$\mathcal{N} = 4$ SYM Loop Notes

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TO DO: THESIS TEMPLATE, REFERENCES, FORMATTING

1 loop review Let's consider a general 1-loop amplitude $\mathcal{A}_n^1$ with n external momenta p_i^μ $(i = 1, ..., n)$

$$\mathcal{A}_n^1 = \mathcal{A}_n^1(p_1, ..., p_n) \tag{1}$$

where the n external momenta have the constraints

$$p_i^\mu p_{\mu,i} = p_i^2 = 0 \tag{2}$$

$$\sum_i p_i^\mu = 0 \tag{3}$$

$\forall i = 1, ..., n$ Namely, (2) is the "on-shell" condition and (3) is the "conservation of momentum" relation. We can trivially satisfy (3) by introducing dual coordinates (region variables) x_i^μ with $(i = 1, ..., n)$ such that

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu = x_{i,i+1}^\mu \tag{4}$$

with $p_n^\mu = x_n^\mu - x_1^\mu$ and periodicity condition $x_{n+1}^\mu = x_1^\mu$. We can relate back every $x_{i,j}$ introduced in (4) to 4-momenta with

$$x_{i,j} = x_i - x_j = p_i + p_{i+1} + \cdots + p_{j-1} \tag{5}$$

Let's now consider a d-dim Minkowskian CFT (defined therefore on $\mathcal{M}^d$) with conformal group $SO(d, 2)$ acting non-linearly on $\mathcal{M}^d$ (clearly $\mathcal{M}^d \ni x^\mu$). To linearise the action of the conformal group on $\mathcal{M}^d$ (Dirac) we consider the linear action of $SO(d, 2)$ on the embedding space, which is a Minkowski spacetime with signature $(d, 2)$ that we denote by $\mathcal{M}^{d,2}$. By the embedding of $\mathcal{M}^d \ni x_i^\mu$ to $\mathcal{M}^{d,2}$ we mean

$$x_i^\mu \to X_i^A = (X^+, X^-, x_i^\mu) \in \mathcal{M}^{d,2} \tag{6}$$

with inner product

$$X \cdot X = \eta_{AB} X^A X^B = -X^+ X^- + \eta_{\mu\nu} X^\mu X^\nu \tag{7}$$

with $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$, $\eta_{00} = -1$ and $\eta_{ii} = 1$. This is done, so that $SO(d, 2)$ acts linearly on $\mathcal{M}^{d,2}$, in contrast to its non-linear action on $\mathcal{M}^d \ni x_i^\mu$ dual Minkowski coordinates.
For $d = 4$, the embedding space is a $\dim(\mathcal{M}^{d,2}) = 6$ Minkowski-space with signature $(4, 2)$ so X^A, $A = (1, ..., \dim(\mathcal{M}^{d,2})) = (1, ..., 6)$, where $X^+ = X^0 + X^5$ and $X^- = X^0 - X^5$. The

condition $X \cdot X = X^2 = 0$ defines an $SO(4,2)$ invariant subspace of $5d$, the null-cone, or light-cone. Then, we obtain $\mathcal{M}^4$ by projectivising the light-cone, that is, by quotienting the light-cone by rescaling $X \sim \lambda X$ with $\mathbb{R} \ni \lambda$. Because projectivising respects Lorentz rotations of the embedding space $\mathcal{M}^{4,2}$, the projective null-cone naturally inherits an action of $SO(4,2)$ on the original $\mathcal{M}^4$ spacetime.
By gauge-fixing the rescaling we can identify the original Minkowski spacetime $\mathcal{M}^4$ with the projective light-cone. The gauge-fixing condition $X^+ = 1$ gives light-cone (projective subspace) vectors of the form

$$X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \tag{8}$$

($x^2 = x^\mu x_\mu$) In this gauge choice we can see that (not confusing the component of y with its squared length)

$$X \cdot Y = \begin{pmatrix} 1 \\ x^2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^T \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ y^2 \\ y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = -\frac{1}{2}x^2 - \frac{1}{2}y^2 + x_\mu y^\mu \tag{9}$$

Trivially, we see that

$$X \cdot X = X^2 = -\frac{1}{2}x^2 - \frac{1}{2}x^2 + x_\mu x^\mu = -x^2 + x^2 = 0 \tag{10}$$

Moreover, for dual vectors x^μ, y^μ we can map their squared difference to the projective light-cone, giving

$$(x-y)^2 \to (X-Y)^2 = (X)^2 + (Y)^2 - 2X \cdot Y = -2X \cdot Y = x^2 + y^2 - 2x_\mu y^\mu = (x-y)^2 \tag{11}$$

It is therefore convenient to introduce the product on the projective space

$$(X_i, X_j) = -2X_i \cdot X_j = (x_i - x_j)^2 \tag{12}$$

Thus, we can express products of momenta as products of null vectors, that is vectos that are on the light-cone. For example, taking 4-momentum p_1^μ

$$p_2^2 = 0 \mapsto x_{21}^2 = (x_2 - x_1)^2 = 0 \mapsto (X_2, X_1) = 0 \tag{13}$$

