$\mathcal{N} = 4$ SYM Loop Notes

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1 loop review Let's consider a general 1-loop amplitude $\mathcal{A}_n^1$ with n external momenta p_i^μ $(i = 1, ..., n)$

$$\mathcal{A}_n^1 = \mathcal{A}_n^1(p_1, ..., p_n) \tag{1}$$

where the n external momenta have the constraints

$$p_i^\mu p_{\mu,i} = p_i^2 = 0 \tag{2}$$

$$\sum_i p_i^\mu = 0 \tag{3}$$

$\forall i = 1, ..., n$ Namely, (2) is the "on-shell" condition and (3) is the "conservation of momentum" relation. We can trivially satisfy (3) by introducing dual Minkowski coordinates (region variables) x_i^μ with $(i = 1, ..., n)$ such that

$$p_i^\mu = x_i^\mu - x_{i-1}^\mu = x_{i,i-1}^\mu \tag{4}$$

with $p_1 = x_1 - x_n$. We exploit the isomorphism of the Lie-algebras $\mathfrak{so}(3,1) \simeq \mathfrak{so}(4,2)$ to map an arbitrary x_i^μ to 6-d projective light cone

$$x_i^\mu \to X_i^A = (1, x^2, x_i^\mu) \tag{5}$$

On this projective light cone we use projective coordinates X^A $(A = 1, ..., 6)$, where the scalar product on the light cone is

$$X^A X_A = \eta_{AB} X^A X^B = -X^+ X^- + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 \tag{6}$$

where $X^+ = X^0 + X^1$ and $X^- = X^0 - X^1$ Thus, we have embedded the loop amplitude problem into 6-d projective space $SO(4,2)$. On this projective space we have the light-cone invariant

$$X^2 = 0 \tag{7}$$

Hence, due to the embedding we can express the following relation between dual Minkowski and projective coordinates

$$(x_i - x_j)^2 = (X_i - X_j)^2 = (X_i)^2 + (X_j)^2 - 2X_i X_j = -2X_i X_j \tag{8}$$

It is therefore convenient to introduce the product on the projective space

$$(X_i, X_j) = -2X_i X_j = (x_i - x_j)^2 \tag{9}$$

More importantly, we can therefore express products of momenta as products defined on the 6-d light cone projective space. For example, taking 4-momenta p_1^μ

$$p_2^2 = 0 \mapsto x_{21}^2 = (x_2 - x_1)^2 = 0 \mapsto (X_2, X_1) = 0 \tag{10}$$

Due to the on-shell constraint, we obtain the relations

$$(X_i, X_{i+1}) = (X_i, X_{i-1}) = 0 \tag{11}$$

Similarly, for a variable momentum l we can assign a dual Minkowski variable x_0 as $l = x_0 - x_4$ (to ensure momentum conservation). To the dual Minkowski representation of the loop momentum we then assign then a 6-d light cone coordinate, as previously. Thus, we map the products as

$$l^2 \mapsto x_{04}^2 \mapsto (X_0, X_4) \tag{12}$$

Generally,

$$l_i^2 \mapsto (x_0 - x_i)^2 = x_{0i}^2 \mapsto (X_0, X_i) \tag{13}$$

Given this embedding, we can express Feynman-integrals on the 6-d light cone projective space. The most general 1-loop n-point integral has the structure

$$I_n = \int \frac{d^4 l P(l)}{D_1 D_2 \cdots D_n} \tag{14}$$

with propagators $D_1, ..., D_n$, which for on shell external momenta exhibits conformal symmetry. Thus, due to the embedding formalism we can use a manifestly conformal representation of the 1-loop integral with a conformal integral defined on $SO(4,2)$.

n=4 (box) 1-loop scalar integral The amplitude has the structure

$$\mathcal{A}_4^1 = \mathcal{A}_4^1(p_1, p_2, p_3, p_4) \tag{15}$$

and arises at the first quantum correction of the 4-point interaction.

$$I_4 = \int d^4 l f(l) = \int \frac{d^4 l}{l^2 (l - p_1)^2 (l - p_1 - p_2)^2 (l + p_4)^2} \tag{16}$$

By transforming to dual Minkowski coordinates, we obtain an integal in the x_0 variable (the transformation is linear so has unit Jacobian)

$$\int d^4 l \mapsto \int d^4 x_0 \tag{17}$$

$$\Rightarrow I_4 = \int \frac{d^4 x_0}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{18}$$

We can now embed the integral in the 6-d light cone projective space. The conformal integal measure ($X_0^2 = 0$) is

$$\int d^4x_0 \mapsto \int \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} \tag{19}$$

Therefore, we can with (18) express any loop integral with conformal coordinates

$$\int d^4l f(l) = \int d^4x_0 f(x_0) = \int \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{20}$$

