$\mathcal{N} = 4$ SYM Loop Notes

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TO DO: THESIS TEMPLATE, REFERENCES, FORMATTING

1 loop review Let's consider a general 1-loop amplitude $\mathcal{A}_n^1$ with n external momenta p_i^μ $(i = 1, ..., n)$

$$\mathcal{A}_n^1 = \mathcal{A}_n^1(p_1, ..., p_n) \tag{1}$$

where the n external momenta have the constraints

$$p_i^\mu p_{\mu,i} = p_i^2 = 0 \tag{2}$$

$$\sum_i p_i^\mu = 0 \tag{3}$$

$\forall i = 1, ..., n$ Namely, (2) is the "on-shell" condition and (3) is the "conservation of momentum" relation. We can trivially satisfy (3) by introducing dual coordinates (region variables) x_i^μ with $(i = 1, ..., n)$ such that

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu = x_{i,i+1}^\mu \tag{4}$$

with $p_n^\mu = x_n^\mu - x_1^\mu$ and periodicity condition $x_{n+1}^\mu = x_1^\mu$. We can relate back every $x_{i,j}$ introduced in (4) to 4-momenta with

$$x_{i,j} = x_i - x_j = p_i + p_{i+1} + \cdots + p_{j-1} \tag{5}$$

Let's now consider a d-dim Minkowskian CFT (defined therefore on $\mathcal{M}^d$) with conformal group $SO(d, 2)$ acting non-linearly on $\mathcal{M}^d$ (clearly $\mathcal{M}^d \ni x^\mu$). To linearise the action of the conformal group on $\mathcal{M}^d$ (Dirac) we consider the linear action of $SO(d, 2)$ on the embedding space, which is a Minkowski spacetime with signature $(d, 2)$ that we denote by $\mathcal{M}^{d,2}$. By the embedding of $\mathcal{M}^d \ni x_i^\mu$ to $\mathcal{M}^{d,2}$ we mean

$$x_i^\mu \to X_i^A = (X^+, X^-, x_i^\mu) \in \mathcal{M}^{d,2} \tag{6}$$

with inner product

$$X \cdot X = \eta_{AB} X^A X^B = -X^+ X^- + \eta_{\mu\nu} X^\mu X^\nu \tag{7}$$

with $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$, $\eta_{00} = -1$ and $\eta_{ii} = 1$. This is done, so that $SO(d, 2)$ acts linearly on $\mathcal{M}^{d,2}$, in contrast to its non-linear action on $\mathcal{M}^d \ni x_i^\mu$ dual Minkowski coordinates.
For $d = 4$, the embedding space is a $\dim(\mathcal{M}^{d,2}) = 6$ Minkowski-space with signature $(4, 2)$ so X^A, $A = (1, ..., \dim(\mathcal{M}^{d,2})) = (1, ..., 6)$, where $X^+ = X^0 + X^5$ and $X^- = X^0 - X^5$. The

condition $X \cdot X = X^2 = 0$ defines an $SO(4,2)$ invariant subspace of $5d$, the null-cone, or light-cone. Then, we obtain $\mathcal{M}^4$ by projectivising the light-cone, that is, by quotienting the light-cone by rescaling $X \sim \lambda X$ with $\mathbb{R} \ni \lambda$. Because projectivising respects Lorentz rotations of the embedding space $\mathcal{M}^{4,2}$, the projective null-cone naturally inherits an action of $SO(4,2)$ on the original $\mathcal{M}^4$ spacetime.
By gauge-fixing the rescaling we can identify the original Minkowski spacetime $\mathcal{M}^4$ with the projective light-cone. The gauge-fixing condition $X^+ = 1$ gives light-cone (projective subspace) vectors of the form

$$X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \tag{8}$$

($x^2 = x^\mu x_\mu$) In this gauge choice we can see that (not confusing the component of y with its squared length)

$$X \cdot Y = \begin{pmatrix} 1 \\ x^2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^T \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y^2 \\ y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = -\frac{1}{2}x^2 - \frac{1}{2}y^2 + x_\mu y^\mu \tag{9}$$

Trivially, we see that

$$X \cdot X = X^2 = -\frac{1}{2}x^2 - \frac{1}{2}x^2 + x_\mu x^\mu = -x^2 + x^2 = 0 \tag{10}$$

Moreover, for dual vectors x^μ, y^μ we can map their squared difference to the projective light-cone, giving

$$(x-y)^2 \to (X-Y)^2 = (X)^2 + (Y)^2 - 2X \cdot Y = -2X \cdot Y = x^2 + y^2 - 2x_\mu y^\mu = (x-y)^2 \tag{11}$$

It is therefore convenient to introduce the product on the projective space

$$(X_i, X_j) = -2X_i \cdot X_j = (x_i - x_j)^2 \tag{12}$$

Thus, we can express products of momenta as products of null vectors, that is vectos that are on the light-cone. For example, taking 4-momentum p_1^μ

$$p_2^2 = 0 \mapsto x_{21}^2 = (x_2 - x_1)^2 = 0 \mapsto (X_2, X_1) = 0 \tag{13}$$

Due to the on-shell constraint, we obtain the relations

$$(X_i, X_{i+1}) = (X_i, X_{i-1}) = 0 \tag{14}$$

Similarly, for a variable momentum l we can assign a dual variable x_0 as $l = x_0 - x_4$ (to ensure momentum conservation). To the dual representation of the loop momentum we then assign then a coordinate on the light-cone as previously. Thus, we map the products as

$$l^2 \mapsto x_{04}^2 \mapsto (X_0, X_4) \tag{15}$$

Generally,

$$l_i^2 \mapsto (x_0 - x_i)^2 = x_{0i}^2 \mapsto (X_0, X_i) \tag{16}$$

Given this embedding, we can express Feynman-integrals on the projective null-cone. The 1-loop n-point integral has the structure

$$I_n = \int \frac{d^4 l P(l)}{D_1 D_2 \cdots D_n} \tag{17}$$

with propagators $D_1, ..., D_n$, which for on shell external momenta exhibits conformal symmetry. Thus, due to the embedding formalism we can use a manifestly conformal representation of the 1-loop integral with a conformal integral defined on the embedding space.

n=4 (box) 1-loop scalar integral The amplitude has the structure

$$\mathcal{A}_4^1 = \mathcal{A}_4^1(p_1, p_2, p_3, p_4) \tag{18}$$

and arises at the first quantum correction of the 4-point boson interaction (with Mandelstam variables s, t).

$$I_4 = \int d^4 l f(l) = \int \frac{st d^4 l}{l^2 (l - p_1)^2 (l - p_1 - p_2)^2 (l + p_4)^2} \tag{19}$$

By transforming to dual coordinates, we obtain an integral in the x_0 variable (the transformation is linear so has unit Jacobian)

$$\int d^4 l \mapsto \int d^4 x_0 \tag{20}$$

$$\Rightarrow I_4 = \int \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{21}$$

Using Feynman/Schwinger parametrisation, we can express this with

$$I_4 = \int \int_0^\infty \int_0^\infty \int_0^\infty d\alpha_2 d\alpha_3 d\alpha_4 \frac{d^4 x_0 x_{13}^2 x_{24}^2}{(x_{01}^2 + \alpha_2 x_{02}^2 + \alpha_3 x_{03}^2 + \alpha_4 x_{04}^2)^4} \tag{22}$$

Under the dilation transformation

$$x_i^\mu \to \lambda x_i^\mu \tag{23}$$

(with $\lambda \in \mathbb{R}$)

$$\frac{(\lambda^4 d^4x_0)(\lambda x_1 - \lambda x_3)^2(\lambda x_2 - \lambda x_4)^2}{(\lambda x_0 - \lambda x_1)^2(\lambda x_0 - \lambda x_2)^2(\lambda x_0 - \lambda x_3)^2(\lambda x_0 - \lambda x_4)^2} = \frac{\lambda^8 d^4x_0 x_{13}^2 x_{24}^2}{\lambda^8 x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \frac{d^4x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{24}$$

the integrand is invariant. Also, it is invariant under inversion transformation

$$x_i^\mu \to \frac{x_i^\mu}{x_i^\mu x_{\mu i}} = \frac{x_i^\mu}{x_i^2} \tag{25}$$

$$\frac{\frac{d^4x_0}{x_0^8}(\frac{x_1}{x_1^2} - \frac{x_3}{x_3^2})^2(\frac{x_2}{x_2^2} - \frac{x_4}{x_4^2})^2}{(\frac{x_0}{x_0^2} - \frac{x_1}{x_1^2})^2(\frac{x_0}{x_0^2} - \frac{x_2}{x_2^2})^2(\frac{x_0}{x_0^2} - \frac{x_3}{x_3^2})^2(\frac{x_0}{x_0^2} - \frac{x_4}{x_4^2})^2} = \frac{\frac{d^4x_0}{x_0^8}\frac{x_{13}^2 x_{24}^2}{x_1^2 x_2^2 x_3^2 x_4^2}}{\frac{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}{x_0^8 x_1^2 x_2^2 x_3^2 x_4^2}} = \frac{d^4x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{26}$$

