$\mathcal{N} = 4$ SYM Loop Notes

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Updated Date: 26.05.2022
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1 loop review Let's consider a general 1-loop amplitude $\mathcal{A}_n^1$ with n external momenta p_i^μ $(i = 1, ..., n)$

$$\mathcal{A}_n^1 = \mathcal{A}_n^1(p_1, ..., p_n) \tag{1}$$

where the n external momenta have the constraints

$$p_i^\mu p_{\mu,i} = p_i^2 = 0 \tag{2}$$

$$\sum_i p_i^\mu = 0 \tag{3}$$

$\forall i = 1, ..., n$ Namely, (2) is the "on-shell" condition and (3) is the "conservation of momentum" relation. We can trivially satisfy (3) by introducing dual coordinates (region variables) x_i^μ with $(i = 1, ..., n)$ such that

$$p_i^\mu = x_i^\mu - x_{i-1}^\mu = x_{i,i-1}^\mu \tag{4}$$

with $p_1 = x_1 - x_n$. Let's now consider a d-dim Minkowskian CFT (defined therefore on $\mathcal{M}^d$) with conformal group $SO(d, 2)$ acting non-linearly on $\mathcal{M}^d$ (clearly $\mathcal{M}^d \ni x^\mu$). To linearise the action of the conformal group on $\mathcal{M}^d$ (Dirac) we consider the linear action of $SO(d, 2)$ on the embedding space, which is a Minkowski spacetime with signature $(d, 2)$ that we denote by $\mathcal{M}^{d,2}$. By the embedding of $\mathcal{M}^d \ni x_i^\mu$ to $\mathcal{M}^{d,2}$ we mean

$$x_i^\mu \to X_i^A = (X^+, X^-, x_i^\mu) \in \mathcal{M}^{d,2} \tag{5}$$

with inner product

$$X \cdot X = \eta_{AB} X^A X^B = -X^+ X^- + \eta_{\mu\nu} X^\mu X^\nu \tag{6}$$

with $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$, $\eta_{00} = -1$ and $\eta_{ii} = 1$. This is done, so that $SO(d, 2)$ acts linearly on $\mathcal{M}^{d,2}$, in contrast to its non-linear action on $\mathcal{M}^d \ni x_i^\mu$ dual Minkowski coordinates. For $d = 4$, the embedding space is a $\dim(\mathcal{M}^{d,2}) = 6$ Minkowski-space with signature $(4, 2)$ so X^A, $A = (1, ..., \dim(\mathcal{M}^{d,2})) = (1, ..., 6)$, where $X^+ = X^0 + X^5$ and $X^- = X^0 - X^5$. The condition $X \cdot X = X^2 = 0$ defines an $SO(4, 2)$ invariant subspace of $5d$, the null-cone, or light-cone. Then, we obtain $\mathcal{M}^4$ by projectivising the light-cone, that is, by quotienting the light-cone by rescaling $X \sim \lambda X$ with $\mathbb{R} \ni \lambda$. Because projectivising respects Lorentz rotations of the embedding space $\mathcal{M}^{4,2}$, the projective null-cone naturally inherits an action of $SO(4, 2)$ on the original $\mathcal{M}^4$ spacetime.

By gauge-fixing the rescaling we can identify the original Minkowski spacetime $\mathcal{M}^4$ with the projective light-cone. The gauge-fixing condition $X^+ = 1$ gives light-cone (projective subspace) vectors of the form

$$X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \tag{7}$$

($x^2 = x^\mu x_\mu$) In this gauge choice we can see that (not confusing the component of y with its squared length)

$$X \cdot Y = \begin{pmatrix} 1 \\ x^2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^T \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y^2 \\ y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = -\frac{1}{2}x^2 - \frac{1}{2}y^2 + x_\mu y^\mu \tag{8}$$

Trivially, we see that

$$X \cdot X = X^2 = -\frac{1}{2}x^2 - \frac{1}{2}x^2 + x_\mu x^\mu = -x^2 + x^2 = 0 \tag{9}$$