Due to the on-shell constraint, we obtain the relations

$$(X_i, X_{i+1}) = (X_i, X_{i-1}) = 0 \tag{14}$$

Similarly, for a variable momentum l we can assign a dual variable x_0 as $l = x_0 - x_4$ (to ensure momentum conservation). To the dual representation of the loop momentum we then assign then a coordinate on the light-cone as previously. Thus, we map the products as

$$l^2 \mapsto x_{04}^2 \mapsto (X_0, X_4) \tag{15}$$

Generally,

$$l_i^2 \mapsto (x_0 - x_i)^2 = x_{0i}^2 \mapsto (X_0, X_i) \tag{16}$$

Given this embedding, we can express Feynman-integrals on the projective null-cone. The 1-loop n-point integral has the structure

$$I_n = \int \frac{d^4 l P(l)}{D_1 D_2 \cdots D_n} \tag{17}$$

with propagators $D_1, ..., D_n$, which for on shell external momenta exhibits conformal symmetry. Thus, due to the embedding formalism we can use a manifestly conformal representation of the 1-loop integral with a conformal integral defined on the embedding space.

n=4 (box) 1-loop scalar integral The amplitude has the structure

$$\mathcal{A}_4^1 = \mathcal{A}_4^1(p_1, p_2, p_3, p_4) \tag{18}$$

and arises at the first quantum correction of the 4-point boson interaction (with Mandelstam variables s, t).

$$I_4 = \int d^4 l f(l) = \int \frac{st d^4 l}{l^2 (l - p_1)^2 (l - p_1 - p_2)^2 (l + p_4)^2} \tag{19}$$

By transforming to dual coordinates, we obtain an integral in the x_0 variable (the transformation is linear so has unit Jacobian)

$$\int d^4 l \mapsto \int d^4 x_0 \tag{20}$$

$$\Rightarrow I_4 = \int \frac{d^4 x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{21}$$

Using Feynman/Schwinger parametrisation, we can express this with

$$I_4 = \int \int_0^\infty \int_0^\infty \int_0^\infty d\alpha_2 d\alpha_3 d\alpha_4 \frac{d^4 x_0 x_{13}^2 x_{24}^2}{(x_{01}^2 + \alpha_2 x_{02}^2 + \alpha_3 x_{03}^2 + \alpha_4 x_{04}^2)^4} \tag{22}$$

Under the dilation transformation

$$x_i^\mu \to \lambda x_i^\mu \tag{23}$$

(with $\lambda \in \mathbb{R}$)

$$\frac{(\lambda^4 d^4x_0)(\lambda x_1 - \lambda x_3)^2(\lambda x_2 - \lambda x_4)^2}{(\lambda x_0 - \lambda x_1)^2(\lambda x_0 - \lambda x_2)^2(\lambda x_0 - \lambda x_3)^2(\lambda x_0 - \lambda x_4)^2} = \frac{\lambda^8 d^4x_0 x_{13}^2 x_{24}^2}{\lambda^8 x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \frac{d^4x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{24}$$

the integrand is invariant. Also, it is invariant under inversion transformation

$$x_i^\mu \to \frac{x_i^\mu}{x_i^\mu x_{\mu i}} = \frac{x_i^\mu}{x_i^2} \tag{25}$$

$$\frac{\frac{d^4x_0}{x_0^8}(\frac{x_1}{x_1^2} - \frac{x_3}{x_3^2})^2(\frac{x_2}{x_2^2} - \frac{x_4}{x_4^2})^2}{(\frac{x_0}{x_0^2} - \frac{x_1}{x_1^2})^2(\frac{x_0}{x_0^2} - \frac{x_2}{x_2^2})^2(\frac{x_0}{x_0^2} - \frac{x_3}{x_3^2})^2(\frac{x_0}{x_0^2} - \frac{x_4}{x_4^2})^2} = \frac{\frac{d^4x_0}{x_0^8}\frac{x_{13}^2 x_{24}^2}{x_1^2 x_2^2 x_3^2 x_4^2}}{\frac{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}{x_0^8 x_1^2 x_2^2 x_3^2 x_4^2}} = \frac{d^4x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \tag{26}$$

So, the integrand in the dual coordinate representation exhibits conformal invariance. We can therefore use the embedding formalism to embed the integral on the light cone. The conformal integral measure ($X_0^2 = 0$) becomes

$$\int d^4l f(l) \mapsto \int d^4x_0 f(x_0) \mapsto \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{27}$$

Consequently, the 4-pt box scalar integral after embedding on the projective light cone takes the form

$$I_4 = \int d^4l f(l) = \int \frac{d^4x_0 x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} = \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} \frac{(X_1, X_3)(X_2, X_4)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \tag{28}$$

So the integral has the structure

$$I_4 = \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{29}$$

Clearly, the integral is invariant under dilation $X_0 \to \lambda X_0$

$$\int \frac{\lambda^6 d^6X_0 \delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} \frac{\lambda^4}{\lambda^4 (X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)} \tag{30}$$

$$\implies \int \frac{\lambda^6 d^6X_0 \delta(\lambda X_0^2)}{Vol(GL(1))} f(\lambda X_0) = \int \frac{d^6X_0 \delta(X_0^2)}{Vol(GL(1))} f(X_0) \tag{31}$$

