
$\mathcal{N} = 4$ SYM Loop Notes

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1 loop review Let's consider a general 1-loop amplitude \mathcal{A}_n^1 with n external momenta p_i^μ ($i = 1, \dots, n$)

$$\mathcal{A}_n^1 = \mathcal{A}_n^1(p_1, \dots, p_n) \quad (1)$$

where the n external momenta have the constraints

$$p_i^\mu p_{\mu,i} = p_i^2 = 0 \quad (2)$$

$$\sum_i p_i^\mu = 0 \quad (3)$$

$\forall i = 1, \dots, n$ Namely, (2) is the "on-shell" condition and (3) is the "conservation of momentum" relation. We can trivially satisfy (3) by introducing dual coordinates (region variables) x_i^μ with ($i = 1, \dots, n$) such that

$$p_i^\mu = x_i^\mu - x_{i-1}^\mu = x_{i,i-1}^\mu \quad (4)$$

with $p_1 = x_1 - x_n$. Let's now consider a d -dim Minkowskian CFT (defined therefore on \mathcal{M}^d) with conformal group $SO(d, 2)$ acting non-linearly on \mathcal{M}^d (clearly $\mathcal{M}^d \ni x^\mu$). To linearise the action of the conformal group on \mathcal{M}^d (Dirac) we consider the linear action of $SO(d, 2)$ on the embedding space, which is a Minkowski spacetime with signature $(d, 2)$ that we denote by $\mathcal{M}^{d,2}$. By the embedding of $\mathcal{M}^d \ni x_i^\mu$ to $\mathcal{M}^{d,2}$ we mean

$$x_i^\mu \rightarrow X_i^A = (X^+, X^-, x_i^\mu) \in \mathcal{M}^{d,2} \quad (5)$$

with inner product

$$X \cdot X = \eta_{AB} X^A X^B = -X^+ X^- + \eta_{\mu\nu} X^\mu X^\nu \quad (6)$$

with $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$, $\eta_{00} = -1$ and $\eta_{ii} = 1$. This is done, so that $SO(d, 2)$ acts linearly on $\mathcal{M}^{d,2}$, in contrast to its non-linear action on $\mathcal{M}^d \ni x_i^\mu$ dual Minkowski coordinates.

For $d = 4$, the embedding space is a $\dim(\mathcal{M}^{d,2}) = 6$ Minkowski-space with signature $(4, 2)$ so X^A , $A = (1, \dots, \dim(\mathcal{M}^{d,2})) = (1, \dots, 6)$, where $X^+ = X^0 + X^5$ and $X^- = X^0 - X^5$. The condition $X \cdot X = X^2 = 0$ defines an $SO(4, 2)$ invariant subspace of $5d$, the null-cone, or light-cone. Then, we obtain \mathcal{M}^4 by projectivising the light-cone, that is, by quotienting the light-cone by rescaling $X \sim \lambda X$ with $\mathbb{R} \ni \lambda$. Because projectivising respects Lorentz rotations of the embedding space $\mathcal{M}^{4,2}$, the projective null-cone naturally inherits an action of $SO(4, 2)$ on the original \mathcal{M}^4 spacetime.

By gauge-fixing the rescaling we can identify the original Minkowski spacetime \mathcal{M}^4 with the projective light-cone. The gauge-fixing condition $X^+ = 1$ gives light-cone (projective subspace) vectors of the form

$$X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \quad (7)$$

($x^2 = x^\mu x_\mu$) In this gauge choice we can see that (not confusing the component of y with its squared length)

$$X \cdot Y = \begin{pmatrix} 1 \\ x^2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^T \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y^2 \\ y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = -\frac{1}{2}x^2 - \frac{1}{2}y^2 + x_\mu y^\mu \quad (8)$$

Trivially, we see that

$$X \cdot X = X^2 = -\frac{1}{2}x^2 - \frac{1}{2}x^2 + x_\mu x^\mu = -x^2 + x^2 = 0 \quad (9)$$

