$\mathcal{N} = 4$ SYM Loop Notes

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1 loop review Let's consider a general 1-loop amplitude $\mathcal{A}_n^1$ with n external momenta p_i^μ $(i = 1, ..., n)$

$$\mathcal{A}_n^1 = \mathcal{A}_n^1(p_1, ..., p_n) \tag{1}$$

where the n external momenta have the constraints

$$p_i^\mu p_{\mu,i} = p_i^2 = 0 \tag{2}$$

$$\sum_i p_i^\mu = 0 \tag{3}$$

$\forall i = 1, ..., n$ Namely, (2) is the "on-shell" condition and (3) is the "conservation of momentum" relation. We can trivially satisfy (3) by introducing dual coordinates (region variables) x_i^μ with $(i = 1, ..., n)$ such that

$$p_i^\mu = x_i^\mu - x_{i-1}^\mu = x_{i,i-1}^\mu \tag{4}$$

with $p_1 = x_1 - x_n$. Let's now consider a d-dim Euclidean CFT (defined therefore on $\mathbb{R}^d$) with conformal group $SO(d + 1, 1)$ acting non-linearly on $\mathbb{R}^d$ (clearly $\mathbb{R} \ni x^\mu$). To linearise the action of the conformal group on $\mathbb{R}^d$ (Dirac) we consider the linear action of $SO(d + 1, 1)$ on the embedding space $\mathbb{R}^{d+1,1}$ by the embedding of x_i^μ to $\mathbb{R}^{d+1,1}$

$$x_i^\mu \to X_i^A = (X^+, X^-, x_i^\mu) \in \mathbb{R}^{d+1,1} \tag{5}$$

with inner product

$$X \cdot X = \eta_{AB} X^A X^B = -X^+ X^- + \eta_{\mu\nu} X^\mu X^\nu \tag{6}$$

with $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$ and $\eta_{\mu\mu} = 1$
For $d = 4$, the embedding space is $\mathbb{R}^{5,1}$ so X^A, $A = (1, ..., dim(\mathbb{R}^{5,1})) = (1, ..., 6)$, where $X^+ = X^0 + X^5$ and $X^- = X^0 - X^5$. The condition $X \cdot X = X^2 = 0$ defines an $SO(5, 1)$ invariant subspace of $5d$, the null-cone, or light-cone. Then, we obtain $\mathbb{R}^4$ by projectivising the light-cone, that is, by quotienting the light-cone by rescaling $X \sim \lambda X$ with $\mathbb{R} \ni \lambda$. Because projectivising respects Lorentz rotations of the embedding space $\mathbb{R}^{5,1}$, the projective null-cone naturally inherits an action of $SO(5, 1) \simeq SO(4, 2)$ on the original $\mathbb{R}^4$ spacetime.
By gauge-fixing the rescaling we can identify the original Eucledian spacetime $\mathbb{R}^d$ with

the projective light-cone. The gauge-fixing condition $X^+ = 1$ gives light-cone (projective subspace) vectors of the form

$$X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \tag{7}$$

($x^2 = x^\mu x_\mu$) In this gauge choice we can see that (not confusing the component of y with its squared length)

$$X \cdot Y = \begin{pmatrix} 1 \\ x^2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^T \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y^2 \\ y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = -\frac{1}{2}x^2 - \frac{1}{2}y^2 + x_\mu y^\mu \tag{8}$$

Trivially, we see that

$$X \cdot X = X^2 = -\frac{1}{2}x^2 - \frac{1}{2}x^2 + x_\mu x^\mu = -x^2 + x^2 = 0 \tag{9}$$

Moreover, for dual vectors x^μ, y^μ we can map their squared difference to the projective light-cone, giving

$$(x-y)^2 \to (X-Y)^2 = (X)^2 + (Y)^2 - 2X \cdot Y = -2X \cdot Y = x^2 + y^2 - 2x_\mu y^\mu = (x-y)^2 \tag{10}$$

It is therefore convenient to introduce the product on the projective space

$$(X_i, X_j) = -2X_i \cdot X_j = (x_i - x_j)^2 \tag{11}$$

Thus, we can express products of momenta as products of null vectors, that is vectos that are on the light-cone. For example, taking 4-momentum p_1^μ

$$p_2^2 = 0 \mapsto x_{21}^2 = (x_2 - x_1)^2 = 0 \mapsto (X_2, X_1) = 0 \tag{12}$$