Moreover, for dual vectors x^μ, y^μ we can map their squared difference to the projective light-cone, giving

$$(x-y)^2 \to (X-Y)^2 = (X)^2 + (Y)^2 - 2X \cdot Y = -2X \cdot Y = x^2 + y^2 - 2x_\mu y^\mu = (x-y)^2 \tag{10}$$

It is therefore convenient to introduce the product on the projective space

$$(X_i, X_j) = -2X_i \cdot X_j = (x_i - x_j)^2 \tag{11}$$

Thus, we can express products of momenta as products of null vectors, that is vectos that are on the light-cone. For example, taking 4-momentum p_1^μ

$$p_2^2 = 0 \mapsto x_{21}^2 = (x_2 - x_1)^2 = 0 \mapsto (X_2, X_1) = 0 \tag{12}$$

Due to the on-shell constraint, we obtain the relations

$$(X_i, X_{i+1}) = (X_i, X_{i-1}) = 0 \tag{13}$$

Similarly, for a variable momentum l we can assign a dual variable x_0 as $l = x_0 - x_4$ (to ensure momentum conservation). To the dual representation of the loop momentum we then assign then a coordinate on the light-cone as previously. Thus, we map the products as

$$l^2 \mapsto x_{04}^2 \mapsto (X_0, X_4) \tag{14}$$

Generally,

$$l_i^2 \mapsto (x_0 - x_i)^2 = x_{0i}^2 \mapsto (X_0, X_i) \tag{15}$$

Given this embedding, we can express Feynman-integrals on the projective null-cone. The 1-loop n-point integral has the structure

$$I_n = \int \frac{d^4 l P(l)}{D_1 D_2 \cdots D_n} \tag{16}$$

with propagators $D_1, ..., D_n$, which for on shell external momenta exhibits conformal symmetry. Thus, due to the embedding formalism we can use a manifestly conformal representation of the 1-loop integral with a conformal integral defined on the embedding space.

n=4 (box) 1-loop scalar integral The amplitude has the structure

$$\mathcal{A}_4^1 = \mathcal{A}_4^1(p_1, p_2, p_3, p_4) \tag{17}$$

and arises at the first quantum correction of the 4-point boson interaction (with Mandelstam variables s, t).

$$I_4 = \int d^4 l f(l) = \int \frac{st d^4 l}{l^2 (l - p_1)^2 (l - p_1 - p_2)^2 (l + p_4)^2} \tag{18}$$

By transforming to dual coordinates, we obtain an integral in the x_0 variable (the transformation is linear so has unit Jacobian)

$$\int d^4 l \mapsto \int d^4 x_0 \tag{19}$$

$$\Rightarrow I_4 = \int \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{20}$$

Under the dilation transformation

$$x_i^\mu \to \lambda x_i^\mu \tag{21}$$

(with $\lambda \in \mathbb{R}$)

$$\frac{(\lambda^4 d^4 x_0)(\lambda x_1 - \lambda x_3)^2 (\lambda x_2 - \lambda x_4)^2}{(\lambda x_0 - \lambda x_1)^2 (\lambda x_0 - \lambda x_2)^2 (\lambda x_0 - \lambda x_3)^2 (\lambda x_0 - \lambda x_4)^2} = \frac{\lambda^8 d^4 x_0 x_{13}^2 x_{24}^2}{\lambda^8 x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{22}$$

the integrand is invariant. Also, it is invariant under inversion transformation

$$x_i^\mu \to \frac{x_i^\mu}{x_i^\mu x_{\mu i}} = \frac{x_i^\mu}{x_i^2} \tag{23}$$

