UNIVERSITY OF COPENHAGEN
DEPARTMENT OF COMPUTER SCIENCE

M.Sc. Thesis

Alexander Boccaletti

Multi-loop Feynman integrals in maximally symmetric gauge theory

Advisors: Chi Zhang Matthias Wilhelm

This thesis has been submitted to the...

Abstract

[English version of the Abstract]

Acknowledgements

[Acknowledgments]

Table of Contents

Chapter 1

Introduction

[General introduction Chapter.]

Chapter 2

Planar $\mathcal{N} = 4$ sYM theory

In this chapter we describe the framework of the loop computation subject of this paper, namely planar $\mathcal{N} = 4$ sYM theory. As introduced in the previous section, $\mathcal{N} = 4$ supersymmetric Yang-Mills (sYM) theory in the 't Hooft $N \to \infty$ limit of colours, in $n = 4$ spacetime dimensions, is among the most extraordinary ones [1]. The chapter initially introduces the properties of this theory, then outlines concepts relevant to the understanding of the overall computation.

2.1 Supersymmetry

Let's first elaborate on the concept of supersymmetry (SUSY), how it relates to other concepts in QFT, and how it is relevant to the computation being presented in this paper. Supersymmetry is a continuous spacetime symmetry realised by a set of transformations (or one such transformation) that map(s) bosons into fermions and fermions into bosons. Different supersymmetric theories contain different numbers of these transformations; we can classify SUSY theories by the number of transformation generators that transform spin-1 gauge fields to spin-1/2 fermionic fields and vice-versa. These operators relate particle states with different numbers of bosons and fermions. As known from the consideration of spacetime symmetries in QFT, the Poincaré-symmetries of 4-dimensional Minkowski-space $\mathcal{M}^4$ with metric $\eta_{\mu\nu} = \text{diag}(+---)$ are generated by spacetime translations T having infinitesimal generators P^μ and $O(1,3) \ni \Lambda$ Lorentz-transformations having infinitesimal generators $J_{\mu\nu} = -J_{\nu\mu}$. The Lorentz group is the group of transformations acting on $\mathcal{M}^4$ that preserves the lengths of 4-vectors, i.e. $O(1,3) = \{GL(4, \mathbf{R}) \ni \Lambda |\ (\Lambda x, \Lambda x) = (x, x), \forall x \in \mathcal{M}^4\}$. We restrict our consideration to the group of Lorentz-transformations that preserve the orientation of space and time; the proper ortochronous subgroup of $O(1,3)$, namely $SO(1,3)_+ = \{O(1,3) \ni \Lambda |\ {\Lambda_0}^0 \geq 1 \ and \ \det(\Lambda) = 1\}$ and refer to it from now on as the Lorentz group.

We can express any translation transformation in terms of the infinitesimal generator as: $T = \exp(ia^\mu P_\mu)$. Similarly for any Lorentz-transformation: $\Lambda = \exp(i\omega^{\mu\nu} J_{\mu\nu})$. The Poincaré-algebra is then generated by the translation generators and Lorentz- transformation generators through the commutator relations

$$[P_\mu, P_\nu] = 0 \tag{2.1}$$

$$[P_\mu, J_{\rho\sigma}] = -i(P_\rho \eta_{\mu\sigma} - P_\sigma \eta_{\mu\rho}) \tag{2.2}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(J_{\mu\rho}\eta_{\nu\sigma} - J_{\mu\sigma}\eta_{\nu\rho} - J_{\nu\rho}\eta_{\mu\sigma} + J_{\nu\sigma}\eta_{\mu\rho}) \tag{2.3}$$

These expressions can be further decomposed in a relativistically noncovarinat manner, at the level of the quantum mechanically conserved (commuting with $H = P^0$) momentum 3-vector P_i generating 3 translations and angular momentum 3-vector J_i generating 3 rotations and the nonconserved (H noncommuting) Lorentz-boosts 3-vector generating the boosts in the 3 spatial directions.[2] In supersymmetric extensions, the Poincaré-algebra is extended by the inclusion of the infinitesimal generators of supersymmetry (SUSY) transformations, the so-called spinor-supercharges (spinor operators) [3]

