High-Dimensional Probability: Answers, Theorems, and Definitions

Last revised on August 24, 2021

- Companion notes for *High-Dimensional Probability*, by Roman Vershynin. Link to book (PDF available online): www.math.uci.edu/~rvershyn/papers/HDP-book/HDP-book.html.
- **Disclaimer:** These notes compile my answers to the exercises, and lift the required theorems and definitions from the book. I wrote these notes to aid my personal study of the book. Read them at your own risk![*]

Contents

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0 Appetizer: Using probability to cover a geometric set

A point $x \in \mathbb{R}^n$ is a **convex combination** of points $x_1, ..., x_m \in \mathbb{R}^n$ if

$$x = \sum_{i=1}^{m} \lambda_i x_i \qquad \text{with each } \lambda_i \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{m} \lambda_i = 1.$$

The **convex hull** of $T \subseteq \mathbb{R}^n$, $\mathrm{conv}(T)$, is the set of all convex combinations of T.

Theorem 0.0.1 (Catheodory's Theorem)**.** Let $x \in \mathrm{conv}(T)$. There exists $k \leq n+1$ points $x_1, ..., x_k \in T$ such that x is a convex combination of $x_1, ..., x_k$.

The result says we can obtain any point in the convex hull of T using at most a dimension-dependent number of points. Let the **diameter** of a set T be defined as $\mathrm{diam}(T) = \sup\{\|x - y\|_2 : x, y \in T\}$.

Theorem 0.0.2 (Approximate Catheodory's Theorem)**.** Let $\mathrm{diam}(T) = 1$. Let $x \in \mathrm{conv}(T)$. For any k, there exists k points $x_1, ..., x_k \in T$ such that

$$\left\| x - \frac{1}{k} \sum_{j=1}^{k} x_j \right\|_2 \leq \frac{1}{\sqrt{k}}$$

Proof. Suppose $|T| = m$. WLOG we can assume T is bounded by 1 in $\|\cdot\|_2$. We write $x = \sum_{i=1}^{m} \lambda_i x_i$, and interpret λ_i as probabilities. We define the random variable

$$X = x_i \text{ with probability } \lambda_i$$

for $i = 1, ..., m$. We can verify that $\mathbb{E}X = \sum_{i=1}^{m} \lambda_i x_i = x$. Taking $X_1, ..., X_k \overset{\text{iid}}{\sim} X$. It remains to analyse the quantity $\mathbb{E}\|x - \frac{1}{k}\sum_{j=1}^{k} X_j\|_2^2$.

$$\begin{aligned}
\mathbb{E}\left\| x - \frac{1}{k}\sum_{j=1}^{k} X_j \right\|_2^2 &\leq \frac{1}{k^2}\mathbb{E}\left\| \sum_{j=1}^{k} X_j - x \right\|_2^2 \\
&= \frac{1}{k^2}\sum_{j=1}^{k} \mathbb{E}\left\| X_j - x \right\|_2^2 && \text{(by Exercise 0.0.3 (a))} \\
&= \frac{1}{k}\mathbb{E}\|X - x\|_2^2
\end{aligned}$$

Applying the result of Exercise 0.0.3 (b), we obtain

$$\mathbb{E}\|X - x\|_2^2 = \mathbb{E}\|X\|_2^2 - \|\mathbb{E}X\|_2^2 \leq \mathbb{E}\|X\|_2^2 \leq 1$$

Plugging this in above, we obtain the desired bound in expectation, hence there must exist a realization of the X_j, $x_1, ..., x_k$, such that the bound holds. $\square$

Exercise 0.0.3. Check the following identities for random vectors.

(a) Let $X_1, ..., X_k$ be independent, mean zero random vectors in $\mathbb{R}^n$. Show that

$$\mathbb{E}\left\| \sum_{j=1}^{k} X_j \right\|_2^2 = \mathbb{E}\sum_{j=1}^{k} \|X_j\|_2^2$$

Answer.

$$\mathbb{E}\left\|\sum_{j=1}^{k} X_j\right\|_2^2 = \sum_{i=1}^{n}\mathbb{E}\left(\sum_{j=1}^{m} X_j^{(i)}\right)^2 = \sum_{i=1}^{n}\mathrm{Var}\left(\sum_{j=1}^{m} X_j^{(i)}\right) \qquad \text{(by mean zero)}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\mathrm{Var}\left(X_j^{(i)}\right) \qquad \text{(by independence)}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\mathbb{E}\left(X_j^{(i)}\right)^2 \qquad \text{(by mean zero)}$$
$$= \mathbb{E}\sum_{j=1}^{m}\|X_j\|_2^2$$

$\square$

Among other things, this result implies that the expected squared distance of a random walk (starting from the origin) is equal to sum of the expected squared distances of each step.

(b) Let X be a random vector in $\mathbb{R}^n$. Show that

$$\mathbb{E}\|X - \mathbb{E}X\|_2^2 = \mathbb{E}\|X\|_2^2 - \|\mathbb{E}X\|_2^2$$

Answer.

$$\mathbb{E}\|X - \mathbb{E}X\|_2^2 = \mathbb{E}\sum_{i=1}^{n}\left(X^{(i)} - (\mathbb{E}X)^{(i)}\right)^2 = \sum_{i=1}^{n}\mathrm{Var}(X^{(i)}) = \sum_{i=1}^{n}\mathbb{E}\left(X^{(i)}\right)^2 - \left(\mathbb{E}X^{(i)}\right)^2$$
$$= \mathbb{E}\|X\|_2^2 - \|\mathbb{E}X\|_2^2$$

$\square$

Corollary 0.0.4 (Covering polytopes by balls)**.** Let $P \subseteq \mathbb{R}^n$ be a polytope with $\mathrm{diam}(P) = 1$. Let m be the number of vertices of P. Let $\varepsilon > 0$. We can cover P with m^k balls of radius ε for $k \geq \lceil 1/\varepsilon^2 \rceil$.

