

# *High-Dimensional Probability: Answers, Theorems, and Definitions*

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- Companion notes for *High-Dimensional Probability*, by Roman Vershynin. Link to book (PDF available online): [www.math.uci.edu/~rvershyn/papers/HDP-book/HDP-book.html](http://www.math.uci.edu/~rvershyn/papers/HDP-book/HDP-book.html).
- **Disclaimer:** These notes compile my answers to the exercises, and lift the required theorems and definitions from the book. I wrote these notes to aid my personal study of the book. Read them at your own risk!\*

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\*Scribe: Alex Bie, [alexbie98@gmail.com](mailto:alexbie98@gmail.com).

## 0 Appetizer: Using probability to cover a geometric set

A point  $x \in \mathbb{R}^n$  is a **convex combination** of points  $x_1, \dots, x_m \in \mathbb{R}^n$  if

$$x = \sum_{i=1}^m \lambda_i x_i \quad \text{with each } \lambda_i \geq 0 \quad \text{and} \quad \sum_{i=1}^m \lambda_i = 1.$$

The **convex hull** of  $T \subseteq \mathbb{R}^n$ ,  $\text{conv}(T)$ , is the set of all convex combinations of  $T$ .

**Theorem 0.0.1** (Carathéodory's Theorem). Let  $x \in \text{conv}(T)$ . There exists  $k \leq n + 1$  points  $x_1, \dots, x_k \in T$  such that  $x$  is a convex combination of  $x_1, \dots, x_k$ .

The result says we can obtain any point in the convex hull of  $T$  using at most a dimension-dependent number of points. Let the **diameter** of a set  $T$  be defined as  $\text{diam}(T) = \sup\{\|x - y\|_2 : x, y \in T\}$ .

**Theorem 0.0.2** (Approximate Carathéodory's Theorem). Let  $\text{diam}(T) = 1$ . Let  $x \in \text{conv}(T)$ . For any  $k$ , there exists  $k$  points  $x_1, \dots, x_k \in T$  such that

$$\left\| x - \frac{1}{k} \sum_{j=1}^k x_j \right\|_2 \leq \frac{1}{\sqrt{k}}$$

*Proof.* Suppose  $|T| = m$ . WLOG we can assume  $T$  is bounded by 1 in  $\|\cdot\|_2$ . We write  $x = \sum_{i=1}^m \lambda_i x_i$ , and interpret  $\lambda_i$  as probabilities. We define the random variable

$$X = x_i \text{ with probability } \lambda_i$$

for  $i = 1, \dots, m$ . We can verify that  $\mathbb{E}X = \sum_{i=1}^m \lambda_i x_i = x$ . Taking  $X_1, \dots, X_k \stackrel{\text{iid}}{\sim} X$ . It remains to analyse the quantity  $\mathbb{E}\|x - \frac{1}{k} \sum_{j=1}^k X_j\|_2^2$ .

$$\begin{aligned} \mathbb{E} \left\| x - \frac{1}{k} \sum_{j=1}^k X_j \right\|_2^2 &\leq \frac{1}{k^2} \mathbb{E} \left\| \sum_{j=1}^k X_j - x \right\|_2^2 \\ &= \frac{1}{k^2} \sum_{j=1}^k \mathbb{E} \|X_j - x\|_2^2 && \text{(by Exercise 0.0.3 (a))} \\ &= \frac{1}{k} \mathbb{E} \|X - x\|_2^2 \end{aligned}$$

Applying the result of Exercise 0.0.3 (b), we obtain

$$\mathbb{E} \|X - x\|_2^2 = \mathbb{E} \|X\|_2^2 - \|\mathbb{E}X\|_2^2 \leq \mathbb{E} \|X\|_2^2 \leq 1$$

Plugging this in above, we obtain the desired bound in expectation, hence there must exist a realization of the  $X_j$ ,  $x_1, \dots, x_k$ , such that the bound holds.  $\square$

**Exercise 0.0.3.** Check the following identities for random vectors.

(a) Let  $X_1, \dots, X_k$  be independent, mean zero random vectors in  $\mathbb{R}^n$ . Show that

$$\mathbb{E} \left\| \sum_{j=1}^k X_j \right\|_2^2 = \mathbb{E} \sum_{j=1}^k \|X_j\|_2^2$$