$$A = 1, 2, ..., \mathcal{N} \quad \begin{cases} Q^A_\alpha & \alpha = 1, 2 \\ \overline{Q}_{\dot{\alpha}A} = (Q^A_\alpha)^\dagger & \dot{\alpha} = 1, 2 \end{cases} \tag{2.4}$$

where $\mathcal{N}$ denotes the number of supercharges in the supersymmetric extension of the Poincaré-algebra, α is a left spinor index and $\dot{\alpha}$ right spinor index. For $\mathcal{N} = 1$ we have a so-called simple SUSY, whereas for $\mathcal{N} > 1$ the supersymmetry is called an extended supersymmetry[4]. The supercharges introduced here transform according to the spinor representation of the $SO(1, 3)$ Lorentz-group, commute with the Hamiltonian (properties of spinors are unchanged by time-translations) and obey the following anti-commutator relations

$$\{Q^A_\alpha, \overline{Q}_{\dot{\beta}B}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^A_B \tag{2.5}$$

where $\sigma^\mu_{\alpha\dot{\beta}}$ are the 4 2×2 Pauli matrices ($\mu = 0, 1, 2, 3$ and $\alpha, \dot{\beta} = 1, 2$) and have the anti-commutators

$$\{Q^A_\alpha, Q^B_\beta\} = 2\varepsilon_{\alpha\beta} Z^{AB} \tag{2.6}$$

with central charges Z^{AB} that are anti-symmetric in SUSY indices, i.e $Z^{AB} = -Z^{BA}$ (we couldn't have symmetric indices as that would make the expression 0). The central charges commute with every supersymmetry generator and for a simple SUSY must have that $Z = 0$. Similarly, for the adjoint spinor operator the anti-commutator is $\{\overline{Q}^A_{\dot{\alpha}}, \overline{Q}^B_{\dot{\beta}}\} = 2\varepsilon_{\dot{\alpha}\dot{\beta}} Z^{AB}$. The Poincaré-algebra sector of the super-Poincaré algebra holds and we can find out the new commutator relations in the supersymmetrically extended algebra. Since Q_α transforms according to the spinor representation of $SO(1, 3)$ we must have that

$$Q'_\alpha = \left(\exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\right)_\alpha^{\ \beta} Q_\beta \simeq \left(1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta = Q_\alpha - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu} Q_\alpha \tag{2.7}$$

where $\sigma^{\mu\nu} = \sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu$ and $\sigma^\mu = (\mathbf{1}, \vec{\sigma})$ with parity conjugate $\overline{\sigma}^\mu = (\mathbf{1}, -\vec{\sigma})$. On the other hand, for a representation of the super-Poincare algebra on a Hilbert space $\mathcal{H}$ we have the unitary transformation on the supercharge operator given by (no change on spinor indices as this is a unitary QM transformation on an operator, not a Poincaré-transformation)

$$Q'_\alpha = U^\dagger Q_\alpha U \simeq \left(1 - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) Q_\alpha \left(1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right) = Q_\alpha - \frac{i}{2}\omega_{\mu\nu}[J^{\mu\nu}, Q_\alpha] \tag{2.8}$$

Equating the two expressions for Q'_α we obtain the following super-Poincaré commutator relation

$$[Q_\alpha, J^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta \tag{2.9}$$

Following the same line of thought for $\overline{Q}_{\dot{\alpha}A}$ we obtain that

$$[\overline{Q}^{\dot{\alpha}}, J^{\mu\nu}] = (\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \overline{Q}^{\dot{\beta}} \tag{2.10}$$

With the commutator relations between the supersymmetry operators and the Lorentz-transformation generators given, we can now consider the translation infinitesimal generator sector of the supersymmetrically extended Poincaré-algebra. According to their structure, it follows that

$$[Q_\alpha, P^\mu] = c(\sigma^\mu)_{\alpha\dot{\alpha}} \overline{Q}^{\dot{\alpha}} \tag{2.11}$$

Similarly, for the adjoint supercharge

$$[\overline{Q}^{\dot{\alpha}}, P^\mu] = c^*(\overline{\sigma}^\mu)^{\dot{\alpha}\alpha} Q_\alpha \tag{2.12}$$