So by embedding the integral on the projective light-cone, we have made the conformal invariance manifest. In other words, we have found a more natural representation that explicitly exhibits the conformal symmetry of the loop integral considered. For planar theories, we can decompose $n > 4$ loop integrals to a basis of the 4-point integrals discussed in this section. Moreover we can also define an anti-symmetric product on the projective light-cone with

$$\langle X_i X_j X_k X_l X_m X_p\rangle = \varepsilon^{ABCDFG} X_{iA} X_{jB} X_{kC} X_{lD} X_{mF} X_{pG} \tag{32}$$

where ϵ^{ABCDFG} is the totally anti-symmetric tensor. This will be used in the next section, for the reduction of the $n = 5$ (and above) 1-loop case.

n=5 (pentagonal) 1-loop integral Let's consider a general 5-point 1-loop amplitude. By transforming to dual coordinates with variable x_0 and general dual vector w we obtain the representation

$$I_5 = \int \frac{d^4 x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \tag{33}$$

Similarly as before, we can embed to the projective light-cone. Since we are in $6d$ we can hence write an arbitrary $\mathcal{M}^{4,2} \ni W$ as

$$W = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5 + rR = c_i X_i + rR \tag{34}$$

with $(X_i, R) = 0$ and $(R, R) = 1$. Then

$$I_5 = \int \frac{d^4 x_0 (x_0 - w)^2}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2 x_{50}^2} \mapsto \mathcal{I}_5 = \int \frac{d^6 X_0 (X_0, W)}{Vol(GL(1))(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \tag{35}$$

So

$$\mathcal{I}_5 = \int \frac{d^6 X_0}{Vol(GL(1))} \frac{c_i (X_0, X_i) + r(X_0, R)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} \tag{36}$$

Remarkably, the (X_0, R) term does not contribute. This can be seen using the method of Feynman/Schwinger parametrisation

$$\int [d^4 X_0] \frac{r(X_0, R)}{(X_1, X_0)(X_2, X_0)(X_3, X_0)(X_4, X_0)(X_5, X_0)} = \tag{37}$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 \frac{[d^4 X_0] r(X_0, R)}{((X_1, X_0) + \alpha_2 (X_2, X_0) + \alpha_3 (X_3, X_0) + \alpha_4 (X_4, X_0) + \alpha_5 (X_5, X_0))^5} \tag{38}$$

By defining $\mathcal{W} = X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5$, we have

$$\Pi_{i=2}^5 (\int_0^\infty d\alpha_i) \int [d^4 X_0] \frac{(X_0, R)}{(\mathcal{W}, X_0)^5} \sim \Pi_{i=2}^5 (\int_0^\infty d\alpha_i) \int [d^4 X_0] (rR) \partial_{\mathcal{W}} \frac{1}{(\mathcal{W}, X_0)^4} \sim \tag{39}$$

$$\sim \Pi_{i=2}^5 (\int_0^\infty d\alpha_i)(rR) \partial_{\mathcal{W}} \frac{1}{(\mathcal{W}, \mathcal{W})^2} \sim \Pi_{i=2}^5 (\int_0^\infty d\alpha_i)(rR) \frac{\mathcal{W}}{(\mathcal{W}, \mathcal{W})^3} \tag{40}$$

Since $(R, X_i) = 0 \implies (R, \mathcal{W}) = 0$ we obtain

$$\Pi_{i=2}^{5}(\int_0^\infty d\alpha_i)\frac{r(R,\mathcal{W})}{(\mathcal{W},\mathcal{W})^3} = 0 \tag{41}$$

Thus

$$\mathcal{I}_5 = \int \frac{d^6X_0}{Vol(GL(1))}\frac{c_i(X_i, X_0)}{(X_1,X_0)(X_2,X_0)(X_3,X_0)(X_4,X_0)(X_5,X_0)} \tag{42}$$

The coefficients of the base expansion can be expressed using the anti-symmetric product. For example, we can get c_1 as

$$W = c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + rR \tag{43}$$

$$\langle WX_2X_3X_4X_5R\rangle = \langle (c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + rR)X_2X_3X_4X_5\rangle \tag{44}$$

$$\langle WX_2X_3X_4X_5R\rangle = \langle c_1X_1X_2X_3X_4X_5R\rangle \implies c_1 = \frac{\langle WX_2X_3X_4X_5R\rangle}{\langle X_1X_2X_3X_4X_5R\rangle} \tag{45}$$

Similarly, for c_2

$$\langle WX_1X_3X_4X_5R\rangle = \langle c_2X_2X_1X_3X_4X_5R\rangle \implies c_2 = \frac{\langle WX_1X_3X_4X_5R\rangle}{\langle X_2X_1X_3X_4X_5R\rangle} \tag{46}$$