Moreover, for dual vectors x^μ, y^μ we can map their squared difference to the projective light-cone, giving

$$(x - y)^2 \rightarrow (X - Y)^2 = (X)^2 + (Y)^2 - 2X \cdot Y = -2X \cdot Y = x^2 + y^2 - 2x_\mu y^\mu = (x - y)^2 \quad (10)$$

It is therefore convenient to introduce the product on the projective space

$$(X_i, X_j) = -2X_i \cdot X_j = (x_i - x_j)^2 \quad (11)$$

Thus, we can express products of momenta as products of null vectors, that is vectors that are on the light-cone. For example, taking 4-momentum p_1^μ

$$p_2^2 = 0 \mapsto x_{21}^2 = (x_2 - x_1)^2 = 0 \mapsto (X_2, X_1) = 0 \quad (12)$$

Due to the on-shell constraint, we obtain the relations

$$(X_i, X_{i+1}) = (X_i, X_{i-1}) = 0 \quad (13)$$

Similarly, for a variable momentum l we can assign a dual variable x_0 as $l = x_0 - x_4$ (to ensure momentum conservation). To the dual representation of the loop momentum we then assign then a coordinate on the light-cone as previously. Thus, we map the products as

$$l^2 \mapsto x_{04}^2 \mapsto (X_0, X_4) \quad (14)$$

Generally,

$$l_i^2 \mapsto (x_0 - x_i)^2 = x_{0i}^2 \mapsto (X_0, X_i) \quad (15)$$

Given this embedding, we can express Feynman-integrals on the projective null-cone. The 1-loop n -point integral has the structure

$$I_n = \int \frac{d^4 l P(l)}{D_1 D_2 \cdots D_n} \quad (16)$$

with propagators D_1, \dots, D_n , which for on shell external momenta exhibits conformal symmetry. Thus, due to the embedding formalism we can use a manifestly conformal representation of the 1-loop integral with a conformal integral defined on the embedding space.

n=4 (box) 1-loop scalar integral The amplitude has the structure

$$\mathcal{A}_4^1 = \mathcal{A}_4^1(p_1, p_2, p_3, p_4) \quad (17)$$

and arises at the first quantum correction of the 4-point boson interaction (with Mandelstam variables s, t).

$$I_4 = \int d^4 l f(l) = \int \frac{std^4 l}{l^2(l-p_1)^2(l-p_1-p_2)^2(l+p_4)^2} \quad (18)$$

By transforming to dual coordinates, we obtain an integral in the x_0 variable (the transformation is linear so has unit Jacobian)

$$\int d^4 l \mapsto \int d^4 x_0 \quad (19)$$

$$\Rightarrow I_4 = \int \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \quad (20)$$

Under the dilation transformation

$$x_i^\mu \rightarrow \lambda x_i^\mu \quad (21)$$

(with $\lambda \in \mathbb{R}$)

$$\frac{(\lambda^4 d^4 x_0)(\lambda x_1 - \lambda x_3)^2(\lambda x_2 - \lambda x_4)^2}{(\lambda x_0 - \lambda x_1)^2(\lambda x_0 - \lambda x_2)^2(\lambda x_0 - \lambda x_3)^2(\lambda x_0 - \lambda x_4)^2} = \frac{\lambda^8 d^4 x_0 x_{13}^2 x_{24}^2}{\lambda^8 x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \quad (22)$$

the integrand is invariant. Also, it is invariant under inversion transformation

$$x_i^\mu \rightarrow \frac{x_i^\mu}{x_i^\mu x_{\mu i}} = \frac{x_i^\mu}{x_i^2} \quad (23)$$

$$\frac{\frac{d^4 x_0}{x_0^8} \left(\frac{x_1}{x_1^2} - \frac{x_3}{x_3^2}\right)^2 \left(\frac{x_2}{x_2^2} - \frac{x_4}{x_4^2}\right)^2}{\left(\frac{x_0}{x_0^2} - \frac{x_1}{x_1^2}\right)^2 \left(\frac{x_0}{x_0^2} - \frac{x_2}{x_2^2}\right)^2 \left(\frac{x_0}{x_0^2} - \frac{x_3}{x_3^2}\right)^2 \left(\frac{x_0}{x_0^2} - \frac{x_4}{x_4^2}\right)^2} = \frac{\frac{d^4 x_0}{x_0^8} \frac{x_{13}^2 x_{24}^2}{x_1^2 x_2^2 x_3^2 x_4^2}}{\frac{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}{x_0^8 x_1^2 x_2^2 x_3^2 x_4^2}} = \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \quad (24)$$