By using the Jacobi-identity for the commutators involving $P^\mu, Q_\alpha, \overline{Q}^{\dot{\alpha}}$ we get

$$\begin{aligned} 0 &= [P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] + [Q_\alpha, [P^\mu, P^\nu]] = \\ &= -c(\sigma^\nu)_{\alpha\dot{\alpha}}[P^\mu, \overline{Q}^{\dot{\alpha}}] + c(\sigma^\mu)_{\alpha\dot{\alpha}}[P^\nu, \overline{Q}^{\dot{\alpha}}] = \\ &= |c|^2(\sigma^\nu)_{\alpha\dot{\alpha}}(\overline{\sigma}^\mu)^{\dot{\alpha}\beta}Q_\beta - |c|^2(\sigma^\mu)_{\alpha\dot{\alpha}}(\overline{\sigma}^\nu)^{\dot{\alpha}\beta}Q_\beta = \\ &= |c|^2(\sigma^{\nu\mu})_\alpha^{\ \beta}Q_\beta \quad \Longrightarrow \quad c = 0 \end{aligned} \tag{2.13}$$

This result is consistent with the fact that the properties of spinors don't change if they are translated in space by an arbitrary space translation. So we obtain that

$$[Q_\alpha, P^\mu] = [\overline{Q}_{\dot{\alpha}}, P^\mu] = 0 \tag{2.14}$$

The relations obtained this way expand the original symmetry algebra to an extended symmetry algebra allowing us to relate fermionic particle states $|F\rangle$ and bosonic particle states $|B\rangle$ through a newly introduced symmetry of spacetime. By considering the massless particle representations of the super-Poincaré algebra, it can be understood that the supercharge operator(s) Q^A_α decrease helicity by $-1/2$ and the adjoint(s) increase it by $1/2$. So in general

$$Q|F\rangle \sim |B\rangle \qquad \overline{Q}|F\rangle \sim |B\rangle \tag{2.15}$$

$$Q|B\rangle \sim |F\rangle \qquad \overline{Q}|B\rangle \sim |F\rangle \tag{2.16}$$

Let's now consider the relationship of the newly introduced SUSY generators with another class of transformation generators in QFT, namely generators of internal symmetries. Internal symmetries are symmetries only acting on fields that don't transform points of spacetime and leave the Lagrangian of the theory and the physical results invariant to the transformation. As examples, we can list the global $U(1)$ symmetry of the complex scalar field theory, the local gauge symmetry of QED and the local $SU(3)$ gauge symmetry of QCD which we will describe with more detail in the next section. By a global internal symmetry, we mean that the symmetry operator of the transformation is independent of spacetime, so for example in global $U(1)$ symmetry $U(1) \ni e^{i\alpha}$ the α is constant in spacetime. On the other hand, for a local symmetry the operator in spacetime dependent, so for example in local $U(1)$ symmetry $U(1) \ni e^{i\alpha(x)}$ the $\alpha = \alpha(x)$ depends on spacetime.

Internal symmetry generators commute with all Poincaré-algebra generators. Furthermore, M_a are the generators of some internal transformation, then

$$[M_a, Q^A_\alpha] = 0 \tag{2.17}$$

$$[M_a, \overline{Q}_{\dot{\alpha}A}] = 0 \tag{2.18}$$

so internal symmetry generators commute with all of the super-Poincaré generators. This means that the spectrum for a SUSY theory must be arranged in pairs of the form: (boson, fermion) with the paired bosons and fermions having equal masses and internal quantum numbers. However, let's consider the following global $U(1)$ transformation on the SUSY operators with some real α

$$Q^A_\alpha \to e^{i\alpha} Q^A_\alpha \tag{2.19}$$

$$\overline{Q}_{\dot{\alpha}A} \to e^{-i\alpha}\overline{Q}_{\dot{\alpha}A} \tag{2.20}$$