So, the integrand in the dual coordinate representation exhibits conformal invariance. We can therefore use the embedding formalism to embed the integral on the light cone. The conformal integral measure ($X_0^2 = 0$) becomes

$$\int d^4l f(l) \mapsto \int d^4x_0 f(x_0) \mapsto \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{27}$$

Consequently, the 4-pt box scalar integral after embedding on the projective light cone takes the form

$$I_4 = \int d^4l f(l) = \int \frac{d^4x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} \frac{(X_1, X_3)(X_2, X_4)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \tag{28}$$

So the integral has the structure

$$I_4 = \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{29}$$

Clearly, the integral is invariant under dilation $X_0 \to \lambda X_0$

$$\int \frac{\lambda^6 d^6X_0 \delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} \frac{\lambda^4}{\lambda^4 (X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \tag{30}$$

$$\implies \int \frac{\lambda^6 d^6X_0 \delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{31}$$

So by embedding the integral on the projective light-cone, we have made the conformal invariance manifest. In other words, we have found a more natural representation that explicitly exhibits the conformal symmetry of the loop integral considered. For planar theories, we can decompose $n > 4$ loop integrals to a basis of the 4-point integrals discussed in this section. Moreover we can also define an anti-symmetric product on the projective light-cone with

$$\langle X_i X_j X_k X_l X_m X_p \rangle = \varepsilon^{ABCDFG} X_{iA} X_{jB} X_{kC} X_{lD} X_{mF} X_{pG} \tag{32}$$

where ϵ^{ABCDFG} is the totally anti-symmetric tensor. This will be used in the next section, for the reduction of the $n = 5$ (and above) 1-loop case.

n=5 (pentagonal) 1-loop integral Let's consider a general 5-point 1-loop amplitude. By transforming to dual coordinates with variable x_0 and general dual vector w we obtain the representation

$$I_5 = \int \frac{d^4 x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \tag{33}$$

Similarly as before, we can embed to the projective light-cone. Since we are in $6d$ we can hence write an arbitrary $\mathcal{M}^{4,2} \ni W$ as

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR = c_i X_i + rR \tag{34}$$

with $(X_i, R) = 0$ and $(R, R) = 1$. Then

$$I_5 = \int \frac{d^4 x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \mapsto \mathcal{I}_5 = \int \frac{d^6 X_0 (X_0, W)}{Vol(GL(1))(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \tag{35}$$

So

$$\mathcal{I}_5 = \int \frac{d^6 X_0}{Vol(GL(1))} \frac{c_i (X_0, X_i) + r(X_0, R)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \tag{36}$$

Remarkably, the (X_0, R) term does not contribute. This can be seen using the method of Feynman/Schwinger parametrisation

$$\mathcal{I}_5 = \int [d^4 X_0] \frac{r(X_0, R)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} = \tag{37}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 \int \frac{[d^4 X_0] r(X_0, R)}{((X_1, X_0) + \alpha_2 (X_2, X_0) + \alpha_3 (X_3, X_0) + \alpha_4 (X_4, X_0) + \alpha_5 (X_5, X_0)} \tag{38}$$

By defining $\mathcal{W} = X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5$, we have

$$\Pi_{i=2}^5 (\int_0^\infty d\alpha_i) \int [d^4 X_0] \frac{(X_0, R)}{(\mathcal{W}, X_0)^5} \sim \Pi_{i=2}^5 (\int_0^\infty d\alpha_i) \int [d^4 X_0] (rR) \partial_{\mathcal{W}} \frac{1}{(\mathcal{W}, X_0)^4} \sim \tag{39}$$

$$\sim \Pi_{i=2}^5 (\int_0^\infty d\alpha_i)(rR) \partial_{\mathcal{W}} \frac{1}{(\mathcal{W}, \mathcal{W})^2} \sim \Pi_{i=2}^5 (\int_0^\infty d\alpha_i)(rR) \frac{\mathcal{W}}{(\mathcal{W}, \mathcal{W})^3} \tag{40}$$

Since $(R, X_i) = 0 \implies (R, \mathcal{W}) = 0$ we obtain

$$\Pi_{i=2}^{5}(\int_0^\infty d\alpha_i)\frac{r(R,\mathcal{W})}{(\mathcal{W},\mathcal{W})^3} = 0 \tag{41}$$

Thus

$$\mathcal{I}_5 = \int \frac{d^6 X_0}{Vol(GL(1))}\frac{c_i(X_i, X_0)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \tag{42}$$

The coefficients of the base expansion can be expressed using the anti-symmetric product. For example, we can get c_1 as

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR \tag{43}$$

$$\langle W X_2 X_3 X_4 X_5 R\rangle = \langle (c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR) X_2 X_3 X_4 X_5\rangle \tag{44}$$

$$\langle W X_2 X_3 X_4 X_5 R\rangle = \langle c_1 X_1 X_2 X_3 X_4 X_5 R\rangle \implies c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R\rangle}{\langle X_1 X_2 X_3 X_4 X_5 R\rangle} \tag{45}$$

Similarly, for c_2

$$\langle W X_1 X_3 X_4 X_5 R\rangle = \langle c_2 X_2 X_1 X_3 X_4 X_5 R\rangle \implies c_2 = \frac{\langle W X_1 X_3 X_4 X_5 R\rangle}{\langle X_2 X_1 X_3 X_4 X_5 R\rangle} \tag{46}$$

Let's consider the c_1 term further with the definition of the anti-symmetric product

$$c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R\rangle}{\langle X_1 X_2 X_3 X_4 X_5 R\rangle} = \frac{\langle W X_2 X_3 X_4 X_5 R\rangle\langle X_1 X_2 X_3 X_4 X_5 R\rangle}{\langle X_1 X_2 X_3 X_4 X_5 R\rangle\langle X_1 X_2 X_3 X_4 X_5 R\rangle} = \tag{47}$$

$$= \frac{\varepsilon^{ABCDFG}\varepsilon_{PQRSTV} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_G X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V}{\varepsilon^{HJKLMN}\varepsilon_{PQRSTV} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V} \tag{48}$$

Using the product identity for the $6d$ Levi-Civita tensor

$$\varepsilon^{ABCDFG}\varepsilon_{PQRSTV} = \delta^{ABCDFG}_{PQRSTV} = \begin{vmatrix} \delta^A_P & \delta^A_Q & \delta^A_R & \delta^A_S & \delta^A_T & \delta^A_V \\ \delta^B_P & \delta^B_Q & \delta^B_R & \delta^B_S & \delta^B_T & \delta^B_V \\ \delta^C_P & \delta^C_Q & \delta^C_R & \delta^C_S & \delta^C_T & \delta^C_V \\ \delta^D_P & \delta^D_Q & \delta^D_R & \delta^D_S & \delta^D_T & \delta^D_V \\ \delta^F_P & \delta^F_Q & \delta^F_R & \delta^F_S & \delta^F_T & \delta^F_V \\ \delta^G_P & \delta^G_Q & \delta^G_R & \delta^G_S & \delta^G_T & \delta^G_V \end{vmatrix} \tag{49}$$

$$\tag{50}$$

we obtain

$$c_1 = \frac{\begin{vmatrix} \delta^A_P & \delta^A_Q & \delta^A_R & \delta^A_S & \delta^A_T & \delta^A_V \\ \delta^B_P & \delta^B_Q & \delta^B_R & \delta^B_S & \delta^B_T & \delta^B_V \\ \delta^C_P & \delta^C_Q & \delta^C_R & \delta^C_S & \delta^C_T & \delta^C_V \\ \delta^D_P & \delta^D_Q & \delta^D_R & \delta^D_S & \delta^D_T & \delta^D_V \\ \delta^F_P & \delta^F_Q & \delta^F_R & \delta^F_S & \delta^F_T & \delta^F_V \\ \delta^G_P & \delta^G_Q & \delta^G_R & \delta^G_S & \delta^G_T & \delta^G_V \end{vmatrix} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_G X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V}{\begin{vmatrix} \delta^H_P & \delta^H_Q & \delta^H_R & \delta^H_S & \delta^H_T & \delta^H_V \\ \delta^J_P & \delta^J_Q & \delta^J_R & \delta^J_S & \delta^J_T & \delta^J_V \\ \delta^K_P & \delta^K_Q & \delta^K_R & \delta^K_S & \delta^K_T & \delta^K_V \\ \delta^L_P & \delta^L_Q & \delta^L_R & \delta^L_S & \delta^L_T & \delta^L_V \\ \delta^M_P & \delta^M_Q & \delta^M_R & \delta^M_S & \delta^M_T & \delta^M_V \\ \delta^N_P & \delta^N_Q & \delta^N_R & \delta^N_S & \delta^N_T & \delta^N_V \end{vmatrix} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V} = \tag{51}$$