Proof. Take T to be the vertex set of P. $|T| = m$. Note that for any $x \in P$, $x \in \mathrm{conv}(T)$. By Theorem 0.0.2, taking $k \geq \lceil 1/\varepsilon^2 \rceil$, we can find $x_1, ..., x_k \in T$ such that

$$\left\|x - \frac{1}{k}\sum_{j=1}^{k} x_j\right\| \leq \frac{1}{\sqrt{k}} \leq \varepsilon$$

The number of ball centres obtained from selecting a set of k points out of m with repetition is bounded by m^k (possibly repeating orders). Hence we have an ε-cover sufficient to cover P. $\square$

Exercise 0.0.5 (Bionomial coefficient inequality)**.** Show that for $1 \leq r \leq n$

$$\left(\frac{n}{r}\right)^r \leq \binom{n}{r} \leq \sum_{k=0}^{r}\binom{n}{k} \leq \left(\frac{en}{r}\right)^r$$

Answer. For the first inequality, consider

$$\frac{\left(\frac{n}{r}\right)^r}{\binom{n}{r}} = \overbrace{\frac{\frac{n}{r}}{\frac{n}{r}} \cdot \frac{\frac{n}{r}}{\frac{n-1}{r-1}} \cdot .... \cdot \frac{\frac{n}{r}}{\frac{n-r+1}{1}}}^{r} \leq 1 \cdot 1 \cdot .... \cdot 1 = 1$$

The second inequality follows immediately. To justify the last inequality, write

$$\begin{aligned}
\left(\frac{en}{r}\right)^r = e^r \cdot \left(\frac{n}{r}\right)^r &= \sum_{k=0}^{\infty} \frac{r^k}{k!} \cdot \left(\frac{n}{r}\right)^r && \text{(Maclaurin series for } e^x\text{)} \\
&\geq \sum_{k=0}^{r} \frac{r^k}{k!} \cdot \left(\frac{n}{r}\right)^r \\
&= \sum_{k=0}^{r} \frac{n^k \cdot n^{r-k}}{k! \cdot r^{r-k}} \\
&\geq \sum_{k=0}^{r} \frac{n^k}{k!} && \text{(by } n \geq r\text{)} \\
&\geq \sum_{k=0}^{r} \binom{n}{r}
\end{aligned}$$

□

Exercise 0.0.6 (Improved covering)**.** Show that in the setting of Corollary 0.0.4, for $k \geq \lceil 1/\varepsilon^2 \rceil$

$$(C + C\varepsilon^2 m)^k$$

balls suffice for a suitable constant C.

Answer. We can give a tighter bound than given in the proof of Corollary 0.0.4 on the number of ball centres obtained from selecting a set of k points out of m with repetition (since computing the mean of k is order-invariant with respect to input points). By the "stars-and-bars"† argument, this quantity is given by

$$\binom{m+k-1}{k-1}$$

Note that $\min\{k-1, m\} = k-1 \leq \min\{k, m-1\}$, so looking at row $m+k-1$ of Pascal's triangle

$$\binom{m+k-1}{k-1} \leq \binom{m+k-1}{k}$$

Then, using Exercise 0.0.5

$$\binom{m+k-1}{k} \leq \left(\frac{e(m+k-1)}{k}\right)^k = \left(e\frac{k-1}{k} + e\frac{1}{k}m\right)^k \leq (e + e\varepsilon^2 m)^k$$

□

†https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics)

1 Preliminaries on random variables

1.1 Basic quantities

The **expection** of a random variable X is denoted as $\mathbb{E}X$, and **variance** is denoted as $\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2$. (We note that the expectation operator $\mathbb{E}$ can be directly defined as the Lebesgue integral of the random variable (measurable function) $X : \Omega \to \mathbb{R}$ in the probability space $(\Omega, \mathcal{M}, \mathbb{P})$.

The **moment generating function** of X is given by

$$M_X(t) = \mathbb{E}e^{tX} \qquad \text{for all } t \in \mathbb{R}$$

The **p-th moment** of X is given by $\mathbb{E}X^p$. We also let $\|X\|_p = (\mathbb{E}X^p)^{\frac{1}{p}}$ denote the **p-norm** of X. For $p = \infty$, we have

$$\|X\|_\infty = \operatorname{ess\,sup} X$$

recalling that the **essential supremum** of a function f is the "smallest value γ such that $\{\omega \in \Omega : |f(\omega)| > \gamma\}$ has measure 0".

From this, we can define the ***L^p* spaces**[‡], given a probability space $(\Omega, \mathcal{M}, \mathbb{P})$

$$L^p = \{X : \|X\|_p < \infty\}$$

Results from measure and integration theory tell us that the $(L^p, \|\cdot\|_p)$ are complete. In the case of L^2, we have that with the inner product

$$\begin{aligned}\langle X, Y\rangle &= \int_\Omega XY(\omega)\mu(\omega) \\ &= \mathbb{E}XY\end{aligned}$$

$(L^2, \langle\cdot,\cdot\rangle)$ is a Hilbert space. In this case we can express the **standard deviation** of X as $\sqrt{\mathrm{Var}(X)} = \|X - \mathbb{E}X\|_2$, and the **covariance** of random variable X and Y as

$$\mathrm{Cov}(X, Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y) = \langle X - \mathbb{E}X, Y - \mathbb{E}Y\rangle)$$

In this setting, considering random variables as vectors in L^2, the covariance between X and Y can be interpreted as the *alignment* between the vectors $X - \mathbb{E}X$ and $Y - \mathbb{E}Y$.

1.2 Some classical inequalities

We say $f : \mathbb{R} \to \mathbb{R}$ is **convex** if

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y) \qquad \text{for all } x, y \in \mathbb{R} \text{ and } t \in [0, 1]$$

Jensen's inequality states that for any random variable X and a convex function f, we get

$$f(\mathbb{E}X) \le \mathbb{E}(f(X))$$

A corollary of Jensen's inequality implies that[§]

$$\|X\|_p \le \|X\|_q \qquad \text{for all } 1 \le p \le q \le \infty$$

Minkowski's inequality asserts that the triangle inequality holds for the L_p spaces

$$\|X + Y\|_p \le \|X\|_p + \|Y\|_p \qquad \text{for all } X, Y \in L^p$$

In L^2, we have the **Cauchy-Schwarz inequality**, which states that $|\mathbb{E}XY| \le \mathbb{E}|XY| \le \|X\|_2\|Y\|_2$. **Holder's inequality** additionally asserts that for $1/p + 1/q = 1$

$$|\mathbb{E}XY| \le \|XY\|_1 \le \|X\|_p\|Y\|_q$$

[‡] A technical note is that the objects of L_p are actually equivalence classes of functions $[X]$ with equality almost everywhere, otherwise $\|\cdot\|_p$ is only a semi-norm.