This is a $U(1)$ symmetry between the supercharges, which is generated by some operator R that generates this internal symmetry. By transforming the operators according to this $U(1)_R$ transformation we obtain

$$Q^A_\alpha \to e^{-iR\alpha} Q^A_\alpha e^{iR\alpha} \tag{2.21}$$

Expanding and keeping only terms in $\mathcal{O}(\alpha)$ we obtain

$$Q^A_\alpha \to (1 - iR\alpha) Q^A_\alpha (1 + iR\alpha) \sim i\alpha Q^A_\alpha R - i\alpha R Q^A_\alpha \tag{2.22}$$

for this transformation to be a symmetry, we get the commutator condition

$$[Q^A_\alpha, R] = Q^A_\alpha \tag{2.23}$$

Proceeding similarly for the adjoint, we obtain

$$[\overline{Q}_{\dot{\alpha}A}, R] = -\overline{Q}_{\dot{\alpha}A} \tag{2.24}$$

Thus, the introduction of the SUSY generators has created an internal symmetry between them, known as R symmetry. In the case of extended supersymmetries with $\mathcal{N} > 1$, the theory also has a global $SU(N)_R$ R-symmetry, as the different supercharges indexed by $A = 1, ..., \mathcal{N}$ can be rotated into one another while leaving the theory invariant.

2.2 Supersymmetric Yang-Mills theory

By briefly presenting the supersymmetric extension of the Poincaré algebra, we are now in a position to discuss the possible supersymmetric extensions of Yang-Mills theories. Let's first however discuss what Yang-Mills theories are and what internal properties they possess.

We have briefly mentioned in the previous chapter that internal symmetries leave the action and physical results of the theory invariant. In the case of Yang-Mills theory, this internal symmetry is a local $SU(N)$ symmetry; the theory is invariant under local $SU(N)$ transformations of the fermionic and gauge fields of the theory, where

$$SU(N) = \{U \in GL(\mathbb{C}, N) |\ U^\dagger = U^{-1}\ and\ \det(U) = 1\} \tag{2.25}$$

The $SU(N)$ invariant Lagrangian is that of Yang-Mills theory and is given by

$$\mathcal{L}_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_{i,j=1}^{N} \overline{\Psi}_i \left(\delta_{ij} i\partial\!\!\!/ - \delta_{ij} m + g A\!\!\!/^a T^a_{ij}\right) \Psi_j \tag{2.26}$$

with non-Abelian field strength tensor given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{2.27}$$

Given spin-1/2 fermionic fields Ψ_j with $= 1, 2, ...N$, the Lagrangian (and therefore the action) is invariant under the local transformation

$$\Psi_j(x) \to [U(x)]_j{}^k \Psi_k(x) \qquad j = 1, 2, ..., N \tag{2.28}$$

where $U(x) = \exp(i\alpha(x)^a T^a)$ is an $SU(N)$ matrix and the Hermitian infinitesimal generators T^a of the transformation are $N \times N$ matrices with $a = 1, 2, ..., N^2 - 1$. Under the gauge transformation, the fermionic fields are transformed into each other by this $N \times N$ unitary matrix. The theory also contains gauge fields that mediate interactions between these fermionic fields. Since we demand the theory to be invariant under local $\Psi(x) \to U(x)\Psi(x)$ transformations, we must transform the gauge fields accordingly to maintain the invariance. This transformation on the massless spin-1 gauge fields is

$$A^a_\mu T^a \to U(x) A^a_\mu T^a U^{-1}(x) - \frac{i}{g}(\partial_\mu U(x))U^{-1} \tag{2.29}$$

Since $\mathfrak{su}(N) \ni T^a$, the transformation generators satisfy

$$[T^a, T^b] = i f^{abc} T^c \tag{2.30}$$

where f^{abc} are the structure constants of $\mathfrak{su}(N)$ and the infinitesimal generators of the transformation are normalised as $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$. For $N = 2$, we have 3 generators spanning the

Lie-algebra $\mathfrak{su}(2)$ in the fundamental representation, that are related to the Pauli-matrices as $T^a = \frac{1}{2}\sigma^a$. For $N=3$, there are 8 generators $T^a = \frac{1}{2}\lambda^a$, with λ^a being the Gell-Mann matrices known from QCD.