$$\frac{\frac{d^4 x_0}{x_0^8} \left(\frac{x_1}{x_1^2} - \frac{x_3}{x_3^2}\right)^2 \left(\frac{x_2}{x_2^2} - \frac{x_4}{x_4^2}\right)^2}{\left(\frac{x_0}{x_0^2} - \frac{x_1}{x_1^2}\right)^2 \left(\frac{x_0}{x_0^2} - \frac{x_2}{x_2^2}\right)^2 \left(\frac{x_0}{x_0^2} - \frac{x_3}{x_3^2}\right)^2 \left(\frac{x_0}{x_0^2} - \frac{x_4}{x_4^2}\right)^2} = \frac{\frac{d^4 x_0}{x_0^8} \frac{x_{13}^2 x_{24}^2}{x_1^2 x_2^2 x_3^2 x_4^2}}{\frac{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}{x_0^8 x_1^2 x_2^2 x_3^2 x_4^2}} = \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{24}$$

So, the integrand in the dual coordinate representation exhibits conformal invariance. We can therefore use the embedding formalism to embed the integral on the light cone. The conformal integral measure ($X_0^2 = 0$) becomes

$$\int d^4 l f(l) \mapsto \int d^4 x_0 f(x_0) \mapsto \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{25}$$

Consequently, the 4-pt box scalar integral after embedding on the projective light cone takes the form

$$I_4 = \int d^4 l f(l) = \int \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} \frac{(X_1, X_3)(X_2, X_4)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \tag{26}$$

So the integral has the structure

$$I_4 = \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{27}$$

Clearly, the integral is invariant under dilation $X_0 \to \lambda X_0$

$$\int \frac{\lambda^6 d^6 X_0 \delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} \frac{\lambda^4}{\lambda^4 (X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \tag{28}$$

$$\Longrightarrow \int \frac{\lambda^6 d^6 X_0 \delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6 X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{29}$$

So by embedding the integral on the projective light-cone, we have made the conformal invariance manifest. In other words, we have found a more natural representation that explicitly exhibits the conformal symmetry of the loop integral considered. For planar theories, we can decompose $n > 4$ loop integrals to a basis of the 4-point integrals discussed in this section. Moreover we can also define an anti-symmetric product on the projective light-cone with

$$\langle X_i X_j X_k X_l X_m X_p \rangle = \varepsilon^{ABCDFG} X_{iA} X_{jB} X_{kC} X_{lD} X_{mF} X_{pG} \tag{30}$$

where ϵ^{ABCDFG} is the totally anti-symmetric tensor. This will be used in the next section, for the reduction of the $n = 5$ (and above) 1-loop case.

n=5 (pentagonal) 1-loop integral Let's consider a general 5-point 1-loop amplitude. By transforming to dual coordinates with variable x_0 and general dual vector w we obtain the representation

$$I_5 = \int \frac{d^4 x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \tag{31}$$

Similarly as before, we can embed to the projective light-cone. Since we are in $6d$ we can hence write an arbitrary $\mathcal{M}^{4,2} \ni W$ as

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR = c_i X_i + rR \tag{32}$$

with $(X_i, R) = 0$ and $(R, R) = 1$. Then

$$I_5 = \int \frac{d^4 x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \mapsto \mathcal{I}_5 = \int \frac{d^6 X_0 (X_0, W)}{Vol(GL(1))(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \tag{33}$$

So

$$\mathcal{I}_5 = \int \frac{d^6 X_0}{Vol(GL(1))} \frac{c_i(X_0, X_i) + r(X_0, R)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \tag{34}$$

Remarkably, the (X_0, R) term does not contribute. This can be seen using the method of Feynman/Schwinger parametrisation

$$\int [d^4 X_0] \frac{r(X_0, R)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} = \tag{35}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 \frac{[d^4 X_0] r(X_0, R)}{((X_1, X_0) + \alpha_2 (X_2, X_0) + \alpha_3 (X_3, X_0) + \alpha_4 (X_4, X_0) + \alpha_5 (X_5, X_0))^5} \tag{36}$$

So

$$\Pi_{i=2}^5 (\int_0^\infty d\alpha_i) \int [d^4 X_0] \frac{(X_0, R)}{((W, X_0))^5} \sim \Pi_{i=2}^5 (\int_0^\infty d\alpha_i) \int [d^4 X_0] (rR) \partial_w \frac{1}{((W, X_0))^4} \sim \tag{37}$$

$$\sim \Pi_{i=2}^5 (\int_0^\infty d\alpha_i)(rR) \partial_w \frac{1}{((W, W))^2} \tag{38}$$