*Answer.*

$$\begin{aligned}
\mathbb{E} \left\| \sum_{j=1}^k X_j \right\|_2^2 &= \sum_{i=1}^n \mathbb{E} \left( \sum_{j=1}^m X_j^{(i)} \right)^2 = \sum_{i=1}^n \text{Var} \left( \sum_{j=1}^m X_j^{(i)} \right) && \text{(by mean zero)} \\
&= \sum_{i=1}^n \sum_{j=1}^m \text{Var} \left( X_j^{(i)} \right) && \text{(by independence)} \\
&= \sum_{i=1}^n \sum_{j=1}^m \mathbb{E} \left( X_j^{(i)} \right)^2 && \text{(by mean zero)} \\
&= \mathbb{E} \sum_{j=1}^m \|X_j\|_2^2
\end{aligned}$$

□

Among other things, this result implies that the expected squared distance of a random walk (starting from the origin) is equal to sum of the expected squared distances of each step.

(b) Let  $X$  be a random vector in  $\mathbb{R}^n$ . Show that

$$\mathbb{E} \|X - \mathbb{E}X\|_2^2 = \mathbb{E} \|X\|_2^2 - \|\mathbb{E}X\|_2^2$$

*Answer.*

$$\begin{aligned}
\mathbb{E} \|X - \mathbb{E}X\|_2^2 &= \mathbb{E} \sum_{i=1}^n \left( X^{(i)} - (\mathbb{E}X)^{(i)} \right)^2 = \sum_{i=1}^n \text{Var}(X^{(i)}) = \sum_{i=1}^n \mathbb{E} \left( X^{(i)} \right)^2 - \left( \mathbb{E}X^{(i)} \right)^2 \\
&= \mathbb{E} \|X\|_2^2 - \|\mathbb{E}X\|_2^2
\end{aligned}$$

□

**Corollary 0.0.4** (Covering polytopes by balls). Let  $P \subseteq \mathbb{R}^n$  be a polytope with  $\text{diam}(P) = 1$ . Let  $m$  be the number of vertices of  $P$ . Let  $\varepsilon > 0$ . We can cover  $P$  with  $m^k$  balls of radius  $\varepsilon$  for  $k \geq \lceil 1/\varepsilon^2 \rceil$ .

*Proof.* Take  $T$  to be the vertex set of  $P$ .  $|T| = m$ . Note that for any  $x \in P$ ,  $x \in \text{conv}(T)$ . By Theorem 0.0.2, taking  $k \geq \lceil 1/\varepsilon^2 \rceil$ , we can find  $x_1, \dots, x_k \in T$  such that

$$\left\| x - \frac{1}{k} \sum_{j=1}^k x_j \right\| \leq \frac{1}{\sqrt{k}} \leq \varepsilon$$

The number of ball centres obtained from selecting a set of  $k$  points out of  $m$  with repetition is bounded by  $m^k$  (possibly repeating orders). Hence we have an  $\varepsilon$ -cover sufficient to cover  $P$ . □

**Exercise 0.0.5** (Binomial coefficient inequality). Show that for  $1 \leq r \leq n$

$$\left( \frac{n}{r} \right)^r \leq \binom{n}{r} \leq \sum_{k=0}^r \binom{n}{k} \leq \left( \frac{en}{r} \right)^r$$

*Answer.* For the first inequality, consider

$$\frac{\left( \frac{n}{r} \right)^r}{\binom{n}{r}} = \frac{\frac{n}{r} \cdot \frac{n}{r} \cdot \dots \cdot \frac{n}{r}}{\frac{n}{r} \cdot \frac{n-1}{r-1} \cdot \dots \cdot \frac{n-r+1}{1}} \leq 1 \cdot 1 \cdot \dots \cdot 1 = 1$$

The second inequality follows immediately. To justify the last inequality, write

$$\begin{aligned}
\left(\frac{en}{r}\right)^r &= e^r \cdot \left(\frac{n}{r}\right)^r = \sum_{k=0}^{\infty} \frac{r^k}{k!} \cdot \left(\frac{n}{r}\right)^r && \text{(Maclaurin series for } e^x\text{)} \\
&\geq \sum_{k=0}^r \frac{r^k}{k!} \cdot \left(\frac{n}{r}\right)^r \\
&= \sum_{k=0}^r \frac{n^k \cdot n^{r-k}}{k! \cdot r^{r-k}} \\
&\geq \sum_{k=0}^r \frac{n^k}{k!} && \text{(by } n \geq r\text{)} \\
&\geq \sum_{k=0}^r \binom{n}{k}
\end{aligned}$$

□

**Exercise 0.0.6** (Improved covering). Show that in the setting of Corollary 0.0.4, for  $k \geq \lceil 1/\varepsilon^2 \rceil$

$$(C + C\varepsilon^2 m)^k$$

balls suffice for a suitable constant  $C$ .