So, the integrand in the dual coordinate representation exhibits conformal invariance. We can therefore use the embedding formalism to embed the integral on the light cone. The conformal integral measure ($X_0^2 = 0$) becomes

$$\int d^4 l f(l) \mapsto \int d^4 x_0 f(x_0) \mapsto \int \frac{d^6 X_0 \delta(X_0^2)}{\text{Vol}(GL(1))} f(X_0) \quad (25)$$

Consequently, the 4-pt box scalar integral after embedding on the projective light cone takes the form

$$I_4 = \int d^4 l f(l) = \int \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \int \frac{d^6 X_0 \delta(X_0^2)}{\text{Vol}(GL(1))} \frac{(X_1, X_3)(X_2, X_4)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \quad (26)$$

So the integral has the structure

$$I_4 = \int \frac{d^6 X_0 \delta(X_0^2)}{\text{Vol}(GL(1))} f(X_0) \quad (27)$$

Clearly, the integral is invariant under dilation $X_0 \rightarrow \lambda X_0$

$$\int \frac{\lambda^6 d^6 X_0 \delta(\lambda X_0^2)}{\text{Vol}(GL(1))} f(\lambda X_0) = \int \frac{d^6 X_0 \delta(X_0^2)}{\text{Vol}(GL(1))} \frac{\lambda^4}{\lambda^4 (X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \quad (28)$$

$$\Rightarrow \int \frac{\lambda^6 d^6 X_0 \delta(\lambda X_0^2)}{\text{Vol}(GL(1))} f(\lambda X_0) = \int \frac{d^6 X_0 \delta(X_0^2)}{\text{Vol}(GL(1))} f(X_0) \quad (29)$$

So by embedding the integral on the projective light-cone, we have made the conformal invariance manifest. In other words, we have found a more natural representation that explicitly exhibits the conformal symmetry of the loop integral considered. For planar theories, we can decompose $n > 4$ loop integrals to a basis of the 4-point integrals discussed in this section. Moreover we can also define an anti-symmetric product on the projective light-cone with

$$\langle X_i X_j X_k X_l X_m X_p \rangle = \epsilon^{ABCDFG} X_{iA} X_{jB} X_{kC} X_{lD} X_{mF} X_{pG} \quad (30)$$

where ϵ^{ABCDFG} is the totally anti-symmetric tensor. This will be used in the next section, for the reduction of the $n = 5$ (and above) 1-loop case.

n=5 (pentagonal) 1-loop integral Let's consider a general 5-point 1-loop amplitude. By transforming to dual coordinates with variable x_0 and general dual vector w we obtain the representation

$$I_5 = \int \frac{d^4 x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \quad (31)$$

Similarly as before, we can embed to the projective light-cone. Since we are in $6d$ we can hence write an arbitrary $\mathcal{M}^{4,2} \ni W$ as

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + r R = c_i X_i + r R \quad (32)$$

with $(X_i, R) = 0$ and $(R, R) = 1$. Then

$$I_5 = \int \frac{d^4 x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \mapsto \mathcal{I}_5 = \int \frac{d^6 X_0(X_0, W)}{\text{Vol}(GL(1))(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \quad (33)$$

So

$$\mathcal{I}_5 = \int \frac{d^6 X_0}{\text{Vol}(GL(1))} \frac{c_i(X_0, X_i) + r(X_0, R)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \quad (34)$$

The coefficients of the base expansion can be expressed using the anti-symmetric product. For example, we can get c_1 as