But $(W, W) = (R, R) = 1$, thus

$$\Pi_{i=2}^5 (\int_0^\infty d\alpha_i)(rR) \partial_w \frac{1}{((W, W))^2} = \Pi_{i=2}^5 (\int_0^\infty d\alpha_i)(rR) \partial_w \frac{1}{1} = 0 \tag{39}$$

The coefficients of the base expansion can be expressed using the anti-symmetric product. For example, we can get c_1 as

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR \tag{40}$$

$$\langle W X_2 X_3 X_4 X_5 R \rangle = \langle (c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR) X_2 X_3 X_4 X_5 \rangle \tag{41}$$

$$\langle W X_2 X_3 X_4 X_5 R \rangle = \langle c_1 X_1 X_2 X_3 X_4 X_5 R \rangle \implies c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R \rangle} \tag{42}$$

Similarly, for c_2

$$\langle W X_1 X_3 X_4 X_5 R \rangle = \langle c_2 X_2 X_1 X_3 X_4 X_5 R \rangle \implies c_2 = \frac{\langle W X_1 X_3 X_4 X_5 R \rangle}{\langle X_2 X_1 X_3 X_4 X_5 R \rangle} \tag{43}$$

Let's consider the c_1 term further with the definition of the anti-symmetric product

$$c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R\rangle}{\langle X_1 X_2 X_3 X_4 X_5 R\rangle} = \frac{\langle W X_2 X_3 X_4 X_5 R\rangle \langle X_1 X_2 X_3 X_4 X_5 R\rangle}{\langle X_1 X_2 X_3 X_4 X_5 R\rangle \langle X_1 X_2 X_3 X_4 X_5 R\rangle} = \tag{44}$$

$$= \frac{\varepsilon^{ABCDFG} \varepsilon_{PQRSTV} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_G X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V}{\varepsilon^{HJKLMN} \varepsilon_{PQRSTV} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V} \tag{45}$$

Using the product identity for the $6d$ Levi-Civita tensor

$$\varepsilon^{ABCDFG} \varepsilon_{PQRSTV} = \delta^{ABCDFG}_{PQRSTV} = \begin{vmatrix} \delta^A_P & \delta^A_Q & \delta^A_R & \delta^A_S & \delta^A_T & \delta^A_V \\ \delta^B_P & \delta^B_Q & \delta^B_R & \delta^B_S & \delta^B_T & \delta^B_V \\ \delta^C_P & \delta^C_Q & \delta^C_R & \delta^C_S & \delta^C_T & \delta^C_V \\ \delta^D_P & \delta^D_Q & \delta^D_R & \delta^D_S & \delta^D_T & \delta^D_V \\ \delta^F_P & \delta^F_Q & \delta^F_R & \delta^F_S & \delta^F_T & \delta^F_V \\ \delta^G_P & \delta^G_Q & \delta^G_R & \delta^G_S & \delta^G_T & \delta^G_V \end{vmatrix} \tag{46}$$

