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Supplementary Material: Stable Volume Dissipation for High-Order Finite-Difference Methods with the Generalized Summation-by-Parts Property

Alex Bercik · David A. Craig Penner · David W. Zingg

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Contents

1 Volume-Dissipation Operators

In this section, we present the $\tilde{\mathsf{D}}_s$ matrices introduced in the main article. We first provide the structure of the operators, then provide the entries. Entries are not explicitly provided for CSBP operators, since for those operators we simply repeat the interior stencil at the boundaries, which reproduces the $\tilde{\mathsf{D}}_s$ operators found in [7]. We include factors of Δx in the following definitions, which although typically are simply $\Delta x = 1$, can also account for appropriate scaling of the default internal stencil (e.g. as with Mattsson operators).

A. Bercik
University of Toronto Institute for Aerospace Studies, Toronto, Canada
E-mail: alex.bercik@mail.utoronto.ca

D. A. Craig Penner
University of Toronto Institute for Aerospace Studies, Toronto, Canada
E-mail: david.a.craigpenner@nasa.gov
Present address: Science and Technology Corporation, NASA Ames Research Center, Moffett Field, United States

D. W. Zingg
University of Toronto Institute for Aerospace Studies, Toronto, Canada
E-mail: david.zingg@utoronto.ca

1.1 Finite-Difference Volume-Dissipation Operator Structures

First-Order Accurate Dissipation ($s = 1$)

$$\tilde{\mathsf{D}}_1 = \begin{bmatrix} -1 & 1 & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix} \qquad \mathsf{B} = \mathrm{diag}(0, 1, \ldots, 1).$$

Third-Order Accurate Dissipation ($s = 2$)

$$\mathsf{B}_2 = \mathrm{diag}(1, \ldots, 1) \quad \text{or} \quad \mathsf{B}_2 = \mathrm{diag}(0, 1, \ldots, 1, 0)$$

$$\tilde{\mathsf{D}}_2 = \frac{1}{\Delta x^2} \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & & & & \\ d_{2,1} & d_{2,2} & d_{2,3} & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & -d_{2,3} & -d_{2,2} & -d_{2,1} \\ & & & & -d_{1,3} & -d_{1,2} & -d_{1,1} \end{bmatrix}$$

Fifth-Order Accurate Dissipation ($s = 3$)

$$\mathsf{B}_3 = \mathrm{diag}(0, 1, \ldots, 1) \quad \text{or} \quad \mathsf{B}_3 = \mathrm{diag}(0, 0, 1, \ldots, 1, 0)$$

$$\tilde{\mathsf{D}}_3 = \frac{1}{\Delta x^3} \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & & & & & \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & & & & & \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & & & & & \\ & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} & & & & \\ & & -1 & 3 & -3 & 1 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & -1 & 3 & -3 & 1 & \\ & & & & & -d_{4,5} & -d_{4,4} & -d_{4,3} & -d_{4,2} \\ & & & & & & -d_{3,4} & -d_{3,3} & -d_{3,2} & -d_{3,1} \\ & & & & & & -d_{2,4} & -d_{2,3} & -d_{2,2} & -d_{2,1} \end{bmatrix}$$

Seventh-Order Accurate Dissipation ($s = 4$)

$$\mathsf{B}_4 = \operatorname{diag}(1, \dots, 1) \quad \text{or} \quad \mathsf{B}_4 = \operatorname{diag}(0, 0, 1, \dots, 1, 0, 0)$$

$$\tilde{\mathsf{D}}_4 = \frac{1}{\Delta x^4} \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & & & & & & & & \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} & & & & & & & & \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} & & & & & & & & \\ & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} & d_{4,6} & & & & & & & \\ & & d_{5,3} & d_{5,4} & d_{5,5} & d_{5,6} & d_{5,7} & & & & & & \\ & & & 1 & -4 & 6 & -4 & 1 & & & & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & & & 1 & -4 & 6 & -4 & 1 & & & \\ & & & & & & d_{5,7} & d_{5,6} & d_{5,5} & d_{5,4} & d_{5,3} & & \\ & & & & & & & d_{4,6} & d_{4,5} & d_{4,4} & d_{4,3} & d_{4,2} & \\ & & & & & & & & d_{3,5} & d_{3,4} & d_{3,3} & d_{3,2} & d_{3,1} \\ & & & & & & & & d_{2,5} & d_{2,4} & d_{2,3} & d_{2,2} & d_{2,1} \\ & & & & & & & & d_{1,5} & d_{1,4} & d_{1,3} & d_{1,2} & d_{1,1} \end{bmatrix}$$

Ninth-Order Accurate Dissipation ($s = 5$)

$$\mathsf{B}_5 = \operatorname{diag}(0, 1, \dots, 1) \quad \text{or} \quad \mathsf{B}_5 = \operatorname{diag}(0, 0, 0, 1, \dots, 1, 0, 0)$$

$$\tilde{\mathsf{D}}_5 = \frac{1}{\Delta x^5}\begin{bmatrix}
d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & & & & & & & & & & \\
d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} & d_{2,6} & & & & & & & & & & \\
d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} & d_{3,6} & & & & & & & & & & \\
d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} & d_{4,6} & & & & & & & & & & \\
 & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} & d_{5,6} & d_{5,7} & & & & & & & & & \\
 & & d_{6,3} & d_{6,4} & d_{6,5} & d_{6,6} & d_{6,7} & d_{6,8} & & & & & & & & \\
 & & & d_{7,4} & d_{7,5} & d_{7,6} & d_{7,7} & d_{7,8} & d_{7,9} & & & & & & & \\
 & & & & -1 & 5 & -10 & 10 & -5 & 1 & & & & & & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & & & & & -1 & 5 & -10 & 10 & -5 & 1 & & & & \\
 & & & & & & & -d_{7,9} & -d_{7,8} & -d_{7,7} & -d_{7,6} & -d_{7,5} & -d_{7,4} & & & \\
 & & & & & & & & -d_{6,8} & -d_{6,7} & -d_{6,6} & -d_{6,5} & -d_{6,4} & -d_{6,3} & & \\
 & & & & & & & & & -d_{5,7} & -d_{5,6} & -d_{5,5} & -d_{5,4} & -d_{5,3} & -d_{5,2} & \\
 & & & & & & & & & & -d_{4,6} & -d_{4,5} & -d_{4,4} & -d_{4,3} & -d_{4,2} & -d_{4,1} \\
 & & & & & & & & & & -d_{3,6} & -d_{3,5} & -d_{3,4} & -d_{3,3} & -d_{3,2} & -d_{3,1} \\
 & & & & & & & & & & -d_{2,6} & -d_{2,5} & -d_{2,4} & -d_{2,3} & -d_{2,2} & -d_{2,1}
\end{bmatrix}$$

1.2 Spectral-Element Volume-Dissipation Operator Structures

The results of Theorem 6 indicate that the dissipation operators provided in this section are unique. For this reason, no B matrices are provided in this section either.