$$= \frac{\begin{vmatrix} W\cdot X_1 & W\cdot X_2 & W\cdot X_3 & W\cdot X_4 & W\cdot X_5 & W\cdot R \\ X_2\cdot X_1 & X_2\cdot X_2 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R \\ X_3\cdot X_1 & X_3\cdot X_2 & X_3\cdot X_3 & X_3\cdot X_4 & X_3\cdot X_1 & X_3\cdot R \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & X_4\cdot X_4 & X_4\cdot X_5 & X_4\cdot R \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & X_5\cdot X_5 & X_5\cdot R \\ R\cdot X_1 & R\cdot X_2 & R\cdot X_3 & R\cdot X_4 & R\cdot X_5 & R\cdot R \end{vmatrix}}{\begin{vmatrix} X_1\cdot X_1 & X_1\cdot X_2 & X_1\cdot X_3 & X_1\cdot X_4 & X_1\cdot X_5 & X_1\cdot R \\ X_2\cdot X_1 & X_2\cdot X_2 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R \\ X_3\cdot X_1 & X_3\cdot X_2 & X_3\cdot X_3 & X_3\cdot X_4 & X_3\cdot X_1 & X_3\cdot R \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & X_4\cdot X_4 & X_4\cdot X_5 & X_4\cdot R \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & X_5\cdot X_5 & X_5\cdot R \\ R\cdot X_1 & R\cdot X_2 & R\cdot X_3 & R\cdot X_4 & R\cdot X_5 & R\cdot R \end{vmatrix}} = \tag{52}$$

$$= \frac{\begin{vmatrix} W\cdot X_1 & W\cdot X_2 & W\cdot X_3 & W\cdot X_4 & W\cdot X_5 & 1 \\ X_2\cdot X_1 & 0 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & 0 \\ X_3\cdot X_1 & X_3\cdot X_2 & 0 & X_3\cdot X_4 & X_3\cdot X_1 & 0 \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & 0 & X_4\cdot X_5 & 0 \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & X_1\cdot X_2 & X_1\cdot X_3 & X_1\cdot X_4 & X_1\cdot X_5 & 0 \\ X_2\cdot X_1 & 0 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & 0 \\ X_3\cdot X_1 & X_3\cdot X_2 & 0 & X_3\cdot X_4 & X_3\cdot X_1 & 0 \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & 0 & X_4\cdot X_5 & 0 \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}} \tag{53}$$

We can obtain the other coefficients similarly. Before continuing to higher-external leg cases, we introduce the following notation for the integration measure in embedding space

$$[d^4 X_0] = \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} \tag{54}$$

n=6 (hexagonal) 1-loop integral The $n = 6$ 1-loop integral expressed in terms of embedding space coordinates $\mathcal{M}^{d,2} \ni X^A$ has the form

$$\mathcal{I}_6 = \int \frac{[d^4 X_0] T_{AB} X_0^A X_0^B}{\Pi_{i=1}^6 (X_0, X_i)} = \int \frac{[d^4 X_0] T_{AB} X_0^A X_0^B}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{55}$$

By introducing $\mathcal{M}^{d,2} \ni W_1, W_2$, with

$$\mathcal{F}_6(W_1, W_2, X_i) = \frac{(W_1, X_0)(W_2, X_0)}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{56}$$

we see that we can obtain the integrand of $\mathcal{I}_6$ with differentiation of F_6 and contraction with T_{AB}

$$T_{AB} \frac{\partial}{\partial W_1^A} \frac{\partial}{\partial W_2^B} F_6 = \frac{T_{AB} X_0^A X_0^B}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{57}$$

Thus

$$\mathcal{I}_6 = \int [d^4 X_0] T_{AB} \frac{\partial}{\partial W_1^A} \frac{\partial}{\partial W_2^B} \mathcal{F}_6 = \int \frac{[d^4 X_0] T_{AB} X_0^A X_0^B}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{58}$$

n=7 (heptagonal) 1-loop integral According to the previous ($n = 6$) example, we have the integral embedded to $\mathcal{M}^{4,2}$ of the form

$$\mathcal{I}_7 = \int \frac{[d^4 X_0] T_{ABC} X_0^A X_0^B X_0^C}{\Pi_{i=1}^7 (X_0, X_i)} = \int \frac{[d^4 x_0] T_{ABC} X_0^A X_0^B X_0^C}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)(X_0, X_7)} \tag{59}$$

By introducing $\mathcal{M}^{d,2} \ni W_1, W_2, W_3$, with

$$\mathcal{F}_7(W_1, W_2, W_3, X_i) = \frac{(W_1, X_0)(W_2, X_0)(W_3, X_0)}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{60}$$

the integrand is then obtained similarly T_{ABC}

$$T_{ABC} \frac{\partial}{\partial W_1^A} \frac{\partial}{\partial W_2^B} \frac{\partial}{\partial W_3^C} = \frac{T_{ABC} X_0^A X_0^B X_0^C}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)(X_0, X_7)} \tag{61}$$

Thus

$$\mathcal{I}_7 = \int [d^4 X_0] T_{ABC} \frac{\partial}{\partial W_1^A} \frac{\partial}{\partial W_2^B} \frac{\partial}{\partial W_3^C} \mathcal{F}_7 = \int \frac{[d^4 X_0] T_{AB} X_0^A X_0^B X_0^C}{\Pi_{i=1}^7 (X_0, X_i)} \tag{62}$$

General n external leg (n-gon) 1-loop integral Generalising the previous discussions, we can consider the case of an n-gon, so n external leg 1-loop integral. By embedding the integral to $\mathcal{M}^{4,2}$ we obtain

$$\mathcal{I}_n = \int \frac{[d^4 X_0] T_{a_1 a_2 \cdots a_{n-4}} X_0^{a_1} X_0^{a_2} \cdots X_0^{a_{n-4}}}{\Pi_{i=1}^{n}(X_0, X_i)} \tag{63}$$

By introducing $\mathcal{M}^{4,2} \ni W_1, W_2, \cdots, W_{n-4}$, with

$$\mathcal{F}_n(W_1, W_2, \cdots, W_{n-4}, X_i) = \frac{(W_1, X_0)(W_2, X_0) \cdots (W_{n-4}, X_0)}{\Pi_{i=1}^{n}(X_0, X_i)} \tag{64}$$

the integrand is then obtained similarly T_{ABC}

$$T_{a_1 a_2 \cdots a_{n-4}} \frac{\partial}{\partial W_1^{a_1}} \frac{\partial}{\partial W_2^{a_2}} \cdots \frac{\partial}{\partial W_{n-4}^{a_{n-4}}} \mathcal{F}_n = \frac{T_{a_1 a_2 \cdots a_{n-4}} X_0^{a_1} X_0^{a_2} \cdots X_0^{a_{n-4}}}{\Pi_{i=1}^{n}(X_0, X_i)} \tag{65}$$

Thus

$$\mathcal{I}_n = \int [d^4 X_0] T_{a_1 a_2 \cdots a_{n-4}} \frac{\partial}{\partial W_1^{a_1}} \frac{\partial}{\partial W_2^{a_2}} \cdots \frac{\partial}{\partial W_{n-4}^{a_{n-4}}} \mathcal{F}_n = \int \frac{[d^4 X_0] T_{a_1 a_2 \cdots a_{n-4}} X_0^{a_1} X_0^{a_2} \cdots X_0^{a_{n-4}}}{\Pi_{i=1}^{n}(X_0, X_i)} \tag{66}$$

Momentum-space twistors In the case of massless particles, we can introduce spinor-helicity variables, so that the on-shell momentum constraint $p_i^2 = 0$ ($\forall i = 1, ..., n$) is trivialised. Firslty, we map $4 - \textit{momenta}$ to $2x2$-hermitian matrices (4 d.f = 4 d.f) by using the Paulis. Since $\det(p_i) = -p^2 = 0$, we can express the $2x2$ matrices as the product of 2 spinors

$$p_i^\mu \to (p_i)^{\alpha\dot{\alpha}} = p_i^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \tag{67}$$

This is useful, since in the circumstance of massless particles the gauge interactions conserve helicity, which is exploited by computing in the helicity bases. Hence, we work $2d$ quantities, instead of $4d$-momenta. In $2d$ space we cannot have 3 linearly independent spinors. This is reflected by the Schouten-identity

$$\tilde{\lambda}_i^{\dot{\alpha}} (\tilde{\lambda}_{\dot{\beta},j} \tilde{\lambda}_k^{\dot{\beta}}) + \tilde{\lambda}_j^{\dot{\alpha}} (\tilde{\lambda}_{\dot{\beta},k} \tilde{\lambda}_i^{\dot{\beta}}) + \tilde{\lambda}_k^{\dot{\alpha}} (\tilde{\lambda}_{\dot{\beta},i} \tilde{\lambda}_j^{\dot{\beta}}) = 0 \tag{68}$$

The identity is identical for λ spinors. Although the spinor variables $\lambda, \tilde{\lambda}$ trivialise the on-shell relation $\forall i = 1, 2, ...n$, it does not give information about momentum-conservation (3). So it would be most useful to work with a mathematical object that simultaneously satisfies (2) and (3). Such an object $\exists$ and is called the momentum-space twistor (introduced by Hodges).