Let's consider the c_1 term further with the definition of the anti-symmetric product

$$c_1 = \frac{\langle WX_2X_3X_4X_5R\rangle}{\langle X_1X_2X_3X_4X_5R\rangle} = \frac{\langle WX_2X_3X_4X_5R\rangle\langle X_1X_2X_3X_4X_5R\rangle}{\langle X_1X_2X_3X_4X_5R\rangle\langle X_1X_2X_3X_4X_5R\rangle} = \tag{47}$$

$$= \frac{\varepsilon^{ABCDFG}\varepsilon_{PQRSTV}W_AX_{2B}X_{3C}X_{4D}X_{5F}R_GX_{1P}X_{2Q}X_{3R}X_{4S}X_{5T}R_V}{\varepsilon^{HJKLMN}\varepsilon_{PQRSTV}X_{1H}X_{2J}X_{3K}X_{4L}X_{5M}R_NX_{1P}X_{2Q}X_{3R}X_{4S}X_{5T}R_V} \tag{48}$$

Using the product identity for the $6d$ Levi-Civita tensor

$$\varepsilon^{ABCDFG}\varepsilon_{PQRSTV} = \delta^{ABCDFG}_{PQRSTV} = \begin{vmatrix} \delta^A_P & \delta^A_Q & \delta^A_R & \delta^A_S & \delta^A_T & \delta^A_V \\ \delta^B_P & \delta^B_Q & \delta^B_R & \delta^B_S & \delta^B_T & \delta^B_V \\ \delta^C_P & \delta^C_Q & \delta^C_R & \delta^C_S & \delta^C_T & \delta^C_V \\ \delta^D_P & \delta^D_Q & \delta^D_R & \delta^D_S & \delta^D_T & \delta^D_V \\ \delta^F_P & \delta^F_Q & \delta^F_R & \delta^F_S & \delta^F_T & \delta^F_V \\ \delta^G_P & \delta^G_Q & \delta^G_R & \delta^G_S & \delta^G_T & \delta^G_V \end{vmatrix} \tag{49}$$

(50)

we obtain

$$c_1 = \frac{\begin{vmatrix} \delta_P^A & \delta_Q^A & \delta_R^A & \delta_S^A & \delta_T^A & \delta_V^A \\ \delta_P^B & \delta_Q^B & \delta_R^B & \delta_S^B & \delta_T^B & \delta_V^B \\ \delta_P^C & \delta_Q^C & \delta_R^C & \delta_S^C & \delta_T^C & \delta_V^C \\ \delta_P^D & \delta_Q^D & \delta_R^D & \delta_S^D & \delta_T^D & \delta_V^D \\ \delta_P^F & \delta_Q^F & \delta_R^F & \delta_S^F & \delta_T^F & \delta_V^F \\ \delta_P^G & \delta_Q^G & \delta_R^G & \delta_S^G & \delta_T^G & \delta_V^G \end{vmatrix} W_A X_{2B} X_{3C} X_{4D} X_{5F} R_G X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V}{\begin{vmatrix} \delta_P^H & \delta_Q^H & \delta_R^H & \delta_S^H & \delta_T^H & \delta_V^H \\ \delta_P^J & \delta_Q^J & \delta_R^J & \delta_S^J & \delta_T^J & \delta_V^J \\ \delta_P^K & \delta_Q^K & \delta_R^K & \delta_S^K & \delta_T^K & \delta_V^K \\ \delta_P^L & \delta_Q^L & \delta_R^L & \delta_S^L & \delta_T^L & \delta_V^L \\ \delta_P^M & \delta_Q^M & \delta_R^M & \delta_S^M & \delta_T^M & \delta_V^M \\ \delta_P^N & \delta_Q^N & \delta_R^N & \delta_S^N & \delta_T^N & \delta_V^N \end{vmatrix} X_{1H} X_{2J} X_{3K} X_{4L} X_{5M} R_N X_{1P} X_{2Q} X_{3R} X_{4S} X_{5T} R_V} = \tag{51}$$

$$= \frac{\begin{vmatrix} W\cdot X_1 & W\cdot X_2 & W\cdot X_3 & W\cdot X_4 & W\cdot X_5 & W\cdot R \\ X_2\cdot X_1 & X_2\cdot X_2 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R \\ X_3\cdot X_1 & X_3\cdot X_2 & X_3\cdot X_3 & X_3\cdot X_4 & X_3\cdot X_1 & X_3\cdot R \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & X_4\cdot X_4 & X_4\cdot X_5 & X_4\cdot R \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & X_5\cdot X_5 & X_5\cdot R \\ R\cdot X_1 & R\cdot X_2 & R\cdot X_3 & R\cdot X_4 & R\cdot X_5 & R\cdot R \end{vmatrix}}{\begin{vmatrix} X_1\cdot X_1 & X_1\cdot X_2 & X_1\cdot X_3 & X_1\cdot X_4 & X_1\cdot X_5 & X_1\cdot R \\ X_2\cdot X_1 & X_2\cdot X_2 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & X_2\cdot R \\ X_3\cdot X_1 & X_3\cdot X_2 & X_3\cdot X_3 & X_3\cdot X_4 & X_3\cdot X_1 & X_3\cdot R \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & X_4\cdot X_4 & X_4\cdot X_5 & X_4\cdot R \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & X_5\cdot X_5 & X_5\cdot R \\ R\cdot X_1 & R\cdot X_2 & R\cdot X_3 & R\cdot X_4 & R\cdot X_5 & R\cdot R \end{vmatrix}} = \tag{52}$$