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR \quad (35)$$

$$\langle W X_2 X_3 X_4 X_5 R \rangle = \langle (c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR) X_2 X_3 X_4 X_5 \rangle \quad (36)$$

$$\langle W X_2 X_3 X_4 X_5 R \rangle = \langle c_1 X_1 X_2 X_3 X_4 X_5 R \rangle \implies c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R \rangle} \quad (37)$$

Similarly, for c_2

$$\langle W X_1 X_3 X_4 X_5 R \rangle = \langle c_2 X_2 X_1 X_3 X_4 X_5 R \rangle \implies c_2 = \frac{\langle W X_1 X_3 X_4 X_5 R \rangle}{\langle X_2 X_1 X_3 X_4 X_5 R \rangle} \quad (38)$$

Let's consider the c_1 term further with the definition of the anti-symmetric product

$$c_1 = \frac{\langle W X_2 X_3 X_4 X_5 R \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R \rangle} = \frac{\langle W X_2 X_3 X_4 X_5 R \rangle \langle X_1 X_2 X_3 X_4 X_5 R \rangle}{\langle X_1 X_2 X_3 X_4 X_5 R \rangle \langle X_1 X_2 X_3 X_4 X_5 R \rangle} = \quad (39)$$

$$= \frac{\varepsilon^{ABCD FG} \varepsilon_{PQRSTV} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_G X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V}{\varepsilon^{HJKL MN} \varepsilon_{PQRSTV} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V} \quad (40)$$

Using the product identity for the 6d Levi-Civita tensor

$$\varepsilon^{ABCD FG} \varepsilon_{PQRSTV} = \delta_{PQRSTV}^{ABCD FG} = \begin{vmatrix} \delta_P^A & \delta_Q^A & \delta_R^A & \delta_S^A & \delta_T^A & \delta_V^A \\ \delta_P^B & \delta_Q^B & \delta_R^B & \delta_S^B & \delta_T^B & \delta_V^B \\ \delta_P^C & \delta_Q^C & \delta_R^C & \delta_S^C & \delta_T^C & \delta_V^C \\ \delta_P^D & \delta_Q^D & \delta_R^D & \delta_S^D & \delta_T^D & \delta_V^D \\ \delta_P^E & \delta_Q^E & \delta_R^E & \delta_S^E & \delta_T^E & \delta_V^E \\ \delta_P^F & \delta_Q^F & \delta_R^F & \delta_S^F & \delta_T^F & \delta_V^F \\ \delta_P^G & \delta_Q^G & \delta_R^G & \delta_S^G & \delta_T^G & \delta_V^G \end{vmatrix} \quad (41)$$

$$(42)$$

we obtain

$$c_1 = \frac{\begin{vmatrix} \delta_P^A & \delta_Q^A & \delta_R^A & \delta_S^A & \delta_T^A & \delta_V^A \\ \delta_P^B & \delta_Q^B & \delta_R^B & \delta_S^B & \delta_T^B & \delta_V^B \\ \delta_P^C & \delta_Q^C & \delta_R^C & \delta_S^C & \delta_T^C & \delta_V^C \\ \delta_P^D & \delta_Q^D & \delta_R^D & \delta_S^D & \delta_T^D & \delta_V^D \\ \delta_P^F & \delta_Q^F & \delta_R^F & \delta_S^F & \delta_T^F & \delta_V^F \\ \delta_P^G & \delta_Q^G & \delta_R^G & \delta_S^G & \delta_T^G & \delta_V^G \end{vmatrix} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_G X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V}{\begin{vmatrix} \delta_P^H & \delta_Q^H & \delta_R^H & \delta_S^H & \delta_T^H & \delta_V^H \\ \delta_P^J & \delta_Q^J & \delta_R^J & \delta_S^J & \delta_T^J & \delta_V^J \\ \delta_P^K & \delta_Q^K & \delta_R^K & \delta_S^K & \delta_T^K & \delta_V^K \\ \delta_P^L & \delta_Q^L & \delta_R^L & \delta_S^L & \delta_T^L & \delta_V^L \\ \delta_P^M & \delta_Q^M & \delta_R^M & \delta_S^M & \delta_T^M & \delta_V^M \\ \delta_P^N & \delta_Q^N & \delta_R^N & \delta_S^N & \delta_T^N & \delta_V^N \end{vmatrix} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V} = \quad (43)$$