Givem a momentum-space twistor

$$Z_A = (\tilde{\lambda}, \mu) \tag{69}$$

its components are related by the incidence relation

$$\mu^\alpha = x^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}} \tag{70}$$

So for a twistor to be a twistor it has to have the form

$$Z_A = (\tilde{\lambda}, \mu) = (\tilde{\lambda}, x\tilde{\lambda}) \tag{71}$$

A twistor Z has components of a point in $\mathbb{C}^4$, but since (68) cant tell the difference between Z and scaled twistor tZ the space is projectivised; thus, these twistors form a line in $\mathbb{CP}^3$. Using helicity spinors, we can re-write the dual-coordinate defining relation (4) as

$$p_i^\mu \to p_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} \tag{72}$$

Multiplying both sides of (70) with $\tilde{\lambda}_{\dot{\alpha},i}$ and re-arranging yields

$$(x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}})\tilde{\lambda}_{\dot{\alpha},i} = 0 \implies x_i^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i} = x_{i+1}^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i} = \mu_i^\alpha \tag{73}$$

Thus, we can naturally define the incidence relation (68) through (71). $\forall x_i, x_{i+1}$ pair of dual coordinates (zone variables) satisfying (71) the dual coordinates x_i, x_{i+1} are null-separated by the momentum vector

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = p_i^{\alpha\dot{\alpha}} \to p_i^\mu \tag{74}$$

This implies that the line in dual-coordinate space determined by null-separated coordinates x_i, x_{i+1} defines a point $Z_A = (\tilde{\lambda}, x\tilde{\lambda}) = (\tilde{\lambda}, \mu)$ in twistor space, through the incidence relation (71).

Furthermore, let's consider the dual coordinate $x_i^{\alpha\dot{\alpha}}$, which according to (71) appears in the coincidence relations

$$x_i^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i} = \mu_i^\alpha \tag{75}$$

and (by shifting $i+1 \to i$ in the dual coordinate)

$$x_i^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i-1} = \mu_{i-1}^\alpha \tag{76}$$

On the otehr hand

$$-x_{\alpha\dot{\beta},i}\tilde{\lambda}_i^{\dot{\beta}} = \mu_{\alpha,i} \tag{77}$$

Acting on (74) with $\tilde{\lambda}_i^{\dot{\beta}}$ and on (73) with $\tilde{\lambda}_{i-1}^{\dot{\beta}}$ and combining we obtain

$$\tilde{\lambda}_i^{\dot{\beta}}\mu_{i-1}^\alpha - \tilde{\lambda}_{i-1}^{\dot{\beta}}\mu_i^\alpha = x_i^{\alpha\dot{\alpha}}(\tilde{\lambda}_i^{\dot{\beta}}\tilde{\lambda}_{\dot{\alpha},i-1} - \tilde{\lambda}_{i-1}^{\dot{\beta}}\tilde{\lambda}_{\dot{\alpha},i}) = x_i^{\alpha\dot{\beta}}\langle i-1, i\rangle \tag{78}$$

where $\langle i-1, i\rangle = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\alpha},i-1}\tilde{\lambda}_{\dot{\beta},i}$ With $\dot{\alpha} = \dot{\beta}$, we obtain

$$x_i^{\alpha\dot{\alpha}} = \frac{\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i-1} - \tilde{\lambda}_{i-1}^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i}}{\langle i-1, i\rangle} \tag{79}$$

Hence, for a pair of twistors on the same line in twistor space we can assign a point in dual-coordinate space.

By combining the two findings we see that (from (71))
line in dual-coordinate space $\iff$ point in twistor space
and that (from (76))
line in twistor space $\iff$ point in dual-coordinate space

Since he conformal group in $4d$ Minkowski-spacetime is $SO(4,2)$, so in order make the group action linear we embed in $\mathcal{M}^{4,2}$ with $\dim(\mathcal{M}^{4,2}) = 6$. The $X \cdot X = 0$ constraint and projective nature $X \sim rX$ make the d.o.f equal to $6-2=4$ and since $SO(4,2) \sim SU(2,2)$ we can express the embedding-space $6d$ vector X as a bi-spinor $X^{IJ} = -X^{JI}$, i.e. the anti-symmetric tensor representation of $SU(2,2)$(Skinner). Henceforth, the projective constraint in this representation can be expressed as

$$X^2 = X \cdot X = \frac{1}{2}\epsilon_{IJKL}X^{IJ}X^{KL} = 0 \tag{80}$$

$\iff$ X^{IJ} is a rank-2 tensor. This implies we can write the tensor as

$$X^{IJ} = Z_i^{[I}Z_j^{J]} = Z_i^I Z_j^J - Z_i^J Z_j^I \tag{81}$$

using twistor pairs (Z_i, Z_j). Furthermore, X is defined projectively $\implies$ Z is projective $\implies$ **twistor space** $= \mathbb{CP}^3$. So we see why changing from zone variables to momentum twistors is useful since they transform linearly under dual conformal transformations. Every twistor carries $SU(2,2)$ dual confomal indices indices $I = (\alpha, \dot{\alpha})$, so we can construct a dual-conformal invaraint by defining the DCI-invariant 4-bracket

$$\langle i,j,k,l\rangle = \epsilon_{ABCD}Z_i^A Z_j^B Z_k^C Z_l^D = (\tilde{\lambda}_{\dot{\alpha},i}\tilde{\lambda}_j^{\dot{\alpha}})(\mu_{\beta,k}\mu_l^{\beta}) + (\tilde{\lambda}_{\dot{\alpha},i}\tilde{\lambda}_k^{\dot{\alpha}})(\mu_{\beta,l}\mu_j^{\beta}) + (\tilde{\lambda}_{\dot{\alpha},i}\tilde{\lambda}_l^{\dot{\alpha}})(\mu_{\beta,j}\mu_k^{\beta}) + \tag{82}$$

$$+(\tilde{\lambda}_{\dot{\alpha},k}\tilde{\lambda}_l^{\dot{\alpha}})(\mu_{\beta,i}\mu_j^{\beta}) + (\tilde{\lambda}_{\dot{\alpha},l}\tilde{\lambda}_j^{\dot{\alpha}})(\mu_{\beta,i}\mu_k^{\beta}) + (\tilde{\lambda}_{\dot{\alpha},j}\tilde{\lambda}_k^{\dot{\alpha}})(\mu_{\beta,i}\mu_l^{\beta}) = \tag{83}$$

$$= \langle ij\rangle[\mu_k\mu_l] + \langle ik\rangle[\mu_l\mu_j] + \langle il\rangle[\mu_j\mu_k] + \langle kl\rangle[\mu_i\mu_j] + \langle lj\rangle[\mu_i\mu_k] + \langle jk\rangle[\mu_i\mu_l] \tag{84}$$

where we defined

$$\langle ab\rangle = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\alpha},a}\tilde{\lambda}_{\dot{\beta},b} \tag{85}$$

and

$$[\mu_a\mu_b] = \epsilon_{\alpha\beta}\mu_a^{\alpha}\mu_b^{\beta} \tag{86}$$

Also

$$[\mu_i|^{\alpha} = \langle i|_{\dot{\alpha}}x_i^{\alpha\dot{\alpha}} = \langle i|_{\dot{\alpha}}x_{i+1}^{\alpha\dot{\alpha}} \tag{87}$$

and

$$|\mu_i]_{\alpha} = -(x_i)_{\alpha\dot{\alpha}}|i\rangle^{\dot{\alpha}} = -(x_{i+1})_{\alpha\dot{\alpha}}|i\rangle^{\dot{\alpha}} \tag{88}$$

Considering the case where $\langle k, j-1, j, r\rangle$ we obtain using the incidence relation.

$$\langle k, j-1, j, r\rangle = \langle k, j-1\rangle[\mu_j\mu_r] + \langle kj\rangle[\mu_r\mu_{j-1}] + \langle kr\rangle[\mu_{j-1}\mu_j] + \tag{89}$$

$$+\langle jr\rangle[\mu_k\mu_{j-1}] + \langle r, j-1\rangle[\mu_k\mu_j] + \langle j-1, k\rangle[\mu_k\mu_r] = \tag{90}$$

$$= -\langle k, j-1\rangle\langle j|x_jx_r|r\rangle - \langle kj\rangle\langle r|x_rx_j|j-1\rangle - \langle kr\rangle\langle j-1|x_jx_j|j\rangle - \tag{91}$$

$$-\langle jr\rangle\langle k|x_kx_j|j-1\rangle - \langle r, j-1\rangle\langle k|x_kx_j|j\rangle - \langle j, j-1\rangle\langle k|x_kx_r|r\rangle = \tag{92}$$

$$= -(\langle k, j-1\rangle\langle j| + \langle jk\rangle\langle j-1|)x_jx_r|r\rangle - \langle k|x_kx_j(\langle jr\rangle|j-1\rangle + \langle r, j-1\rangle|j\rangle) - \tag{93}$$

$$-\langle kr\rangle\langle j-1|x_j^2|j\rangle - \langle j-1, j\rangle\langle k|x_kx_r|r\rangle = \tag{94}$$

$$= \langle j-1, j\rangle\langle k|x_jx_r|r\rangle - \langle kr\rangle\langle j-1|x_j^2|j\rangle + \langle j-1, j\rangle\langle k|x_kx_j|r\rangle - \langle j-1, j\rangle\langle k|x_kx_r|r\rangle = \tag{95}$$

$$= \langle j-1, j\rangle\langle k|x_kx_r|r\rangle - \langle j-1, j\rangle\langle k|x_j^2|r\rangle + \langle j-1, j\rangle\langle k|x_kx_j|r\rangle - \langle j-1, j\rangle\langle k|x_kx_r|r\rangle = \tag{96}$$

$$= \langle j-1, j\rangle\langle k|(x_k - x_j)(x_j - x_r)|r\rangle = \langle j-1, j\rangle\langle k|x_{kj}x_{jr}|r\rangle \tag{97}$$