*Answer.* We can give a tighter bound than given in the proof of Corollary 0.0.4 on the number of ball centres obtained from selecting a set of  $k$  points out of  $m$  with repetition (since computing the mean of  $k$  is order-invariant with respect to input points). By the “stars-and-bars”<sup>†</sup> argument, this quantity is given by

$$\binom{m+k-1}{k-1}$$

Note that  $\min\{k-1, m\} = k-1 \leq \min\{k, m-1\}$ , so looking at row  $m+k-1$  of Pascal’s triangle

$$\binom{m+k-1}{k-1} \leq \binom{m+k-1}{k}$$

Then, using Exercise 0.0.5

$$\binom{m+k-1}{k} \leq \left(\frac{e(m+k-1)}{k}\right)^k = \left(e\frac{k-1}{k} + e\frac{1}{k}m\right)^k \leq (e + e\varepsilon^2 m)^k$$

□

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<sup>†</sup>[https://en.wikipedia.org/wiki/Stars\\_and\\_bars\\_\(combinatorics\)](https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics))

# 1 Preliminaries on random variables

## 1.1 Basic quantities

The **expectation** of a random variable  $X$  is denoted as  $\mathbb{E}X$ , and **variance** is denoted as  $\text{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2$ . (We note that the expectation operator  $\mathbb{E}$  can be directly defined as the Lebesgue integral of the random variable (measurable function)  $X : \Omega \rightarrow \mathbb{R}$  in the probability space  $(\Omega, \mathcal{M}, \mathbb{P})$ .)

The **moment generating function** of  $X$  is given by

$$M_X(t) = \mathbb{E}e^{tX} \quad \text{for all } t \in \mathbb{R}$$

The **p-th moment** of  $X$  is given by  $\mathbb{E}X^p$ . We also let  $\|X\|_p = (\mathbb{E}X^p)^{\frac{1}{p}}$  denote the **p-norm** of  $X$ . For  $p = \infty$ , we have

$$\|X\|_\infty = \text{ess sup } X$$

recalling that the **essential supremum** of a function  $f$  is the "smallest value  $\gamma$  such that  $\{\omega \in \Omega : |f(\omega)| > \gamma\}$  has measure 0".

From this, we can define the  **$L^p$  spaces**<sup>†</sup>, given a probability space  $(\Omega, \mathcal{M}, \mathbb{P})$

$$L^p = \{X : \|X\|_p < \infty\}$$

Results from measure and integration theory tell us that the  $(L^p, \|\cdot\|_p)$  are complete. In the case of  $L^2$ , we have that with the inner product

$$\begin{aligned} \langle X, Y \rangle &= \int_{\Omega} XY(\omega) \mu(\omega) \\ &= \mathbb{E}XY \end{aligned}$$

$(L^2, \langle \cdot, \cdot \rangle)$  is a Hilbert space. In this case we can express the **standard deviation** of  $X$  as  $\sqrt{\text{Var}(X)} = \|X - \mathbb{E}X\|_2$ , and the **covariance** of random variable  $X$  and  $Y$  as

$$\text{Cov}(X, Y) = \mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y) = \langle X - \mathbb{E}X, Y - \mathbb{E}Y \rangle$$

In this setting, considering random variables as vectors in  $L^2$ , the covariance between  $X$  and  $Y$  can be interpreted as the *alignment* between the vectors  $X - \mathbb{E}X$  and  $Y - \mathbb{E}Y$ .