$$= \frac{\begin{vmatrix} W \cdot X_1 & W \cdot X_2 & W \cdot X_3 & W \cdot X_4 & W \cdot X_5 & W \cdot R \\ X_2 \cdot X_1 & X_2 \cdot X_2 & X_2 \cdot X_3 & X_2 \cdot X_4 & X_2 \cdot X_5 & X_2 \cdot R \\ X_3 \cdot X_1 & X_3 \cdot X_2 & X_3 \cdot X_3 & X_3 \cdot X_4 & X_3 \cdot X_5 & X_3 \cdot R \\ X_4 \cdot X_1 & X_4 \cdot X_2 & X_4 \cdot X_3 & X_4 \cdot X_4 & X_4 \cdot X_5 & X_4 \cdot R \\ X_5 \cdot X_1 & X_5 \cdot X_2 & X_5 \cdot X_3 & X_5 \cdot X_4 & X_5 \cdot X_5 & X_5 \cdot R \\ R \cdot X_1 & R \cdot X_2 & R \cdot X_3 & R \cdot X_4 & R \cdot X_5 & R \cdot R \end{vmatrix}}{\begin{vmatrix} X_1 \cdot X_1 & X_1 \cdot X_2 & X_1 \cdot X_3 & X_1 \cdot X_4 & X_1 \cdot X_5 & X_1 \cdot R \\ X_2 \cdot X_1 & X_2 \cdot X_2 & X_2 \cdot X_3 & X_2 \cdot X_4 & X_2 \cdot X_5 & X_2 \cdot R \\ X_3 \cdot X_1 & X_3 \cdot X_2 & X_3 \cdot X_3 & X_3 \cdot X_4 & X_3 \cdot X_5 & X_3 \cdot R \\ X_4 \cdot X_1 & X_4 \cdot X_2 & X_4 \cdot X_3 & X_4 \cdot X_4 & X_4 \cdot X_5 & X_4 \cdot R \\ X_5 \cdot X_1 & X_5 \cdot X_2 & X_5 \cdot X_3 & X_5 \cdot X_4 & X_5 \cdot X_5 & X_5 \cdot R \\ R \cdot X_1 & R \cdot X_2 & R \cdot X_3 & R \cdot X_4 & R \cdot X_5 & R \cdot R \end{vmatrix}} = \quad (44)$$

$$= \frac{\begin{vmatrix} W \cdot X_1 & W \cdot X_2 & W \cdot X_3 & W \cdot X_4 & W \cdot X_5 & W \cdot R \\ X_2 \cdot X_1 & 0 & X_2 \cdot X_3 & X_2 \cdot X_4 & X_2 \cdot X_5 & X_2 \cdot R \\ X_3 \cdot X_1 & X_3 \cdot X_2 & 0 & X_3 \cdot X_4 & X_3 \cdot X_5 & X_3 \cdot R \\ X_4 \cdot X_1 & X_4 \cdot X_2 & X_4 \cdot X_3 & 0 & X_4 \cdot X_5 & X_4 \cdot R \\ X_5 \cdot X_1 & X_5 \cdot X_2 & X_5 \cdot X_3 & X_5 \cdot X_4 & 0 & X_5 \cdot R \\ R \cdot X_1 & R \cdot X_2 & R \cdot X_3 & R \cdot X_4 & R \cdot X_5 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & X_1 \cdot X_2 & X_1 \cdot X_3 & X_1 \cdot X_4 & X_1 \cdot X_5 & X_1 \cdot R \\ X_2 \cdot X_1 & 0 & X_2 \cdot X_3 & X_2 \cdot X_4 & X_2 \cdot X_5 & X_2 \cdot R \\ X_3 \cdot X_1 & X_3 \cdot X_2 & 0 & X_3 \cdot X_4 & X_3 \cdot X_5 & X_3 \cdot R \\ X_4 \cdot X_1 & X_4 \cdot X_2 & X_4 \cdot X_3 & 0 & X_4 \cdot X_5 & X_4 \cdot R \\ X_5 \cdot X_1 & X_5 \cdot X_2 & X_5 \cdot X_3 & X_5 \cdot X_4 & 0 & X_5 \cdot R \\ R \cdot X_1 & R \cdot X_2 & R \cdot X_3 & R \cdot X_4 & R \cdot X_5 & 1 \end{vmatrix}} \quad (45)$$