[§] For $q < \infty$, the result follows by applying Jensen's inequality for $f(x) = x^{\frac{q}{p}}$. Otherwise, $\|X\|_\infty = \gamma = (\mathbb{E}\gamma^p)^{\frac{1}{p}} = \|\gamma\|_p \ge \|X\|_p$.

which also holds for $p = 1, q = \infty$.

The **cumulative distribution function** of X is defined as

$$F_X(t) = \mathbb{P}\{X \leq t\} = \mathbb{P}(X^{-1}(-\infty, t]) \qquad \text{for all } t \in \mathbb{R}$$

and we refer to $\mathbb{P}\{X > t\} = 1 - F_X(t)$ as the **tail** of X.

Lemma 1.2.1 (Integral identity)**.** Let $X \geq 0$ be a random variable. Then

$$\mathbb{E}X = \int_0^\infty \mathbb{P}\{X > t\} dt$$

with left side $= \infty$ iff right side $= \infty$.

Exercise 1.2.2 (Generalization of integral identity)**.** Show that Lemma can be extended to be valid for any X

$$\mathbb{E}X = \int_0^\infty \mathbb{P}\{X > t\} dt - \int_{-\infty}^0 \mathbb{P}\{X < t\} dt$$

Answer. For not necessary non-negative X, $\mathbb{E}X := \mathbb{E}X^+ - \mathbb{E}X^-$ when they exist and are both $< \infty$, where

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad X^- = \begin{cases} -X & \text{if } X \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Applying Lemma 1.2.1 to the terms yields the result. For the second term

$$\mathbb{E}X^- = \int_0^\infty \mathbb{P}\{X^- > t\} dt = \int_0^\infty \mathbb{P}\{X < -t\} dt = \int_{-\infty}^0 \mathbb{P}\{X < t\} dt$$

□

Exercise 1.2.3 (p-th moment via the tail)**.** Let X be a random variable and $0 < p < \infty$. Show that

$$\mathbb{E}|X|^p = \int_0^\infty p t^{p-1} \mathbb{P}\{|X| > t\} dt$$

whenever the right side is $< \infty$.

Answer. On the right side, substitute $u = t^p$, so $du = pt^{p-1} dt$ and

$$\int_0^\infty p t^{p-1} \mathbb{P}\{|X| > t\} dt = \int_0^\infty \mathbb{P}\{|X| > u^{\frac{1}{p}}\} du = \int_0^\infty \mathbb{P}\{|X|^p > u\} du = \mathbb{E}|X|^p$$

where the last equality comes from applying Lemma 1.2.1 to the random variable $|X|^p \geq 0$. □

Proposition 1.2.4 (Markov's inequality)**.** Let $X \geq 0$ with $\mathbb{E}X < \infty$. Then for $t > 0$

$$\mathbb{P}\{X \geq t\} \leq \frac{\mathbb{E}X}{t}$$

Proof. Fix $t > 0$. Applying Lemma 1.2.1

$$\mathbb{E}X = \int_0^\infty \mathbb{P}\{X \geq x\} dx \geq \int_0^t \mathbb{P}\{X \geq x\} dx \geq \int_0^t \mathbb{P}\{X \geq t\} dx = t \cdot \mathbb{P}\{X \geq t\}$$

□

Corollary 1.2.5 (Chebyshev's inequality)**.** Let X have $\mathbb{E}X < \infty$ and $\mathrm{Var}(X) < \infty$. Then for $t > 0$

$$\mathbb{P}\{|X - \mathbb{E}X| > t\} \leq \frac{\mathrm{Var}(X)}{t^2}$$

Exercise 1.2.6. Give a proof of Chebyshev's inequality using Markov's inequality.

Answer. The random variable $|X - \mathbb{E}X|^2$ is well-defined (by $\mathbb{E}X < \infty$), non-negative, with finite expectation. Applying Markov's inequality with $t^2 > 0$ yields

$$\mathbb{P}\{|X - \mathbb{E}X| \geq t\} = \mathbb{P}\{|X - \mathbb{E}X|^2 \geq t^2\} \leq \frac{\mathrm{Var}(X)}{t^2}$$

□

1.3 Limits theorems

For independent and identically distributed variables $X_1, ..., X_N$, the sample mean $\frac{1}{N}\sum_{i=1}^N X_i$ has

$$\mathrm{Var}(\frac{1}{N}\sum_{i=1}^N X_i) = \frac{\mathrm{Var}(X_1)}{N} \to 0 \text{ as } N \to \infty$$

so we should expect it to concentrate around the true mean.

Theorem 1.3.1 (Strong law of large numbers)**.** Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables with $\mathbb{E}X_1 < \infty$. Then the averaged partial sums

$$\frac{S_N}{N} = \frac{1}{N}\sum_{i=1}^N X_i \to \mathbb{E}X_1 \quad \text{almost surely}$$

where random variables $(Y_N)_{N=1}^\infty$ are said to **converge almost surely** to a random variable Y if there exists measurable $Z \in \mathcal{M}$ with $\mathbb{P}(Z) = 0$ and

$$\lim_{N\to\infty} Y_N(\omega) = Y(\omega) \quad \text{for every } \omega \in \Omega \setminus Z$$