Due to the on-shell constraint, we obtain the relations

$$(X_i, X_{i+1}) = (X_i, X_{i-1}) = 0 \tag{13}$$

Similarly, for a variable momentum l we can assign a dual variable x_0 as $l = x_0 - x_4$ (to ensure momentum conservation). To the dual representation of the loop momentum we then assign then a coordinate on the light-cone as previously. Thus, we map the products as

$$l^2 \mapsto x_{04}^2 \mapsto (X_0, X_4) \tag{14}$$

Generally,

$$l_i^2 \mapsto (x_0 - x_i)^2 = x_{0i}^2 \mapsto (X_0, X_i) \tag{15}$$

Given this embedding, we can express Feynman-integrals on the projective null-cone. The 1-loop n-point integral has the structure

$$I_n = \int \frac{d^4 l P(l)}{D_1 D_2 \cdots D_n} \tag{16}$$

with propagators $D_1, ..., D_n$, which for on shell external momenta exhibits conformal symmetry. Thus, due to the embedding formalism we can use a manifestly conformal representation of the 1-loop integral with a conformal integral defined on the embedding space.

n=4 (box) 1-loop scalar integral The amplitude has the structure

$$\mathcal{A}_4^1 = \mathcal{A}_4^1(p_1, p_2, p_3, p_4) \tag{17}$$

and arises at the first quantum correction of the 4-point boson interaction (with Mandelstam varaibles s, u).

$$I_4 = \int d^4 l f(l) = \int \frac{s u d^4 l}{l^2 (l - p_1)^2 (l - p_1 - p_2)^2 (l + p_4)^2} \tag{18}$$

By transforming to dual coordinates, we obtain an integral in the x_0 variable (the transformation is linear so has unit Jacobian)

$$\int d^4 l \mapsto \int d^4 x_0 \tag{19}$$

$$\Rightarrow I_4 = \int \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{20}$$

Under the dilation transformation

$$x_i^\mu \to \lambda x_i^\mu \tag{21}$$

(with $\lambda \in \mathbb{R}$)

$$\frac{(\lambda^4 d^4 x_0)(\lambda x_1 - \lambda x_3)^2 (\lambda x_2 - \lambda x_4)^2}{(\lambda x_0 - \lambda x_1)^2 (\lambda x_0 - \lambda x_2)^2 (\lambda x_0 - \lambda x_3)^2 (\lambda x_0 - \lambda x_4)^2} = \frac{\lambda^8 d^4 x_0 x_{13}^2 x_{24}^2}{\lambda^8 x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{22}$$

the integrand is invariant. Also, it is invariant under inversion transformation

$$x_i^\mu \to \frac{x_i^\mu}{x_i^\mu x_{\mu i}} = \frac{x_i^\mu}{x_i^2} \tag{23}$$

$$\frac{\frac{d^4 x_0}{x_0^8} \left(\frac{x_1}{x_1^2} - \frac{x_3}{x_3^2}\right)^2 \left(\frac{x_2}{x_2^2} - \frac{x_4}{x_4^2}\right)^2}{\left(\frac{x_0}{x_0^2} - \frac{x_1}{x_1^2}\right)^2 \left(\frac{x_0}{x_0^2} - \frac{x_2}{x_2^2}\right)^2 \left(\frac{x_0}{x_0^2} - \frac{x_3}{x_3^2}\right)^2 \left(\frac{x_0}{x_0^2} - \frac{x_4}{x_4^2}\right)^2} = \frac{\frac{d^4 x_0}{x_0^8} \frac{x_{13}^2 x_{24}^2}{x_1^2 x_2^2 x_3^2 x_4^2}}{\frac{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}{x_0^8 x_1^2 x_2^2 x_3^2 x_4^2}} = \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{24}$$

So, the integrand in the dual coordinate representation exhibits conformal invariance. We can therefore use the embedding formalism to embed the integral on the light cone. The conformal integral measure ($X_0^2 = 0$) becomes

$$\int d^4 l f(l) \mapsto \int d^4 x_0 f(x_0) \mapsto \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{25}$$

Consequently, the 4-pt box scalar integral after embedding on the projective light cone takes the form

$$I_4 = \int d^4 l f(l) = \int \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} \frac{(X_1, X_3)(X_2, X_4)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \tag{26}$$