We can obtain the other coefficients similarly. Before continuing to higher-external leg

cases, we introduce the following notation for the integration measure in embedding space

$$[d^4 x_0] = \frac{d^6 X_0 \delta(X_0^2)}{\text{Vol}(GL(1))} \quad (46)$$

n=6 (hexagonal) 1-loop integral The $n = 6$ 1-loop integral expressed in terms of embedding space coordinates $\mathcal{M}^{d,2} \ni X^A$ has the form

$$\mathcal{I}_6 = \int \frac{[d^4 X_0] T_{AB} X_0^A X_0^B}{\Pi_{i=1}^6(X_0, X_i)} = \int \frac{[d^4 x_0] T_{AB} X_0^A X_0^B}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \quad (47)$$

By introducing $\mathcal{M}^{d,2} \ni W_1, W_2$, with

$$\mathcal{F}_6(W_1, W_2, X_i) = \frac{(W_1, X_0)(W_2, X_0)}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \quad (48)$$

we see that we can obtain the integrand of \mathcal{I}_6 with differentiation of \mathcal{F}_6 and contraction with T_{AB}

$$T_{AB} \frac{\partial}{\partial W_1^A} \frac{\partial}{\partial W_2^B} \mathcal{F}_6 = \frac{T_{AB} X_0^A X_0^B}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \quad (49)$$

Thus

$$\mathcal{I}_6 = \int [d^4 x_0] T_{AB} \frac{\partial}{\partial W_1^A} \frac{\partial}{\partial W_2^B} \mathcal{F}_6 = \int \frac{[d^4 X_0] T_{AB} X_0^A X_0^B}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \quad (50)$$

n=7 (heptagonal) 1-loop integral According to the previous ($n = 6$) example, we have the integral embedded to $\mathcal{M}^{4,2}$ of the form

$$\mathcal{I}_7 = \int \frac{[d^4 x_0] T_{ABC} X_0^A X_0^B X_0^C}{\Pi_{i=1}^7(X_0, X_i)} = \int \frac{[d^4 x_0] T_{ABC} X_0^A X_0^B X_0^C}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)(X_0, X_7)} \quad (51)$$

By introducing $\mathcal{M}^{d,2} \ni W_1, W_2, W_3$, with

$$\mathcal{F}_7(W_1, W_2, W_3, X_i) = \frac{(W_1, X_0)(W_2, X_0)(W_3, X_0)}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \quad (52)$$

the integrand is then obtained similarly T_{ABC}

$$T_{ABC} \frac{\partial}{\partial W_1^A} \frac{\partial}{\partial W_2^B} \frac{\partial}{\partial W_3^C} \mathcal{F}_7 = \frac{T_{ABC} X_0^A X_0^B X_0^C}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)(X_0, X_7)} \quad (53)$$

Thus

$$\mathcal{I}_7 = \int [d^4 x_0] T_{ABC} \frac{\partial}{\partial W_1^A} \frac{\partial}{\partial W_2^B} \frac{\partial}{\partial W_3^C} \mathcal{F}_7 = \int \frac{[d^4 x_0] T_{ABC} X_0^A X_0^B X_0^C}{\Pi_{i=1}^7(X_0, X_i)} \quad (54)$$