Having discussed the internal properties of Yang-Mills theories, we now consider the supersymmetric extension of YM and describe what happens when including SUSY in the picture. It is evident at this point that the theory in focus contains $\mathcal{N}=4$ spinor operators Q^A_α converting bosonic states into fermionic states. We know from the previous section that the central charges Z^{AB} commute with all symmetry generators, so we must have for any sYM theory that

$$[T^a, Z^{AB}] = 0 \tag{2.31}$$

Moreover, as seen earlier in the previous chapter we also have in general that

$$[Q^A_\alpha, internal\ generators] = [\overline{Q_{\dot{\alpha}}A}, internal\ generators] = 0 \tag{2.32}$$

The same applies for the Lie-algebra generators $\mathfrak{su}(N) \ni T^a$

$$[T^a, Q^A_\alpha] = [T^a, \overline{Q}_{\dot{\alpha}A}] = 0 \tag{2.33}$$

Consequently, we must have that superpartner particles carry the same colour charge and have and same other internal quantum numbers, but different spin. We only limit ourselves to the construction of a supersymmetric theory with spin states having spin ≤ 1. The motivation for this restriction is the No-Go theorem from Weinberg and Witten, stating that: a QFT defined in $n = 3+1$ spacetime dimensions with a non-0, conserved, Poincaré-covariant and gauge-invaraint (for unfixed gauges) stress-energy tensor $T^a_{\mu\nu}$ does not allow massless states with helicities $|h| > 1$. This requirement constrains the form of the theory uniquely, up to a choice of gauge group [6]. Remarkably, the maximal amount of SUSY generators we can have in $n=4$ dimensions to constrain the theory to maximum spin-1 is: $\mathcal{N}=4$. Hence, $\mathcal{N}=4$ sYM is the maximally supersymmetric $SU(N)$ gauge theory in 4 dimensions. The massless representations of the theory can be examined through (2.5) by choosing the Lorentz frame in which $P^\mu = (E,0,0,E)$ with $E > 0$. With maximal supersymmetry, one finds that there are 16 massless helicity states corresponding to 11 massless particles in this theory; the $\mathcal{N}=4$ gauge multiplet (A_μ λ^a_α φ^I) containing $1\times$ A_μ spin-1 gauge field, $4\times$ left Weyl-spinors labeled by $SU(4)$ indices $a = 1,2,3,4$ (we can "rotate" the A_μ into $\mathcal{N}=4$ different spinors as we have $4 \times Q_\alpha$) and $6\times$ real scalar fields φ^I with $I = 1,2,...,6$ labeling the $SO(6)_R \sim SU(4)_R$ global R-symmetry. The Lagrangian for a maximally supersymmetric $SU(N)$ gauge theory is unique and is given by

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}(&\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - i\overline{\lambda}^a\overline{\sigma}^\mu D_\mu\lambda_a - D_\mu\varphi^I D^\mu\varphi^I + \\ &+ gC^{ab}_I\lambda_a[\varphi^I, \lambda_b] + g\overline{C}_{I,ab}\overline{\lambda}^a[\varphi^I, \overline{\lambda}^b] + \frac{g^2}{2}[\varphi^I, \varphi^J]^2)\end{aligned} \tag{2.34}$$

where g, θ are couplings and C_I^{ab} represent the structure constants of the symmetry group $SU(4)_R$.

Let's now consider the symmetries of this Lagrangian and its action. Since the mass dimensions are: $[dx] = -1, [\partial] = [A_\mu] = [\varphi_I] = 1, [\lambda_a] = 3/2, [g] = [\theta] = 0$, each term in (2.34) has a mass dimension of 4 $\implies$ $[S] = 0$ as $S = \int d^4x\mathcal{L}$. Thus, the theory is scale-invariant, which when combined together with Poincaré-invariance discussed earlier yields that the theory is conformal invariant; is invariant with respect to $SO(2,4) \sim SU(2,2)$ conformal transformations. So this symmetry is generated by the usual Poincaré-generator of translations P^μ and Lorentz transformations $J^{\mu\nu}$, the dilation generator D and special conformal transformation generator K^μ given by

$$D = X_\mu P^\mu \tag{2.35}$$

$$K^\mu = -2X^\mu D + X^2 P^\mu + 2J^{\mu\nu} X_\nu \tag{2.36}$$

generating inverted translation transformations. As seen earlier, the Lagrangian has a $SU(4)_R$ symmetry which is manifest from the presence of the structure constants in the Weyl-scalar terms. Also, we have the Poincaré supersymmetry generated by the spinor supercharges Q_α^A. The supercharges don't commute with the generators of special conformal transformations, as

$$[K^\mu, Q_\alpha] = i(\sigma^\mu)_{\alpha\dot{\beta}} \overline{S}^{\dot{\beta}} \tag{2.37}$$

$$[K^\mu, \overline{Q}^{\dot{\alpha}}] = i(\overline{\sigma}^\mu)^{\dot{\alpha}\beta} S_\beta \tag{2.38}$$

