$, done, else can easily extend to an $(r', B') \leq (r, B)$ in which $|r'| = n$.) Then since G is generic, $G \cap D_n \neq \varnothing$ for each n. Pick any (r_n, B_n) from each of these, and define $x_G(n) = r_n(n)$. Clearly this will be the same for any (r_n, B_n) chosen from the intersection, since otherwise we would have two incompatible elements from G, which is presumably a filter. Thus each filter defines a unique 'real', which we call a **Mathias real**. Conversely, given a 'real' $x \in [\omega]^\omega$, we can define a collection of conditions G_x which is always a filter (but may or may not be generic), in the following way: For each n, deposit all conditions of the form $(\{x_1, ..., x_n\}, A)$ where $x_n < min(A)$ and for some $n_0 \geq n$, $x_m \in A$ for all $m > n_0$ (i.e. all bodies which contain a tail of the real). To see that this is a filter, suppose $(\{x_1, ..., x_n\}, A), (\{x_1, ..., x_m\}, B) \in G_x$, and wlog $m \geq n$. Then $(\{x_1, ..., x_m\}, \{x_{m+1}, x_{m+2}, ...\})$ exists below both of these sets and is clearly in G_x, so any two conditions are compatible. Furthermore given any $(s, A) \geq (\{x_1, ..., x_m\}, A')$ with the latter in G_x, it of course must then be the case that $s = \{x_1, ..., x_n\}$ for some $n \leq m$, and $A' \subseteq A$, but then $\{x_{m+1}, x_{m+2}, ...\} \subseteq A$, so clearly $(s, A) \in G_x$. Thus every real defines a unique filter.

Next we would like to prove the Mathias property, which gives a convenient feature of generics under the Mathias forcing. First, we need the following lemma:

Lemma 0.7. *Let $D \subseteq (\mathbb{P}, \leq)$ be dense below $p = (s, A) \in \mathbb{P}$. Then there is an $A' \subseteq A$ such that for any $f \in [A']^\omega$ there is an n such that $(s \cup f \restriction n, f - (f \restriction n))$ is below a condition in D. (Here we identify an increasing function with its range.)*

Proof. $\square$

Lemma 0.8 (Mathias property)**.** *Let M be a transitive model of ZF and $\mathbb{P}^M$ be the Mathias forcing as defined in M. Then a filter $G \subseteq \mathbb{P}$ is generic for $\mathbb{P}$ over M iff for every $A \in ([\omega]^\omega)^M$ we have that x_G is eventually in A or x_G is eventually in $\omega - A$. In particular, if $x \in [\omega]^\omega$ is generic over M, then so is any subsequence*

Subsets of natural numbers can be naturally associated with oracles, by virtue of each number being naturally associated with a string. Alternatively, each natural number can be seen as representing a position in the infinite binary string coding the oracle. Either way, for what follows we will see oracles as first and foremost, subsets of ω. Our next goal is to create a set $C \subseteq \omega$ such that for all $D \subseteq C$, $\boldsymbol{NP}^D \neq \boldsymbol{coNP}^D$. That is to say, we want a homogeneous set relative to which $\boldsymbol{NP}$ is not equal to $\boldsymbol{coNP}$.

1 Relationship between forcing notions

Consider the set P consisting of equivalence classes of Borel sets in Cantor space modulo the meager sets. Define $A \leq B$ if $A \subseteq B$. Since every Borel set is open modulo a meager set, canonical representatives can be taken as open. Furthermore, every open set contains a basic open set, so the partial order of basic open sets ordered by inclusion densely embeds into the partial order of equivalence classes we've described, and thus creates the same class of generics. Consider a generic G over this partial order. We claim there is a unique $x \in \mathcal{C}$ associated with this. Suppose that $B_p \in G$ for some finite string p. Then $B_q \in G$ for all $q \geq p$, i.e. if $p = 1010$, then B_1, B_{10}, and B_{101} are all in G as well. Furthermore no filter over this partial order can contain B_p and B_q for any p, q of the same length with $p \neq q$. The reason for this is that since G is a filter, there must be a B_r such that $B_r \subseteq B_q$ and $B_r \subseteq B_p$, but B_p and B_q are disjoint so this is impossible. Therefore the condition of being a filter enforces that it is a descending linear sequence of basic open sets defined by initial segments which extend one another. Note then that the condition of simply being a filter here is enough to guarantee that the intersection of all elements is a single point. It could be the case that, say, B_{0101} and B_{010100} are in the filter, but B_{01010} is not, but this has no bearing on that claim. Thus filters can be thought of as points, as can generics. Take the element x_G filtered out by a generic G. We claim that this belongs to every comeager set. It suffices to show that x_G belongs to every dense open set. Let D be such a set, and consider the collection $\mathcal{D} = \{\text{basic open sets } B : B \in D\}$. We claim that this is dense in the sense of the partial order. This is easy to see. If B_p is any basic open set in $\mathcal{C}$, then since D is open and dense, $B_p \cap D$ contains a basic open set B, and this is of course in $\mathcal{D}$. Thus $B \subseteq B_p$, and so $\mathcal{D}$ is dense. Since G is generic then, $G \cap D \neq \varnothing$. Pick B_p to be a basic open set in the intersection. Then since $x_G \in B$ for all $B \in G$, $x_G \in B_p \subseteq D$.

Conversely, suppose that Next consider the equivalence classes of Borel sets of positive measure in Cantor space modulo the measure 0 sets. Again order by inclusion. Now since Cantor space is closed, it can be shown that every such set contains a compact set of positive measure, so again that partial order densely embeds into the one we defined, and we consider that one. Let F be a filter. Note that it is no longer clear that it codes a single point in $\mathcal{C}$, and I don't think that it does. A generic filter does however, and just as before this point x_G is the intersection of all points of the generic. To see this, note that for each n, $D_n = \{\text{basic open sets contained in sets of measure } \frac{1}{n}\}$. This set is dense for every n: given a compact set B of positive measure, it certainly contains a set of measure smaller than any desired $\frac{1}{n}$. Thus G must intersect D_n, confirming that the set G, which is still a linear chain of nested sets just as before, by virtue of being a filter, has a subsequence of compact sets with positive measure approaching 0.