$$= \frac{\begin{vmatrix} W\cdot X_1 & W\cdot X_2 & W\cdot X_3 & W\cdot X_4 & W\cdot X_5 & 1 \\ X_2\cdot X_1 & 0 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & 0 \\ X_3\cdot X_1 & X_3\cdot X_2 & 0 & X_3\cdot X_4 & X_3\cdot X_1 & 0 \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & 0 & X_4\cdot X_5 & 0 \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & X_1\cdot X_2 & X_1\cdot X_3 & X_1\cdot X_4 & X_1\cdot X_5 & 0 \\ X_2\cdot X_1 & 0 & X_2\cdot X_3 & X_2\cdot X_4 & X_2\cdot X_5 & 0 \\ X_3\cdot X_1 & X_3\cdot X_2 & 0 & X_3\cdot X_4 & X_3\cdot X_1 & 0 \\ X_4\cdot X_1 & X_4\cdot X_2 & X_4\cdot X_3 & 0 & X_4\cdot X_5 & 0 \\ X_5\cdot X_1 & X_5\cdot X_2 & X_5\cdot X_3 & X_5\cdot X_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}} \tag{53}$$

We can obtain the other coefficients similarly. Before continuing to higher-external leg cases, we introduce the following notation for the integration measure in embedding space

$$[d^4X_0] = \frac{d^6X_0\delta(X_0^2)}{Vol(GL(1))} \tag{54}$$

n=6 (hexagonal) 1-loop integral The $n = 6$ 1-loop integral expressed in terms of embedding space coordinates $\mathcal{M}^{d,2} \ni X^A$ has the form

$$\mathcal{I}_6 = \int \frac{[d^4X_0]T_{AB}X_0^AX_0^B}{\Pi_{i=1}^6(X_0, X_i)} = \int \frac{[d^4X_0]T_{AB}X_0^AX_0^B}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{55}$$

By introducing $\mathcal{M}^{d,2} \ni W_1, W_2$, with

$$\mathcal{F}_6(W_1, W_2, X_i) = \frac{(W_1, X_0)(W_2, X_0)}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{56}$$

we see that we can obtain the integrand of $\mathcal{I}_6$ with differentiation of F_6 and contraction with T_{AB}

$$T_{AB}\frac{\partial}{\partial W_1^A}\frac{\partial}{\partial W_2^B}F_6 = \frac{T_{AB}X_0^AX_0^B}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{57}$$

Thus

$$\mathcal{I}_6 = \int [d^4X_0]T_{AB}\frac{\partial}{\partial W_1^A}\frac{\partial}{\partial W_2^B}\mathcal{F}_6 = \int \frac{[d^4X_0]T_{AB}X_0^AX_0^B}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{58}$$

n=7 (heptagonal) 1-loop integral According to the previous ($n = 6$) example, we have the integral embedded to $\mathcal{M}^{4,2}$ of the form

$$\mathcal{I}_7 = \int \frac{[d^4X_0]T_{ABC}X_0^AX_0^BX_0^C}{\Pi_{i=1}^7(X_0, X_i)} = \int \frac{[d^4x_0]T_{ABC}X_0^AX_0^BX_0^C}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)(X_0, X_7)} \tag{59}$$

By introducing $\mathcal{M}^{d,2} \ni W_1, W_2, W_3$, with

$$\mathcal{F}_7(W_1, W_2, W_3, X_i) = \frac{(W_1, X_0)(W_2, X_0)(W_3, X_0)}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)} \tag{60}$$

the integrand is then obtained similarly T_{ABC}

$$T_{ABC}\frac{\partial}{\partial W_1^A}\frac{\partial}{\partial W_2^B}\frac{\partial}{\partial W_3^C} = \frac{T_{ABC}X_0^AX_0^BX_0^C}{(X_0, X_1)(X_0, X_2)(X_0, X_3)(X_0, X_4)(X_0, X_5)(X_0, X_6)(X_0, X_7)} \tag{61}$$

Thus

$$\mathcal{I}_7 = \int [d^4X_0]T_{ABC}\frac{\partial}{\partial W_1^A}\frac{\partial}{\partial W_2^B}\frac{\partial}{\partial W_3^C}\mathcal{F}_7 = \int \frac{[d^4X_0]T_{AB}X_0^AX_0^BX_0^C}{\Pi_{i=1}^7(X_0, X_i)} \tag{62}$$

General n external leg (n-gon) 1-loop integral Generalising the previous discussions, we can consider the case of an n-gon, so n external leg 1-loop integral. By embedding the integral to $\mathcal{M}^{4,2}$ we obtain

$$\mathcal{I}_n = \int \frac{[d^4X_0]T_{a_1a_2\cdots a_{n-4}}X_0^{a_1}X_0^{a_2}\cdots X_0^{a_{n-4}}}{\Pi_{i=1}^n(X_0,X_i)} \tag{63}$$