General n external leg (n-gon) 1-loop integral Generalising the previous discussions, we can consider the case of an n -gon, so n external leg 1-loop integral. By embedding the integral to $\mathcal{M}^{4,2}$ we obtain

$$\mathcal{I}_n = \int \frac{[d^4 x_0] T_{a_1 a_2 \dots a_{n-4}} X_0^{a_1} X_0^{a_2} \dots X_0^{a_{n-4}}}{\Pi_{i=1}^n(X_0, X_i)} \quad (55)$$

By introducing $\mathcal{M}^{4,2} \ni W_1, W_2, \dots, W_{n-4}$, with

$$\mathcal{F}_n(W_1, W_2, \dots, W_{n-4}, X_i) = \frac{(W_1, X_0)(W_2, X_0) \dots (W_{n-4}, X_0)}{\Pi_{i=1}^n(X_0, X_i)} \quad (56)$$

the integrand is then obtained similarly T_{ABC}

$$T_{a_1 a_2 \dots a_{n-4}} \frac{\partial}{\partial W_1^{a_1}} \frac{\partial}{\partial W_2^{a_2}} \dots \frac{\partial}{\partial W_{n-4}^{a_{n-4}}} \mathcal{F}_n = \frac{T_{a_1 a_2 \dots a_{n-4}} X_0^{a_1} X_0^{a_2} \dots X_0^{a_{n-4}}}{\Pi_{i=1}^n(X_0, X_i)} \quad (57)$$

Thus

$$\mathcal{I}_n = \int [d^4 x_0] T_{a_1 a_2 \dots a_{n-4}} \frac{\partial}{\partial W_1^{a_1}} \frac{\partial}{\partial W_2^{a_2}} \dots \frac{\partial}{\partial W_{n-4}^{a_{n-4}}} \mathcal{F}_n = \int \frac{[d^4 x_0] T_{a_1 a_2 \dots a_{n-4}} X_0^{a_1} X_0^{a_2} \dots X_0^{a_{n-4}}}{\Pi_{i=1}^n(X_0, X_i)} \quad (58)$$

Momentum-space twistors We have seen that for an $\mathcal{M}^{4,2} \ni X$

$$X \cdot X = \eta_{AB} X^A X^B = -X^+ X^- + \eta_{\mu\nu} X^\mu X^\nu = -(X^0)^2 + (X^5)^2 - (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = 0 \quad (59)$$

In this way we describe the conformally compactified Minkowski space with \mathbb{RP}^5 coordinates, so (59) defines an equation in \mathbb{RP}^5 . After complexification of space-time, the coordinates representing $\mathcal{M}^{4,2} \ni X$ are in \mathbb{CP}^3 . Since $SO(4, 2) \simeq SU(2, 2) \simeq SU(4)$, we can choose the anti-symmetric tensor representation of $SU(4)$. By doing this we can package the compactified space-time coordinates in $X_{\alpha\beta} = -X_{\beta\alpha}$ ($\alpha, \beta = 0, 1, 2, 3$) and express (59) as

$$\varepsilon^{\alpha\beta\gamma\delta} X_{\alpha\beta} X_{\gamma\delta} = 0 \quad (60)$$

So for an arbitrary X satisfying (60)

$$\iff X_{\alpha\beta} = A_{[\alpha} B_{\beta]} \quad (61)$$

for $\mathbb{T} \ni A, B$ (A, B in twistor space, which can be regarded as a copy of \mathbb{CP}^3). In this way we can represent a point in conformally compactified and complexified Minkowski space with twistors A, B . More specifically we can represent a point in conformally compactified spacetime with a line in \mathbb{CP}^3 defined by twistors A, B (61). On the other hand, given an arbitrary $\mathbb{CP}^3 \ni W_\alpha$ that is on the line $A \wedge B$

$$X_{[\alpha\beta} W_{\gamma]} = 0 \quad (62)$$

which thus means that W_α is a linear combination of A_α, B_α . Therefore, points of conformally compactified, complexified space-time correspond to holomorphic lines $\mathbb{CP}^1 \subset \mathbb{CP}^3$ defined by the twistor pair (Z_A, Z_B)