Another special case useful for re-expressing the propagators in the integrands in terms of the $SU(2,2)$ invariant 4-bracket

$$\langle j-1, j, k-1, k\rangle = \langle j-1, j\rangle[\mu_{k-1}\mu_k] + \langle j-1, k-1\rangle[\mu_k\mu_j] + \langle j-1, k\rangle[\mu_j\mu_{k-1}] + \tag{98}$$

$$+\langle k-1, k\rangle[\mu_{j-1}\mu_j] + \langle k, j\rangle[\mu_{j-1}\mu_{k-1}] + \langle j, k-1\rangle[\mu_{j-1}\mu_k] = -\langle j-1, j\rangle\langle k-1|y_k^2|k\rangle - \tag{99}$$

$$-\langle j-1, k-1\rangle\langle k|y_ky_j|j\rangle - \langle j-1, k\rangle\langle j|y_jy_k|k-1\rangle - \langle k-1, k\rangle\langle j-1|y_j^2|j\rangle - \tag{100}$$

$$-\langle kj\rangle\langle j|y_jy_k|k-1\rangle - \langle j, k-1\rangle\langle j-1|y_jy_k|k\rangle = -\langle j-1, j\rangle\langle k-1|y_k^2|k\rangle - \tag{101}$$

$$-(\langle k-1, j-1\rangle\langle j| + \langle j, k-1\rangle\langle j-1|)y_jy_k|k\rangle - (\langle j-1, k\rangle\langle j| + \langle kj\rangle\langle j-1|)y_jy_k|k-1\rangle - \tag{102}$$

$$-\langle k-1, k\rangle\langle j-1|y_j^2|j\rangle = -\langle j-1, j\rangle\langle k-1|y_k^2|k\rangle + \langle j-1, j\rangle\langle k-1|y_jy_k|k\rangle + \tag{103}$$

$$+\langle j-1, j\rangle\langle k-1|y_ky_j|k\rangle - \langle k-1, k\rangle\langle j-1|y_j^2|j\rangle = \tag{104}$$

$$= \langle j-1, j\rangle\langle k-1|(y_ky_j - y_j^2 + y_jy_k - y_k^2)|k\rangle = \langle j-1, j\rangle\langle k-1|y_{kj}y_{jk}|k\rangle \tag{105}$$

$$\langle j-1, j, k-1, k\rangle = \langle j-1, j\rangle\langle k-1|y_{jk}y_{kj}|k\rangle = \langle j-1, j\rangle\langle k-1, k\rangle y_{jk}^2 \tag{106}$$

$$\implies y_{jk}^2 = \frac{\langle j-1, j, k-1, k\rangle}{\langle j-1, j\rangle\langle k-1, k\rangle} \tag{107}$$

Thus, we can express the on-shell condition with the 4-bracket as

$$y_{jk}^2 = (y_j - y_k)^2 = 0 \implies \langle j-1, j, k-1, k\rangle = 0 \tag{108}$$

which is the statement that $\mathbb{CP}^1 \subset \mathbb{CP}^3$ lines $(j-1, j)$ and $(k-1, k)$ are co-planar
Let's now consider n momentum twistors, $\mathbb{CP}^3 \ni Z_i, \forall i = 1, ..., n$ that are not subject to constraints and consider the n lines defined between the consecutive points $(Z_i, Z_{i+1}) = (i, i+1)$ in $\mathbb{CP}^3$ where the nth line is defined by $(n, 1)$ We can map each of these lines between the n points in twistor space to dual-coordinates through

$$y_i^{\dot{\alpha}\alpha} = \frac{|i\rangle^{\dot{\alpha}}[\mu_{i-1}|^{\alpha} - |i-1\rangle^{\dot{\alpha}}[\mu_i|^{\alpha}}{\langle i-1, 1\rangle} \tag{109}$$

with the constraint on the dual-coordinates guaranteeing on-shell momenta

$$\langle i|_{\dot{\alpha}}(y_i - y_{i+1})^{\dot{\alpha}\alpha} = 0 \implies (y_i - y_{i+1})^2 = 0 \tag{110}$$

The lines defined by $(Z_i, Z_{i+1}) = (i, i+1)$ by definition close a contour in $\mathbb{CP}^3$, so momentum is conserved as $y_{n+1} = y_1$ (as the contour is closed in dual coordinate space as well) Hence, we can express the on-shell condition and momentum conservation geometrically as the intersection of n-lines $(Z_i, Z_{i+1}) = (i, i+1)$ at n-points $(i) = Z_i$ in $\mathbb{CP}^3$

The intersection point $\mathbb{CP}^3 \ni Z_q = (q)$ of the line passing through $(Z_i, Z_j) = (i, j)$ and the plane spanned by $(Z_k, Z_l, Z_m) = (k, m, l)$ is

$$(q) = (i, j) \cap (k, l, m) = Z_i\langle j, k, l, m\rangle - Z_j\langle i, k, l, m\rangle \tag{111}$$

The line (p, q) formed by the intersection of planes spanned by (i, j, k) and (l, m, n) is

$$(p, q) = (i, j, k) \cap (l, m, n) = (i, j)\langle k, l, m, n\rangle + (j, k)\langle i, l, m, n\rangle + (k, i)\langle j, l, m, n\rangle \tag{112}$$

We can realise an isomorphism

$$\mathcal{M}^{4,2} \longleftrightarrow SU(2, 2) \tag{113}$$

with the $6d$ chiral gamma matrices. As discussed earlier, the number of independent components in X_m equals the number of independent components in X_{AB} ($\dim(\mathcal{M}^{4,2}) = \dim(SU(2, 2))$. Given $\mathcal{M}^{4,2} \ni X_m$ and $SU(2, 2) \ni X_{AB}$ we have

$$X_m \mapsto X_{AB} = \Gamma^m_{AB} X_m \tag{114}$$

where the explicit form of the Γ^m_{AB} is

$$\Gamma^+_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 2i\epsilon^{\alpha\beta} \end{pmatrix} \qquad \Gamma^-_{AB} = \begin{pmatrix} -2i\epsilon_{\dot\alpha\dot\beta} & 0 \\ 0 & 0 \end{pmatrix} \qquad \Gamma^\mu_{AB} = \begin{pmatrix} 0 & \sigma^\mu_{\dot\alpha\gamma}\epsilon^{\gamma\beta} \\ -\overline{\sigma}^{\mu,\alpha\dot\gamma}\epsilon_{\dot\gamma\dot\beta} & 0 \end{pmatrix} \tag{115}$$

In order to realise the inverse mapping, we first use the totally anti-symmetric tensor ϵ^{ABCD} with $SU(2,2)$ indices to invert the Γ matrices

$$\tilde\Gamma^{m,AB} = \frac{1}{2}\epsilon^{ABCD}\Gamma^m_{CD} \tag{116}$$

$$X_{AB} \mapsto X^m = \frac{1}{2}\tilde\Gamma^{m,AB}X_{AB} \tag{117}$$

Thus, we see that the invariant inner product on $\mathcal{M}^{4,2}$ is represnted in terms of the $SU(2,2)$ product as

$$\eta_{nm}X^nY^m = X_mY^m = \tilde\Gamma^{AB}_m X_{AB}\tilde\Gamma^{m,CD}Y_{CD} = \epsilon^{ABCD}X_{AB}Y_{CD} \tag{118}$$

using $\tilde\Gamma^{AB}_m\tilde\Gamma^{m,CD} = 2\epsilon^{ABCD}$. So making (80) explicit as

$$\eta_{nm}X^nX^m = X_mX^m = \tilde\Gamma^{AB}_m X_{AB}\tilde\Gamma^{m,CD}X_{CD} = \epsilon^{ABCD}X_{AB}X_{CD} = 0 \tag{119}$$

From now on we denote $SU(2,2)$ indices with: $\alpha, \beta, ...$ Now in terms of embedding coordinates, we have seen that the anti-symmetric $X_{\alpha\beta}$ corresponds to a point in space-time $\iff X_{\alpha\beta} = K_{[\alpha}L_{\beta]}$ with twistors K_α, L_β. Such an $X_{\alpha\beta}$ realises the anti-symmetric tensor representation of $SU(2,2)$ with the null-condition expressed as

$$X \cdot X = \varepsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}X_{\gamma\delta} = 0 \tag{120}$$