Zeroth-Order Accurate Dissipation ($s = 1$) for $p = 1$ Operators

$$\tilde{\mathsf{D}}_1 = \begin{bmatrix} d_{1,1} & d_{1,2} \\ d_{1,1} & d_{1,2} \end{bmatrix}$$

First-Order Accurate Dissipation ($s = 2$) for $p = 2$ Operators

$$\tilde{\mathsf{D}}_2 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{1,1} & d_{1,2} & d_{1,3} \\ d_{1,1} & d_{1,2} & d_{1,3} \end{bmatrix}$$

Second-Order Accurate Dissipation ($s = 3$) for $p = 3$ Operators

$$\tilde{\mathsf{D}}_3 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} \end{bmatrix}$$

Third-Order Accurate Dissipation ($s = 4$) for $p = 4$ Operators

$$\tilde{\mathsf{D}}_4 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} \end{bmatrix}$$

Fourth-Order Accurate Dissipation ($s = 5$) for $p = 5$ Operators

$$\tilde{\mathsf{D}}_5 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} \end{bmatrix}$$

Fifth-Order Accurate Dissipation ($s = 6$) for $p = 6$ Operators

$$\tilde{\mathsf{D}}_6 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} \end{bmatrix}$$

Sixth-Order Accurate Dissipation ($s = 7$) for $p = 7$ Operators

$$\tilde{\mathsf{D}}_7 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} \end{bmatrix}$$

Seventh-Order Accurate Dissipation ($s = 8$) for $p = 8$ Operators

$$\tilde{\mathsf{D}}_8 = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} \\ d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} \end{bmatrix}$$

1.3 Volume-Dissipation Operator Entries

In this section we provide the values of the entries shown in §1.1 and §1.2. All operators are defined on a reference nodal distribution $\tilde{\boldsymbol{x}} \in [0, N-1]$ for N nodes, which is critical to their definition as 'undivided'. We provide references for definitions of the divided difference operators D and associated nodal distributions.

Fifth-Order Accurate Dissipation ($s = 3$) for Second-Degree Hybrid-Gauss-Trapezoidal-Lobatto Operator ($p = 2$)

Consider the nodal distribution and difference operators as provided in [2].

$$
\begin{aligned}
&\Delta x = 1 \\
&d_{1,1} = -1.0833333333333333, && d_{1,2} = 2.9060846560846561, \\
&d_{1,3} = -2.7857142857142857, && d_{1,4} = 9.6296296296296296\text{e}{-1}, \\
&d_{2,1} = d_{3,1} = d_{1,1}, && d_{2,2} = d_{3,2} = d_{1,2}, \\
&d_{2,3} = d_{3,3} = d_{1,3}, && d_{2,4} = d_{3,4} = d_{1,4}, \\
&d_{4,2} = -8.7182539682539683\text{e}{-1}, && d_{4,3} = 2.7857142857142857, \\
&d_{4,4} = -2.8888888888888889, && d_{4,5} = 9.7500000000000000\text{e}{-1}.
\end{aligned}
$$

Seventh-Order Accurate Dissipation ($s = 4$) for Third-Degree Hybrid-Gauss-Trapezoidal-Lobatto Operator ($p = 3$)

Consider the nodal distribution and difference operators as provided in [2].

$$
\begin{aligned}
&\Delta x = 1 \\
&d_{1,1} = 1.3129403102461554, && d_{1,2} = -3.6414785702115980, \\
&d_{1,3} = 4.8785522269403464, && d_{1,4} = -3.4629092233170398, \\
&d_{1,5} = 9.1289525634213598\text{e}{-1}, && d_{2,1} = d_{3,1} = d_{1,1}, \\
&d_{2,2} = d_{3,2} = d_{1,2}, && d_{2,3} = d_{3,3} = d_{1,3}, \\
&d_{2,4} = d_{3,4} = d_{1,4}, && d_{2,5} = d_{3,5} = d_{1,5}, \\
&d_{4,2} = 6.6986132731292683\text{e}{-1}, && d_{4,3} = -3.1476025984370158, \\
&d_{4,4} = 5.1943638349755597, && d_{4,5} = -3.6515810253685439, \\
&d_{4,6} = 9.3495846151707318\text{e}{-1}, && d_{5,3} = 9.2264307437286137\text{e}{-1}, \\
&d_{5,4} = -3.8492663572591594, && d_{5,5} = 5.8847788041224216, \\
&d_{5,6} = -3.9484606671899034, && d_{5,7} = 9.9030514595377980\text{e}{-1}.
\end{aligned}
$$

Ninth-Order Accurate Dissipation ($s = 5$) for Fourth-Degree Hybrid-Gauss-Trapezoidal-Lobatto Operator ($p = 4$)

Consider the nodal distribution as provided in [2]. However, the difference operator provided in [2] is incorrect, so a new operator was derived for this work and can be found at https://github.com/alexbercik/ESSBP. Note that the nodal distribution provided in [3] is also not accurate to double precision.

$$
\begin{aligned}
&\Delta x = 1 \\
&d_{1,1} = -1.7004187548796678, && d_{1,2} = 4.6263210261950949, \\
&d_{1,3} = -6.9899154979123421, && d_{1,4} = 7.2462818598409470, \\
&d_{1,5} = -4.0430217689261414, && d_{1,6} = 8.6075313568210942\text{e}{-1}, \\
&d_{2,1} = d_{3,1} = d_{4,1} = d_{1,1}, && d_{2,2} = d_{3,2} = d_{4,2} = d_{1,2}, \\
&d_{2,3} = d_{3,3} = d_{4,3} = d_{1,3}, && d_{2,4} = d_{3,4} = d_{4,4} = d_{1,4}, \\
&d_{2,5} = d_{3,5} = d_{4,5} = d_{1,5}, && d_{2,6} = d_{3,6} = d_{4,6} = d_{1,6}, \\
&d_{5,2} = -5.6085982148173551\text{e}{-1}, && d_{5,3} = 3.0519467297780998, \\
&d_{5,4} = -7.1631104716152893, && d_{5,5} = 8.0860435378522829, \\
&d_{5,6} = -4.3037656784105471, && d_{5,7} = 8.8974570387718928\text{e}{-1}, \\
&d_{6,3} = -6.9263352155911833\text{e}{-1}, && d_{6,4} = 4.1615497073217719, \\
&d_{6,5} = -9.0327972752461722, && d_{6,6} = 9.3633904284106928, \\
&d_{6,7} = -4.7612559253537453, && d_{6,8} = 9.6174658642657110\text{e}{-1}, \\
&d_{7,4} = -9.6144205853679095\text{e}{-1}, && d_{7,5} = 4.9148933227922590, \\
&d_{7,6} = -9.9141627908269083, && d_{7,7} = 9.9426109901364758, \\
&d_{7,8} = -4.9784482003901688, && d_{7,9} = 9.9654873682513319\text{e}{-1}.
\end{aligned}
$$

Fifth-Order Accurate Dissipation ($s = 3$) for Second-Degree Hybrid-Gauss-Trapezoidal Operator ($p = 2$)

Consider the nodal distribution and difference operators as provided in [2].

$$
\begin{aligned}
&\Delta x = 1 \\
&d_{1,1} = -1.5430012813588106, && d_{1,2} = 3.8811068744647719, \\
&d_{1,3} = -3.4264884568651276, && d_{1,4} = 1.0883828637591663, \\
&d_{2,1} = d_{3,1} = d_{1,1}, && d_{2,2} = d_{3,2} = d_{1,2}, \\
&d_{2,3} = d_{3,3} = d_{1,3}, && d_{2,4} = d_{3,4} = d_{1,4}, \\
&d_{4,2} = -1.0256035731987477, && d_{4,3} = 3.0417306413686773, \\
&d_{4,4} = -3.0207212025059216, && d_{4,5} = 1.0045941343359920.
\end{aligned}
$$

Seventh-Order Accurate Dissipation ($s = 4$) for Third-Degree Hybrid-Gauss-Trapezoidal Operator ($p = 3$)