$$\tag{47}$$

we obtain

$$c_1 = \frac{\begin{vmatrix} \delta^A_P & \delta^A_Q & \delta^A_R & \delta^A_S & \delta^A_T & \delta^A_V \\ \delta^B_P & \delta^B_Q & \delta^B_R & \delta^B_S & \delta^B_T & \delta^B_V \\ \delta^C_P & \delta^C_Q & \delta^C_R & \delta^C_S & \delta^C_T & \delta^C_V \\ \delta^D_P & \delta^D_Q & \delta^D_R & \delta^D_S & \delta^D_T & \delta^D_V \\ \delta^F_P & \delta^F_Q & \delta^F_R & \delta^F_S & \delta^F_T & \delta^F_V \\ \delta^G_P & \delta^G_Q & \delta^G_R & \delta^G_S & \delta^G_T & \delta^G_V \end{vmatrix} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_G X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V}{\begin{vmatrix} \delta^H_P & \delta^H_Q & \delta^H_R & \delta^H_S & \delta^H_T & \delta^H_V \\ \delta^J_P & \delta^J_Q & \delta^J_R & \delta^J_S & \delta^J_T & \delta^J_V \\ \delta^K_P & \delta^K_Q & \delta^K_R & \delta^K_S & \delta^K_T & \delta^K_V \\ \delta^L_P & \delta^L_Q & \delta^L_R & \delta^L_S & \delta^L_T & \delta^L_V \\ \delta^M_P & \delta^M_Q & \delta^M_R & \delta^M_S & \delta^M_T & \delta^M_V \\ \delta^N_P & \delta^N_Q & \delta^N_R & \delta^N_S & \delta^N_T & \delta^N_V \end{vmatrix} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V} = \tag{48}$$

$$= \frac{\begin{vmatrix} W\cdot X_1 & W\cdot X_2 & W\cdot X_3 & W\cdot X_4 & W\cdot X_5 & W\cdot R \\ X_2\cdot X_1 & X_2\cdot X_2 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R \\ X_3\cdot X_1 & X_3\cdot X_2 & X_3\cdot X_3 & X_3\cdot X_4 & X_3\cdot X_1 & X_3\cdot R \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & X_4\cdot X_4 & X_4\cdot X_5 & X_4\cdot R \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & X_5\cdot X_5 & X_5\cdot R \\ R\cdot X_1 & R\cdot X_2 & R\cdot X_3 & R\cdot X_4 & R\cdot X_5 & R\cdot R \end{vmatrix}}{\begin{vmatrix} X_1\cdot X_1 & X_1\cdot X_2 & X_1\cdot X_3 & X_1\cdot X_4 & X_1\cdot X_5 & X_1\cdot R \\ X_2\cdot X_1 & X_2\cdot X_2 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R \\ X_3\cdot X_1 & X_3\cdot X_2 & X_3\cdot X_3 & X_3\cdot X_4 & X_3\cdot X_1 & X_3\cdot R \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & X_4\cdot X_4 & X_4\cdot X_5 & X_4\cdot R \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & X_5\cdot X_5 & X_5\cdot R \\ R\cdot X_1 & R\cdot X_2 & R\cdot X_3 & R\cdot X_4 & R\cdot X_5 & R\cdot R \end{vmatrix}} = \tag{49}$$

$$= \frac{\begin{vmatrix} W\cdot X_1 & W\cdot X_2 & W\cdot X_3 & W\cdot X_4 & W\cdot X_5 & W\cdot R \\ X_2\cdot X_1 & 0 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R \\ X_3\cdot X_1 & X_3\cdot X_2 & 0 & X_3\cdot X_4 & X_3\cdot X_1 & X_3\cdot R \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & 0 & X_4\cdot X_5 & X_4\cdot R \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & 0 & X_5\cdot R \\ R\cdot X_1 & R\cdot X_2 & R\cdot X_3 & R\cdot X_4 & R\cdot X_5 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & X_1\cdot X_2 & X_1\cdot X_3 & X_1\cdot X_4 & X_1\cdot X_5 & X_1\cdot R \\ X_2\cdot X_1 & 0 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R \\ X_3\cdot X_1 & X_3\cdot X_2 & 0 & X_3\cdot X_4 & X_3\cdot X_1 & X_3\cdot R \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & 0 & X_4\cdot X_5 & X_4\cdot R \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & 0 & X_5\cdot R \\ R\cdot X_1 & R\cdot X_2 & R\cdot X_3 & R\cdot X_4 & R\cdot X_5 & 1 \end{vmatrix}} \tag{50}$$