By introducing $\mathcal{M}^{4,2} \ni W_1, W_2, \cdots, W_{n-4}$, with

$$\mathcal{F}_n(W_1,W_2,\cdots,W_{n-4},X_i) = \frac{(W_1,X_0)(W_2,X_0)\cdots(W_{n-4},X_0)}{\Pi_{i=1}^n(X_0,X_i)} \tag{64}$$

the integrand is then obtained similarly T_{ABC}

$$T_{a_1a_2\cdots a_{n-4}}\frac{\partial}{\partial W_1^{a_1}}\frac{\partial}{\partial W_2^{a_2}}\cdots\frac{\partial}{\partial W_{n-4}^{a_{n-4}}}\mathcal{F}_n = \frac{T_{a_1a_2\cdots a_{n-4}}X_0^{a_1}X_0^{a_2}\cdots X_0^{a_{n-4}}}{\Pi_{i=1}^n(X_0,X_i)} \tag{65}$$

Thus

$$\mathcal{I}_n = \int [d^4X_0]T_{a_1a_2\cdots a_{n-4}}\frac{\partial}{\partial W_1^{a_1}}\frac{\partial}{\partial W_2^{a_2}}\cdots\frac{\partial}{\partial W_{n-4}^{a_{n-4}}}\mathcal{F}_n = \int \frac{[d^4X_0]T_{a_1a_2\cdots a_{n-4}}X_0^{a_1}X_0^{a_2}\cdots X_0^{a_{n-4}}}{\Pi_{i=1}^n(X_0,X_i)} \tag{66}$$

Momentum-space twistors In the case of massless particles, we can introduce spinor-helicity variables, so that the on-shell momentum constraint $p_i^2 = 0$ ($\forall i = 1, ..., n$) is trivialised. Firslty, we map $4 - momenta$ to $2x2$-hermitian matrices (4 d.f = 4 d.f) by using the Paulis. Since $\det(p_i) = -p^2 = 0$, we can express the $2x2$ matrices as the product of 2 spinors

$$p_i^\mu \to (p_i)^{\alpha\dot{\alpha}} = p_i^\mu(\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda_i^\alpha\tilde{\lambda}_i^{\dot{\alpha}} \tag{67}$$

This is useful, since in the circumstance of massless particles the gauge interactions conserve helicity, which is exploited by computing in the helicity bases. Hence, we work $2d$ quantities, instead of $4d$-momenta. In $2d$ space we cannot have 3 linearly independent spinors. This is reflected by the Schouten-identity

$$\tilde{\lambda}_i^{\dot{\alpha}}(\tilde{\lambda}_{\dot{\beta},j}\tilde{\lambda}_k^{\dot{\beta}}) + \tilde{\lambda}_j^{\dot{\alpha}}(\tilde{\lambda}_{\dot{\beta},k}\tilde{\lambda}_i^{\dot{\beta}}) + \tilde{\lambda}_k^{\dot{\alpha}}(\tilde{\lambda}_{\dot{\beta},i}\tilde{\lambda}_j^{\dot{\beta}}) = 0 \tag{68}$$

The identity is identical for λ spinors. Although the spinor variables $\lambda, \tilde{\lambda}$ trivialise the on-shell relation $\forall i = 1, 2, ...n$, it does not give information about momentum-conservation (3). So it would be most useful to work with a mathematical object that simultaneously satisfies (2) and (3). Such an object $\exists$ and is called the momentum-space twistor (introduced by Hodges).

Givem a momentum-space twistor

$$Z_A = (\tilde{\lambda}, \mu) \tag{69}$$

its components are related by the incidence relation

$$\mu^{\alpha} = x^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}} \tag{70}$$

So for a twistor to be a twistor it has to have the form

$$Z_A = (\tilde{\lambda}, \mu) = (\tilde{\lambda}, x\tilde{\lambda}) \tag{71}$$

A twistor Z has components of a point in $\mathbb{C}^4$, but since (68) cant tell the difference between Z and scaled twistor tZ the space is projectivised; thus, these twistors form a line in $\mathbb{CP}^3$. Using helicity spinors, we can re-write the dual-coordinate defining relation (4) as

$$p_i^{\mu} \to p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} \tag{72}$$

Multiplying both sides of (70) with $\tilde{\lambda}_{\dot{\alpha},i}$ and re-arranging yields

$$(x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}})\tilde{\lambda}_{\dot{\alpha},i} = 0 \implies x_i^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i} = x_{i+1}^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i} = \mu_i^{\alpha} \tag{73}$$

Thus, we can naturally define the incidence relation (68) through (71). $\forall x_i, x_{i+1}$ pair of dual coordinates (zone variables) satisfying (71) the dual coordinates x_i, x_{i+1} are null-separated by the momentum vector

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = p_i^{\alpha\dot{\alpha}} \to p_i^{\mu} \tag{74}$$

This implies that the line in dual-coordinate space determined by null-separated coordinates x_i, x_{i+1} defines a point $Z_A = (\tilde{\lambda}, x\tilde{\lambda}) = (\tilde{\lambda}, \mu)$ in twistor space, through the incidence relation (71).