One could consider the nodal distribution and difference operators as provided in [2]. However, a new operator was derived for this work that possesses better spectral radius properties. It can be found at https://github.com/

alexbercik/ESSBP.

$$
\begin{aligned}
&\Delta x = 1 \\
&d_{1,1} = 1.6368547635124326, && d_{1,2} = -5.1312368178539326, \\
&d_{1,3} = 6.7432475762095972, && d_{1,4} = -4.3056604939976920, \\
&d_{1,5} = 1.0567949721295947, && d_{2,1} = d_{3,1} = d_{1,1}, \\
&d_{2,2} = d_{3,2} = d_{1,2}, && d_{2,3} = d_{3,3} = d_{1,3}, \\
&d_{2,4} = d_{3,4} = d_{1,4}, && d_{2,5} = d_{3,5} = d_{1,5}, \\
&d_{4,2} = 1.0049004737140893, && d_{4,3} = -3.9967058527768872, \\
&d_{4,4} = 5.9890573495302516, && d_{4,5} = -3.9967414465915728, \\
&d_{4,6} = 9.9948947612411912\text{e}{-1}, && d_{5,3} = 9.9497625273298166\text{e}{-1}, \\
&d_{5,4} = -3.9903454645655616, && d_{5,5} = 5.9927503494231235, \\
&d_{5,6} = -3.9967766348396608, && d_{5,7} = 9.9939549724911724\text{e}{-1}.
\end{aligned}
$$

Ninth-Order Accurate Dissipation ($s = 5$) for Fourth-Degree Hybrid-Gauss-Trapezoidal Operator ($p = 4$)

One could consider the nodal distribution and difference operators as provided in [3]. However, a new operator was derived for this work that possesses better spectral radius properties. It can be found at https://github.com/alexbercik/ESSBP. Note that the nodal distribution and operators provided in [2] are not accurate to double precision.

$$
\begin{aligned}
&\Delta x = 1 \\
&d_{1,1} = -1.7800529914525139, && d_{1,2} = 6.1689468366190433, \\
&d_{1,3} = -1.0401887638412772\text{e}{+1}, && d_{1,4} = 1.0057677324478505\text{e}{+1}, \\
&d_{1,5} = -5.0576859101306045, && d_{1,6} = 1.0130023788983420, \\
&d_{2,1} = d_{3,1} = d_{4,1} = d_{1,1}, && d_{2,2} = d_{3,2} = d_{4,2} = d_{1,2}, \\
&d_{2,3} = d_{3,3} = d_{4,3} = d_{1,3}, && d_{2,4} = d_{3,4} = d_{4,4} = d_{1,4}, \\
&d_{2,5} = d_{3,5} = d_{4,5} = d_{1,5}, && d_{2,6} = d_{3,6} = d_{4,6} = d_{1,6}, \\
&d_{5,2} = -9.0869447206580021\text{e}{-1}, && d_{5,3} = 4.5591261092532523, \\
&d_{5,4} = -9.3816871792381694, && d_{5,5} = 9.6177960638442605, \\
&d_{5,6} = -4.8656932994311105, && d_{5,7} = 9.7915277763756729\text{e}{-1}, \\
&d_{6,3} = -9.2668399084729638\text{e}{-1}, && d_{6,4} = 4.8183131640805096, \\
&d_{6,5} = -9.8042690934729287, && d_{6,6} = 9.8728931193944579, \\
&d_{6,7} = -4.9527163075477525, && d_{6,8} = 9.9246310839301013\text{e}{-1}, \\
&d_{7,4} = -9.9426338308002477\text{e}{-1}, && d_{7,5} = 4.9874234330127357, \\
&d_{7,6} = -9.9874075960916072, && d_{7,7} = 9.9916015388338977, \\
&d_{7,8} = -4.9968499156661880, && d_{7,9} = 9.9949592299118653\text{e}{-1}.
\end{aligned}
$$

Fifth-Order Accurate Dissipation ($s = 3$) for Second-Degree Mattsson et al.*'s Optimal "Accurate" Operator ($p = 2$)*

Consider the nodal distribution and difference operators as provided in [6] and https://bitbucket.org/martinalmquist/optimized_sbp_operators.

$$
\begin{aligned}
\Delta x &= \frac{N-1}{2\xi_n + N - 5} \\
d_{1,1} &= -1.7277463987989540, & d_{1,2} &= 3.7021976718569106, \\
d_{1,3} &= -2.9870306597013296, & d_{1,4} &= 1.0125793866433730, \\
d_{2,1} &= d_{3,1} = d_{1,1}, & d_{2,2} &= d_{3,2} = d_{1,2}, \\
d_{2,3} &= d_{3,3} = d_{1,3}, & d_{2,4} &= d_{3,4} = d_{1,4}, \\
d_{4,2} &= -8.1738495424057284\text{e}{-1}, & d_{4,3} &= 2.6916305216679998, \\
d_{4,4} &= -2.8374616146508248, & d_{4,5} &= 9.6321604722339781\text{e}{-1}.
\end{aligned}
$$

and $\xi_n = 1.8022115125776$ as provided in [6].

Seventh-Order Accurate Dissipation ($s = 4$) for Third-Degree Mattsson et al.*'s Optimal "Accurate" Operator ($p = 3$)*

Consider the nodal distribution and difference operators as provided in [6] and https://bitbucket.org/martinalmquist/optimized_sbp_operators

$$
\begin{aligned}
\Delta x &= \frac{N-1}{2\xi_n + N - 7} \\
d_{1,1} &= 5.7302111593550254, & d_{1,2} &= -1.2521994384708010\text{e}{+1}, \\
d_{1,3} &= 1.1419402572581774\text{e}{+1}, & d_{1,4} &= -5.9442797107084950, \\
d_{1,5} &= 1.3166603634797059, & d_{2,1} &= d_{3,1} = d_{1,1}, \\
d_{2,2} &= d_{3,2} = d_{1,2}, & d_{2,3} &= d_{3,3} = d_{1,3}, \\
d_{2,4} &= d_{3,4} = d_{1,4}, & d_{2,5} &= d_{3,5} = d_{1,5}, \\
d_{4,2} &= 1.4441513881249756, & d_{4,3} &= -4.9292485821428592, \\
d_{4,4} &= 6.7286137322006952, & d_{4,5} &= -4.2974416973035725, \\
d_{4,6} &= 1.0539251591207610, & d_{5,3} &= 1.0466075357768034, \\
d_{5,4} &= -4.0887380427706574, & d_{5,5} &= 6.0658234020941999, \\
d_{5,6} &= -4.0291482544157139, & d_{5,7} &= 1.0054553593153680.
\end{aligned}
$$

and $\xi_n = 2.2638953951239$ as provided in [6].