Theorem 1.3.2 (Lindeberg-Lévy CLT)**.** Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables with $\mathbb{E}X_1 = \mu, \mathrm{Var}(X_1) = \sigma^2 < \infty$. Then the normalized partial sums

$$Z_N = \frac{S_N - \mathbb{E}S_N}{\sqrt{\mathrm{Var}(S_N)}} = \frac{\sum_{i=1}^N X_i - N\mu}{\sigma\sqrt{N}} \to N(0,1) \quad \text{in distribution}$$

where real random variables $(Y_N)_{N=1}^\infty$ are said to **converge in distribution** to a random variable Y if their CDF's $F_{Y_N}(t) := \mathbb{P}\{Y_N \le t\}$, $F_Y(t) := \mathbb{P}\{Y \le t\}$ have

$$\lim_{N\to\infty} F_{Y_N}(t) = F_Y(t) \quad \text{for all } t \in \mathbb{R}$$

Exercise 1.3.3. Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables with $\mu, \sigma^2 < \infty$. Show that

$$\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N X_i - \mu\right| = O(\frac{1}{\sqrt{N}})$$

Answer. Considering the convex function $\phi(x) = x^2$, we can apply Jensen's to get

$$\left(\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N X_i - \mu\right|\right)^2 \le \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^N X_i\right) = \frac{\sigma^2}{N}$$

taking the square root of both sides yields the result. □

Theorem 1.3.4 (Poisson limit theorem)**.** Consider a sequence of N-tuples of independent random variables with entries X_{Ni} for $1 \le i \le N$ with $X_{Ni} \sim \mathrm{Bernoulli}(p_{Ni})$. Let $S_N = \sum_{i=1}^N X_{Ni}$, and suppose that as $N \to \infty$

$$\max_{1\le i\le N} p_{Ni} \to 0 \quad \text{and} \quad \mathbb{E}S_N = \sum_{i=1}^N p_{Ni} \to \lambda$$

Then $S_N \to \mathrm{Poisson}(\lambda)$ in distribution, i.e. the CDF $F_{S_N}(t) = \mathbb{P}\{S_N \le t\}$ has

$$\lim_{N\to\infty} F_{S_N}(t) = \sum_{k=1}^{\lfloor t\rfloor} e^{-\lambda}\frac{\lambda^k}{k!}$$

2 Concentrations of sums of independent random variables

2.1 Why concentration inequalities?

Concentration inequalities quantify the variation of a random variable around its mean, and take the from

$$\mathbb{P}\{|X-\mu| \geq t\} \leq \text{ something small}$$

Proposition 2.1.2 (Tails of the normal distribution)**.** Let $Z \sim N(0,1)$. For $t > 0$

$$\left(\frac{1}{t} - \frac{1}{t^3}\right)\frac{1}{\sqrt{2\pi}}e^{-t^2/2} \leq \mathbb{P}\{Z \geq t\} \leq \frac{1}{t}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$$

In particular, for $t \geq 1$, the tail of Z has

$$\mathbb{P}\{Z \geq t\} \leq \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$$

More loosely, we can say

$$\mathbb{P}\{Z \geq T\} = \Theta\left(\frac{1}{te^{t^2/2}}\right) = \tilde{\Theta}\left(\frac{1}{e^{t^2/2}}\right)$$

Proof. For the upper bound, we substitute $x = y + t$ to get

$$\int_t^\infty \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-y^2/2}e^{-ty}e^{-t^2/2}dy \leq \frac{1}{\sqrt{2\pi}}e^{-t^2/2}\int_0^\infty e^{-ty}dy = \frac{1}{t}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$$

For the lower bound, we make use of the identity

$$\int_t^\infty e^{-x^2/2}dx \geq \int_t^\infty (1-3x^{-4})e^{-x^2/2}dx = \left(\frac{1}{t}-\frac{1}{t^3}\right)e^{-t^2/2}$$

□

Example 2.1.1. Consider $S_N = X_1 + ... + X_N$, each $X_i \sim \text{Bernoulli}(1/2)$. We have $\mathbb{E}S_N = N/2, \text{Var}(S_N) = N/4$. From Chebyshev's inequality (Corollary 1.2.5), we get

$$\mathbb{P}\left\{\left|S_N - \frac{N}{2}\right| \geq \frac{N}{4}\right\} \leq \frac{4}{N} = O\left(\frac{1}{N}\right)$$

i.e. the probability of satisfying our concentration requirements goes to 0 linearly. Is this upper bound tight?

We know by the CLT (Theorem 1.3.2), that our normalized S_N converges in distribution to $N(0,1)$. Then for large N, we should see that

$$\mathbb{P}\left\{\left|S_N - \frac{N}{2}\right| \geq \frac{N}{4}\right\} = \mathbb{P}\left\{\left|\frac{S_N - \frac{N}{2}}{\sqrt{\frac{N}{4}}}\right| \geq \sqrt{\frac{N}{4}}\right\} \approx \mathbb{P}\left\{|Z| \geq \sqrt{\frac{N}{4}}\right\} \leq \frac{1}{\sqrt{2\pi N}}e^{-N/8} = \tilde{O}\left(\frac{1}{e^{N/8}}\right)$$

which is exponentially fast (by Proposition 2.1.2). However this central limit theorem argument can't be made rigourous, since the error in approximating normalized S_N with Z decays too slowly (in fact slower than linearly via Theorem 2.1.3). It turns out that for these sums, we get light tails much faster than we approximate $N(0,1)$.

Theorem 2.1.3 (Berry-Esseen CLT)**.** In the setting of Theorem 1.3.2, for all N

$$|F_{Z_N}(t) - F_Z(t)| \leq \frac{\rho}{\sqrt{N}} \quad \text{for all } t \in \mathbb{R}$$

where $\rho = \mathbb{E}|X_1 - \mu|^3/\sigma^3$.

Note that in comparison to Theorem 1.3.2 it additionally requires the third moment $\mathbb{E}X_1^3 < \infty$, and in turn provides a *quantitative* rate for *uniform* convergence in distribution to $N(0,1)$.