## 1.2 Some classical inequalities

We say  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **convex** if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) \quad \text{for all } x, y \in \mathbb{R} \text{ and } t \in [0, 1]$$

**Jensen's inequality** states that for any random variable  $X$  and a convex function  $f$ , we get

$$f(\mathbb{E}X) \leq \mathbb{E}(f(X))$$

A corollary of Jensen's inequality implies that<sup>§</sup>

$$\|X\|_p \leq \|X\|_q \quad \text{for all } 1 \leq p \leq q \leq \infty$$

**Minkowski's inequality** asserts that the triangle inequality holds for the  $L_p$  spaces

$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p \quad \text{for all } X, Y \in L^p$$

In  $L^2$ , we have the **Cauchy-Schwarz inequality**, which states that  $|\mathbb{E}XY| \leq \mathbb{E}|XY| \leq \|X\|_2\|Y\|_2$ . **Holder's inequality** additionally asserts that for  $1/p + 1/q = 1$

$$|\mathbb{E}XY| \leq \|XY\|_1 \leq \|X\|_p\|Y\|_q$$

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<sup>†</sup>A technical note is that the objects of  $L_p$  are actually equivalence classes of functions  $[X]$  with equality almost everywhere, otherwise  $\|\cdot\|_p$  is only a semi-norm.

<sup>§</sup>For  $q < \infty$ , the result follows by applying Jensen's inequality for  $f(x) = x^{\frac{q}{p}}$ . Otherwise,  $\|X\|_\infty = \gamma = (\mathbb{E}\gamma^p)^{\frac{1}{p}} = \|\gamma\|_p \geq \|X\|_p$ .

which also holds for  $p = 1, q = \infty$ .

The **cumulative distribution function** of  $X$  is defined as

$$F_X(t) = \mathbb{P}\{X \leq t\} = \mathbb{P}(X^{-1}(-\infty, t]) \quad \text{for all } t \in \mathbb{R}$$

and we refer to  $\mathbb{P}\{X > t\} = 1 - F_X(t)$  as the **tail** of  $X$ .

**Lemma 1.2.1** (Integral identity). Let  $X \geq 0$  be a random variable. Then

$$\mathbb{E}X = \int_0^\infty \mathbb{P}\{X > t\} dt$$

with left side  $= \infty$  iff right side  $= \infty$ .

**Exercise 1.2.2** (Generalization of integral identity). Show that Lemma can be extended to be valid for any  $X$

$$\mathbb{E}X = \int_0^\infty \mathbb{P}\{X > t\} dt - \int_{-\infty}^0 \mathbb{P}\{X < t\} dt$$

*Answer.* For not necessary non-negative  $X$ ,  $\mathbb{E}X := \mathbb{E}X^+ - \mathbb{E}X^-$  when they exist and are both  $< \infty$ , where

$$X^+ = \begin{cases} X & \text{if } X \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad X^- = \begin{cases} -X & \text{if } X \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Applying Lemma 1.2.1 to the terms yields the result. For the second term

$$\mathbb{E}X^- = \int_0^\infty \mathbb{P}\{X^- > t\} dt = \int_0^\infty \mathbb{P}\{X < -t\} dt = \int_{-\infty}^0 \mathbb{P}\{X < t\} dt$$

□

**Exercise 1.2.3** ( $p$ -th moment via the tail). Let  $X$  be a random variable and  $0 < p < \infty$ . Show that

$$\mathbb{E}|X|^p = \int_0^\infty pt^{p-1} \mathbb{P}\{|X| > t\} dt$$

whenever the right side is  $< \infty$ .

*Answer.* On the right side, substitute  $u = t^p$ , so  $du = pt^{p-1} dt$  and

$$\int_0^\infty pt^{p-1} \mathbb{P}\{|X| > t\} dt = \int_0^\infty \mathbb{P}\{|X| > u^{\frac{1}{p}}\} du = \int_0^\infty \mathbb{P}\{|X|^p > u\} du = \mathbb{E}|X|^p$$

where the last equality comes from applying Lemma 1.2.1 to the random variable  $|X|^p \geq 0$ .

□

**Proposition 1.2.4** (Markov's inequality). Let  $X \geq 0$  with  $\mathbb{E}X < \infty$ . Then for  $t > 0$

$$\mathbb{P}\{X \geq t\} \leq \frac{\mathbb{E}X}{t}$$

*Proof.* Fix  $t > 0$ . Applying Lemma 1.2.1

$$\mathbb{E}X = \int_0^\infty \mathbb{P}\{X \geq x\} dx \geq \int_0^t \mathbb{P}\{X \geq x\} dx \geq \int_0^t \mathbb{P}\{X \geq t\} dx = t \cdot \mathbb{P}\{X \geq t\}$$