Since K^μ is a symmetry generator and Q_α is a symmetry generator $\implies$ S_α is a symmetry generator, which generates the so-called conformal supersymmetries. Hence, conformal supersymmetries generated by S_α are also a symmetry of the theory. Altogether, these symmetry generators compose a larger symmetry algebra called the $\mathfrak{psu}(2,2|4)$ superalgebra, which determines that $\mathcal{N} = 4$ sYM is invariant under global $PSU(2,2|4)$ superconformal transformations.

Furthermore, a very important property of $\mathcal{N} = 4$ is that it is finite in the UV region of momenta; there are no UV divergences. Ultraviolet (UV) divergences arise when the momentum of virtual particles involved in a scattering process becomes very large. This occurs at high momenta because the virtual particles can probe arbitrarily short distances, which can lead to large fluctuations and divergences in the scattering amplitude. The perturbative quantisation of the theory yields amplitudes that are UV finite at any perturbative order in the coupling. This is reflected in the fact that the $\beta(\mu)$ function from the renormalisation group equation [Callan,Symanzik]

$$\beta(\mu) = \frac{dg(\mu)}{d\mu} = 0 \tag{2.39}$$

for any renormalisation scale μ, as the coupling g is actually constant; there is no running of the coupling. This is an intrinsic property of conformal theories (theories with 0 stress-energy tensor), in which the strength of particle interactions does not depend on the energy of the particles involved. On the other hand, the theory has divergences in the region of low momenta, which are called IR divergences. This happens because at low momenta, the virtual particles that mediate the scattering process can propagate over very long distances, leading to long-range interactions and non-local effects. The most common source of IR divergences is the exchange of massless particles, such as gluons in a theory of strong interactions. When the massless particle is exchanged between two charged particles, the virtual particle can have very low momentum, leading to a logarithmic divergence in the scattering amplitude. Regularisation and resummation techniques can be used in this circumstance to obtain finite and physically meaningful results containing no divergences.

The energy scale of supersymmetry breaking is estimated to be between [3]

$$100\ GeV \leq SUSY\ breaking \leq 1000\ GeV \tag{2.40}$$

Although these scale are accessible by the LHC, no particle associated to $\mathcal{N} = 4$, or other SUSY models has ever been detected. This may be because the energy scale is higher, so the SUSY particle masses are still out of reach. In $\mathcal{N} = 4$ sYM there are no massive particles in the conventional sense because the theory has a superconformal symmetry that restricts the allowed masses of particles to 0. However, the theory has bound states called Bogomolnyi-Prasad-Sommerfield (BPS) states that are stable and can carry non-zero charges and masses as the two are related to the BPS equations. These states that preserve a fraction of the supersymmetry of the theory are protected from decaying into other states by giving a lower bound on the mass they can have with respect to the central charges, called the BPS bound. Having remarked on this, we can mention two interesting examples: the supersymmetric partner of the gluon called the gluino and the superpartner of the up, the sup. The gluino must interact via the strong force as a colour octet meaning that it can carry one of eight possible colour charges along with having 0 electric charge, just like the gluon. However, it has spin-1/2 as it is a Majorana-fermion. The sup is the superpartner of the up quark, namely the sup quark. Based on the above it must have the same quantum numbers as the up, so an electric charge of $2/3e$. Again, it's spin is different as the sup if a 0-spin scalar boson. The masses of superpartner particles depend on the specific details of the theory, such as the coupling strengths and the energy scale at which supersymmetry is broken. On the other hand, it may be even possible that Nature does not make use of supersymmetry. Nevertheless, $\mathcal{N} = 4$ is an outstanding toy model to approach more difficult and challenging problems in QFT [simplest QFT].

2.3 Amplitudes' total quantum number management

Having described the basic structure and properties of $\mathcal{N} = 4$ sYM, we are now in the position to focus on the main objects of interest in this paper, namely scattering amplitudes. First we present some issue associated with the usual off-shell formalism presented in a classic QFT textbook. Then we present some new formalisms for working around these issues and working directly at the level of the S-matrix alone without having to deal with redundant degrees of freedom.