Ninth-Order Accurate Dissipation ($s = 5$) for Fourth-Degree Mattsson et al.*'s Optimal "Accurate" Operator ($p = 4$)*

Consider the nodal distribution and difference operators as provided in [6] and https://bitbucket.org/martinalmquist/optimized_sbp_operators

$$
\begin{aligned}
&\Delta x = \frac{N-1}{2\xi_n + N - 9} \\
&d_{1,1} = -1.7800529914525139, && d_{1,2} = 6.1689468366190433, \\
&d_{1,3} = -1.0401887638412772\text{e}{+1}, && d_{1,4} = 1.0057677324478505\text{e}{+1}, \\
&d_{1,5} = -5.0576859101306045, && d_{1,6} = 1.0130023788983420, \\
&d_{2,1} = d_{3,1} = d_{4,1} = d_{1,1}, && d_{2,2} = d_{3,2} = d_{4,2} = d_{1,2}, \\
&d_{2,3} = d_{3,3} = d_{4,3} = d_{1,3}, && d_{2,4} = d_{3,4} = d_{4,4} = d_{1,4}, \\
&d_{2,5} = d_{3,5} = d_{4,5} = d_{1,5}, && d_{2,6} = d_{3,6} = d_{4,6} = d_{1,6}, \\
&d_{5,2} = -9.0869447206580021\text{e}{-1}, && d_{5,3} = 4.5591261092532523, \\
&d_{5,4} = -9.3816871792381694, && d_{5,5} = 9.6177960638442605, \\
&d_{5,6} = -4.8656932994311105, && d_{5,7} = 9.7915277763756729\text{e}{-1}, \\
&d_{6,3} = -9.2668399084729638\text{e}{-1}, && d_{6,4} = 4.8183131640805096, \\
&d_{6,5} = -9.8042690934729287, && d_{6,6} = 9.8728931193944579, \\
&d_{6,7} = -4.9527163075477525, && d_{6,8} = 9.9246310839301013\text{e}{-1}, \\
&d_{7,4} = -9.9426338308002477\text{e}{-1}, && d_{7,5} = 4.9874234330127357, \\
&d_{7,6} = -9.9874075960916072, && d_{7,7} = 9.9916015388338977, \\
&d_{7,8} = -4.9968499156661880, && d_{7,9} = 9.9949592299118653\text{e}{-1}.
\end{aligned}
$$

and $\xi_n = 3.3192851303204$ as provided in [6].

Dissipation for First-Degree Legendre-Gauss-Lobatto Operator ($s = p = 1$)

$$
d_{1,1} = -1.0000000000000000, \qquad d_{1,2} = 1.0000000000000000.
$$

Dissipation for Second-Degree Legendre-Gauss-Lobatto Operator ($s = p = 2$)

$$
\begin{aligned}
&d_{1,1} = 1.0000000000000000, && d_{1,2} = -2.0000000000000000, \\
&d_{1,3} = 1.0000000000000000.
\end{aligned}
$$

Dissipation for Third-Degree Legendre-Gauss-Lobatto Operator ($s = p = 3$)

$$
\begin{aligned}
&d_{1,1} = -1.1111111111111111, && d_{1,2} = 2.4845199749997663, \\
&d_{1,3} = -2.4845199749997663, && d_{1,4} = 1.1111111111111111.
\end{aligned}
$$

Dissipation for Fourth-Degree Legendre-Gauss-Lobatto Operator ($s = p = 4$)

$$
\begin{aligned}
&d_{1,1} = 1.3125000000000000, && d_{1,2} = -3.0625000000000000, \\
&d_{1,3} = 3.5000000000000000, && d_{1,4} = -3.0625000000000000, \\
&d_{1,5} = 1.3125000000000000.
\end{aligned}
$$

Dissipation for Fifth-Degree Legendre-Gauss-Lobatto Operator ($s = p = 5$)

$$\begin{aligned}
d_{1,1} &= -1.6128000000000000, & d_{1,2} &= 3.8427728888186092,\\
d_{1,3} &= -4.6528303520503175, & d_{1,4} &= 4.6528303520503175,\\
d_{1,5} &= -3.8427728888186092, & d_{1,6} &= 1.6128000000000000.
\end{aligned}$$

Dissipation for Sixth-Degree Legendre-Gauss-Lobatto Operator ($s = p = 6$)

$$\begin{aligned}
d_{1,1} &= 2.0370370370370370, & d_{1,2} &= -4.9114762529253907,\\
d_{1,3} &= 6.1336984751476130, & d_{1,4} &= -6.5185185185185185,\\
d_{1,5} &= 6.1336984751476130, & d_{1,6} &= -4.9114762529253907\\
d_{1,7} &= 2.0370370370370370.
\end{aligned}$$

Dissipation for Seventh-Degree Legendre-Gauss-Lobatto Operator ($s = p = 7$)

$$\begin{aligned}
d_{1,1} &= -2.6254366802947751, & d_{1,2} &= 6.3770113278844387,\\
d_{1,3} &= -8.1140161850595777, & d_{1,4} &= 8.9221772820893049,\\
d_{1,5} &= -8.9221772820893049, & d_{1,6} &= 8.1140161850595777\\
d_{1,7} &= -6.3770113278844387, & d_{1,8} &= 2.6254366802947751.
\end{aligned}$$

Dissipation for Eighth-Degree Legendre-Gauss-Lobatto Operator ($s = p = 8$)

$$\begin{aligned}
d_{1,1} &= 3.4366607666015625, & d_{1,2} &= -8.3884327737390355,\\
d_{1,3} &= 1.0804127773638506\text{e}{+1}, & d_{1,4} &= -1.2136535454001033\text{e}{+1},\\
d_{1,5} &= 1.2568359375000000\text{e}{+1}, & d_{1,6} &= -1.2136535454001033\text{e}{+1}\\
d_{1,7} &= 1.0804127773638506\text{e}{+1}, & d_{1,8} &= -8.3884327737390355\\
d_{1,9} &= 3.4366607666015625.
\end{aligned}$$

Dissipation for First-Degree Legendre-Gauss Operator ($s = p = 1$)

$$d_{1,1} = -1.7320508075688773, \qquad d_{1,2} = 1.7320508075688773.$$

Dissipation for Second-Degree Legendre-Gauss Operator ($s = p = 2$)

$$\begin{aligned}
d_{1,1} &= 1.6666666666666667, & d_{1,2} &= -3.3333333333333333,\\
d_{1,3} &= 1.6666666666666667.
\end{aligned}$$

Dissipation for Third-Degree Legendre-Gauss Operator ($s = p = 3$)

$$\begin{aligned}
d_{1,1} &= -1.6490089040086547, & d_{1,2} &= 4.1767665350122357,\\
d_{1,3} &= -4.1767665350122357, & d_{1,4} &= 1.6490089040086547.
\end{aligned}$$

Dissipation for Fourth-Degree Legendre-Gauss Operator ($s = p = 4$)

$$\begin{aligned} d_{1,1} &= 1.7193488544779225, & d_{1,2} &= -4.8693488544779225,\\ d_{1,3} &= 6.3000000000000000, & d_{1,4} &= -4.8693488544779225,\\ d_{1,5} &= 1.7193488544779225. \end{aligned}$$

Dissipation for Fifth-Degree Legendre-Gauss Operator ($s = p = 5$)

$$\begin{aligned} d_{1,1} &= -1.8757467023635989, & d_{1,2} &= 5.6525734058562505,\\ d_{1,3} &= -8.3331856501567726, & d_{1,4} &= 8.3331856501567726,\\ d_{1,5} &= -5.6525734058562505, & d_{1,6} &= 1.8757467023635989. \end{aligned}$$

Dissipation for Sixth-Degree Legendre-Gauss Operator ($s = p = 6$)

$$\begin{aligned} d_{1,1} &= 2.1221693076139195, & d_{1,2} &= -6.6442719823097040,\\ d_{1,3} &= 1.0575012727605837\text{e+}1, & d_{1,4} &= -1.2105820105820106\text{e+}1,\\ d_{1,5} &= 1.0575012727605837\text{e+}1, & d_{1,6} &= -6.6442719823097040\\ d_{1,7} &= 2.1221693076139195. \end{aligned}$$

