# Greedy Algorithm and Proofs

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**Algorithm 1:** Greedy ROM

**Input:** Set $\theta_{train}$, Force vector $f \in \mathbb{R}^n$, Quantity of interest vector $\ell \in \mathbb{R}^n$, Function $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ such that $A(\theta)y(\theta) = f$, $\alpha > \|A(\theta)^{-1}\| \quad \forall \theta \in \theta_{train}$, $tol \in \mathbb{R}$

**Output:** Reduced order model $V \in \mathbb{R}^{n \times r}$ where $r << n$

Let $\theta_{init}$ be an arbitrary element in $\theta_{train}$;<br>
$\theta_{train} \leftarrow \theta_{train} \setminus \{\theta_{init}\}$;<br>
$V \leftarrow A(\theta_{init})^{-1} f$;<br>
**while** $\theta_{train} \neq \emptyset$ **do**<br>
    $\Delta_{max} \leftarrow 0$;<br>
    $\theta_{max} \leftarrow null$;<br>
    **foreach** $\theta \in \theta_{train}$ **do**<br>
        $A \leftarrow A(\theta)$;<br>
        $\widehat{A} \leftarrow V^T A V$;<br>
        $\widehat{y} \leftarrow \widehat{A}^{-1} V^T f$;<br>
        $\widehat{p} \leftarrow (\widehat{A}^T)^{-1} V^T \ell$;<br>
        $\Delta \leftarrow \alpha * \|f - AV\widehat{y}\| * \|\ell - A^T V \widehat{p}\|$;<br>
        **if** $\Delta > \Delta_{max}$ **then**<br>
            $\Delta_{max} \leftarrow \Delta$;<br>
            $\theta_{max} \leftarrow \theta$;<br>
    **if** $\Delta_{max} < tol$ **then**<br>
        **return** $V$;<br>
    **else**<br>
        $\theta_{train} \leftarrow \theta_{train} \setminus \{\theta_{max}\}$;<br>
        $y \leftarrow A(\theta_{max})^{-1} f$;<br>
        $y_{orthogonal} \leftarrow y - proj_{V_1} y - \cdots - proj_{V_r} y$;<br>
        $V \leftarrow [V \mid y_{orthogonal}]$;<br>
**return** *Error: ROM could not be computed with given tolerance*;

**Theorem 1.** *$y(\theta_1), \ldots, y(\theta_{r+1})$ constructed with greedy selection are independent.*

*Proof.* Since $span(V) = span(\{\theta_1, \ldots \theta_{r+1}\})$ after $r$ iterations, proving that $y(\theta_1), \ldots, y(\theta_{r+1})$ are linearly independent is equivalent to proving that $y(\theta_{r+1}) \notin R(V)$ where $V$ is constructed after $r-1$ iterations.

Assume by contradiction that $y(\theta_{r+1}) \in R(V)$. Therefore, $\exists \widehat{y}(\theta_{r+1})$ such that $V\widehat{y}(\theta_{r+1}) = y(\theta_{r+1})$. Evaluate the $\Delta$ of $\theta_{r+1}$ knowing that $A(\theta_{r+1})y(\theta_{r+1}) = f$ and $V\widehat{y}(\theta_{r+1}) = y(\theta_{r+1})$:

$y(\theta_{r+1}) = A(\theta_{r+1})^{-1}f$, so $V\widehat{y}(\theta_{r+1}) = A(\theta_{r+1})^{-1}f$
Multiply on the left by $V^T A(\theta_{r+1})$: $V^T A(\theta_{r+1})V\widehat{y}(\theta_{r+1}) = V^T A(\theta_{r+1})A(\theta_{r+1})^{-1}f = V^T f$
Therefore, $\widehat{y}$ in the algorithm is equal to $\widehat{y}(\theta_{r+1})$.
$\|f - AV\widehat{y}\| = \|f - Ay\| = \|f - f\| = 0$

Therefore, $\Delta = 0$ which means that $\theta_{r+1}$ would not be selected in this iteration of the greedy algorithm. This is a contradiction to the assumption, so $y(\theta_{r+1}) \notin R(V)$, which means that $y(\theta_1), \ldots, y(\theta_{r+1})$ constructed with greedy selection are independent. $\square$