High-Dimensional Probability: Answers, Theorems, and Definitions

Last revised on May 27, 2021

- Companion notes for *High-Dimensional Probability*, by Roman Vershynin. Link to book (PDF available online): www.math.uci.edu/~rvershyn/papers/HDP-book/HDP-book.html.
- **Disclaimer:** These notes compile my answers to the exercises, and lift the required theorems and definitions from the book. I wrote these notes to aid my personal study of the book. Read them at your own risk!

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0 Appetizer: Using probability to cover a geometric set

A point $z \in \mathbb{R}^n$ is a **convex combination** of points $z_1, z_2, ..., z_m \in \mathbb{R}^n$ if

$$z = \sum_{i=1}^{m} \lambda_i z_i \quad \text{with each } \lambda_i \geq 0 \text{ and } \sum_{i=1}^{m} \lambda_i = 1.$$

The **convex hull** of $T \subseteq \mathbb{R}^n$ is the set of all convex combinations of T.

Theorem 1. 0.0.1 (Catheodory's Theorem) Every point in the convex hull of a set $T \subseteq \mathbb{R}^n$ can be expressed as a convex combination of $\leq n+1$ points from T

1 Preliminaries on random variables

1.1 Basic quantities

The **expection** of a random variable X is denoted as $\mathbb{E}\,X$, and **variance** is denoted as $\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}\,X)^2$. (We note that the expectation operator $\mathbb{E}$ can be directly defined as the Lebesgue integral of the random variable $X : \Omega \to \mathbb{R}$ in the probability space (Ω, M, μ).

The **p-th moment** of X is given by $\mathbb{E}\,X^p$. We also let $\|X\|_p = (\mathbb{E}\,X^p)^{\frac{1}{p}}$ denote the **p-norm** of X. For $p = \infty$, we have:

$$\|X\|_\infty = \operatorname{ess\,sup} X$$

recalling that the **essential supremum** of a function f is the "smallest value γ such that $\{\omega \in \Omega : |f(\omega)| > \gamma\}$ has measure 0".

From this, we can define the $\boldsymbol{L^p}$ **spaces**, given a probability space (Ω, M, μ):

$$L^p = \{X : \|X\|_p < \infty\}^*$$

Results from measure and integration theory tell us that the $(L^p, \|\cdot\|_p)$ are complete. In the case of L^2, we have that with the inner product:

$$\begin{aligned}\langle X, Y\rangle &= \int_\Omega XY(\omega)\mu(\omega)\\ &= \mathbb{E}\,XY\end{aligned}$$

$(L^2, \langle\cdot,\cdot\rangle)$ is a Hilbert space. In this case we can express the **standard deviation** of X as $\sqrt{\mathrm{Var}(X)} = \|X - \mathbb{E}\,X\|_2$, and the **covariance** of random variable X and Y as

$$\mathrm{Cov}(X, Y) = \mathbb{E}(X - \mathbb{E}\,X)(Y - \mathbb{E}\,Y) = \langle X - \mathbb{E}\,X, Y - \mathbb{E}\,Y\rangle)$$

Remark 1. In this setting, considering random variables as vectors in L^2, the covariance between X and Y can be interpreted as the *alignment* between the vectors $X - \mathbb{E}\,X$ and $Y - \mathbb{E}\,Y$.

1.2 Inequalities

1.3 Limits of random variables

*A technical note is that the objects of L_p are actually equivalence classes of functions $[X]$ with equality almost everywhere, otherwise $\|\cdot\|_p$ is only a semi-norm.