Dissipation for Seventh-Degree Legendre-Gauss Operator ($s = p = 7$)

$$\begin{aligned} d_{1,1} &= -2.4719527372145800, & d_{1,2} &= 7.9371558399434804,\\ d_{1,3} &= -1.3269472683483800\text{e+}1, & d_{1,4} &= 1.6485779742174750\text{e+}1,\\ d_{1,5} &= -1.6485779742174750\text{e+}1, & d_{1,6} &= 1.3269472683483800\text{e+}1\\ d_{1,7} &= -7.9371558399434804, & d_{1,8} &= 2.4719527372145800. \end{aligned}$$

Dissipation for Eighth-Degree Legendre-Gauss Operator ($s = p = 8$)

$$\begin{aligned} d_{1,1} &= 2.9482381832990194, & d_{1,2} &= -9.6340325643962811,\\ d_{1,3} &= 1.6656389129737205\text{e+}1, & d_{1,4} &= -2.1840711936139944\text{e+}1,\\ d_{1,5} &= 2.3740234375000000\text{e+}1, & d_{1,6} &= -2.1840711936139944\text{e+}1\\ d_{1,7} &= 1.6656389129737205\text{e+}1, & d_{1,8} &= -9.6340325643962811\\ d_{1,9} &= 2.9482381832990194. \end{aligned}$$

2 Selecting the Dissipation Coefficient ε

The selection of the dissipation coefficient ε will, in general, be a problem-dependent task. However, we can use a heuristic approach of matching the coefficient to that of upwind stencils as a suitable starting point [4]. As an example, consider the fourth-order centered difference stencil

$$\frac{u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}}{12\Delta x} \approx \frac{\partial u}{\partial x} - \frac{1}{30}\left(\Delta x^4\right)\frac{\partial^5 u}{\partial x^5} + \dots \tag{1}$$

This stencil is found in the interior of $p = 2$ finite-difference SBP operators. We could, as an alternative to using a volume dissipation model, introduce dissipation by employing an upwind stencil shifted by one node,

$$\frac{u_{j-2} - 6u_{j-1} + 3u_j + 2u_{j+1}}{6\Delta x} = \frac{\partial u}{\partial x} + \frac{1}{12}\left(\Delta x^3\right)\frac{\partial^4 u}{\partial x^4} - \frac{1}{30}\left(\Delta x^4\right)\frac{\partial^5 u}{\partial x^5} + \dots \quad (2)$$

This is the stencil found in the interior of the $p_u = 3$ upwind finite-difference (UFD) SBP operator of [5]. By subtracting (1) from (2), we can show that the upwind stencil (2) consists of an antisymmetric portion (1), plus a symmetric (dissipative) portion

$$\frac{u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2}}{12\Delta x} \approx \frac{1}{12}\left(\Delta x^3\right)\frac{\partial^4 u}{\partial x^4} + \dots \quad (3)$$

Therefore, the interior stencil of the UFD operator is (in the context of a linear PDE) equivalent to the application of the fourth-order central difference operator (1) plus third-order dissipation (3). If we use the convention of undivided difference operators, this corresponds to a dissipation coefficient of $\varepsilon = 1/12$. We can also use an upwind stencil shifted by two nodes,

$$\frac{u_{j-2} - 4u_{j-1} + 3u_j}{2\Delta x} = \frac{\partial u}{\partial x} - \frac{1}{3}\left(\Delta x^2\right)\frac{\partial^3 u}{\partial x^3} + \frac{1}{4}\left(\Delta x^3\right)\frac{\partial^4 u}{\partial x^4} + \dots \quad (4)$$

This is the stencil found in the interior of the $p_u = 2$ UFD SBP operator of [5]. We can also decompose this into an antisymmetric portion, this time a wide, second-order approximation to the derivative of u, plus the same symmetric portion found in (3), but now with coefficient $\varepsilon = 1/4$.

Consider comparable UFD and standard SBP discretizations with volume dissipation from this work with $s = p+1$ (such that each has a provable error convergence rate of $p+1$). Let $u^{(n)}$ denote the n-th partial derivative of u with respect to x. It is possible to show that

- Even UFD discretizations with $p_u = 2p$ have interior stencils with
  - upwind stencils shifted by 2 nodes approximating $u^{(1)} + \mathcal{O}\left(\Delta x^{2p}\right)$
  - a wide antisymmetric portion approximating $u^{(1)} + \mathcal{O}\left(\Delta x^{2p}\right)$
  - a minimal-width symmetric portion approximating $\varepsilon\left(\Delta x^{2p+1}\right)u^{(2p+2)}$ where $\varepsilon = \left[p_u\binom{p_u}{p_u/2}\right]^{-1} = \left[2p\binom{2p}{p}\right]^{-1} = \frac{1}{2p}\frac{(p!)^2}{(2p)!}$
- Odd UFD discretizations with $p_u = 2p+1$ have interior stencils with
  - upwind stencils shifted by 1 node approximating $u^{(1)} + \mathcal{O}\left(\Delta x^{2p+1}\right)$
  - a minimal-width antisymmetric portion approximating $u^{(1)} + \mathcal{O}\left(\Delta x^{2p+2}\right)$
  - a minimal-width symmetric portion approximating $\varepsilon\left(\Delta x^{2p+1}\right)u^{(2p+2)}$ where $\varepsilon = \left[2p_u\binom{p_u-1}{(p_u-1)/2}\right]^{-1} = \left[(4p+2)\binom{2p}{p}\right]^{-1} = \frac{1}{2}\frac{(p!)^2}{(2p+1)!}$
- Typical finite-difference SBP discretizations with volume dissipation $s = p+1$ have interior stencils with
  - a minimal-width antisymmetric portion approximating $u^{(1)} + \mathcal{O}\left(\Delta x^{2p}\right)$
  - a minimal-width symmetric portion approximating $\varepsilon\left(\Delta x^{2p+1}\right)u^{(2p+2)}$

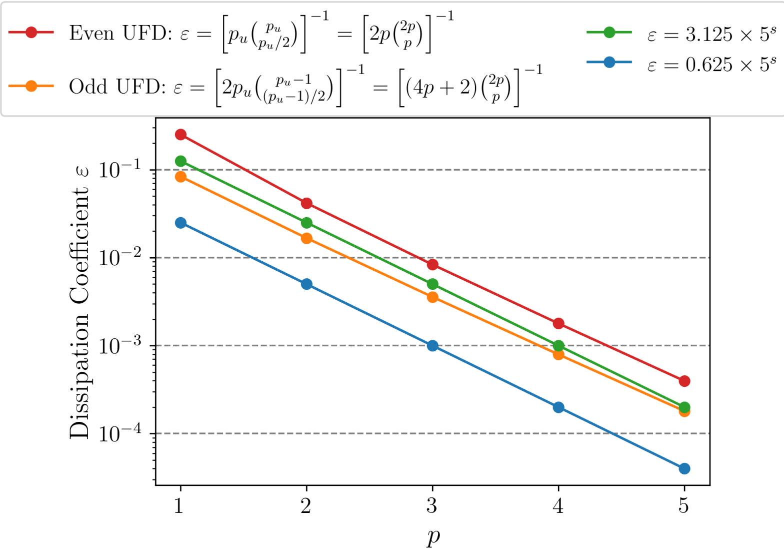

Fig. 1: A comparison of the dissipation coefficients ε recovered from upwind stencils (e.g. on the interiors of the UFD operators of [5]) and the chosen ε for the current volume dissipation operators with $s = p + 1$.