Given any $\mathbb{CP}^3 \ni W_\gamma$, the twistor W_γ lies on the line $(K \wedge L)$ between $K_\alpha, L_\beta \iff$

$$X_{[\alpha\beta}W_{\gamma]} = 0 \tag{121}$$

with bi-twistor $X_{\alpha\beta} = K_{[\alpha}L_{\beta]}$ Furthermore, given two points $\mathcal{M}^{4,2} \ni X, Y$ with corresponding bi-spinors $X_{\alpha\beta} = A_{[\alpha}B_{\beta]}, Y_{\gamma\delta} = C_{[\gamma}D_{\delta]}$, the null-separation condition is

$$\varepsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}Y_{\gamma\delta} = 0 \tag{122}$$

which is the statement of intersection of lines $(A \wedge B), (C \wedge D)$ in $\mathbb{CP}^3$. However, due to the scaling equivalence $W_\alpha \sim rW_\alpha$ there is no natural scale for any non-zero $X^{\alpha\beta}Y_{\alpha\beta}$. In order give a natural scale, the conformal group has to be broken to the Poincaré-group, so we introduce the infinity twistor

$$I_{\alpha\beta} = \begin{pmatrix} \varepsilon^{\dot a\dot b} & 0 \\ 0 & 0 \end{pmatrix} \tag{123}$$

in order to do that. With $I^{\alpha\beta}$ we can define the metric

$$(x-y)^2 = \frac{X^{\alpha\beta}Y_{\alpha\beta}}{I_{\gamma\delta}X^{\gamma\delta}I_{\rho\sigma}Y^{\rho\sigma}} \tag{124}$$

which allows us to have a meaningful scale. Using the totally anti-symmetric tensor we can define the dual infinity twistor

$$I^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}I_{\gamma\delta} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^{ab} \end{pmatrix} \tag{125}$$

Moreover, the infinity twistor twistor plays the role of the projection operator acting on $\mathbb{CP}^3 \ni W_\alpha$

$$I^{\alpha\beta}W_\beta = [\mu|^a \tag{126}$$

In order to evaluate integrals expressed in terms of momentum twistors, we need to define the integration measure d^4x_0 first. Integrating over all x_0 in dual-coordinate space is equivalent to integrating over all $\mathbb{CP}^1 \subset \mathbb{CP}^3$ (Arkani-Hamed). The integral over Z_A and Z_B can be decomposed, into the integral over all $\mathbb{CP}^1$ lines (A, B), and the integral over the twistors Z_A and Z_B moving along a particular line (A, B) (Elvang). Acting with a $GL(2)$ transformation on (Z_A, Z_B), the new twistor pair $(Z_{A'}, Z_{B'})$ defines the same line in $\mathbb{CP}^1$ as (Z_A, Z_B) as the $GL(2, \mathbb{C})$ transformation on indices A, B leaves the line (A, B) invariant. We can parametrise the movement of Z_A and Z_B along (A, B) with the $GL(2)$ transformation. Separating the $GL(2)$ part out we obtain

$$\int d^4Z_A d^4Z_B = \int \frac{d^4Z_A d^4Z_B}{Vol(GL(2))}\int_{GL(2)} \tag{127}$$

The integral over the pairs (Z_A, Z_B) modulo $GL(2)$ is the Grassmannian $G(2, 4)$ that can be parametrised by the $2x4$ matrix

$$\begin{pmatrix} Z_A^1 & Z_A^2 & Z_A^3 & Z_A^4 \\ Z_B^1 & Z_B^2 & Z_B^3 & Z_B^4 \end{pmatrix} = \begin{pmatrix} \langle A|_{\dot{1}} & \langle A|_{\dot{2}} & [\mu_A|^1 & [\mu_A|^2 \\ \langle B|_{\dot{1}} & \langle B|_{\dot{2}} & [\mu_B|^1 & [\mu_B|^2 \end{pmatrix} \tag{128}$$

The $GL(2)$ invariant measure is obtained by taking the integral over all Z_A,Z_B with the product of minors of the $2x4$ matrix (?), which break the conformal invariance to Lorentz-invariance in order to relate the integral of the dual-coordinate measure d^4x_0 with the integral over all twistor lines (A, B) in $\mathbb{CP}^3$

$$\int d^4x_0 \iff \int \frac{d^4Z_A d^4Z_B}{Vol(GL(2))\langle AB\rangle^4} \tag{129}$$

With (123) we can now express integrals over dual-coordinates with momentum-twistor integrals. Recall that

$$x_{jk}^2 = (x_j - x_k)^2 = \frac{\langle j-1, j, k-1, k\rangle}{\langle j-1, j\rangle\langle k-1, k\rangle} \tag{130}$$

To the dual-coordinate variable x_0 corresponds the (A, B) line passing through twistor variables Z_A, Z_B, so

$$x_{0k}^2 = (x_0 - x_k)^2 = \frac{\langle A, B, k-1, k\rangle}{\langle AB\rangle\langle k-1, k\rangle} \tag{131}$$

With this given, we can express the $n = 4$ scalar (box) integral in terms of momentum twistors

$$\int d^4x_0 \frac{x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \iff \tag{132}$$

$$\int \frac{d^4Z_A d^4Z_B}{Vol(GL(2))\langle AB\rangle^4} \frac{\langle AB\rangle^4\langle 41\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 4123\rangle\langle 1234\rangle}{\langle AB41\rangle\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle 41\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle} = \tag{133}$$

$$= \int \frac{d^4Z_A d^4Z_B}{Vol(GL(2))} \frac{\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle} \tag{134}$$

The conformal invariance breaking term $\langle AB\rangle$ disappears, since the integral is conformal invariant as demonstrated earlier. We briefly note that in this context the problem of finding the singular points of the integrand is the problem of finding all lines (A, B) which intersect the lines (12),(23),(34),(41) in $\mathbb{CP}^3$ (Schubert problem). For brevity, we introduce the notation

$$\int_{AB} \iff \int \frac{d^4Z_A d^4Z_B}{Vol(GL(2))} \tag{135}$$

Polylogarithms In order to define polylogs, first we need to consider iterated integrals and their properties. In 1D the iterated integral can be recursively defined as

$$\int_a^b f_1(t)dt \circ f_2(t)dt \circ ... \circ f_n(t)dt = \int_a^b \left(\int_a^t f_1(u)du \circ f_2(u)du \circ ... \circ f_{n-1}(u)du\right) f_n(t)dt \tag{136}$$

Let's consider this definition more generally. If $\mathbb{V}$ is a linear space, a 1-form on $\mathbb{V}$ is a linear functional on $\mathbb{V}$, where a linear functional α on $\mathbb{V}$ is a linear transformation

$$\alpha : \mathbb{V} \to \mathbb{R} \implies \alpha(a\mathbf{v} + b\mathbf{w}) = a\alpha(\mathbf{v}) + b\alpha(\mathbf{w}) \tag{137}$$

On a manifold X the 1-form is defined as a mapping from the tangent bundle of X ($TM = \sqcup_{x\in M} T_xM$, where T_x is the tangent space to X at x) to $\mathbb{R}$ such that by constraining the 1-form on each T_xM we have a linear functional $T_xM \to \mathbb{R}$. So on a manifold X a 1-form is

$$\alpha : TM \to \mathbb{R}, \qquad \alpha_x = \alpha|_{T_xM} : T_xM \to \mathbb{R} \tag{138}$$

Now, given a manifold X, and n 1-forms $\alpha_1, \alpha_2, ..., \alpha_n$ on X, we define the iterated integral on the path γ ($\gamma : [0, 1] \to X$) as

$$\int_\gamma \alpha_1 \circ \alpha_2 \circ ... \circ \alpha_n = \int_0^1 \gamma^*\alpha_1 \circ \gamma^*\alpha_2 \circ ... \circ \gamma^*\alpha_n \tag{139}$$

expressed with the pullbacks $\gamma^*\alpha_i$. If for example $w = \Sigma_i f_i dx^i$ is a 1-form on $X \implies \gamma^*\alpha_i = \Sigma_i f_i \frac{dx^i}{dt} dt$
Furthermore, given a path γ on X we define the functional F of the path as

$$F[\gamma] = \int_\gamma \alpha_1 \circ \alpha_2 \circ ... \circ \alpha_n \tag{140}$$

If F is independent of the path γ and we consider the functional as a function of the endpoint $\gamma(1)$ we obtain

$$dF[\gamma] = \alpha_n \int_\gamma \alpha_1 \circ \alpha_2 \circ ... \circ \alpha_{n-1} \tag{141}$$

Using (128), the $n = 2$ iterated integral is

$$\int_a^b f_1(t)dt \circ f_2(t)dt = \int_a^b \left(\int_a^t f_1(u)du \right) f_2(t)dt \tag{142}$$

With

$$\alpha_1 = \frac{dt}{1-t} \qquad\qquad \alpha_2 = \frac{dt}{t} \tag{143}$$

We obtain the definition of the dilogarithm $\mathrm{Li}_2(z)$

$$\mathrm{Li}_2(z) = \int_0^z \frac{dt}{1-t} \circ \frac{dt}{t} = \int_0^z \left(\int_0^t \frac{du}{1-u} \right) \frac{dt}{t} = -\int_0^z \frac{\ln(1-t)}{t} = -\int_0^z \frac{dt}{t} \mathrm{Li}_1(t) \tag{144}$$

where $\mathrm{Li}_1(t) = -\ln(1-t)$. We can define any n-weight polylogarithm using the relation

$$\mathrm{Li}_n(z) = \int_0^z \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ ... \circ \frac{dt}{t}}_{n-1} = \int_0^z \frac{dt}{t} \mathrm{Li}_{n-1}(t) \tag{145}$$