Furthermore, let's consider the dual coordinate $x_i^{\alpha\dot{\alpha}}$, which according to (71) appears in the coincidence relations

$$x_i^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i} = \mu_i^{\alpha} \tag{75}$$

and (by shifting $i+1 \to i$ in the dual coordinate)

$$x_i^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i-1} = \mu_{i-1}^{\alpha} \tag{76}$$

On the otehr hand

$$-x_{\alpha\dot{\beta},i}\tilde{\lambda}_i^{\dot{\beta}} = \mu_{\alpha,i} \tag{77}$$

Acting on (74) with $\tilde{\lambda}_i^{\dot{\beta}}$ and on (73) with $\tilde{\lambda}_{i-1}^{\dot{\beta}}$ and combining we obtain

$$\tilde{\lambda}_i^{\dot{\beta}}\mu_{i-1}^{\alpha} - \tilde{\lambda}_{i-1}^{\dot{\beta}}\mu_i^{\alpha} = x_i^{\alpha\dot{\alpha}}(\tilde{\lambda}_i^{\dot{\beta}}\tilde{\lambda}_{\dot{\alpha},i-1} - \tilde{\lambda}_{i-1}^{\dot{\beta}}\tilde{\lambda}_{\dot{\alpha},i}) = x_i^{\alpha\dot{\beta}}\langle i-1, i\rangle \tag{78}$$

where $\langle i-1, i\rangle = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\alpha},i-1}\tilde{\lambda}_{\dot{\beta},i}$ With $\dot{\alpha} = \dot{\beta}$, we obtain

$$x_i^{\alpha\dot{\alpha}} = \frac{\tilde{\lambda}_i^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i-1} - \tilde{\lambda}_{i-1}^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha},i}}{\langle i-1, i\rangle} \tag{79}$$

Hence, for a pair of twistors on the same line in twistor space we can assign a point in dual-coordinate space.

By combining the two findings we see that (from (71))
line in dual-coordinate space $\iff$ point in twistor space
and that (from (76))
line in twistor space $\iff$ point in dual-coordinate space

Since he conformal group in $4d$ Minkowski-spacetime is $SO(4,2)$, so in order make the group action linear we embed in $\mathcal{M}^{4,2}$ with $\dim(\mathcal{M}^{4,2}) = 6$. The $X \cdot X = 0$ constraint and projective nature $X \sim rX$ make the d.o.f equal to $6-2 = 4$ and since $SO(4,2) \sim SU(2,2)$ we can express the embedding-space $6d$ vector X as a bi-spinor $X^{IJ} = -X^{JI}$, i.e. the anti-symmetric tensor representation of $SU(2,2)$(Skinner). Henceforth, the projective constraint in this representation can be expressed as

$$X^2 = X \cdot X = \frac{1}{2}\epsilon_{IJKL}X^{IJ}X^{KL} = 0 \tag{80}$$

$\iff$ X^{IJ} is a rank-2 tensor. This implies we can write the tensor as

$$X^{IJ} = Z_i^{[I}Z_j^{J]} = Z_i^I Z_j^J - Z_i^J Z_j^I \tag{81}$$

using twistor pairs (Z_i, Z_j). Furthermore, X is defined projectively $\implies$ Z is projective $\implies$ **twistor space** $= \mathbb{CP}^3$. So we see why changing from zone variables to momentum twistors is useful since they transform linearly under dual conformal transformations. Every twistor carries $SU(2,2)$ dual confomal indices indices $I = (\alpha, \dot{\alpha})$, so we can construct a dual-conformal invaraint by defining the DCI-invariant 4-bracket

$$\langle i,j,k,l\rangle = \epsilon_{ABCD}Z_i^A Z_j^B Z_k^C Z_l^D = (\tilde{\lambda}_{\dot{\alpha},i}\tilde{\lambda}_j^{\dot{\alpha}})(\mu_{\beta,k}\mu_l^{\beta}) + (\tilde{\lambda}_{\dot{\alpha},i}\tilde{\lambda}_k^{\dot{\alpha}})(\mu_{\beta,l}\mu_j^{\beta}) + (\tilde{\lambda}_{\dot{\alpha},i}\tilde{\lambda}_l^{\dot{\alpha}})(\mu_{\beta,j}\mu_k^{\beta}) + \tag{82}$$

$$+(\tilde{\lambda}_{\dot{\alpha},k}\tilde{\lambda}_l^{\dot{\alpha}})(\mu_{\beta,i}\mu_j^{\beta}) + (\tilde{\lambda}_{\dot{\alpha},l}\tilde{\lambda}_j^{\dot{\alpha}})(\mu_{\beta,i}\mu_k^{\beta}) + (\tilde{\lambda}_{\dot{\alpha},j}\tilde{\lambda}_k^{\dot{\alpha}})(\mu_{\beta,i}\mu_l^{\beta}) = \tag{83}$$

$$= \langle ij\rangle[\mu_k\mu_l] + \langle ik\rangle[\mu_l\mu_j] + \langle il\rangle[\mu_j\mu_k] + \langle kl\rangle[\mu_i\mu_j] + \langle lj\rangle[\mu_i\mu_k] + \langle jk\rangle[\mu_i\mu_l] \tag{84}$$

where we defined

$$\langle ab\rangle = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\alpha},a}\tilde{\lambda}_{\dot{\beta},b} \tag{85}$$

and

$$[\mu_a\mu_b] = \epsilon_{\alpha\beta}\mu_a^{\alpha}\mu_b^{\beta} \tag{86}$$

Also

$$[\mu_i|^{\alpha} = \langle i|_{\dot{\alpha}}x_i^{\alpha\dot{\alpha}} = \langle i|_{\dot{\alpha}}x_{i+1}^{\alpha\dot{\alpha}} \tag{87}$$

and

$$|\mu_i]_{\alpha} = -(x_i)_{\alpha\dot{\alpha}}|i\rangle^{\dot{\alpha}} = -(x_{i+1})_{\alpha\dot{\alpha}}|i\rangle^{\dot{\alpha}} \tag{88}$$