In each of the above 3 cases, the minimal-width symmetric portion of the interior stencil is identical - that is the interior dissipation is identical, up to the value of ε. Note that although the interior of the odd UFD operators are more accurate than the other two, the boundaries are only accurate to order (Δx^p), leading to provable convergence rates of $p + 1$.

When selecting ε, therefore, a reasonable choice would be to match it to the coefficient recovered by odd upwind stencils, although this can be decreased if less dissipation is desired. In the current work, for $s = p + 1$, we experimentally chose a value of $\varepsilon = 3.125 \times 5^s = \frac{5^{1-p}}{8}$ to approximate the values recovered from the odd upwind stencils, while still yielding simple decimal values. A much smaller value of $\varepsilon = 0.625 \times 5^s = \frac{1}{5}\frac{5^{1-p}}{8}$ was also used. We plot a comparison of these values in Figure 1, which demonstrates that for $p \leq 5$, the chosen values closely mimic the values recovered by the upwind stencil.

3 Further Discussion on Equivalent Formulations for Minimum-Width Traditional SBP Dissipation operators

The application of dissipation in the form

$$\mathsf{A}_\mathsf{D} = -\varepsilon \mathsf{H}^{-1} \tilde{\mathsf{D}}_s^\mathrm{T} \mathsf{C} \tilde{\mathsf{D}}_s \tag{5}$$

along with the choice

$$\tilde{\mathsf{D}}_s = \tilde{\mathsf{D}}^s = \Delta x^s \mathsf{D}^s = \Delta x^s \underbrace{\mathsf{D}\mathsf{D}\ldots\mathsf{D}}_{s \text{ terms}} \tag{6}$$

leads to dissipation operators whose interior stencils are wider than necessary for a given order of resultant interior stencil accuracy, which unnecessarily increases computational cost through additional floating-point operations. In contrast, the use of (5) with $\tilde{\mathsf{D}}_s$ as described in this work naturally leads to dissipation operators A_D of minimum width. Using a different approach, minimum-width dissipation operators were constructed in [1] by decomposing A_D into the difference between volume dissipation based on the undivided first-derivative SBP operator (6) and correction terms as follows:

$$\mathsf{A}_\mathsf{D} = -\varepsilon \mathsf{H}^{-1} \left[(\tilde{\mathsf{D}}^s)^\mathrm{T} \tilde{\mathsf{H}} \mathsf{B} \tilde{\mathsf{D}}^s - \mathsf{R} \right], \tag{7}$$

where R is constructed such that the interior stencil of A_D is of minimum width. The general form of R holds for even-order interior stencil operators that are arbitrarily high-order, is an extension of [2, 8], and is given by

$$\mathsf{R} = \sum_{i=p+s}^{2ps} \alpha_{p,i} \tilde{\mathsf{D}}_{\mathsf{R},i}^\mathrm{T} \mathsf{B}_{p,i} \tilde{\mathsf{D}}_{\mathsf{R},i}, \tag{8}$$

where the $\alpha_{p,i}$ coefficients depend on both the accuracy of the dissipation and the degree of the operator. The construction procedure to find these coefficients $\alpha_{p,i}$ and operators $\tilde{\mathsf{D}}_{\mathsf{R},i}$ is explained in [1, 2]. Although this formulation appears different to (5), the resulting dissipation operators are in fact largely equivalent. To see this, first consider the interior of the constant-coefficient operator, at least $s+1$ nodes away from the boundaries. Since the interior stencil of A_D must approximate $(-1)^{s+1} d^{2s}/dx^{2s}$ multiplied by a constant, and the minimum-width stencil contains only $2s+1$ distinct nodes, this stencil is unique (up to a multiplicative constant). The operator in (7) is specifically constructed to match this stencil. Alternatively, if using (5) with $\mathsf{D}_s \approx d^s/dx^s$ containing a repeating interior stencil of width $s+1$ and $\mathsf{C} = \mathrm{diag}(\mathbf{1})$, A_D will also contain an interior stencil of width $2s+1$. Since the interior of D^T approximates $(-1)^{s+1} d^s/dx^s$, which follows from the general property $\left[\mathsf{D}_s^\mathrm{T}\right]_{i,i+j} = \left[\mathsf{D}_s\right]_{i+j,i} = \left[\mathsf{D}_s\right]_{i,i-j}$ of repeating interior stencils, the interior of A_D must again be the unique stencil recovered earlier.

Showing that the boundaries of the dissipation operators in (5) and (7) are equivalent, however, is more difficult since it depends on the choices of B, $\mathsf{B}_{p,i}$, and $\tilde{\mathsf{D}}_{\mathsf{R},i}$, which introduce degrees of freedom at the boundary. Similarly, the averaging at half-nodes for a variable coefficient A introduces many nuances that are difficult to account for in generality. Therefore, instead of providing a direct proof of this equivalence, we demonstrate the connection between between (5) and (7) via two specific examples. Both examples assume an equispaced nodal distribution, and do not include $\tilde{\mathsf{H}}$, i.e. use $\mathsf{C} = \mathsf{B}\mathsf{A}$. We also consider only finite-difference operators, since spectral-element operators are naturally of minimum width and are uniquely defined.

3.1 First-Order Accurate Volume Dissipation

Consider first-order accurate dissipation for equispaced nodes (called second-order accurate in [7]):

$$\mathsf{A}_{\mathsf{D},1} = -\mathsf{H}^{-1}\tilde{\mathsf{D}}_1^{\mathrm{T}}\mathsf{B}\tilde{\mathsf{D}}_1, \tag{9}$$

where

$$\tilde{\mathsf{D}}_1 = \begin{bmatrix} -1 & 1 & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{B} = \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}. \tag{10}$$

This dissipation is called first-order since the interior rows have first-order truncation using Taylor series expansions.

$$\mathsf{A}_{\mathsf{D},1} = \frac{1}{\Delta x}\begin{bmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{bmatrix} = \begin{bmatrix} 2\left(\frac{\partial u}{\partial x}\right)_j + \Delta x\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \mathcal{O}(\Delta x^2) \\ \Delta x\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{\Delta x^3}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_j + \mathcal{O}(\Delta x^4) \\ \vdots \\ \Delta x\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{\Delta x^3}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_j + \mathcal{O}(\Delta x^4) \\ -2\left(\frac{\partial u}{\partial x}\right)_j + \Delta x\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \mathcal{O}(\Delta x^2) \end{bmatrix}$$

3.1.1 Option 1: Minimum Width Without Averging

The first option for a variable coefficient problem uses the approach (5) adopted in this work, along with $\mathsf{C} = \mathsf{BA}$ and the simple choice of defining the variable coefficient at the nodes, i.e.

$$\mathsf{A} = \operatorname{diag}\left(a_1, a_2, \ldots, a_N\right), \tag{11}$$

Then

$$\mathsf{A}_{\mathsf{D},1} = -\mathsf{H}^{-1}\tilde{\mathsf{D}}_1^{\mathrm{T}}\mathsf{BA}\tilde{\mathsf{D}}_1 = \frac{1}{\Delta x}\begin{bmatrix} -2a_2 & 2a_2 & & & \\ a_2 & -(a_2+a_3) & a_3 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N-1} & -(a_{N-1}+a_N) & a_N \\ & & & 2a_N & -2a_N \end{bmatrix}. \tag{12}$$