We can obtain the other coefficients similarly. Before continuing to higher-external leg cases, we introduce the following notation for the integration measure in embedding space

$$[d^4X_0] = \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} \tag{51}$$

n=6 (hexagonal) 1-loop integral The $n = 6$ 1-loop integral expressed in terms of embedding space coordinates $\mathcal{M}^{d,2} \ni X^A$ has the form

$$\mathcal{I}_6 = \int \frac{[d^4X_0]T_{AB}X_0^AX_0^B}{\Pi_{i=1}^6(X_0,X_i)} = \int \frac{[d^4x_0]T_{AB}X_0^AX_0^B}{(X_0,X_1)(X_0,X_2)(X_0,X_3)(X_0,X_4)(X_0,X_5)(X_0,X_6)} \tag{52}$$

By introducing $\mathcal{M}^{d,2} \ni W_1, W_2$, with

$$\mathcal{F}_6(W_1,W_2,X_i) = \frac{(W_1,X_0)(W_2,X_0)}{(X_0,X_1)(X_0,X_2)(X_0,X_3)(X_0,X_4)(X_0,X_5)(X_0,X_6)} \tag{53}$$

we see that we can obtain the integrand of $\mathcal{I}_6$ with differentiation of F_6 and contraction with T_{AB}

$$T_{AB}\frac{\partial}{\partial W_1^A}\frac{\partial}{\partial W_2^B}F_6 = \frac{T_{AB}X_0^AX_0^B}{(X_0,X_1)(X_0,X_2)(X_0,X_3)(X_0,X_4)(X_0,X_5)(X_0,X_6)} \tag{54}$$

Thus

$$\mathcal{I}_6 = \int [d^4X_0]T_{AB}\frac{\partial}{\partial W_1^A}\frac{\partial}{\partial W_2^B}\mathcal{F}_6 = \int \frac{[d^4X_0]T_{AB}X_0^AX_0^B}{(X_0,X_1)(X_0,X_2)(X_0,X_3)(X_0,X_4)(X_0,X_5)(X_0,X_6)} \tag{55}$$

n=7 (heptagonal) 1-loop integral According to the previous ($n = 6$) example, we have the integral embedded to $\mathcal{M}^{4,2}$ of the form

$$\mathcal{I}_7 = \int \frac{[d^4X_0]T_{ABC}X_0^AX_0^BX_0^C}{\Pi_{i=1}^7(X_0,X_i)} = \int \frac{[d^4x_0]T_{ABC}X_0^AX_0^BX_0^C}{(X_0,X_1)(X_0,X_2)(X_0,X_3)(X_0,X_4)(X_0,X_5)(X_0,X_6)(X_0,X_7)} \tag{56}$$

By introducing $\mathcal{M}^{d,2} \ni W_1, W_2, W_3$, with

$$\mathcal{F}_7(W_1,W_2,W_3,X_i) = \frac{(W_1,X_0)(W_2,X_0)(W_3,X_0)}{(X_0,X_1)(X_0,X_2)(X_0,X_3)(X_0,X_4)(X_0,X_5)(X_0,X_6)} \tag{57}$$

the integrand is then obtained similarly T_{ABC}

$$T_{ABC}\frac{\partial}{\partial W_1^A}\frac{\partial}{\partial W_2^B}\frac{\partial}{\partial W_3^C} = \frac{T_{ABC}X_0^AX_0^BX_0^C}{(X_0,X_1)(X_0,X_2)(X_0,X_3)(X_0,X_4)(X_0,X_5)(X_0,X_6)(X_0,X_7)} \tag{58}$$

Thus

$$\mathcal{I}_7 = \int [d^4X_0]T_{ABC}\frac{\partial}{\partial W_1^A}\frac{\partial}{\partial W_2^B}\frac{\partial}{\partial W_3^C}\mathcal{F}_7 = \int \frac{[d^4X_0]T_{AB}X_0^AX_0^BX_0^C}{\Pi_{i=1}^7(X_0,X_i)} \tag{59}$$