Considering the case where $\langle k, j-1, j, r\rangle$ we obtain using the incidence relation.

$$\begin{aligned}
\langle k, j-1, j, r\rangle &= \langle k, j-1\rangle[\mu_j\mu_r] + \langle kj\rangle[\mu_r\mu_{j-1}] + \langle kr\rangle[\mu_{j-1}\mu_j]+ && (89)\\
&+\langle jr\rangle[\mu_k\mu_{j-1}] + \langle r, j-1\rangle[\mu_k\mu_j] + \langle j-1, k\rangle[\mu_k\mu_r] = && (90)\\
&= -\langle k, j-1\rangle\langle j|x_jx_r|r\rangle - \langle kj\rangle\langle r|x_rx_j|j-1\rangle - \langle kr\rangle\langle j-1|x_jx_j|j\rangle- && (91)\\
&-\langle jr\rangle\langle k|x_kx_j|j-1\rangle - \langle r, j-1\rangle\langle k|x_kx_j|j\rangle - \langle j, j-1\rangle\langle k|x_kx_r|r\rangle = && (92)\\
&= -(\langle k, j-1\rangle\langle j| + \langle jk\rangle\langle j-1|)x_jx_r|r\rangle - \langle k|x_kx_j(\langle jr\rangle|j-1\rangle + \langle r, j-1\rangle|j\rangle)- && (93)\\
&-\langle kr\rangle\langle j-1|x_j^2|j\rangle - \langle j-1, j\rangle\langle k|x_kx_r|r\rangle = && (94)\\
&= \langle j-1, j\rangle\langle k|x_jx_r|r\rangle - \langle kr\rangle\langle j-1|x_j^2|j\rangle + \langle j-1j\rangle\langle k|x_kx_j|r\rangle - \langle j-1j\rangle\langle k|x_kx_r|r\rangle && (95)
\end{aligned}$$

(Same with other notation, don't know which one is better for thesis?)

$$\begin{aligned}
\langle k, j-1, j, r\rangle &= (\tilde\lambda_k\tilde\lambda_{j-1})(\mu_j\mu_r) + (\tilde\lambda_k\tilde\lambda_j)(\mu_r\mu_{j-1}) + (\tilde\lambda_k\tilde\lambda_r)(\mu_{j-1}\mu_j)+ && (96)\\
&+(\tilde\lambda_j\tilde\lambda_r)(\mu_k\mu_{j-1}) + (\tilde\lambda_r\tilde\lambda_{j-1})(\mu_k\mu_j) + (\tilde\lambda_{j-1}\tilde\lambda_j)(\mu_k\mu_r) = && (97)\\
&= -(\tilde\lambda_k\tilde\lambda_{j-1})(x_j\tilde\lambda_jx_r\tilde\lambda_r) - (\tilde\lambda_k\tilde\lambda_j)(x_r\tilde\lambda_rx_j\tilde\lambda_{j-1}) - (\tilde\lambda_k\tilde\lambda_r)(x_j\tilde\lambda_{j-1}x_j\tilde\lambda_j)- && (98)\\
&-(\tilde\lambda_j\tilde\lambda_r)(x_k\tilde\lambda_kx_j\tilde\lambda_{j-1}) - (\tilde\lambda_r\tilde\lambda_{j-1})(x_k\tilde\lambda_kx_j\tilde\lambda_j) - (\tilde\lambda_{j-1}\tilde\lambda_j)(x_k\tilde\lambda_kx_j\tilde\lambda_r) = && (99)\\
&= -((\tilde\lambda_k\tilde\lambda_{j-1})\tilde\lambda_j + (\tilde\lambda_j\tilde\lambda_k)\tilde\lambda_{j-1})x_jx_r\tilde\lambda_r - x_k\tilde\lambda_kx_j((\tilde\lambda_j\tilde\lambda_r)\tilde\lambda_{j-1} + (\tilde\lambda_r\tilde\lambda_{j-1})\tilde\lambda_j)- && (100)\\
&-(\tilde\lambda_k\tilde\lambda_r)(x_j\tilde\lambda_{j-1}x_j\tilde\lambda_j) - (\tilde\lambda_{j-1}\tilde\lambda_j)(x_k\tilde\lambda_kx_r\tilde\lambda_r) = && (101)\\
&= (\tilde\lambda_{j-1}\tilde\lambda_j)(x_j\tilde\lambda_kx_r\tilde\lambda_r) - (\tilde\lambda_{j-1}\tilde\lambda_j)(x_k\tilde\lambda_kx_r\tilde\lambda_r) + (\tilde\lambda_{j-1}\tilde\lambda_k)(x_k\tilde\lambda_kx_j\tilde\lambda_r) - (\tilde\lambda_k\tilde\lambda_r)(x_j\tilde\lambda_{j-1}x_j\tilde\lambda_j) && (102)
\end{aligned}$$