Scattering amplitudes are central objects in fundamental physics: they are crucial for connecting theory to experiments in particle accelerators such as Large Hadron Collider and play a central role in discovering new structures of Quantum Field Theory (QFT), even gravity[Chi]. The amplitudes are defined by evaluating the scattering matrix (S-matrix) between the initial and final particle states of positive energy. Since collider experiments measure the differential cross section $d\sigma/d\Omega$ of some process and

$$\frac{d\sigma}{d\Omega} \sim |M|^2 \tag{2.41}$$

their importance is hard to overstate. Let's now briefly consider the structure of these scattering amplitudes. They are functions of all incoming n particle momenta and they can be expressed completely symmetrically in terms of external particles (legs) as

$$M^{a_1 \cdots a_n}(p_1, ..., p_n) \tag{2.42}$$

with a_i denoting little group indices. A brief remark: the group of transformations leaving the momentum of on-shell particles invariant is called the Wigner little group. Hence, these indices are associated with the irreducible $E \geq 0$ unitary representations of the Poincaré-group leaving the on-shell momentum invariant, where the mass eigenstates are used to label these irreducible representations of the little group elements. If p_1 denotes the momentum of particle-1, then $D[W(\Lambda, p_1)]_a^b$ denotes the irreducible unitary representation of the Lorentz-group leaving the *on-shell* momentum of particle-1 invariant. Thus, under some Lorentz-transformation Λ (change of frame in $\mathcal{M}^4$) the amplitude must transform as '

$$M^{a_1 a_2 \cdots a_n}(p_1, p_2, ..., p_n) = D[W(\Lambda, p_1)]_{b_1}^{a_1} D[W(\Lambda, p_2)]_{b_2}^{a_2} \cdots D[W(\Lambda, p_n)]_{b_n}^{a_n} M^{b_1 b_2 \cdots b_n}(\Lambda p_1, \Lambda p_2, ..., \Lambda p_n) \tag{2.43}$$

Unfortunately, amplitudes calculated using the usual Feynman rules do not evaluate to amplitudes of the form (2.42). The bias in the description of manifesting Lorentz invariance requires us to describe a massless spin-1 particle with some field A_μ (as it has to transform properly). Although it has 2 possible polarisation states in 4d, we instead create a redundant description to satisfy this bias of making Lorentz invariance manifest. Then, gauge redundancies are needed to cancel out these unphysical degrees of freedom to obtain only the relevant physical ones. We can consider as an example a naively calculated amplitude involving massless spin-1 particles. The amplitude will have Lorentz indices: $M^{\mu_1\mu_2\cdots\mu_n}$. It is impossible to introduce objects with one Lorentz and one little group index, so instead we can define objects that transform properly under Lorentz transformations but become 0 when contracted with $M^{\mu_1\mu_2\cdots\mu_n}$. This introduces redundancy in the description, which needs to be fixed in order to be eliminated. We already have seen a case of such a gauge redundancy in (2.29), where the YM Lagrangian was invariant under gauge transformations of the gluon field. The discussion may falsely indicate that this redundancy originates from spin, but in reality, it originates from the action principle. Seemingly different actions may actually lead to same S-matrix. To see an example of this, let's consider the following real massless scalar field theory defined by the Lagrangian

$$\mathcal{L} = W(\phi)\partial_\mu\phi\partial^\mu\phi, \qquad W(\phi) = 1 + \lambda_1\phi + \frac{\lambda_2}{2!}\phi^2 + \cdots + \frac{\lambda_k}{k!}\phi^k \tag{2.44}$$

We calculate the 4-point amplitude using the usual path integral formalism. Assuming small coupling g and using the notation $D(x-y) = D_{xy}, J(x) = J_x, D_{xy}J_y = \int d^4y D(x-y)J(y)$

$$\begin{aligned} -iM_4 = -i\langle\Omega|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\Omega\rangle = \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4}\left(\frac{\delta}{\delta J_x}\right)^2\left(\partial\frac{\delta}{\delta J_x}\right)^2 e^{-\frac{1}{2}J_z D_{zy} J_y} = \\ = g(D_{x_1}D_{x_2}\partial D_{x_3}\partial D_{x_4} + D_{x_2}D_{x_3}\partial D_{x_4}\partial D_{x_1} + D_{x_3}D_{x_4}\partial D_{x_1}\partial D_{x_2} + \\ + D_{x_2}D_{x_3}\partial D_{x_4}\partial D_{x_1} + D_{x_3}D_{x_4}\partial D_{x_1}\partial D_{x_2} + D_{x_4}D_{x_1}\partial D_{x_2}\partial D_{x_2}) \end{aligned} \tag{2.45}$$