General n external leg (n-gon) 1-loop integral Generalising the previous discussions, we can consider the case of an n-gon, so n external leg 1-loop integral. By embedding the integral to $\mathcal{M}^{4,2}$ we obtain

$$\mathcal{I}_n = \int \frac{[d^4X_0]T_{a_1a_2\cdots a_{n-4}}X_0^{a_1}X_0^{a_2}\cdots X_0^{a_{n-4}}}{\Pi_{i=1}^n(X_0,X_i)} \tag{60}$$

By introducing $\mathcal{M}^{4,2} \ni W_1, W_2, \cdots, W_{n-4}$, with

$$\mathcal{F}_n(W_1,W_2,\cdots,W_{n-4},X_i) = \frac{(W_1,X_0)(W_2,X_0)\cdots(W_{n-4},X_0)}{\Pi_{i=1}^n(X_0,X_i)} \tag{61}$$

the integrand is then obtained similarly T_{ABC}

$$T_{a_1a_2\cdots a_{n-4}}\frac{\partial}{\partial W_1^{a_1}}\frac{\partial}{\partial W_2^{a_2}}\cdots\frac{\partial}{\partial W_{n-4}^{a_{n-4}}}\mathcal{F}_n = \frac{T_{a_1a_2\cdots a_{n-4}}X_0^{a_1}X_0^{a_2}\cdots X_0^{a_{n-4}}}{\Pi_{i=1}^n(X_0,X_i)} \tag{62}$$

Thus

$$\mathcal{I}_n = \int [d^4X_0]T_{a_1a_2\cdots a_{n-4}}\frac{\partial}{\partial W_1^{a_1}}\frac{\partial}{\partial W_2^{a_2}}\cdots\frac{\partial}{\partial W_{n-4}^{a_{n-4}}}\mathcal{F}_n = \int \frac{[d^4X_0]T_{a_1a_2\cdots a_{n-4}}X_0^{a_1}X_0^{a_2}\cdots X_0^{a_{n-4}}}{\Pi_{i=1}^n(X_0,X_i)} \tag{63}$$

Momentum-space twistors We have seen that for an $\mathcal{M}^{4,2} \ni X$

$$X\cdot X = \eta_{AB}X^AX^B = -X^+X^- + \eta_{\mu\nu}X^\mu X^\nu = -(X^0)^2+(X^5)^2-(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2 = 0 \tag{64}$$

In this way we describe the conformally compactified Minkowski space with $\mathbb{RP}^5$ coordinates, so (59) defines an equation in $\mathbb{RP}^5$. After complexification of space-time, the coordinates representing $\mathcal{M}^{4,2} \ni X$ are in $\mathbb{CP}^3$. Since $SO(4,2) \simeq SU(2,2) \simeq SU(4)$, we can choose the anti-symmetric tensor representation of $SU(4)$. By doing this we can package the compactified space-time coordinates in $X_{\alpha\beta} = -X_{\beta\alpha}$ $(\alpha,\beta = 0,1,2,3)$ and express (59) as

$$\varepsilon^{\alpha\beta\gamma\delta}X_{\alpha\beta}X_{\gamma\delta} = 0 \tag{65}$$

So for an arbitrary X satisfying (60)

$$\iff X_{\alpha\beta} = A_{[\alpha}B_{\beta]} \tag{66}$$

for $\mathbb{T} \ni A, B$ (A, B in twistor space, which can be regarded as a copy of $\mathbb{CP}^3$). In this way we can represent a point in conformally compactified and complexified Minkowski space with twistors A, B. More specifically we can represent a point in comformally compactified spacetime with a line in $\mathbb{CP}^3$ defined by twistors A, B (61). On the other hand, given an arbitrary $\mathbb{CP}^3 \ni W_\alpha$ that is on the line $A \wedge B$

$$X_{[\alpha\beta}W_{\gamma]} = 0 \tag{67}$$

which thus means that W_α is a linear combination of A_α, B_α. Therefore, points of conformally compactified, complexified space-time correspond to holomorphic lines $\mathbb{CP}^1 \subset \mathbb{CP}^3$ defined by the twistor pair (Z_A, Z_B)