As explained in the main document, the issue with this approach is that the variable coefficient does not appear symmetrically with respect to the anti-diagonal, i.e., a_1 does *not* appear but a_2 through a_N do. To see the meaning of this, consider recasting the dissipation operator $\mathsf{A}_{\mathsf{D},1}$ in flux difference form

$$\mathsf{A}_{\mathsf{D},1} = -\mathsf{H}^{-1} \begin{bmatrix} \tilde{\mathsf{D}}_1^{\mathrm{T}} \mathsf{B} \ \mathbf{0} \end{bmatrix} \begin{bmatrix} a_1 & & & & & \\ & a_2 & & & & \\ & & \ddots & & & \\ & & & a_{N-1} & & \\ & & & & a_N & \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathsf{D}}_1 \\ \mathbf{0}^{\mathrm{T}} \end{bmatrix}. \tag{13}$$

This is in flux difference form since

$$\begin{bmatrix} \tilde{\mathsf{D}}_1^{\mathrm{T}} \mathsf{B} \ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1 \ \ 0 \end{bmatrix}. \tag{14}$$

In this case, the interior coefficients of the variable coefficient matrix, a_2 through a_N, correspond to the variable coefficient values at the interior flux points or half-nodes, and choosing them to be nodal values introduces an unjustified bias into the selection of the variable coefficients at the half-nodes.

3.1.2 Option 2: Minimum Width With Averaging

To remove the bias from the selection of the variable coefficient at the half-nodes, one could use the simple average introduced in the main document,

$$\mathsf{A} = \operatorname{diag}\left(0, \frac{1}{2}(a_1 + a_2), \frac{1}{2}(a_1 + a_2), \ldots, \frac{1}{2}(a_{N-1} + a_N)\right). \tag{15}$$

This yields the dissipation operator

$$\mathsf{A}_{\mathsf{D},1} = \begin{bmatrix} -(a_1 + a_2) & a_1 + a_2 & & & \\ \frac{a_1}{2} + \frac{a_2}{2} & -\left(\frac{a_1}{2} + a_2 + \frac{a_3}{2}\right) & \frac{a_2}{2} + \frac{a_3}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{a_{N-2}}{2} + \frac{a_{N-1}}{2} & -\left(\frac{a_{N-2}}{2} + a_{N-1} + \frac{a_N}{2}\right) & \frac{a_{N-1}}{2} + \frac{a_N}{2} \\ & & & a_{N-1} + a_N & -(a_{N-1} + a_N) \end{bmatrix}, \tag{16}$$

where the variable coefficient appears in an unbiased manner.

3.1.3 Option 3: Correction Procedure

For another perspective, we can consider decomposing the dissipation operator into the difference between the first derivative operator and correction terms such that the resultant operator is of minimum width. For the first-order dissipation, (7) becomes

$$\mathsf{A}_{\mathsf{D},1} = -\mathsf{H}^{-1}(\tilde{\mathsf{D}}^{\mathrm{T}}\mathsf{B}\mathsf{A}\tilde{\mathsf{D}} + \frac{1}{4}\tilde{\mathsf{D}}_{\mathsf{R},2}^{\mathrm{T}}\mathsf{B}_{\mathsf{R},2}\tilde{\mathsf{D}}_{\mathsf{R},2}), \tag{17}$$

where

$$\tilde{\mathsf{D}} = \begin{bmatrix} -1 & 1 & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{B} = \begin{bmatrix} \frac{1}{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{1}{2} \end{bmatrix},$$

and A is defined without any averaging, i.e. (11). The correction matrices are

$$\tilde{\mathsf{D}}_{\mathsf{R},2} = \begin{bmatrix} -1 & 2 & -1 & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & -1 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \mathsf{B}_{\mathsf{R},2} = \begin{bmatrix} 0 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 0 \end{bmatrix}.$$

Interestingly, expanding (17) yields exactly the same as (16), but here we did not need to think about half-nodes at all.

3.2 Third Order Accurate Volume Dissipation

Consider now the third order accurate dissipation from [7] (called fourth order accurate in [7]):

$$\mathsf{A}_{\mathsf{D},2} = -\mathsf{H}^{-1}\tilde{\mathsf{D}}_2^{\mathrm{T}}\mathsf{B}\tilde{\mathsf{D}}_2, \tag{18}$$

where

$$
\tilde{\mathsf{D}}_2 = \begin{bmatrix} -1 & 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 \\ & & & & -1 & 2 & -1 \end{bmatrix} \quad \text{and} \quad \mathsf{B} = \begin{bmatrix} 0 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 0 \end{bmatrix}. \tag{19}
$$

This dissipation is called third order since the interior rows have third order truncation using Taylor series expansions.

$$
\begin{aligned}
\mathsf{A}_{\mathsf{D},2} &= -\mathsf{H}^{-1}\tilde{\mathsf{D}}_2^{\mathrm{T}}\mathsf{B}_2\tilde{\mathsf{D}}_2 \\
&= \frac{1}{\Delta x}\begin{bmatrix} -2 & 4 & -2 & & & & & \\ 2 & -5 & 4 & -1 & & & & \\ -1 & 4 & -6 & 4 & -1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & -1 & 4 & -6 & 4 & -1 \\ & & & -1 & 4 & -5 & 2 \\ & & & & -2 & 4 & -2 \end{bmatrix} = \begin{bmatrix} -2\Delta x\left(\frac{\partial^2 u}{\partial x^2}\right)_j - 2\Delta x^2\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \mathcal{O}(\Delta x^3) \\ \Delta x\left(\frac{\partial^2 u}{\partial x^2}\right)_j - \Delta x^2\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \mathcal{O}(\Delta x^3) \\ -\Delta x^3\left(\frac{\partial^4 u}{\partial x^4}\right)_j - \frac{\Delta x^5}{6}\left(\frac{\partial^6 u}{\partial x^6}\right)_j + \mathcal{O}(\Delta x^7) \\ \vdots \\ -\Delta x^3\left(\frac{\partial^4 u}{\partial x^4}\right)_j - \frac{\Delta x^5}{6}\left(\frac{\partial^6 u}{\partial x^6}\right)_j + \mathcal{O}(\Delta x^7) \\ \Delta x\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \Delta x^2\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \mathcal{O}(\Delta x^3) \\ -2\Delta x\left(\frac{\partial^2 u}{\partial x^2}\right)_j + 2\Delta x^2\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \mathcal{O}(\Delta x^3) \end{bmatrix}
\end{aligned} \tag{20}
$$

3.2.1 Option 1: Minimum Width Without Averaging

Once again choosing the variable coefficient to be defined at the nodes as in (11), we obtain the operator

$$
\begin{aligned}
\mathsf{A}_{\mathsf{D},2} &= -\mathsf{H}^{-1}\tilde{\mathsf{D}}_2^{\mathrm{T}}\mathsf{B}\mathsf{A}\tilde{\mathsf{D}}_2 \\
&= \frac{1}{\Delta x}\begin{bmatrix} -2a_2 & 4a_2 & -2a_2 & & & & \\ 2a_2 & -(4a_2+a_3) & 2(a_2+a_3) & -a_3 & & & \\ -a_2 & 2(a_2+a_3) & -(a_2+4a_3+a_{N-2}) & 2(a_3+a_{N-2}) & -a_{N-2} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & -a_{N-3} & 2(a_{N-3}+a_{N-2}) & -(a_3+4a_{N-2}+a_{N-1}) & 2(a_{N-2}+a_{N-1}) & -a_{N-1} & \\ & & -a_{N-2} & 2(a_{N-2}+a_{N-1}) & -(a_{N-2}+4a_{N-1}+a_N) & 2(a_{N-1}+a_N) & -a_N \\ & & & -a_{N-1} & 2(a_{N-1}+a_N) & -(a_{N-1}+4a_N) & 2a_N \\ & & & & -2a_N & 4a_N & -2a_N \end{bmatrix}.
\end{aligned}
$$

Similar to before, the issue with this approach is that the variable coefficients at the first and last nodes, a_1 and a_N, do not appear.