Exercise 2.1.4 (Truncated normal distribution)**.** Let $Z \sim N(0,1)$. Show that for all $t > 0$

$$\mathbb{E}Z^2\mathbb{1}_{\{Z\geq t\}} = t\frac{1}{\sqrt{2\pi}}e^{-t^2/2} + \mathbb{P}(Z \geq t) \leq \left(t + \frac{1}{t}\right)\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$$

Answer. To prove the equality

$$\mathbb{E}Z^2\mathbb{1}_{\{Z\geq t\}} := \int_t^\infty z^2\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = \frac{1}{\sqrt{2\pi}}\left(\left[-ze^{-z^2/2}\right]_t^\infty + \int_t^\infty e^{-z^2/2}dz\right) = t\frac{1}{\sqrt{2\pi}}e^{-t^2/2} + \mathbb{P}\{Z \geq t\}$$

The inequality follows from the tail upper bound from Proposition 2.1.2. □

2.2 Hoeffding's inequality

Definition 2.2.1 (Rademacher distribution). We say a random variable X has **Rademacher** distribution if

$$\mathbb{P}\{X = -1\} = \mathbb{P}\{X = 1\} = \frac{1}{2}$$

Theorem 2.2.2 (Hoeffding's inequality). Let $X_1, ..., X_N$ be independent Rademacher random variables. Let $a = (a_1, ..., a_n) \in \mathbb{R}^N$. For $t \geq 0$

$$\mathbb{P}\left\{\sum_{i=1}^{N} a_i X_i \geq t\right\} \leq \exp\left(\frac{-t^2}{2\|a\|_2^2}\right)$$

Proof. WLOG assume $\|a\|_2^2 = 1$. If we prove this version of the theorem, then for any $b = ca \in \mathbb{R}^N$,

$$\mathbb{P}\left\{\sum_{i=1}^{N} b_i X_i \geq t\right\} = \mathbb{P}\left\{\sum_{i=1}^{N} a_i X_i \geq t/c\right\} \leq \exp\left(\frac{-t^2}{2c^2\|a\|_2^2}\right) = \exp\left(\frac{-t^2}{2\|b\|_2^2}\right)$$

We apply Markov's inequality to the MGF of $\sum_{i=1}^{N} a_i X_i$

$$\mathbb{P}\left\{\sum_{i=1}^{N} a_i X_i \geq t\right\} = \mathbb{P}\left\{\exp\left(\lambda \sum_{i=1}^{N} a_i X_i\right) \geq \exp(\lambda t)\right\} \leq \frac{\mathbb{E}\exp\left(\lambda \sum_{i=1}^{N} a_i X_i\right)}{\exp(\lambda t)}$$

Examining the numerator of the right side of the inequality

$$\begin{aligned}
\mathbb{E}\exp\left(\lambda \sum_{i=1}^{N} a_i X_i\right) &= \prod_{i=1}^{N} \mathbb{E}\exp(\lambda a_i X_i) && \text{(by independence of } X_i\text{)} \\
&= \prod_{i=1}^{N} \cosh(\lambda a_i) && \text{(by definition of } \mathbb{E} \text{ of Rademacher RVs)} \\
&\leq \prod_{i=1}^{N} \exp(\lambda^2 a_i^2/2) && \text{(by Exercise 2.2.3)} \\
&= \exp(\lambda^2/2) && \text{(since } \|a\|_2^2 = 1\text{)}
\end{aligned}$$

To complete the proof we optimize λ to minimize the right hand side of the obtained tail bound inequality, $\exp(\lambda^2/2 - \lambda t)$. Setting $d(\lambda^2/2 - \lambda t)/d\lambda = \lambda - t = 0$ yields the minimum $\lambda = t$. This yields the desired inequality

$$\mathbb{P}\left\{\sum_{i=1}^{N} a_i X_i \geq t\right\} \leq \exp(-t^2/2\|a\|_2^2)$$

□

Exercise 2.2.3 (Bounding the hyperbolic cosine). Show that

$$\cosh(x) \leq \exp\left(x^2/2\right) \qquad \text{for all } x \in \mathbb{R}$$

Answer. Recalling that $e^x = \sum_{k=0}^{\infty} x^k/k!$ for all $x \in \mathbb{R}$, we can compute Taylor expansions that converge on $\mathbb{R}$

$$\begin{aligned}
\cosh(x) &= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!} + \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \\
e^{x^2/2} &= \sum_{k=0}^{\infty} \frac{(x^2/2)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!2^k}
\end{aligned}$$

Note that for $k = 0$, the terms match. For $k \geq 1$,

$$\frac{x^{2k}}{(2k)!} \leq \frac{x^{2k}}{k!2^k} \iff (2k)! \geq k!2^k \iff 2k \cdot ... \cdot k+1 \geq \underbrace{2 \cdot ... \cdot 2}_{k \text{ times}}$$

where the last statement holds if $k + 1 \geq 2 \iff k \geq 1$. Hence for all $x \in \mathbb{R}$, the partial sums of the expansion of $\cosh(x)$ are upper bounded by the partial sums of the expansion of $e^{x^2/2}$, which implies the same for their limits. □

Remark 2.2.4. We can use Hoeffding's to analyze the N coin flips from Example 2.1.1, achieving the desired (non-asymptotic) exponentially decaying tail probabilities.

$$\mathbb{P}\left\{\sum_{i=1}^{N} X_i \geq 3N/4\right\} = \mathbb{P}\left\{\sum_{i=1}^{N} 2X_i - 1 \geq N/2\right\} \leq \exp\left(-(N/2)^2/2N\right) = \exp\left(-N/8\right)$$

Theorem 2.2.6 (Hoeffding's inequality for general bounded RVs)**.** Let $X_1, ..., X_N$ be independent random variables, with each X_i's support $[m_i, M_i]$. For $t > 0$

$$\mathbb{P}\left\{\sum_{i=1}^{N}(X_i - \mathbb{E}X_i) \geq t\right\} \leq \exp\left(\frac{-2t^2}{\sum_{i=1}^{N}(M_i - m_i)^2}\right)$$

Exercise 2.2.7. Prove Theorem 2.2.6, possibly with some absolute constant instead of 2 in the tail.

Answer. We consider X_i with mean 0. For X_i without mean 0, we set $Y_i = X_i - \mathbb{E}X_i$ and proceed in the proof with Y_i which have the same support length as the X_i. The argument follows as in the proof of Theorem 2.2.2 differing only at the part where we obtain a bound for the MGF of the individual X_i.