□

**Corollary 1.2.5** (Chebyshev's inequality). Let  $X$  have  $\mathbb{E}X < \infty$  and  $\text{Var}(X) < \infty$ . Then for  $t > 0$

$$\mathbb{P}\{|X - \mathbb{E}X| > t\} \leq \frac{\text{Var}(X)}{t^2}$$

**Exercise 1.2.6.** Give a proof of Chebyshev's inequality using Markov's inequality.

*Answer.* The random variable  $|X - \mathbb{E}X|^2$  is well-defined (by  $\mathbb{E}X < \infty$ ), non-negative, with finite expectation. Applying Markov's inequality with  $t^2 > 0$  yields

$$\mathbb{P}\{|X - \mathbb{E}X| \geq t\} = \mathbb{P}\{|X - \mathbb{E}X|^2 \geq t^2\} \leq \frac{\text{Var}(X)}{t^2}$$

□

### 1.3 Limits theorems

For independent and identically distributed variables  $X_1, \dots, X_N$ , the sample mean  $\frac{1}{N} \sum_{i=1}^N X_i$  has

$$\text{Var}\left(\frac{1}{N} \sum_{i=1}^N X_i\right) = \frac{\text{Var}(X_1)}{N} \rightarrow 0 \text{ as } N \rightarrow \infty$$

so we should expect it to concentrate around the true mean.

**Theorem 1.3.1** (Strong law of large numbers). Let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables with  $\mathbb{E}X_1 < \infty$ . Then the averaged partial sums

$$\frac{S_N}{N} = \frac{1}{N} \sum_{i=1}^N X_i \rightarrow \mathbb{E}X_1 \text{ almost surely}$$

where random variables  $(Y_N)_{N=1}^\infty$  are said to **converge almost surely** to a random variable  $Y$  if there exists measurable  $Z \in \mathcal{M}$  with  $\mathbb{P}(Z) = 0$  and

$$\lim_{N \rightarrow \infty} Y_N(\omega) = Y(\omega) \text{ for every } \omega \in \Omega \setminus Z$$

**Theorem 1.3.2** (Lindeberg-Lévy CLT). Let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables with  $\mathbb{E}X_1 = \mu$ ,  $\text{Var}(X_1) = \sigma^2 < \infty$ . Then the normalized partial sums

$$Z_N = \frac{S_N - \mathbb{E}S_N}{\sqrt{\text{Var}(S_N)}} = \frac{\sum_{i=1}^N X_i - N\mu}{\sigma\sqrt{N}} \rightarrow N(0, 1) \text{ in distribution}$$

where real random variables  $(Y_N)_{N=1}^\infty$  are said to **converge in distribution** to a random variable  $Y$  if their CDF's  $F_{Y_N}(t) := \mathbb{P}\{Y_N \leq t\}$ ,  $F_Y(t) := \mathbb{P}\{Y \leq t\}$  have

$$\lim_{N \rightarrow \infty} F_{Y_N}(t) = F_Y(t) \text{ for all } t \in \mathbb{R}$$

**Exercise 1.3.3.** Let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables with  $\mu, \sigma^2 < \infty$ . Show that

$$\mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N X_i - \mu \right| = O\left(\frac{1}{\sqrt{N}}\right)$$

*Answer.* Considering the convex function  $\phi(x) = x^2$ , we can apply Jensen's to get

$$\left( \mathbb{E} \left| \frac{1}{N} \sum_{i=1}^N X_i - \mu \right| \right)^2 \leq \text{Var} \left( \frac{1}{N} \sum_{i=1}^N X_i \right) = \frac{\sigma^2}{N}$$

taking the square root of both sides yields the result.  $\square$

**Theorem 1.3.4** (Poisson limit theorem). Consider a sequence of  $N$ -tuples of independent random variables with entries  $X_{Ni}$  for  $1 \leq i \leq N$  with  $X_{Ni} \sim \text{Bernoulli}(p_{Ni})$ . Let  $S_N = \sum_{i=1}^N X_{Ni}$ , and suppose that as  $N \rightarrow \infty$

$$\max_{1 \leq i \leq N} p_{Ni} \rightarrow 0 \text{ and } \mathbb{E}S_N = \sum_{i=1}^N p_{Ni} \rightarrow \lambda$$

Then  $S_N \rightarrow \text{Poisson}(\lambda)$  in distribution, i.e. the CDF  $F_{S_N}(t) = \mathbb{P}\{S_N \leq t\}$  has

$$\lim_{N \rightarrow \infty} F_{S_N}(t) = \sum_{k=1}^{\lfloor t \rfloor} e^{-\lambda} \frac{\lambda^k}{k!}$$