3.2.2 Option 2: Correction Procedure

Before exploring the averaging method, this time let us examine the correction procedure defined by (7). For third order dissipation with $p = 1$ and $s = 2$, (7) becomes

$$\mathsf{A}_{\mathsf{D},2} = -\mathsf{H}^{-1}((\tilde{\mathsf{D}}\tilde{\mathsf{D}})^{\mathrm{T}}\mathsf{B}\tilde{\mathsf{D}}\tilde{\mathsf{D}} + \frac{1}{2}\tilde{\mathsf{D}}_{\mathsf{R},3}^{\mathrm{T}}\mathsf{B}_{\mathsf{R},3}\tilde{\mathsf{D}}_{\mathsf{R},3} + \frac{1}{16}\tilde{\mathsf{D}}_{\mathsf{R},4}^{\mathrm{T}}\mathsf{B}_{\mathsf{R},4}\tilde{\mathsf{D}}_{\mathsf{R},4}), \tag{21}$$

where

$$\tilde{\mathsf{D}} = \begin{bmatrix} -1 & 1 & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{B} = \begin{bmatrix} 1 & & & & & & \\ & \frac{4}{3} & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \frac{4}{3} & \\ & & & & & & 1 \end{bmatrix},$$

Also,

$$\tilde{\mathsf{D}}_{\mathsf{R},3} = \begin{bmatrix} -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & & \\ -1 & 3 & -3 & 1 & & & \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & & \\ & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & \\ & & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ & & & -1 & 3 & -3 & 1 \\ & & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathsf{B}_{\mathsf{R},3} = \begin{bmatrix} 0 & & & & & & \\ & \frac{1}{3} & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \frac{1}{3} & \\ & & & & & & 0 \end{bmatrix}$$

and

$$\tilde{\mathsf{D}}_{\mathsf{R},4} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & 6 & -4 & 1 & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & 1 & -4 & 6 & -4 & 1 \\ & & 1 & -4 & 6 & -4 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{B}_{\mathsf{R},3} = \begin{bmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix}$$

Expanding out (21) reveals that for the constant-coefficient case this is the same as (20). For the variable coefficient case, as before we use A defined at the nodes with (11) and obtain the following

$$\mathsf{A}_{\mathsf{D},2} = \begin{bmatrix} -\left(\frac{a_1}{2}+a_2+\frac{a_3}{2}\right) & a_1+2a_2+a_3 & -\left(\frac{a_1}{2}+a_2+\frac{a_3}{2}\right) & & & \\ \frac{a_1}{2}+a_2+\frac{a_3}{2} & -\left(a_1+\frac{9a_2}{4}+\frac{3a_3}{2}+\frac{a_4}{4}\right) & \frac{1}{2}\left(a_1+3a_2+3a_3+a_4\right) & -\left(\frac{a_2}{4}+\frac{a_3}{2}+\frac{a_4}{4}\right) & & \\ -\left(\frac{a_1}{4}+\frac{a_2}{2}+\frac{a_3}{4}\right) & \frac{1}{2}\left(a_1+3a_2+3a_3+a_4\right) & -\left(\frac{a_1}{4}-\frac{3a_2}{2}-\frac{5a_3}{2}-\frac{3a_4}{2}-\frac{c_5}{4}\right) & \frac{1}{2}\left(a_2+3a_3+3a_4+c_5\right) & -\left(\frac{a_3}{4}+\frac{a_4}{2}+\frac{c_5}{4}\right) & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{22}$$

Only the upper triangular portion is shown since the lower portion is now a mirror reflection. The stencil width is the same (5 points on interior), but the variable coefficient at the first and last node appears in a consistent way.

3.2.3 Option 3: Minimum Width With Averaging

To remove the bias from the selection of the variable coefficient at the half-nodes, we could again employ the simple average (15). Instead, however, consider the following average:

$$\mathsf{A} = \operatorname{diag}\left(\tfrac{1}{4}\left(a_1+a_2+a_3\right), \tfrac{1}{4}\left(a_1+a_2+a_3\right), \tfrac{1}{4}\left(a_2+a_3+a_4\right), \ldots, \tfrac{1}{4}\left(a_{N-2}+a_{N-1}+a_N\right)\right). \tag{23}$$

Then, using $\tilde{\mathsf{D}}_2$ and B as defined in (19), the operator $\mathsf{A}_{\mathsf{D},2} = -\mathsf{H}^{-1}\tilde{\mathsf{D}}_2^{\mathrm{T}}\mathsf{B}\mathsf{A}\tilde{\mathsf{D}}_2$ will be the exact same as (22). This example introduces the suggestion that perhaps a simple average at half-nodes is not the ideal average to use. The alternative form of expressing dissipation operators in terms of the undivided SBP operator and correction terms may therefore be useful in terms of guiding how the variable coefficient should appear in the dissipation operator. More sophisticated averages may result in more desirable dissipation operators. Further investigation of this idea however is left to future work.

References

1. Craig Penner, D.A., Zingg, D.W.: High-order artificial dissipation operators possessing the summation-by-parts property. In: 2018 Fluid Dynamics Conference. American Institute of Aeronautics and Astronautics, Atlanta, Georgia (2018). DOI 10.2514/6.2018-4165. AIAA Paper 2018-4165
2. Del Rey Fernández, D.C., Zingg, D.W.: Generalized summation-by-parts operators for the second derivative. SIAM Journal on Scientific Computing **37**(6), A2840–A2864 (2015). DOI 10.1137/140992205
3. Del Rey Fernández, D.C., Boom, P.D., Zingg, D.W.: Corner-corrected diagonal-norm summation-by-parts operators for the first derivative with increased order of accuracy. Journal of Computational Physics **330**, 902–923 (2017). DOI 10.1016/j.jcp.2016.10.051
4. Lomax, H., Pulliam, T.H., Zingg, D.W.: Fundamentals of Computational Fluid Dynamics, 1 edn. Scientific Computation. Springer Berlin, Heidelberg (2001). DOI 10.1007/978-3-662-04654-8
5. Mattsson, K.: Diagonal-norm upwind sbp operators. Journal of Computational Physics **335**, 283–310 (2017). DOI 10.1016/j.jcp.2017.01.042

6. Mattsson, K., Almquist, M., van der Weide, E.: Boundary optimized diagonal-norm sbp operators. Journal of Computational Physics **374**, 1261–1266 (2018). DOI 10.1016/j.jcp.2018.06.010
7. Mattsson, K., Svärd, M., Nordström, J.: Stable and accurate artificial dissipation. Journal of Scientific Computing **21**(1), 57–79 (2004). DOI 10.1023/B:JOMP.0000027955.75872.3f
8. Mattsson, K., Svärd, M., Shoeybi, M.: Stable and accurate schemes for the compressible Navier–Stokes equations. Journal of Computational Physics **227**(4), 2293–2316 (2008). DOI 10.1016/j.jcp.2007.10.018