Claim (Hoeffding's lemma)**.** For a bounded random variable $X \in [m, M]$ with mean 0, we have

$$\mathbb{E}\exp(\lambda X) \leq \exp(\lambda^2(M - m)^2/2)$$

With the claim we arrive at the inequality

$$\mathbb{P}\left\{\sum_{i=1}^{N} X_i \geq t\right\} \leq \exp\left((\lambda^2/2)\sum_{i=1}^{N}(M_i - m_i)^2 - \lambda t\right)$$

Optimizing λ yields $\lambda = t/\sum_{i=1}^{N}(M_i - m_i)^2$ and we get

$$\mathbb{P}\left\{\sum_{i=1}^{N} X_i \geq t\right\} \leq \exp\left(\frac{-(1/2)t^2}{\sum_{i=1}^{N}(M_i - m_i)^2}\right)$$

Note that the 1/2 in the tail is looser than the 2 in the theorem statement, which we can get if we prove a tighter version of Hoeffding's lemma (with /8 instead of /2) using Taylor approximations. For the proof of the claim, we follow an argument by symmetrization presented in the proof of Lemma 5 from these CS229 lecture notes[¶] by John Duchi. Consider X' an independent copy of X. We have

$$\mathbb{E}\exp\left(\lambda X\right) = \mathbb{E}_X \exp\left(\lambda(X - \mathbb{E}_{X'}X')\right) \leq \mathbb{E}_X\mathbb{E}_{X'}\exp\left(\lambda(X - X')\right) = \mathbb{E}_X\mathbb{E}_{X',S}\exp\left(\lambda S(X - X')\right)$$

Where the second inequality is by Jensen's. Since $X - X'$ has a symmetric distribution, it has the same distribution as $S(X - X')$, where S is a Rademacher random variable, giving us the last equality.

$$\begin{aligned}
\mathbb{E}_X\mathbb{E}_{X',S}\exp\left(\lambda S(X - X')\right) &= \mathbb{E}_{X,X'}\cosh\left(\lambda(X - X')\right) && \text{(by definition of } \mathbb{E}_S \text{ of Rademacher RVs)}\\
&\leq \mathbb{E}_{X,X'}\exp\left(\lambda^2(X - X')^2/2\right) && \text{(by Exercise 2.2.3)}\\
&\leq \exp\left(\lambda^2(M - m)^2/2\right) && \text{(since } |X - X'| \leq |M - m|\text{)}
\end{aligned}$$

□

Exercise 2.2.8 (Boosting randomized algorithms)**.** Suppose we have a randomized algorithm for a decision problem that is correct with probability $1/2 + \varepsilon$ for some $\varepsilon > 0$. Show that running the algorithm N times independently and taking the majority yields the correct answer with probability $\geq 1 - \delta$ for $N \geq (1/2\varepsilon^2)\log(1/\delta)$.

Answer. Suppose the input to our algorithm A is a YES instance, and define the random variable X_i

$$X_i = \begin{cases} 1 & \text{if } i\text{th run of } A \text{ outputs YES w.p. } 1/2 + \varepsilon \\ -1 & \text{if } i\text{th run of } A \text{ outputs NO w.p. } 1/2 - \varepsilon \end{cases}$$

[¶] http://cs229.stanford.edu/extra-notes/hoeffding.pdf

Then we have

$$\mathbb{P}\{\text{Majority}(X_1, ..., X_N) = -1\} = \mathbb{P}\left\{\sum_{i=1}^{N} X_i \leq 0\right\} = \mathbb{P}\left\{\sum_{i=1}^{N}(X_i - 2\varepsilon) \leq -2N\varepsilon\right\} \leq \exp\left(-2N\varepsilon^2\right)$$

where the last inequality is obtained by Hoeffding's inequality (Theorem 2.2.6) applied to the bounded random variables $-X_i$'s. Finally we note that

$$N \geq \frac{1}{2\varepsilon^2}\log\frac{1}{\delta} \iff 2N\varepsilon^2 \geq \log\frac{1}{\delta} \iff \exp\left(-2N\varepsilon^2\right) \leq \delta$$

Therefore our algorithm is correct on YES instances with probability $\geq 1 - \delta$. We can use the same argument to conclude the same on NO instances, completing the proof. $\square$

Exercise 2.2.9 (Robust mean estimation). Suppose we want to estimate the mean μ of a random variable X from $X_1, ..., X_N$ copies drawn independently. We want an ε-accurate estimate (falls within $(\varepsilon - \mu, \varepsilon + \mu)$).

(a) Show a sample size $N = O(\sigma^2/\varepsilon^2)$ is sufficient for an ε-accurate estimate w.p. $\geq 3/4$, where $\sigma^2 = \text{Var}(X)$.

Answer. Note that we can't directly apply Hoeffding's since we don't know if X is bounded. However we can apply Chebyshev's (Corollary 1.2.5)

$$\mathbb{P}\left\{\left|\frac{1}{N}\sum_{i=1}^{N} X_i - \mu\right| \geq \varepsilon\right\} \leq \text{Var}\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right)/\varepsilon^2 = \sigma^2/N\varepsilon^2$$

and $\sigma^2/N\varepsilon^2 \leq 1/4 \iff N \geq 4(\sigma^2/\varepsilon^2)$, giving our $\geq 3/4$ success probability with $N = O(\sigma^2/\varepsilon^2)$ samples. $\square$

(b) Show a sample size $N = O(\log(1/\delta)\sigma^2/\varepsilon^2)$ is sufficient for an ε-accurate estimate w.p. $\geq 1 - \delta$.