So the integral has the structure

$$I_4 = \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{27}$$

Clearly, the integral is invariant under dilation $X_0 \to \lambda X_0$

$$\int \frac{\lambda^6 d^6 X_0 \delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} \frac{\lambda^4}{\lambda^4 (X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \tag{28}$$

$$\implies \int \frac{\lambda^6 d^6 X_0 \delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{29}$$

So by embedding the integral on the projective light-cone, we have made the conformal invariance manifest. In other words, we have found a more natural representation that explicitly exhibits the conformal symmetry of the loop integral considered. For planar theories, we can decompose $n > 4$ loop integrals to a basis of the 4-point integrals discussed in this section. Moreover we can also define an anti-symmetric product on the projective light-cone with

$$\langle X_i X_j X_k X_l X_m X_p \rangle = \varepsilon^{ABCDFG} X_{iA} X_{jB} X_{kC} X_{lD} X_{mF} X_{pG} \tag{30}$$

where ϵ^{ABCDFG} is the totally anti-symmetric tensor. This will be used in the next section, for the reduction of the $n = 5$ (and above) 1-loop case.

n=5 (pentagonal) 1-loop integral Let's consider a general 5-point 1-loop amplitude. By transforming to dual coordinates with variable x_0 and general dual vector w we obtain the representation

$$I_5 = \int \frac{d^4 x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \tag{31}$$

Similarly as before, we can embed to the projective light-cone. Since we are in $6d$ we can hence write an arbitrary $\mathbb{R}^{5,1} \ni W$ as

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR = c_i X_i + rR \tag{32}$$

with $(X_i, R) = 0$ and $(R, R) = 1$. Then

$$I_5 = \int \frac{d^4x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \mapsto \mathcal{I}_5 = \int \frac{d^6 X_0 (X_0, W)}{Vol(GL(1))(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \tag{33}$$

So

$$\mathcal{I}_5 = \int \frac{d^6 X_0}{Vol(GL(1))} \frac{c_i(X_0, X_i) + r(X_0, R)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \tag{34}$$

The coefficients of the base expansion can be expressed using the anti-symmetric product. For example, we can get c_1 as

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR \tag{35}$$

$$\langle W X_2 X_3 X_4 X_5 R \rangle = \langle (c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR) X_2 X_3 X_4 X_5 \rangle \tag{36}$$

$$\langle W X_2 X_3 X_4 X_5 R \rangle = \langle c_1 X_1 X_2 X_3 X_4 X_5 R \rangle \implies c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R \rangle} \tag{37}$$

Similarly, for c_2

$$\langle W X_1 X_3 X_4 X_5 R \rangle = \langle c_2 X_2 X_1 X_3 X_4 X_5 R \rangle \implies c_2 = \frac{\langle W X_1 X_3 X_4 X_5 R \rangle}{\langle X_2 X_1 X_3 X_4 X_5 R \rangle} \tag{38}$$

Let's consider the c_1 term further with the definition of the anti-symmetric product

$$c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R \rangle} = \frac{\langle W X_2 X_3 X_4 X_5 R \rangle \langle X_1 X_2 X_3 X_4 X_5 R \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R \rangle \langle X_1 X_2 X_3 X_4 X_5 R \rangle} = \tag{39}$$

$$= \frac{\varepsilon^{ABCDFG} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_G \varepsilon_{PQRSTV} X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V}{\varepsilon^{HJKLMN} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N \varepsilon_{PQRSTV} X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V} \tag{40}$$

Using the product identity for the $6d$ Levi-Civita tensors

$$\varepsilon^{ABCDFG} \varepsilon_{PQRSTV} = \delta^{ABCDFG}_{PQRSTV} = \begin{vmatrix} \delta^A_P & \delta^A_Q & \delta^A_R & \delta^A_S & \delta^A_T & \delta^A_V \\ \delta^B_P & \delta^B_Q & \delta^B_R & \delta^B_S & \delta^B_T & \delta^B_V \\ \delta^C_P & \delta^C_Q & \delta^C_R & \delta^C_S & \delta^C_T & \delta^C_V \\ \delta^D_P & \delta^D_Q & \delta^D_R & \delta^D_S & \delta^D_T & \delta^D_V \\ \delta^F_P & \delta^F_Q & \delta^F_R & \delta^F_S & \delta^F_T & \delta^F_V \\ \delta^G_P & \delta^G_Q & \delta^G_R & \delta^G_S & \delta^G_T & \delta^G_V \end{vmatrix} \tag{41}$$

$$\tag{42}$$