Due to the integral representation, we see that we can produce lower weight polylogs from higher weight polylogs with

$$d\,\mathrm{Li}_2(z) = -\frac{\ln(1-z)}{z} dz = \frac{\mathrm{Li}_1(z)}{z} dz \implies z \frac{d}{dz} \mathrm{Li}_2(z) = \mathrm{Li}_1(z) \tag{146}$$

We can define more general iterated integrals dependent on arbitrary complex parameters a_i, such as Goncharov-polylogarithms. The Goncharov-polylog is defined recursicely starting with $\alpha_1 = \frac{dt}{t-a_1}$ with parameter a_1 and $G(,z) = 1$

$$G(a_1, z) = \int_0^z \frac{dt}{t-a_1} G(,z) = \ln\left(\frac{a_1 - z}{a_1} \right) \tag{147}$$

Following this, we define the next Goncharov with parameters a_1, a_2 with

$$G(a_1, a_2, z) = \int_0^z \frac{dt}{t-a_1} G(a_2, t) = \int_0^z \frac{dt}{t-a_1} \ln\left(\frac{a_2 - t}{a_2} \right) \tag{148}$$

Generally, for the n weight Goncharov-polylog

$$G(a_1,a_2,...,a_n,z)=\int_0^z \frac{dt}{t-a_1}\circ\frac{dt}{t-a_2}\circ...\circ\frac{dt}{t-a_n}=\int_0^z\frac{dt}{t-a_1}G(a_2,a_3,...,a_n,t) \tag{149}$$

The definition of $\mathrm{Li}_1(z)$ can be obtained from $G(a_1,z)$

$$-G(1,z)=-G(\frac{1}{z},1)=-\ln\left(\frac{\frac{1}{z}-1}{\frac{1}{z}}\right)=-\ln(1-z)=\int_0^z\frac{dt}{1-t}=\mathrm{Li}_1(z) \tag{150}$$

Similarly, for the dilog $\mathrm{Li}_2(z)$ we have

$$-G(0,1,z)=-G(0,\frac{1}{z},1)=-\int_0^z\frac{dt}{t}\underbrace{G(\frac{1}{t},1)}_{G(1,t)}=-\int_0^z\frac{dt}{t}\ln(1-t)=\int_0^z\frac{dt}{t}\mathrm{Li}_1(t)=\mathrm{Li}_2(z) \tag{151}$$

We can obtain polylogs of arbitrary weight n (transcendentality) following this, i.e. for the n-polylog $\mathrm{Li}_n(z)$ we have from $G(a_1,a_2,...,a_n,z)$

$$\mathrm{Li}_n(z)=-G(\underbrace{0,0,...,0}_{n-1},\frac{1}{z},1)=-\int_0^z\frac{dt}{t}G(\underbrace{0,0,...,0}_{n-2},\frac{1}{t},1)=\int_0^z\frac{1}{t}\mathrm{Li}_{n-1}(t) \tag{152}$$

In order to generalise the result obtained in (143), let's take the total derivative of $G(a_1,a_2,z)$

$$dG(a_1,a_2,z)=\frac{\partial G}{\partial z}dz+\frac{\partial G}{\partial a_1}da_1+\frac{\partial G}{\partial a_2}da_2 \tag{153}$$

$$\frac{\partial G}{\partial z}=\frac{\partial}{\partial z}\int_0^z\frac{dt}{t-a_1}G(a_2,t)=\frac{1}{z-a_1}G(a_2,z) \tag{154}$$

$$\frac{\partial G}{\partial a_1}=\int_0^z\frac{dt}{(t-a_1)^2}G(a_2,t)=-\int_0^z d\left(\frac{1}{t-a_1}\right)G(a_2,t)=-\frac{G(a_2,t)}{t-a_1}\Big|_0^z+\int_0^z\frac{1}{t-a_1}dG(a_2,t)= \tag{155}$$

$$=-\frac{G(a_2,z)}{z-a_1}+\int_0^z\frac{1}{t-a_1}\frac{dt}{t-a_2}=-\frac{G(a_2,z)}{z-a_1}+\frac{1}{a_1-a_2}\int_0^z dt\left(\frac{1}{t-a_1}-\frac{1}{t-a_2}\right)= \tag{156}$$

$$=-\frac{G(a_2,z)}{z-a_1}+\frac{1}{a_1-a_2}\ln\left(\frac{t-a_1}{t-a_2}\right)\Big|_0^z=-\frac{G(a_2,z)}{z-a_1}+\frac{1}{a_1-a_2}\left(\ln\left(\frac{z-a_1}{z-a_2}\right)-\ln\left(\frac{-a_1}{-a_2}\right)\right)= \tag{157}$$

$$=-\frac{G(a_2,z)}{z-a_1}+\frac{1}{a_1-a_2}(G(a_1,z)-G(a_2,z)) \tag{158}$$

$$\frac{\partial G}{\partial a_2} = \int_0^z \frac{dt}{t-a_1}\frac{\partial G(a_2,t)}{\partial a_2} = \int_0^z \frac{dt}{t-a_1}\left(\int_0^t \frac{du}{(u-a_2)^2}\right) = \int_0^z \frac{dt}{t-a_1}\left(-\frac{1}{a_2}-\frac{1}{t-a_2}\right) = \tag{159}$$

$$= -\frac{G(a_1,z)}{a_2} - \frac{1}{a_1-a_2}(G(a_1,z)-G(a_2,z)) \tag{160}$$

So

$$dG(a_1,a_2,z) = dz\frac{G(a_2,z)}{z-a_1} - da_1\frac{G(a_2,z)}{z-a_1} + da_1\frac{G(a_1,z)}{a_1-a_2} - da_1\frac{G(a_2,z)}{a_1-a_2} - \tag{161}$$

$$-da_2\frac{G(a_1,z)}{a_2} - da_2\frac{G(a_1,z)}{a_1-a_2} + da_2\frac{G(a_2,z)}{a_1-a_2} = \tag{162}$$

$$= \left(\frac{dz}{z-a_1} - \frac{da_1}{z-a_1} - \frac{da_1}{a_1-a_2} + \frac{da_2}{a_1-a_2}\right)G(a_2,z) + \left(\frac{da_1}{a_1-a_2} - \frac{da_2}{a_1-a_2} - \frac{da_2}{a_2}\right)G(a_1,z) \tag{163}$$

The structure of the result suggests that we can write the measures more compactly. By taking $a_0 = z$ and $a_3 = 0$ we can write

$$d\ln\left(\frac{a_i-a_{i-1}}{a_i-a_{i+1}}\right) = \frac{a_i-a_{i+1}}{a_i-a_{i-1}}d\left(\frac{a_i-a_{i-1}}{a_i-a_{i+1}}\right) = \tag{164}$$

$$= \frac{a_i-a_{i+1}}{a_i-a_{i-1}}\frac{d(a_i-a_{i-1})(a_i-a_{i+1}) - d(a_i-a_{i+1})(a_i-a_{i-1})}{(a_i-a_{i+1})^2} = \frac{da_i-da_{i-1}}{a_i-a_{i-1}} - \frac{da_i-da_{i+1}}{a_i-a_{i+1}} \tag{165}$$

For $i=1$

$$d\ln\left(\frac{a_1-z}{a_1-a_2}\right) = \frac{dz}{z-a_1} - \frac{da_1}{z-a_1} - \frac{da_2}{a_1-a_2} + \frac{da_1}{a_1-a_2} \tag{166}$$

whereas for $i=2$

$$d\ln\left(\frac{a_2-a_1}{a_2}\right) = \frac{da_1}{a_1-a_2} - \frac{da_2}{a_1-a_2} - \frac{da_2}{a_2} \tag{167}$$

Thus, from (160) we see that the total derivative can be expressed as

$$dG^{(n=2)}(a_1,a_2,z) = \sum_{i=1}^{n=2} G^{(n=1)}(\mathbf{a}_i)d\ln\left(\frac{a_i-a_{i-1}}{a_i-a_{i+1}}\right) \tag{168}$$

The result holds for arbitrary n weight Goncharov-polylog (He, Li, Yang, Tang). We have seen that we can express n weight polylogs $\mathrm{Li}_n(z)$ iteratively as

$$\mathrm{Li}_n(z) = \int_0^z \frac{dt}{1-t}\circ\frac{dt}{t}\circ\ldots\circ\frac{dt}{t} \tag{169}$$

If the integral is independent under small variations of the path, we can define a multivalued function with differential forms $\alpha_1, \alpha_2, ..., \alpha_n$ and their given ordering called *symbol*

$$\mathcal{S}(\mathrm{Li}_n(z)) = -(1-z) \otimes \underbrace{z \otimes z \otimes ... \otimes z}_{n-1} \tag{170}$$

If the differential forms have the form $\alpha_i = d \ln f_i$, then we can express the symbol immediately. The simplest case is

$$\mathcal{S}(\ln(z)) = z \tag{171}$$

However, for other forms we need to use

$$dF[\gamma] = \alpha_n \int_\gamma \alpha_1 \circ \alpha_2 \circ ... \circ \alpha_{n-1} \tag{172}$$

2 loop review Previously on p.4-9, we have expressed integrals in embedding space. Later we briefly considered the interpretation of the integration measure with momentum-twistors and the 4-point integral in terms of momentum-space twistors. Let's now consider the momentum-twistor representations of integrands and integrals more thoroughly as we consider IR regulated integrals.