Answer. Note that plugging in δ into Chebyshev's in (a) would give us a $N = O(\sigma^2/\varepsilon^2\delta)$ sample requirement to achieve the desired success probability, which has much worse dependence on δ. What we can do instead is boost our weak estimator by running it k times and producing the median. Let our weak estimates be $\hat{\mu}_i$ obtained using N samples each for $1 \leq i \leq k$ (for k odd). Note that Median$(\hat{\mu}_1, ..., \hat{\mu}_k)$ is outside of $(\mu-\varepsilon, \mu+\varepsilon)$ iff there are either $\geq (k+1)/2$ estimates $> \mu + \varepsilon$ or $\geq (k+1)/2$ estimates $< \mu - \varepsilon$. In either case, we have that our weak estimator failed $\geq (k+1)/2$ times. Letting $F_i = 1$ when $\hat{\mu}_i \in (\mu - \varepsilon, \mu + \varepsilon)$ and 0 otherwise, we get

$$\begin{aligned}
\mathbb{P}\left\{\text{Median}(\hat{\mu}_1, ..., \hat{\mu}_k) \text{ is not } \varepsilon\text{-accurate}\right\} &\leq \mathbb{P}\left\{\sum_{i=1}^{k} F_i \geq \frac{k+1}{2}\right\} \\
&= \mathbb{P}\left\{\sum_{i=1}^{k}(F_i - \mathbb{E}F_1) \geq \frac{k+1}{2} - k\mathbb{E}F_1\right\} \\
&\leq \mathbb{P}\left\{\sum_{i=1}^{k}(F_i - \mathbb{E}F_1) \geq \frac{k+1}{2} - \frac{k}{4}\right\} && \text{(by } \mathbb{E}F_1 \leq 1/4\text{)} \\
&\leq \exp\left(-2\left(\frac{k+2}{4}\right)^2/k\right) && \text{(Hoeffding's, Thm. 2.2.6)} \\
&\leq \exp\left(-k/8\right) && \text{(since } (k+2)/4 \geq k/4\text{)}
\end{aligned}$$

Taking odd $k \geq 8\log(1/\delta)$ bounds the error probability to $\leq \delta$ and uses $kN \leq (8\log(1/\delta) + 2)4(\sigma^2/\varepsilon^2) = O(\log(1/\delta)\sigma^2/\varepsilon^2)$ samples.

$\square$

Exercise 2.2.10 (Small ball probabilities). Let $X_1, ..., X_N$ be non-negative independent random variables with continuous distributions. Assume their densities are bounded by 1.

(a) Show that the MGF of X_i satisfies

$$\mathbb{E}\exp(-tX_i) \leq \frac{1}{t} \qquad \text{for all } t > 0$$

Answer. For non-negative X_i with bounded density function $0 \le p(x) \le 1$

$$\mathbb{E}\exp(-tX_i) = \int_0^\infty p(x)e^{-tx}dx \le \int_0^\infty e^{-tx}dx = \left[-\frac{1}{t}e^{-tx}\right]_0^\infty = \frac{1}{t}$$

□

(b) Deduce that for any $\varepsilon > 0$ we have

$$\mathbb{P}\left\{\sum_{i=1}^N X_i \le \varepsilon N\right\} \le (e\varepsilon)^N$$

(This result essentially says that for "sufficiently distributed" (at least, with support of measure ≥ 1) non-negative RVs, their sum is unlikely to be bounded by any constant fraction of the number of trials.)

Answer. Consider the MGF for $-\sum_{i=1}^N X_i$, evalutated at $\lambda = 1/\varepsilon$

$$\begin{aligned}
\mathbb{P}\left\{\sum_{i=1}^N X_i \le \varepsilon N\right\} &= \mathbb{P}\left\{\sum_{i=1}^N -X_i \ge -\varepsilon N\right\} \\
&\le \mathbb{P}\left\{\exp\left(-\frac{1}{\varepsilon}\sum_{i=1}^N X_i\right) \le e^{-N}\right\} \\
&\le \left(\prod_{i=1}^N \mathbb{E}\exp\left(-\frac{1}{\varepsilon}X_i\right)\right)/e^{-N} && \text{(by Markov's (Prop. 1.2.4) and indep.)} \\
&\le (e\varepsilon)^N && \text{(applying part (a) with } t = 1/\varepsilon\text{)}
\end{aligned}$$

Note that this choice of λ minimizes the right hand side of the expression, which can be verified by taking derivatives w.r.t λ. □

2.3 Chernoff's inequality

In the case where $X_i \sim \text{Bernoulli}(p_i)$ we can get tighter bounds than Hoeffding's.

Theorem 2.3.1 (Chernoff's inequality)**.** Let $X_1, ..., X_N$ be independent Bernoulli random variables with parameters p_i for $1 \le i \le N$. Denote $\mu = \mathbb{E}\sum_{i=1}^N X_i = \sum_{i=1}^N p_i$. For $t > \mu$

$$\mathbb{P}\left\{\sum_{i=1}^N X_i \ge t\right\} \le e^{-\mu}\left(\frac{e\mu}{t}\right)^t$$

Exercise 2.3.2 (Chernoff's inequality, lower tail)**.** In the setting of Theorem 2.3.1, show that for $t < \mu$

$$\mathbb{P}\left\{\sum_{i=1}^N X_i \le t\right\} \le e^{-\mu}\left(\frac{e\mu}{t}\right)^t$$

Answer. This result can be obtained by modifying the proof of Theorem 2.3.1.

$$\begin{aligned}
\mathbb{P}\left\{\sum_{i=1}^N X_i \le t\right\} &= \mathbb{P}\left\{\exp\left(-\lambda\sum_{i=1}^N X_i\right) \ge e^{-\lambda t}\right\} && \text{(for any } \lambda \ge 0\text{)} \\
&\le \mathbb{E}\exp\left(-\lambda\sum_{i=1}^N X_i\right) \cdot e^{\lambda t} && \text{(by Markov's (Prop. 1.2.4))} \\
&= e^{\lambda t} \cdot \prod_{i=1}^N \mathbb{E}\exp(-\lambda X_i) && \text{(by independence)}
\end{aligned}$$