We know that $D(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik\cdot(x-y)}$ and by imposing the *physical* conservation of momentum and on-shell conditions

$$\sum_i p_i = 0 \tag{2.46}$$

$$p_i^2 = 0 \tag{2.47}$$

we find that the amplitude is

$$M_4 \sim \sum_{i\neq j} p_i \cdot p_j \sim s + t + u = 0 \tag{2.48}$$

with Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{2.49}$$

$$t = (p_2 + p_3)^2 = (p_1 + p_4)^2 \tag{2.50}$$

$$u = (p_1 + p_3)^2 = (p_2 + p_4)^2 \tag{2.51}$$

Interestingly, this is true for all amplitudes in the theory as the theory is the free scalar theory transformed by some smooth function f. With a suitable $\phi \to f(\phi)$ transformation, a non-symmetry of the action is a symmetry of the S-matrix. As this example illustrates, the S-matrix knows more about the symmetry of the theory than the action. Hence, it makes more sense to work directly on the level of the S-matrix with gauge invariant on-shell amplitudes free of these overcomplicated redundancies. That is what we are going to do from now on and for that reason, we introduce two formalisms for working around these redundancies in the description, namely the spinor-helicity formalism and colour-ordering.

The 4-momenta p_i^μ used to express our amplitudes transform under the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz-group. We can equivalently represent these momenta as 2×2 Hermitian matrices as the vector space of the 2×2 Hermitian matrices is isomorphic to $\mathcal{M}^4$

$$p_i^\mu \to (p_i)_{\alpha\dot{\alpha}} = p_i^\mu (\sigma_\mu)_{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 + ip^2 & p^0 + p^3 \end{pmatrix} \tag{2.52}$$

with $\det(p) = p^2 = m^2$, which is a scalar; a Lorentz-invariant quantity. Moreover, in high-energy processes fermions are ultra-relativistic and behave as massless particles so without further ado, we set $m = 0$. Since now $\det(p_i) = p^2 = 0$, we can express the 2×2 Hermitian matrices as the product of $1\times$ Left (L) and $1\times$ Right (R) spinor

$$p_i^\mu \to (p_i)^{\alpha\dot{\alpha}} = p_i^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \tag{2.53}$$

so that the on-shell momentum constraint $p_i^2 = 0$ $(\forall i = 1, ..., n)$ is trivialised. So instead of working with $4d$ quantities transforming under the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz-group, we work with $2 \times 2d$ quantities transforming under the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz-group. At the level of the Dirac-equation

$$(-i\not{\partial} + m)\Psi(x) = 0 \tag{2.54}$$

with $\Psi(x) \sim u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x}$ and setting $m = 0$ we obtain the Weyl-equations

$$\tag{2.55}$$

Chapter 3

Approaching the double-pentagon compuation

3.1 Momentum twistors as the natural set of variables for loop integrals

3.2 Polylogarithms and the symbol

3.3 Properties and representations of one-loop/two-loop Feynman integrals

3.4 Reduction of two loop integrals to one loop integrals

Chapter 4

The computation of I_{dp}

4.1 Reduction from double-pentagon to hexagon

4.2 Passarino-Veltman reduction of the hexagonal integral

4.3 Integration at the symbol level

Chapter 5

Results and conclusion

5.1 Resultant rational letters

5.2 Resultant algebraic letters

5.3 Conclusion

List of Publications

The work presented in this thesis has lead to the following publications.

1. John Doe. Example entry. *Some Awesome Journal*, 2023.
2. John Doe. Example entry. *Some Awesome Journal*, 2023.
3. John Doe. Example entry. *Some Awesome Journal*, 2023.

Bibliography

John Doe. Example entry. *Some Awesome Journal*, 2023.