In the general 1-loop theory the box integral can be expressed in terms of momentum-space twistors with line $l \sim (AB)$ as

$$I_4 = \int d^4 l \frac{\langle i\ i+1\rangle\langle j\ j+1\rangle\langle k\ k+1\rangle\langle m\ m+1\rangle}{\langle l12\rangle\langle l23\rangle\langle l34\rangle\langle l41\rangle} \tag{173}$$

In loop integrands, combinations of bitwistors Y^{IJ} are present in every expression, reflecting the fact that it is the line $l \sim (AB)$ is being integrated over. As seen earlier, given bitwistors X^{IJ}, Y^{KL} we can express the conformally invariant product as $\langle XY\rangle = \epsilon_{IJKL} X^{IJ} Y^{KL}$. By the Passarino-Veltman reduction discussed at the $n = 5$ point integrand on p.5, we have seen that a non-trivial numerator structure emerges for the integrand. The reduction relied on the fact that the embedding space $\mathcal{M}^{4,2}$ had $\dim(\mathcal{M}^{4,2}) = 6$. Hence, bitwistor have 6 degrees of freedom and the Passarino-Veltman reduction relies on this. The $n = 5$ point integral in the momentum twistor representation is with generic bitwistor Y, which is

$$I_5 = \int d^4 l \frac{1}{\langle l I_\infty\rangle^{4-(n-m)}} \frac{\langle lY\rangle}{\langle l12\rangle\langle l23\rangle\langle l34\rangle\langle l52\rangle\langle l51\rangle} = \int d^4 l \frac{\langle lY\rangle}{\langle l12\rangle\langle l23\rangle\langle l34\rangle\langle l52\rangle\langle l51\rangle} \tag{174}$$

with $n - m = 4$ for manifest DCI, which holds for all $\mathcal{N} = 4$ SYM integrands. For the general n-point we have based on the P-V reduction and manifest DCI

$$I_n = \int d^4 l \frac{\langle lY_1\rangle\langle lY_1\rangle...\langle lY_m\rangle}{\langle l12\rangle\langle l23\rangle...\langle n1\rangle} \tag{175}$$

Let's now consider. It is important to note that the reduction to box-integrals is not valid at the level of the integrand. However, the reduction to boxes and pentagons is. The pentagons arising are however not scalar (as scalar pentagons would need a numerator factor of $lI_\infty\rangle$ to break the DCI), but the non-trivial numerator structures can be used to cure the IR divergences arising from the masslessness of $\mathcal{N}=4$ SYM. When an integral has the property that the residues associated to at least one of its Schubert problems are not the same, we say that the integral is chiral. Chiral integrals have a central role in certain 2-loop computations, such as the 2, 3-loop MHV and the 2-loop NMHV amplitudes. By making legs i and j massless, we can express the chiral pentagon again with line $l \sim (AB)$ and with numerator given by the intersection of $\mathbb{CP}^3 \supset \mathbb{CP}^2$ planes $\bar{i}$ and $\bar{j}$ as

$$\psi_1(i,j,I) = \int d^4l \frac{\langle l\bar{i}\cap\bar{j}\rangle\langle Iij\rangle}{\langle li-1i\rangle\langle lii+1\rangle\langle lj-1j\rangle\langle ljj+1\rangle\langle lI\rangle} \tag{176}$$

with reference dual point $(x_I) \sim I$. We can parametrise. We can expand the numerator with $l \sim (AB)$ so that the manifest dependence on the two planes is preserved at the cost of breaking the manifest dependence on the line (AB).

$$\langle l\bar{i}\cap\bar{j}\rangle = \langle Ai-1ii+1\rangle\langle Bj-1jj+1\rangle - \langle Bi-1ii+1\rangle\langle Aj-1jj+1\rangle \tag{177}$$

By defining $X = Z_{i-1} - \tau_X Z_{i+1}$ we can introduce the following parametrisation

$$\frac{1}{\langle li-1i\rangle\langle lii+1\rangle} = \int_0^\infty \frac{d\tau_X}{(\langle lii-1\rangle - \tau_X(\langle lii+1\rangle)^2} = \int_0^\infty d\tau_X \frac{1}{\langle liX\rangle^2} \tag{178}$$

By similarly defining $Y = Z_{j-1} - \tau_Y Z_{j+1}$ we can express (176) as

$$\int d^4l \int_0^\infty \frac{d\tau_X d\tau_Y \langle l\bar{i}\cap\bar{j}\rangle\langle Iij\rangle}{(\langle lii-1\rangle - \tau_X(\langle lii+1\rangle)^2(\langle ljj-1\rangle - \tau_Y(\langle ljj+1\rangle)^2\langle lI\rangle} = \int d^4l \int_0^\infty d^2\tau \frac{\langle l\bar{i}\cap\bar{j}\rangle\langle Iij\rangle}{\langle liX\rangle^2\langle ljY\rangle^2\langle lI\rangle} \tag{179}$$

Thus, so far

$$\psi_1(i,j,I) = \int d^4l \int_0^\infty d^2\tau \frac{\langle l\bar{i}\cap\bar{j}\rangle\langle Iij\rangle}{\langle liX\rangle^2\langle ljY\rangle^2\langle lI\rangle} \tag{180}$$

In order to rewrite the integrand using Feynman/Schwinger representations let's express the chiral pentagon in terms of embedding representation. With $w \sim I$ and Schubert problem solutions $z \sim (ij)$(from term $\langle Iij\rangle$) and $\bar{z} \sim (\bar{i}\cap\bar{j})$(from term $\langle l\bar{i}\cap\bar{j}\rangle$) we have the embedding space representation (with bold quantities $\in \mathcal{M}^{4,2}$)

$$\int d^4l \frac{(l-\bar{z})^2(w-z)^2}{(l-x)^4(l-y)^4(y-w)^2} \mapsto \int [d^4l] \frac{(\mathbf{l},\bar{\mathbf{z}})(\mathbf{w},\mathbf{z})}{(\mathbf{l},\mathbf{x})(\mathbf{l},\mathbf{x})(\mathbf{l},\mathbf{y})(\mathbf{l},\mathbf{y})(\mathbf{y},\mathbf{w})} \tag{181}$$

By introducing Feynman parameters α_1, α_2 and $\mathbf{w}$ being the "remainder" linear combination term we have

$$\int [d^4l] \frac{(\mathbf{l},\bar{\mathbf{z}})(\mathbf{w},\mathbf{z})}{(\mathbf{l},\mathbf{x})(\mathbf{l},\mathbf{x})(\mathbf{l},\mathbf{y})(\mathbf{l},\mathbf{y})(\mathbf{y},\mathbf{w})} = \int_0^\infty d\alpha_1 d\alpha_2 \int [d^4l] \frac{\alpha_1\alpha_2(\mathbf{l},\bar{\mathbf{z}})(\mathbf{w},\mathbf{z})}{(\mathbf{l}\cdot(\alpha_1\mathbf{x}+\alpha_2\mathbf{y}+\mathbf{w}))^5} \tag{182}$$

Defining $\mathbf{W} = \alpha_1 \mathbf{x} + \alpha_2 \mathbf{y} + \mathbf{w}$ and using constraints $(x - \bar{z})^2 = 0$,$(y - \bar{z})^2 = 0$ we obtain that $\bar{\mathbf{z}} \cdot (\alpha_1 \mathbf{x} + \alpha_2 \mathbf{y} + \mathbf{w}) = (\bar{\mathbf{z}}, W) = (\bar{\mathbf{z}}, w)$, so

$$\int_0^\infty d\alpha_1 d\alpha_2 \int [d^4 l] \frac{\alpha_1 \alpha_2 (\mathbf{l}, \bar{\mathbf{z}})(\mathbf{w}, \mathbf{z})}{(\mathbf{l}, \mathbf{W})^5} \sim \int_0^\infty d\alpha_1 d\alpha_2 \int [d^4 l] \bar{\mathbf{z}} \cdot \partial_{\mathbf{W}} \frac{\alpha_1 \alpha_2 (\mathbf{w}, \mathbf{z})}{(\mathbf{l}, \mathbf{W})^4} \sim \tag{183}$$

$$\sim \int_0^\infty d\alpha_1 d\alpha_2 \bar{\mathbf{z}} \cdot \partial_{\mathbf{W}} \frac{\alpha_1 \alpha_2 (\mathbf{w}, \mathbf{z})}{(\mathbf{W}, \mathbf{W})^2} \sim \int_0^\infty d\alpha_1 d\alpha_2 \frac{\alpha_1 \alpha_2 (\mathbf{w}, \mathbf{z})(\bar{\mathbf{z}}, \partial_{\mathbf{W}})}{(\mathbf{W}, \mathbf{W})^2} \tag{184}$$

Since

$$(\mathbf{W}, \mathbf{W}) = 2\alpha_1 \alpha_2 (\mathbf{x}, \mathbf{y}) + 2\alpha_1 (\mathbf{x}, \mathbf{w}) + 2\alpha_2 (\mathbf{y}, \mathbf{w}) \tag{185}$$