For each term, we have

$$\mathbb{E}\exp(-\lambda X_i) = p_i e^{-\lambda} + (1-p_i) = 1 + p_i\left(e^{-\lambda}-1\right) \le \exp\left(p_i\left(e^{-\lambda}-1\right)\right)$$

which yields

$$\begin{aligned}
\mathbb{P}\left\{\sum_{i=1}^N X_i \le t\right\} &\le \exp\left(\lambda t + \mu(e^{-\lambda}-1)\right) && \left(\text{since } \mu = \sum_{i=1}^N p_i\right) \\
&= \exp\left(t\ln\frac{\mu}{t} + t - \mu\right) && \left(\text{plugging in } \lambda = \ln\frac{\mu}{t} > 0 \text{ by } \mu > t\right) \\
&= e^{-\mu}\left(\frac{e\mu}{t}\right)^t
\end{aligned}$$

□

Exercise 2.3.3 (Poisson tails)**.** Let $X \sim \text{Poisson}(\lambda)$. Show that for $t > \lambda$

$$\mathbb{P}\left\{X \ge t\right\} \le e^{-\lambda}\left(\frac{e\lambda}{t}\right)^t$$

Answer. Consider the MGF of X, and apply Markov's inequality (Prop. 1.2.4). For all $c \ge 0$, we have

$$\mathbb{P}\left\{X \ge t\right\} \le \mathbb{P}\left\{e^{cX} \le e^{ct}\right\} \le \mathbb{E}e^{cX}\cdot e^{-ct} = e^{-ct}\cdot\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}e^{ck} = e^{-\lambda}\cdot e^{-ct}\sum_{k=0}^{\infty}\frac{(\lambda e^c)^k}{k!}$$

Taking $c = \ln(t/\lambda) > 0$, since by assumption $t > \lambda$

$$\mathbb{P}\{X \ge t\} \le e^{-\lambda}\cdot\left(\frac{\lambda}{t}\right)^t\cdot e^{\lambda(t/\lambda)} = e^{-\lambda}\left(\frac{e\lambda}{t}\right)^t$$

□

Remark 2.3.4 (Poisson tails)**.** Applying Stirling's formula $k! \approx \sqrt{2\pi k}(k/e)^k$ to the Poisson density yields

$$\mathbb{P}\{X = k\} \approx \frac{1}{\sqrt{2\pi k}}e^{-\lambda}\left(\frac{e\lambda}{k}\right)^k$$

So the tail bound of all the mass $\ge k$ from Exercise 2.3.3 is only a logarithmic factor (w.r.t. main term) larger than the mass assigned to only k.

Exercise 2.3.5 (Chernoff's inequality, small deviations)**.** In the setting of Theorem 2.3.1, for $0 < \delta < 1$

$$\mathbb{P}\left\{\left|\sum_{i=1}^N X_i - \mu\right| \ge \delta\mu\right\} \le 2\exp\left(-c\mu\delta^2\right)$$

where c is some absolute constant. Compared to the Hoeffding bound that has $-c\mu^2\delta^2/N$ in its exponent, we have $-c\mu\delta^2$, a $N/\mu \ge 1$ multiplier which speeds up convergence especially for small $\mu = \sum_{i=1}^N p_i$.

Answer. To bound the upper tail, since $(1+\delta)\mu > \mu$ we can apply Theorem 2.3.1

$$\mathbb{P}\left\{\sum_{i=0}^N X_i - \mu \ge \delta\mu\right\} = \mathbb{P}\left\{\sum_{i=0}^N X_i \ge (1+\delta)\mu\right\} \le e^{-\mu}\left(\frac{e\mu}{(1+\delta)\mu}\right)^{(1+\delta)\mu} = \exp\left(\delta\mu - \ln(1+\delta)(1+\delta)\mu\right)$$

If we can find $c > 0$ such that $\delta - \ln(1+\delta)(1+\delta) \le -c\delta^2$, equivalently that $\ln(1+\delta)(1+\delta) - \delta \ge c\delta^2$ for all $0 < \delta < 1$, then by the previous string of inequalities we get

$$\mathbb{P}\left\{\sum_{i=0}^N X_i - \mu \ge \delta\mu\right\} \le \exp\left(-c\delta^2\mu\right)$$

Using the Maclaourin series for $\ln(1+x)$, we have that for $x \geq 0$

$$\ln(1+x) \geq x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

Plugging this in, we get

$$\begin{aligned}
\ln(1+\delta)(1+\delta) - \delta &\geq (\delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} - \frac{\delta^4}{4})(1+\delta) - \delta \\
&= \frac{\delta^2}{2} - \frac{\delta^3}{6} + \frac{\delta^4}{12} - \frac{\delta^5}{4} \\
&\geq \frac{\delta^2}{12} && \text{(since } -\delta^3, -\delta^4 \geq -\delta^2 \text{ on } \delta \in [0,1])
\end{aligned}$$

So choosing $c = 1/12$ suffices to prove the result for the upper tail. For the lower tail, we are trying to bound

$$\mathbb{P}\left\{\sum_{i=0}^{N} X_i - \mu \leq -\delta\mu\right\} = \mathbb{P}\left\{\sum_{i=0}^{N} X_i \leq (1-\delta)\mu\right\} \leq \exp\left(-c\delta^2\mu\right)$$

Note that since $(1-\delta)\mu < \mu$, we can apply Exercise 2.3.2, and proceed exactly as in the upper tail calculation to arrive at the same bound. The event of observing absolute deviation from the mean $\geq \delta\mu$ is precisely the disjoint union of these two events, so we add the probabilities to get the desired result.

$\square$

Exercise 2.3.6 (Poisson distribution near the mean)**.** Let $X \sim \text{Poisson}(\lambda)$. Show that for $0 < t \leq \lambda$

$$\mathbb{P}\left\{|X - \lambda| \geq t\right\} \leq 2\exp\left(-\frac{ct^2}{\lambda}\right)$$

for some absolute constant c. Note that this, combined with Exercise 2.3.3 tell us that we have gaussian tails near the mean, but further away they are heavier.

Exercise 2.3.8 (Normal approximation to Poisson)**.** Let $X \sim \text{Poisson}(\lambda)$. Show that as $k \to \infty$

$$\frac{X - \lambda}{\sqrt{\lambda}} \to N(0,1) \quad \text{in distribution}$$