Consequently, the 4-pt box scalar integral after embedding in 6-d light cone projective space takes the form

$$I_4 = \int d^4l f(l) = \int \frac{d^4x_0}{x_{01}^2x_{02}^2x_{03}^2x_{04}^2} = \int \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} \frac{1}{(X_1,X_0)(X_2,X_0)(X_3,X_0)(X_4,X_0)} \tag{21}$$

So the integral has the structure

$$I_4 = \int \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{22}$$

Clearly, the integral is invariant under dilation $X_0 \to \lambda X_0$

$$\int \frac{\lambda^6 d^6X_0\delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} \frac{\lambda^4}{\lambda^4(X_1,X_0)(X_2,X_0)(X_3,X_0)(X_4,X_0)} \tag{23}$$

$$\implies \int \frac{\lambda^6 d^6X_0\delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{24}$$

Furthermore, the integral is invariant under inversion $X_0 \to \frac{X_0}{x^2}$.

$$\int \frac{d^6X_0\delta(\frac{X_0^2}{x^4})}{x^{12}\cdot Vol(GL(1))} f(\frac{X_0}{x^2}) = \int \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} \frac{x^8}{x^8\cdot(X_1,X_0)(X_2,X_0)(X_3,X_0)(X_4,X_0)} \tag{25}$$

$$\implies \int \frac{d^6X_0\delta(\frac{X_0^2}{x^4})}{x^{12}\cdot Vol(GL(1))} f(\frac{X_0}{x^2}) = \int \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{26}$$

So by embedding the integral in 6-d projective space, we have made the conformal invariance manifest. In other words, we have found a more natural representation that explicitly exhibits the conformal symmetry of the loop integral considered. For planar theories, we can decompose $n > 4$ loop integrals to a basis of the 4-point integrals discussed in this section. This basis decomposition of the integrand is possible because we have embedded the problem in a 6-d projective space, in which we can decompose a general object to 6 base elements with the appropriate coefficients. Since we are in 6-d, we can hence write an arbitrary $SO(4,2) \ni W$ as

$$W = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + rR \tag{27}$$

with $(X_i, R) = 0$ and $(R, R) = 1$. We can also define an anti-symmetric product on $SO(4,2)$ with

$$\langle X_iX_jX_kX_lX_mX_p\rangle = \epsilon^{ABCDFG}X_{iA}X_{jB}X_{kC}X_{lD}X_{mF}X_{pG} \tag{28}$$

where ϵ^{ABCDFG} is the totally anti-symmetric tensor.

n=5 (pentagonal) 1-loop scalar integral Let's consider a general 5-point 1-loop amplitude

$$I_5 = \int \frac{d^4lP(l)}{l^2(l-p_1)^2(l-p_1-p_2)^2(l+p_4+p_5)^2(l+p_5)^2} \tag{29}$$

By transforming to dual Minkowski coordinates with variable x_0 and general dual vector w we obtain the representation

$$I_5 = \int \frac{d^4x_0(x_0-w)^2}{x_{10}^2x_{20}^2x_{30}^2x_{40}^2x_{50}^2} \tag{30}$$

Similarly as before, we can embed to 6-d light cone projective space (using (5)) to make the integrand manifestly DCI

$$I_5 = \int \frac{d^4x_0(x_0-w)^2}{x_{10}^2x_{20}^2x_{30}^2x_{40}^2x_{50}^2} = \int \frac{d^6X_0(X_0,W)}{Vol(GL(1))(X_1,X_0)(X_2,X_0)(X_3,X_0)(X_4,X_0)(X_5,X_0)} \tag{31}$$

From (27) (sum over i is understood)

$$W = c_iX_i + rR \implies (X_0, W) = c_i(X_0, X_i) + r(X_0, R) \tag{32}$$

which gives

$$I_5 = \int \frac{d^6X_0}{Vol(GL(1))}\frac{c_i(X_0,X_i) + r(X_0,R)}{(X_1,X_0)(X_2,X_0)(X_3,X_0)(X_4,X_0)(X_5,X_0)} \tag{33}$$

The coefficients of the base expansion can be expressed using the anti-symmetric product. For example, we can get c_1 as

$$W = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + rR \tag{34}$$

$$\langle WX_2X_3X_4X_5R\rangle = \langle(c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + rR)X_2X_3X_4X_5\rangle \tag{35}$$

$$\langle WX_2X_3X_4X_5R\rangle = \langle c_1X_1X_2X_3X_4X_5R\rangle \implies c_1 = \frac{\langle WX_2X_3X_4X_5R\rangle}{\langle X_1X_2X_3X_4X_5R\rangle} \tag{36}$$

Similarly, for c_2

$$\langle WX_1X_3X_4X_5R\rangle = \langle c_2X_2X_1X_3X_4X_5R\rangle \implies c_2 = \frac{\langle WX_1X_3X_4X_5R\rangle}{\langle X_2X_1X_3X_4X_5R\rangle} \tag{37}$$